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Simplicity of Neretin’s group of spheromorphisms


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SIMPLICITY OF NERETIN’S GROUP
OF SPHEROMORPHISMS

by Christophe KAPOUDJIAN

Introduction.

Answering a question of I.M. Gelfand on the existence of analogues of highest-weight representations of the diffeomorphism group of the circle in the case of $p$-adic transformation groups, Yu.A. Neretin constructed a group of transformations of the boundary $\partial T_p$ of the regular tree $T_p$ (cf. [12] and [13]): the group $N_p$ of spheromorphisms ($\S$1). When $p$ is a prime integer, the boundary $\partial T_p$ is naturally homeomorphic to the projective line on the field of $p$-adic numbers, and in any case, to a Cantor set.

Roughly speaking, a spheromorphism is a transformation induced in the boundary by a “piecewise” tree automorphism. The spheromorphism group is generated by two groups: on the one hand a Higman-Thompson group ($\S$2), which is countable and almost-acts on the tree, respecting a local orientation of the edges, and on the other hand, the tree automorphism group ($\S$3).

Exploiting simplicity theorems known for the generating two groups, and adapting some arguments of a simplicity theorem of Epstein, we finally prove the simplicity of $N_p$ (the analogue of M.R. Herman’s theorem on the simplicity of the orientation-preserving diffeomorphism group of the circle, cf. [7]), and of some of its subgroups ($\S$4):

Keywords: Cantor set – Higman-Thompson groups – $p$-adic numbers – Simple groups – Spheromorphism – Tree – Tree automorphism group.

THEOREM. — For each integer \( p \geq 2 \), the spheromorphism group \( N_p \) is simple.

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1. The Neretin group of spheromorphisms.

1.1. Let \( T_n \) be the regular tree whose vertices have valence \( n+1 \), with \( n \geq 2 \), and \( \partial T_n \) its boundary, or set of “ends”, see e.g. [14] or [6].

We may describe the boundary \( \partial T_n \) as a compact ultrametric space: choose a vertex \( o \) of the tree \( T_n \). Each end is defined by a unique chain (i.e. a sequence of consecutive vertices \( (o = x_0, x_1, \ldots) \) with \( x_{i+2} \neq x_i \)) starting from the origin \( o \). The metric on \( \partial T_n \) is defined in the following way: Let \( \omega, \omega' \in \partial T_n \) be respectively represented by the chains \( (o = x_0, x_1, \ldots) \) and \( (o = x'_0, x'_1, \ldots) \).

- If the intersection of the supports of the chains is reduced to \( \{o\} \), then declare the distance between \( \omega \) and \( \omega' \) to be equal to 1: \( d(\omega, \omega') = 1 \).

- If \( x_i = x'_i \) for \( i = 0, \ldots, k \) and \( x_{k+1} \neq x'_{k+1} \), then define \( d(\omega, \omega') = \frac{n}{n+1} n^{-k} \).

It follows that a closed ball of radius \( \frac{n}{n+1} n^{-k} \) is the set of all points of \( \partial T_n \) represented by chains containing a fixed finite chain \( (o = x_0, x_1, \ldots, x_k) \), and that it is an open set. In fact, \( \partial T_n \) endowed with the metric \( d \) is a compact ultrametric space, homeomorphic to a Cantor set.

When \( p \) is prime, \( T_p \) is the Bruhat-Tits building of the \( p \)-adic Lie group \( SL_2(\mathbb{Q}_p) \), just as the Poincaré disk \( D \) is the symmetric space of the real group \( SL_2(\mathbb{R}) \). The boundary \( \partial T_p \), which can be identified with \( \mathbb{Q}_p P^1 \), the projective line on \( \mathbb{Q}_p \), may thus be viewed as the \( p \)-adic analogue of the circle.

1.2. Let \( \partial T_n \) still denote the boundary of the tree \( T_n \), \( n \geq 2 \). The group of spheromorphisms \( N_n \) can be defined as the group of transformations of \( \partial T_n \) induced by “piecewise” tree automorphisms:

Take a finite subtree of \( T_n \). Its complementarily has finitely many connected components \( L_1, \ldots, L_k \), called branches, all isomorphic to an infinite \( n \)-ary complete rooted tree. A subset \( \partial L \) of the boundary is
naturally associated to each branch $L$: it consists of all the ends represented by the chains running over this branch. The $k$ disjoint sets $\partial L_j$, $j = 1, \ldots, k$ cover the boundary. We call $(L_1, \ldots, L_k)$ a broom.

**Remark.** — Each ball for the metric $d$ is of the form $\partial L$, and each $\partial L$ is a finite union of balls. The family $\{\partial L : L \text{ branch}\}$ is a basis of closed-open sets for the topology defined by $d$.

Let $(L_1, \ldots, L_k)$ and $(L'_1, \ldots, L'_k)$ be two brooms of $T_n$, $\sigma$ a permutation of $\{1, \ldots, k\}$. Let $\phi_j : L_j \to L'_{\sigma(j)}$ be a rooted tree isomorphism, $j = 1, \ldots, k$. These $k$ mappings induce a bijection $\phi = (\phi_j : \partial L_j \to \partial L'_{\sigma(j)})_{j=1,\ldots,k}$ of the boundary. Such a broom appearing in the definition of $\phi$ is called $\phi$-adapted, and is obviously not uniquely associated to $\phi$. It is clear that the set of all the $\phi$'s defined by this procedure is a group of homeomorphisms of the boundary.

