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A non-abelian tensor product of Leibniz algebra


<http://www.numdam.org/item?id=AIF_1999__49_4_1149_0>
A NON-ABELIAN TENSOR PRODUCT
OF LEIBNIZ ALGEBRAS

by Allahtan V. GNEDBAYE

Introduction.

Let $\mathfrak{g}$ be a Lie algebra and let $M$ be a representation of $\mathfrak{g}$, seen as a right $\mathfrak{g}$-module. Given a $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{g}$, one can endow the $K$-module $M$ with a bracket $[[m, m'] := m^{\mu(m')}$ which is not skew-symmetric but satisfies the Leibniz rule of derivations:

$$[[m, [m', m'']] = [[[m, m'], m'']] - [[[m, m''], m']].$$

Such objects were baptized Leibniz algebras by Jean-Louis Loday and are studied as a non-commutative variation of Lie algebras (see [8]). One of the main examples of Lie algebras comes from the notion of derivations. For the Leibniz algebras, there is an analogue notion of biderivations (see [7]).

The aim of this article is to “integrate” the Leibniz algebra of biderivations by means of a non-abelian tensor product of Leibniz algebras as it is done for Lie algebras.

In the classical case, D. Guin (see [5]) has shown that, given crossed Lie $\mathfrak{g}$-algebras $\mathcal{M}$ and $\mathcal{N}$, the set of derivations $\text{Der}_\mathfrak{g}(\mathcal{M}, \mathcal{N})$ has a structure of pre-crossed Lie $\mathfrak{g}$-algebra. Moreover the functor $\text{Der}_\mathfrak{g}(\mathcal{M}, -)$ is right adjoint to the functor $-\otimes_\mathfrak{g}\mathcal{N}$ where $-\otimes_\mathfrak{g}$ is the non-abelian tensor product of Lie algebras defined by G. J. Ellis (see [3]). D. Guin uses these objects

Keywords: Biderivation - Crossed module - Leibniz algebra - Milnor-type Hochschild homology - Non-abelian Leibniz (co)homology - Non-abelian tensor product.

to construct a non-abelian (co)homology theory for Lie algebras, which enables him to compare the $\mathbb{K}$-modules $HC_1(A)$ and $K_2^{\text{add}}(A)$ where $A$ is an arbitrary associative algebra. We give a non-commutative version of his results, in the sense that Leibniz algebras play the role of Lie algebras, the additive Milnor $K$-theory $K_*^{\text{add}}(A)$ (resp. the cyclic homology $HC_*(A)$) being replaced by the Milnor-type Hochschild homology $HH_*^M(A)$ (resp. the classical Hochschild homology $HH_*(A)$).

To this end, we introduce the notion of (pre)crossed Leibniz $g$-algebra as a simultaneous generalization of notions of representation and two-sided ideal of the Leibniz algebra $g$. Given crossed Leibniz $g$-algebras $M$ and $N$, we equip the set $\text{Bider}_g(M, N)$ of biderivations with a structure of pre-crossed Leibniz $g$-algebra. On the other hand, we construct a non-abelian tensor product $M \ast N$ of Leibniz algebras with mutual actions on one another. When $M$ and $N$ are crossed Leibniz $g$-algebras, this tensor product has also a structure of crossed Leibniz $g$-algebra. It turns out that the functor $- \ast g N$ is left adjoint to the functor $\text{Bider}_g(N, -)$. Another characterization of this tensor product is the following. If the Leibniz algebra $g$ is perfect (and free as a $\mathbb{K}$-module), then the Leibniz algebra $g \ast g$ is the universal central extension of $g$ (see [4]). We give also low-degrees (co)homological interpretations of these objects, which yield an exact sequence of $\mathbb{K}$-modules

$$
A/[A, A] \otimes HH_1(A) \oplus HH_1(A) \otimes A/[A, A] \rightarrow \mathfrak{H}_1(\mathfrak{A}, L(\mathfrak{A}))
$$

$$
\rightarrow \mathfrak{H}_1(\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]) \rightarrow \mathfrak{H}_1(\mathfrak{A}) \rightarrow \mathfrak{H}_1^\mathfrak{M}(\mathfrak{A}) \rightarrow [\mathfrak{A}, \mathfrak{A}] / [\mathfrak{A}, [\mathfrak{A}, \mathfrak{A}]] \rightarrow 0
$$

where $L(A)$ is the $\mathbb{K}$-module $A \otimes A / \text{im}(b_3)$ equipped with a suitable Leibniz bracket (see section 1.2).

Throughout this paper the symbol $\mathbb{K}$ denotes a commutative ring with a unit element and $\otimes$ stands $\otimes_\mathbb{K}$.

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Bibliography
1. Prerequisites on Leibniz algebras.