**DEFINITION 1.1 (Spheromorphism group, [13]).** — For each $n \geq 2$, the set of all bijections $\phi = (\phi_j : \partial L_j \to \partial L'_{\sigma(j)})_{j=1,\ldots,k}$ of the boundary $\partial T_n$ is the spheromorphism group of Neretin, and is denoted $N_n$.

**Remarks.** — 1) In view of this description, the automorphism group $\text{Aut} T_n$ of the tree embeds as a subgroup of $N_n$. The image of $\text{Aut} T_n$ in $N_n$ is the set of spheromorphisms which possess an adapted broom with two branches.

2) When $p$ is a prime integer, $\partial T_p$ is homeomorphic to $\mathbb{Q}_p \mathbb{P}^1$, and $N_p$ contains the group $A_n p$ of locally analytic bijections of $\mathbb{Q}_p \mathbb{P}^1$ (see [13]).

### 2. Higman-Thompson groups.

#### 2.1. Definition of Higman-Thompson groups.**In 1965, R.J. Thompson, interested in finitely presented groups with non-solvable word problem, introduced a group (denoted $G_{2,1}$ in the following) which happened to be the first known example of finitely generated infinite simple group [11]. Thompson’s group was later generalized by G. Higman ([8]). For the description of the Higman-Thompson groups, we refer to [2]. See also [4].

Recall that a finite $n$-ary rooted planar tree is a finite tree $T$ with root $x$ realized in the oriented plane such that
If $T$ is not reduced to $x$, the valence of $x$ is equal to $n$.

The valence of a vertex $v \neq x$ is equal to 1 or $n+1$: if the valence of $v$ is 1, we call $v$ a leaf of the tree; if it is equal to $n+1$, $v$ has $n$ adjacent edges not contained in the geodesic joining the root $x$ to $v$. We realize them by drawing them down from the vertex $v$. We order them from the left to the right and label their terminal vertices (opposite to $v$) $\alpha_0(v), \ldots, \alpha_{n-1}(v)$.

The set of leaves of a finite $n$-ary rooted tree $T$ is called a basis and is denoted $B_T$.

\begin{center}
\begin{tikzpicture}
  \node (x) at (0,0) {$x$};
  \node (a0) at (1,-1) {$\alpha_0(x)$};
  \node (a1) at (-1,-1) {$\alpha_1(x)$};
  \node (a0a1) at (0,-2) {$\alpha_0\alpha_1(x)$};
  \node (a1a1) at (0,-2) {$\alpha_1\alpha_1(x)$};
  \draw (x) -- (a0);
  \draw (x) -- (a1);
  \draw (a0) -- (a0a1);
  \draw (a1) -- (a1a1);
\end{tikzpicture}
\end{center}

**Case $n = 2$**

\begin{definition}
A simple expansion of a finite $n$-ary rooted tree $T$ is any finite $n$-ary rooted tree $T'$ obtained by the following procedure:

- Choose a vertex $v$ in the base $B_T$.
- Make an expansion of $v$ by drawing $n$ edges down from it.

We get a new tree $T'$ whose basis $B_{T'}$ is deduced from $B_T$ by replacing $v$ by $\alpha_0(v), \ldots, \alpha_{n-1}(v)$.

An expansion $T'$ of $T$ is a tree obtained from $T$ by making finitely many successive simple expansions. Any two trees $T_1$ and $T_2$ always possess a common expansion.

The elements of the Higman-Thompson groups will be represented by “symbols”:

**Definition 2.2 (symbols).** Consider a pair $(T_1, T_2)$ of finite $n$-ary rooted trees with basis having the same cardinality. Let $\sigma : B_{T_1} \rightarrow B_{T_2}$ be a bijection from the basis of the first tree to the basis of the second one. We call the triple $(T_1, T_2, \sigma)$ a symbol.

A simple expansion of a symbol $(T_1, T_2, \sigma)$ is any symbol $(T'_1, T'_2, \sigma')$ thus obtained:

- $T'_1$ is a simple expansion of $T_1$, deduced from $T_1$ by expanding a vertex $v \in B_{T_1}$.
- $T'_2$ is the expansion of $T_2$ realized from the vertex $\sigma(v)$.  

• \( \sigma^\prime : B_{T_1} \to B_{T_2} \) is defined by

\[
\sigma^\prime\big|_{B_{T_1}\setminus\{v\}} = \sigma\big|_{B_{T_1}\setminus\{v\}},
\]

\[
\sigma^\prime(\alpha_i(v)) = \alpha_i(\sigma(v)) , i = 0, \ldots, n - 1.
\]

An expansion \((T_1', T_2', \sigma')\) of the symbol \((T_1, T_2, \sigma)\) is obtained from the latter by making finitely many simple expansions.

Declare now that \((T_1, T_2, \sigma)\) and \((T_1', T_2', \sigma')\) are equivalent if they possess a common expansion.

All the necessary vocabulary has been introduced to set the following:

**Definition 2.3 (Higman-Thompson groups).** — The set of equivalence classes of symbols \([[(T_1, T_2, \sigma)]\) form a set \(G_n\) endowed with the following group structure:

Two elements \([[(T_1, T, \sigma)]\) and \([[(T', T_2, \sigma')]\) being given, at the price of making expansions of their representing symbols, it may be supposed that \(T = T'\). Then \(\sigma'\sigma : B_{T_1} \to B_{T_2}\) can be defined, and we set

\[
[[T_1, T, \sigma]] [[T_2, T, \sigma']] = [[T_1, T_2, \sigma'\sigma]],
\]

since it is easy to check that this definition is independent of the chosen symbols.

The neutral element is \([[(T, T, \sigma = \text{id})]\) represented by any symbol \((T, T, \sigma = \text{id})\).

The inverse of \([[(T_1, T_2, \sigma)]\) is \([[(T_2, T_1, \sigma^{-1})]\).

The group \(G_n\) belongs to the family of Higman-Thompson groups.
Example \((n = 2)\).

\[
A = \begin{bmatrix}
1 & , & 1 & , & \sigma(i) = i \\
2 & 3 & 1 & 2 & \\
\end{bmatrix} \quad B = \begin{bmatrix}
1 & , & 1 & , & \sigma'(i) = i \\
2 & 3 & 4 & 2 & 3 \\
\end{bmatrix}
\]

\[
AB = \begin{bmatrix}
1 & , & 1 & , & \sigma''(i) = i \\
2 & 3 & 5 & 4 & \\
\end{bmatrix}
\]

Recall that the leaves of a tree \(T\) (i.e. the vertices in \(B_T\)) are always labelled from the left to the right. Let \((T, T', \sigma)\) be a symbol, and \(\sigma : B_T = \{v_1, \ldots, v_k\} \to B_{T'} = \{v'_1, \ldots, v'_k\}\). There exists a unique permutation \(\tau \in S_k\) such that

\[
\sigma(v_i) = v'_\tau(i) \quad \forall i = 1, \ldots, k.
\]

Then define \(\theta(\sigma) = \epsilon(\tau)\) the signature of \(\tau\). An easy calculation shows that if \((\tilde{T}, \tilde{T}', \tilde{\sigma})\) is a simple expansion of the symbol \((T, T', \sigma)\), then

\[
\theta(\tilde{\sigma}) = \theta(\sigma)(-1)^{n-1},
\]

so that when \(n\) is an odd integer, \(\theta(\sigma)\) is independent of the chosen symbol, and we get the group epimorphism

\[
\theta : G_n \to \mathbb{Z}/2\mathbb{Z}
\]

\[
\theta([\{T, T', \sigma\}]) = \epsilon(\tau).
\]

**Generalization.** Let \(r \geq 1\) be a fixed integer. First consider pairs of \(r\)-uplets of finite \(n\)-ary rooted trees \(((T_1, \ldots, T_r), (T'_1, \ldots, T'_r))\), and bijections \(\sigma\) from \(B_{T_1} \cup \ldots \cup B_{T_r}\) to \(B_{T'_1} \cup \ldots \cup B_{T'_r}\). (We do not ask \(\sigma\) to map \(B_{T_i}\) onto \(B_{T'_i}\)). We always suppose the \(r\)-uplet of trees to be ordered from the left \((T_1)\) to the right \((T_r)\). Any triple \(((T_1, \ldots, T_r), (T'_1, \ldots, T'_r), \sigma)\) is called an \(r\)-symbol. Similarly to the case \(r = 1\), we define the group \(G_{n,r}\) where the elements are represented by \(r\)-symbols. Of course, \(G_{n,1} = G_n\).

As in the case \(r = 1\), the morphism \(\theta : G_{n,r} \to \mathbb{Z}/2\mathbb{Z}\) can be defined provided \(n\) is odd. We set \(G'_{n,r} = \text{Ker}\theta\). If \(n\) is even, we agree that \(G'_{n,r} = G_{n,r}\). We are now ready to cite the simplicity theorem:
THEOREM 2.1 ([2]). — The group \( G'_{n,r} \) is the commutator subgroup of \( G_{n,r} \), and every non-trivial subgroup normalized by \( G'_{n,r} \) contains it. In particular, \( G_{n,r} \) is simple if \( n \) is even, and if \( n \) is odd, \( G_{n,r} \) contains a simple group of index 2, namely \( G'_{n,r} = [G_{n,r}, G_{n,r}] \).

2.2. Embedding of \( G_{n,1} = G_n \) and \( G_{n,2} \) into the Neretin group \( N_n \). The finite \( n \)-ary rooted trees we used in the definition of the Higman-Thompson groups may be canonically embedded in a chosen branch \( L \) of the regular tree \( T_n \), by simply completing the finite tree to an infinite \( n \)-ary rooted tree and then, identifying it to the branch \( L \). Denote by \( L' \) the branch opposite to \( L \) in \( T_n \) (linked to \( L \) by an edge). Each \( g \in G_{n,1} \), defined by a symbol \((T_1, T_2, \sigma)\), induces a spheromorphism \( \tilde{g} \) in an obvious way: if \((v_1^1)\) (resp. \((v_2^2)\)) are the leaves of \( T_1 \) (resp. \( T_2 \)), denote by \( L_1^1 \) (resp. \( L_2^2 \)) the subbranch of \( L \) whose root is \( v_1^1 \) (resp. \( v_2^2 \)). Then \( \tilde{g} \) is induced on \( \partial L \) by the collection \((L_i^1 \cong L_i^2)\); the isomorphisms respecting the left-to-right order of the edges of the branches. On \( \partial L' \), one imposes \( \tilde{g}|_{\partial L'} = \text{id}|_{\partial L'} \). The embedding

\[ G_{n,1} \hookrightarrow N_n \]

is now obtained.

On the other hand, we need the two branches \( L \) and \( L' \) like above to realize \( G_{n,2} \) in \( N_n \). Each \( g \in G_{n,2} \) will induce a spheromorphism by a procedure analogous to the previous one. It will appear in the following that, as far as we are concerned with the Neretin group \( N_n \), \( G_{n,2} \) is more relevant than the group \( G_{n,1} = G_n \) itself.

3. The group \( \text{Aut} T_n \) of automorphisms of the tree \( T_n \), \( n \geq 2 \).

3.1. Simplicity theorem. In [15], the author gave a theorem of simplicity of a class of groups of automorphisms of a tree:

DEFINITION 3.1. — Let \( A \) be a tree, \( G \) be a group of automorphisms of \( A \), \( C \) be a (finite or infinite) chain of \( A \), and \( F \) the fixator of \( C \) in \( G \). For each vertex \( x \) of \( A \), let \( \pi(x) \) be the nearest vertex from \( x \) in \( C \). For each vertex \( s \) of \( C \), the set \( \pi^{-1}(s) \) (which constitutes a subtree of \( A \)) is invariant under the action of \( F \); denote by \( F_s \) the group of permutations of this set induced by \( F \). There is a natural homomorphism

\[ F \rightarrow \prod_{s \in \text{Vert}(C)} F_s, \]
where $\text{Vert}(C)$ denotes the set of vertices of $C$.

We say that the group $G$ possesses the property (P) if the homomorphism (1) is an isomorphism for all chains $C$ (i.e. the actions of $F$ on the sets $\pi^{-1}(s)$ are independent from each other).

For example the group of all automorphisms of $A$ possesses the property (P).

**Theorem 3.1 (J. Tits).** — Let $A$ be a tree, $G$ be a group of automorphisms of $A$, and $G^+$ be the subgroup generated by the stabilizers of the edges of $A$ in $G$. Suppose that $G$ possesses the property (P), conserves no proper non-empty subtree of $A$ and fixes no end of $A$. Then each subgroup of $G$ normalized by $G$ and not reduced to the identity contains $G^+$. In particular, $G^+$ is a simple group or is reduced to the identity.

**Example 1.** — $A = T_n$, $n \geq 2$, $G = \text{Aut} T_n$. It happens that $G^+ = \text{Aut}^+ T_n$ coincides with the group of type-preserving automorphisms of the tree. So $\text{Aut}^+ T_n$ is a simple group, of index 2 in $\text{Aut} T_n$.

**Example 2.** — Equipped Bruhat-Tits trees.

Let $p \geq 2$ be a prime integer. In [13], the author defines an equipment on the tree $T_p$ as the specification, for each vertex $v$, of a labelling of its adjacent edges $(l''_0, \ldots, l''_{p-1}, l''_\infty)$ by the points of $\mathbb{F}_p P^1$. If $v$ and $v'$ are linked by an edge $l = l''_i = l''_j$, there is no reason that $i = j$.

We denote by $\widetilde{T}_p$ such an equipped tree, and define the subgroup $\text{Aut} \widetilde{T}_p$ of $\text{Aut} T_p$ as the set of tree automorphisms such that their restrictions to the adjacent edges of a vertex belong to $\text{PSL}_2(\mathbb{F}_p)$. Since $\text{Aut} \widetilde{T}_p$ obviously satisfies property (P), conserves no proper non-empty subtree of $T_p$ and fixes no end, the group $(\text{Aut} \widetilde{T}_p)^+$ is simple.

Two equipped trees $\widetilde{T}_p^1$ and $\widetilde{T}_p^2$ being given, we use the transitivity of $\text{SL}_2(\mathbb{F}_p)$ on $\mathbb{F}_p P^1$ to construct a tree isomorphism $\widetilde{T}_p^1 \rightarrow \widetilde{T}_p^2$ respecting the equipments. Such an isomorphism conjugates $\text{Aut} \widetilde{T}_p$ and $\text{Aut} \widetilde{T}_p$.

**3.2. A family of subgroups of $N_n$.**

**Definition 3.2.** — If $G$ is a subgroup of $\text{Aut} T_n$ we define

$$(N_n)_G := < G_{n,2}, G^+ >$$

the subgroup of $N_n$ generated by $G_{n,2}$ and $G^+$. 
Example 1. — If $G = \text{Aut} T_n$, $(N_n)_G = N_n$. In this case, we can even show:

**Proposition 3.1.** — The subgroups $[G_n, G_n]$ and $\text{Aut}^+ T_n$ of the group $N_n$, $n \geq 2$, generate the group $N_n$.

**Proof.** — Let us denote by $L$ the chosen branch of the tree $T_n$ where we realized the Higman-Thompson group $G_n$. If $L'$ is the branch opposite to $L$ (i.e., linked with $L$ by an edge), then the boundaries of $L$ and $L'$ partition the whole boundary of the tree: $\partial L \cup \partial L' = \partial T_n$.

**First case.** — Suppose that $\phi \in N_n$ possesses a broom $(L_i)_{i=1, \ldots, I}$ such that $\phi|_{\partial L_i} = \text{id}|_{\partial L_i}$. At the price of making an expansion of $L_1$, one can suppose that $L_1$ and $L'$ have the same type (i.e. their roots have the same type). Then there exists $k \in \text{Aut}^+ T_n$ such that $k(L') = L_1$. So $k^{-1}\phi k|_{\partial L'} = \text{id}|_{\partial L'}$. Let us now consider $k^{-1}\phi k|_{\partial L}$. It may be seen as the composite

$$\partial L \xrightarrow{\tau} \partial L \xrightarrow{\sigma} \partial L$$

with $\tau \in G_n$ and $\sigma \in \text{Aut}^+ T_n$, $\sigma|_{\partial L'} = \text{id}|_{\partial L'}$. Then on the whole boundary $\partial T_n$, $k^{-1}\phi k = \sigma\tau$.

When $n$ is odd, $\text{Aut}^+ T_n \cap (G_n \setminus [G_n, G_n]) \neq \emptyset$, so that it can be supposed that $\tau \in [G_n, G_n]$.

**Second case: general case.** — (a) Suppose there exists $L_i$ in the broom adapted to $\phi$ such that $\partial L_i$ and $\phi(\partial L_i) = \partial L_i'$ have the same type. Then there exists $k \in \text{Aut}^+ T_n$ such that $k\phi(\partial L_i) = \partial L_i$ and $k \circ \phi|_{\partial L_i} = \text{id}|_{\partial L_i}$. The first case enables to conclude.

(b) If not, for all $i$, the types of $\partial L_i$ and $\phi(\partial L_i)$ are opposite. Then we use an element $\tau_0$ of $G_n$ (it is possible to find it of the form $[\tau_1, \tau_2]$) such that for some branch $L_0$, $\tau_0(L_0)$ and $L_0$ have opposite types. At the price of making an expansion of $L_1$ to make $\phi(\partial L_1)$ and $\partial L_0$ have the same type, there exists some $k \in \text{Aut}^+ T_n$ such that $k\phi(\partial L_1) = \partial L_0$. The types of $L_1$ and $L_0$ are still opposite. Then $\tau_0 k\phi(\partial L_1) = \tau_0(\partial L_0) = \partial L_0'$, and the types of $L_1$ and $L_0'$ coincide. Hence $\tau_0 k\phi$ satisfies the condition of case (a).

It follows that $\phi$ may be written as a product of elements of $G_n$ and $\text{Aut}^+ T_n$.

Example 2. — Now $p$ is a prime integer. Let $\widetilde{T}_p$ be any equipment on the tree $T_p$ such that the elements of $G_{p,2}$ are induced by piecewise tree
automorphisms of Aut $\tilde{T}_p$ (cf. §3.1, Example 2).

If $G = \text{Aut } \tilde{T}_p$, then we claim that $(N_p)_G$ is the group denoted $\text{Diff}^+(\tilde{T}_p)$ in [13]:

$$\text{Diff}^+(\tilde{T}_p) = \{ \phi = (\phi_j : L_j \to L'_j)_{j}, \quad \phi_j = \text{restriction of some element of } \text{Aut } \tilde{T}_p \}.$$ 

Indeed, $\text{Diff}^+(\tilde{T}_p)$ contains $G$, and because of the condition on the equipment, it contains $G_{p,2}$. So, $<G, G_{p,2}> \subset \text{Diff}^+(\tilde{T}_p)$. On the other hand, every $\phi \in \text{Diff}^+(\tilde{T}_p)$ can be written $\phi = \psi \circ \tau$, where $\tau = (L_j \to L'_j)_{j}$ belongs to $G_{p,2}$, and $\psi = (\psi_j = L'_j \to L'_j)_{j}$, with $\psi_j$ induced by some element of $G$, which can be modified to be supported in the branch $L'_j$. It follows that $\psi_j \in G^+$, and $\psi = \prod \psi_j \in G^+$. Thus

$$<G, G_{p,2}> \subset \text{Diff}^+(\tilde{T}_p) \subset <G^+, G_{p,2}>,$$

and the inclusions are equalities. Then $(N_p)_{\text{Aut } \tilde{T}_p} = \text{Diff}^+(\tilde{T}_p)$ as claimed.

Remarks. — 1) Any isomorphism of equipped trees $\tilde{T}_p \to \tilde{T}_p'$ conjugates $\text{Diff}^+(\tilde{T}_p)$ and $\text{Diff}^+(\tilde{T}_p')$.

2) If $p = 2$, the group $\text{PSL}_2(\mathbb{F}_2)$ is the full symmetric group $S_3$, so that $\text{Diff}^+(\tilde{T}_2) = N_2$.

4. Simplicity of $(N_p)_G$.

We now give the main theorem of the article, valid for any integer $p \geq 2$:

**Theorem 4.1.** — Let $G$ be a subgroup of Aut $T_p$ such that

1. $G^+$ is simple (e.g. $G$ satisfies the conditions of Theorem 3.1),

2. If $p$ is odd, $G^+ \cap (G_{p,2} \setminus [G_{p,2}, G_{p,2}])$ is non-empty,

3. $G^+$ possesses two non-commuting elements supported in a branch of the tree.

Then the group $(N_p)_G$ is simple.

Condition 2. implies that $(N_p)_G$ is generated by $G^+$ and $G_{p,2}' = [G_{p,2}, G_{p,2}]$, since $G_{p,2}$ is generated by $G_{p,2}'$ together with any element in $G_{p,2} \setminus G_{p,2}'$. 

**Corollary 4.1.** — For each integer $p \geq 2$, the group $N_p$ of all spheromorphisms is simple.

For each prime number $p \geq 3$ and for any choice of equipment of the tree $T_p$, the commutator subgroup $[\text{Diff}^+ (\overline{T}_p), \text{Diff}^+ (\overline{T}_p)]$ is simple, and there is a short exact sequence

$$1 \rightarrow [\text{Diff}^+ (\overline{T}_p), \text{Diff}^+ (\overline{T}_p)] \rightarrow \text{Diff}^+ (\overline{T}_p) \stackrel{\tilde{\theta}}{\rightarrow} \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$  

In other words, $H_1(\text{Diff}^+ (\overline{T}_p), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

**Proof of Corollary 4.1.** — $G = \text{Aut} T_p$ obviously satisfies all the conditions of the theorem above.

As for the statements about $\text{Diff}^+ (\overline{T}_p)$, they can be proven by using a particular equipment, since for different equipments the groups are conjugated. So, remembering that $T_p$ is obtained by gluing by an edge the two branches $L$ and $L'$ appearing in the definition of $G_{p,2}$, define the equipment $\overline{T}_p^0$ in the following way: label the $p$ edges drawn down from a vertex from 0 (on the left) to $p - 1$ (on the right), whereas the edge pointing towards the root of the branch ($L$ or $L'$) is labelled $\infty$. Then setting $G = \text{Aut} \overline{T}_p^0$, we have $(N_p)_G = \text{Diff}^+ (\overline{T}_p^0)$ (cf. §3.2 Example 2). But condition 2 of Theorem 4.1 fails for such $G$. We recalled in Section 2 that when $p$ is odd, there is an epimorphism

$$\theta : G_{p,2} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$$

whose kernel is the simple group $[G_{p,2}, G_{p,2}]$. It happens that $\theta$ may be extended to the group $\text{Diff}^+ (\overline{T}_p^0)$: if $\phi = (\phi_j : L_j \rightarrow L'_{\sigma(j)}j)$, where the indices of the branches label their roots from the left to the right (suppose the branches involved to be subbranches of $L$ or $L'$), $\tilde{\theta}(\phi)$ will be the signature of $\sigma$. Indeed, if we refine some branch $L_j$ into $L_{j_0} \cup L_{j_1} \cup \ldots \cup L_{j_{p-1}}$, then $\phi_j$ induces

$$\phi_{j_i} : L_{j_{i_i}} \rightarrow L'_{\sigma(j)}j_i \quad i = 0, 1, \ldots, p - 1,$$

with $i \in \mathbb{F}_p \rightarrow k_i \in \mathbb{F}_p$ in $B \subset PSL_2(\mathbb{F}_p)$, the stabilizer of $\infty$. Since $B$ lies in the alternating group $A_p$ on a set with $p$ elements, the permutation deduced from $\sigma$ has the same signature as in the case $k_i = i \forall i \in \mathbb{F}_p$. But then we saw (cf. §2) that, since $p$ is odd, the signature remains unchanged. So

$$\tilde{\theta} : \text{Diff}^+ (\overline{T}_p^0) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is a well-defined homomorphism.
It is clear that the kernel of $\tilde{\theta}$ is generated by $[G_{p,2}, G_{p,2}]$ and $(\text{Aut} \, \tilde{T}_p^0)^+$, and the proof of the theorem will show that this group is simple. Now the kernel contains the commutator subgroup $[\text{Diff}^+ (\tilde{T}_p^0), \text{Diff}^+ (\tilde{T}_p^0)]$, which is normal and non-trivial, consequently it coincides with the kernel.

Proof of Theorem 4.1. — Let $H < (N_p)_G$ be a non-trivial normal subgroup of $(N_p)_G$. Then $H \cap G^+$ is normal in $G^+$ and $H \cap [G_{p,2}, G_{p,2}]$ is normal in $[G_{p,2}, G_{p,2}]$. Hence either $H \supset G^+$ or $H \cap G^+ = \{\text{id}\}$, and either $H \supset [G_{p,2}G_{p,2}]$ or $H \cap [G_{p,2}, G_{p,2}] = \{\text{id}\}$.

So we will prove that the cases $H \cap G^+ = \{\text{id}\}$ and $H \cap [G_{p,2}, G_{p,2}] = \{\text{id}\}$ do not occur. We will use some arguments of a theorem of Epstein ([5] and [1]):

THEOREM 4.2 (Epstein, 1970). — Let $X$ be a paracompact Hausdorff topological space, $\Gamma$ a group of homeomorphisms of $X$, and $U$ a basis of open sets for the topology of $X$. The Epstein axioms for the triple $(X, \Gamma, U)$ are:

1. Axiom 1: If $U \in U$ and $g \in \Gamma$, then $gU \in U$.

2. Axiom 2: $\Gamma$ acts transitively on $U$.

3. Axiom 3: Let $g \in \Gamma$, $U \in U$ and $B$ an open covering of $X$; then there exists an integer $n$ and $g_1, \ldots, g_n \in \Gamma$ and $V_1, \ldots, V_n \in B$ such that

   (i) $g = g_n g_{n-1} \cdots g_1$,

   (ii) $\text{supp} (g_i) \subset V_i$,

   (iii) $\text{supp} (g_i) \cup (g_{i-1} \cdots g_1 \bar{U}) \neq X$, $1 \leq i \leq n$.

Suppose the triple $(X, \Gamma, U)$ as above satisfies the Epstein axioms. Then if $H$ is a non-trivial subgroup of $\Gamma$ that is normalized by $[\Gamma, \Gamma]$, then $[\Gamma, \Gamma] \subset H$. In particular, the group $[\Gamma, \Gamma]$ is simple.

The simplicity of $[\text{Diff}^+(S^1), \text{Diff}^+(S^1)]$ was an easy corollary of this theorem. M.R. Herman finally proved $\text{Diff}^+(S^1)$ was perfect, hence simple ([7]). For more details, we suggest the reader to refer to the very interesting book [1].

In the case of a non-connected topological space and a non-trivial group $\Gamma$, axiom 3 can never be satisfied (see [5]). Consequently, we will not be able to use the preceding theorem directly to prove the simplicity of $(N_p)_G$. However, setting $X = \partial T_p$, $U = \{\partial L : L \text{ branch of } T_p\}$ and
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\( \Gamma = (N_p)_G \), it is easy to see that the triple \((\partial T_p, (N_p)_G, \mathcal{U})\) satisfies axiom 2 and a

**“modified axiom 1”**: If \( U \in \mathcal{U} \) and \( g \in \Gamma \), then there exists \( U' \in \mathcal{U} \), \( U' \subset U \), such that \( gU' \in \mathcal{U} \).

Then we can show that two lemmas, which are steps in the proof of the Epstein theorem, still hold in our case:

**LEMMA 4.1** (from 1.4.2 in [5], or Lemma 2.2.5 in [1]). — Let \((X, \Gamma, \mathcal{U})\) be a triple satisfying the modified axiom 1 and axiom 2. Let \( V_0 \in \mathcal{U} \) and \( h \in \Gamma \) with \( \text{supp} \, h \subset V_0 \), and suppose that \( H \triangleleft \Gamma \) is a non-trivial normal subgroup of \( \Gamma \). Then there exists some \( \rho \in H \) such that \( \rho|_{V_0} = h|_{V_0} \).

**Proof.** — Choose any \( \alpha \in H \) with \( \alpha \neq \text{id} \), and find \( x \in X \) such that \( \alpha(x) \neq x \). Choose a small neighborhood \( U \in \mathcal{U} \) of \( x \) such that \( U \cap \alpha^{-1}(U) = \emptyset \). Next, take \( V, W \in \mathcal{U} \) such that \( V \cap W = \emptyset, V \cup W \subset U \), \( x \in V \). Suppose first that \( V_0 = V \). By axiom 2, there exists \( g \in \Gamma \) with \( gW = V \) Define

\[
\rho = [\alpha, [g, h]] = \alpha^{-1} [g, h]^{-1} \alpha [g, h].
\]

Then \( \rho \in \Gamma \) since \( H \triangleleft \Gamma \). We can verify that

\[
\rho = \begin{cases}
   h & \text{on } V, \\
   g^{-1}h^{-1}g & \text{on } W, \\
   \alpha^{-1}h\alpha & \text{on } \alpha^{-1}V, \\
   \alpha^{-1}g^{-1}h^{-1}g\alpha & \text{on } \alpha^{-1}W, \\
   \text{id} & \text{elsewhere}.
\end{cases}
\]

Now if \( V_0 \neq V \), choose \( k \in \Gamma \) (by axiom 2) such that \( k(V) = V_0 \). Then \( \text{supp} \, k^{-1}hk = k^{-1}(\text{supp} \, h) \subset V \), and by the previous case, there exists \( \rho \in H \) such that \( k^{-1}hk|_V = \rho|_V \), so that \( k\rho|_{V_0} = k\rho|_{V_0} \). Since \( kpk^{-1} \in H \), the proof is done.

**LEMMA 4.2** (variation of 1.4.6 in [5] or Lemma 2.2.7 in [1]). — \( \Gamma \) still satisfies the modified axiom 1 and axiom 2. Moreover, it is supposed 2-transitive:

\[
\forall (x_1, x_2), \forall (y_1, y_2), \ x_1 \neq x_2 \text{ and } y_1 \neq y_2 \Rightarrow \exists \phi \in \Gamma \ \phi(x_i) = y_i, \ i = 1, 2.
\]

Let \( h_1, h_2 \in \Gamma \) be such that there exists \( V_0 \in \mathcal{U} \) with \( \text{supp} \, h_i \subset V_0 \), \( i = 1, 2 \). Then \([h_1, h_2]\) belongs to \( H \).

**Proof.** — Let \( x \) be in \( X \). There exist \( \alpha_1, \alpha_2 \) in \( H \) such that \( x, \alpha_1^{-1}(x) \) and \( \alpha_2^{-1}(x) \) are pairwise distinct. Indeed, since \( \alpha \neq \text{id} \in H \),
there exists some $x \in X$ with $\alpha(x) \neq x$. So, in a neighborhood of $x$ there exists $y \neq x$ such that $\alpha(y) \neq y$. Now one can find $\phi \in \Gamma$ with $\phi(x) = y$ and $\phi^{-1}\alpha\phi(x) \neq \alpha(x)$ (which is equivalent to $\alpha(y) \neq \phi\alpha(x)$).

As for the condition $\alpha(y) \neq y$, it is equivalent to $\phi^{-1}\alpha\phi(x) \neq x$. Then one sets $\alpha_1^{-1} = \alpha$, $\alpha_2^{-1} = \phi^{-1}\alpha\phi$. So $\alpha_1$ and $\alpha_2$ belong to $H$, $x$, $\alpha_1^{-1}(x)$ and $\alpha_2^{-1}(x)$ are pairwise distinct. Then choose $U \in \mathcal{U}$ a neighborhood of $x$ such that $U, \alpha_1^{-1}(U)$ and $\alpha_2^{-1}(U)$ are pairwise disjoint. One can also find $g_1, g_2$ in $\Gamma$, and a neighborhood $V \in \mathcal{U}$ of $x$ such that $V$, $g_1^{-1}(V)$ and $g_2^{-1}(V)$ are pairwise disjoint and included in $U$. Suppose first that $\text{supp} h_i \subset V$, $i = 1, 2$. Then apply the previous lemma to $(\alpha_i, g_i, h_i, V, W_i = g_i^{-1}V)$, $i = 1, 2$. One gets $\rho_i|_V = h_i|_V$. The support of $\rho_i$ is included in $V \cup g_i^{-1}(V) \cup \alpha_i^{-1}(V) \cup \alpha_i^{-1}g_i^{-1}(V)$. The seven sets involved are disjoint, so that $[h_1, h_2] = [\rho_1, \rho_2]$.

To conclude, we may assume $V = V_0$, at the price of making some conjugation.

End of the proof of Theorem 4.1. — Choose $V_0 = \partial L_0$ where $L_0$ is some branch of the tree, and by condition 3, find two non-commuting elements $h_1$ and $h_2$ in $G^+$ with supports in $\partial L_0$. Apply Lemma 4.2 to $\Gamma = (N_p)_{\mathcal{G}}$, which is 2-transitive on $\partial T_p$, since $G^p$ itself is 2-transitive. Then $[h_1, h_2] \in G^+ \cap H$, so $H \supset G^+$. Similarly, choose two non-commuting elements $h'_1$ and $h'_2$ in $G_p = G_{p,1} \subset G_{p,2}$ (they are supported in a branch), so that $[h'_1, h'_2] \in [G_p, G_p] \cap H \subset [G_{p,2}, G_{p,2}] \cap H$, and $H \supset [G_{p,2}, G_{p,2}]$. Finally, $H$ contains two groups that generate $(N_p)_{\mathcal{G}}$, so $H = (N_p)_{\mathcal{G}}$.

5. Concluding remarks.

The question of the simplicity of the group $N_n$ is a preamble of a series of homological problems. First the result implies $H_1(N_n, \mathbb{Z}) = 0$. As for the second homology group $H_2(N_n, \mathbb{Z})$, though its complete computation could not be achieved (because the group $N_n$ is very huge), we know it is non-trivial. Indeed, the group $N_n$ possesses a non-trivial central extension by $\mathbb{Z}/2\mathbb{Z}$, called the “Central Geometric Extension” in [9] and [10], a sort of analogue of the Bott-Virasoro extension of $\text{Diff}^+(S^1)$.

On the other hand, K. Brown proved that the groups $G_n$ are all $\mathbb{Q}$-acyclic, i.e. $H_i(G_n, \mathbb{Q}) = 0$ for all $i > 0$ (cf. [3]). By using a description
of $N_n$ as the automorphism group of a free object of some appropriate category, it becomes possible to define an $N_n$-simplicial complex, and to use it to prove the $\mathbb{Q}$-acyclicity of $N_n$ (cf. [9] and [10]).

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