1.1. Leibniz algebras.

A Leibniz algebra is a \( K \)-module \( \mathfrak{g} \) equipped with a bilinear map \([-, -] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \) called bracket and satisfying only the Leibniz identity

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y]
\]

for any \( x, y, z \in \mathfrak{g} \). In the presence of the condition \([x, x] = 0\), the Leibniz identity is equivalent to the so-called Jacobi identity. Therefore Lie algebras are examples of Leibniz algebras.

A morphism of Leibniz algebras is a linear map \( f : \mathfrak{g}_1 \to \mathfrak{g}_2 \) such that

\[
f([x, y]) = [f(x), f(y)]
\]

for any \( x, y \in \mathfrak{g}_1 \). It is clear that Leibniz algebras and their morphisms form a category that we denote by \((\text{Leib})\).

A two-sided ideal of a Leibniz algebra \( \mathfrak{g} \) is a submodule \( \mathfrak{h} \) such that \([x, y] \in \mathfrak{h} \) and \([y, x] \in \mathfrak{h}\) for any \( x \in \mathfrak{h} \) and any \( y \in \mathfrak{g} \). For any two-sided ideal \( \mathfrak{h} \) in \( \mathfrak{g} \), the quotient module \( \mathfrak{g}/\mathfrak{h} \) inherits a structure of Leibniz algebra induced by the bracket of \( \mathfrak{g} \). In particular, let \([x, x]\) be the two-sided ideal in \( \mathfrak{g} \) generated by all brackets \([x, x]\). The Leibniz algebra \( \mathfrak{g}/([x, x]) \) is in fact a Lie algebra, said canonically associated to \( \mathfrak{g} \) and is denoted by \( \mathfrak{g}_{\text{Lie}} \).

Let \( \mathfrak{g} \) be a Leibniz algebra. Denote by \( \mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] \) the submodule generated by all brackets \([x, y]\). The Leibniz algebra \( \mathfrak{g} \) is said to be perfect if \( \mathfrak{g}' = \mathfrak{g} \). It is clear that any submodule of \( \mathfrak{g} \) containing \( \mathfrak{g}' \) is a two-sided ideal in \( \mathfrak{g} \).

1.2. Examples.

Let \( M \) be a representation of a Lie algebra \( \mathfrak{g} \) (the action of \( \mathfrak{g} \) on \( M \) being denoted by \( m^g \) for \( m \in M \) and \( g \in \mathfrak{g} \)). For any \( \mathfrak{g} \)-equivariant map \( \mu : M \to \mathfrak{g} \), the bracket given by \([m, m'] := m^{\mu(m')} \) induces a structure of Leibniz (non-Lie) algebra on \( M \). Observe that any Leibniz algebra \( \mathfrak{g} \) can be obtained in such a way by taking the canonical projection \( \mathfrak{g} \to \mathfrak{g}_{\text{Lie}} \) (which is obviously \( \mathfrak{g}_{\text{Lie}} \)-equivariant).

Let \( A \) be an associative algebra and let \( b_3 : A^\otimes 3 \to A^\otimes 2 \) be the Hochschild boundary that is, the linear map defined by

\[
b_3(a \otimes b \otimes c) := ab \otimes c - a \otimes bc + ca \otimes b, \quad a, b, c \in A.
\]
Then the bracket given by

\[[a \otimes b, c \otimes d] := (ab - ba) \otimes (cd - dc), a, b, c, d \in A,\]

defines a structure of Leibniz algebra on the \(K\)-module \(L(A) := A^{\otimes 2}/ \text{im}(b_3)\). Moreover, we have an exact sequence of \(K\)-modules

\[0 \to \text{HH}_1(A) \to L(A) \xrightarrow{b_2} A \to \text{HH}_0(A)\]

where \(\text{HH}_*(A)\) denotes the Hochschild homology groups and \(b_2(x, y) = [x, y] := xy - yx\) for any \(x, y \in A\).

### 1.3. Free Leibniz algebra.

Let \(V\) be a \(K\)-module and let \(\bar{T}(V) := \bigoplus_{n \geq 1} V^{\otimes n}\) be the reduced tensor module. The bracket defined inductively by

\[\,[x, v] = x \otimes v, \text{ if } x \in \bar{T}(V) \text{ and } v \in V\]

\[\,[x, y \otimes v] = [x, y] \otimes v - [x \otimes v, y], \text{ if } x, y \in \bar{T}(V) \text{ and } v \in V,\]

satisfies the Leibniz identity. The Leibniz algebra so defined is the free Leibniz algebra over \(V\) and is denoted by \(\mathcal{F}(V)\) (see [8]). Observe that one has

\[v_1 \otimes v_2 \otimes \cdots \otimes v_n = \cdots[[v_1, v_2], v_3] \cdots v_n], \forall v_1, \ldots, v_n \in V.\]

Moreover, the free Lie algebra over \(V\) is nothing but the Lie algebra \(\mathcal{F}(V)_{\text{Lie}}\).

### 2. Crossed Leibniz algebras.

#### 2.1. Leibniz action.

Let \(\mathfrak{g}\) and \(\mathfrak{m}\) be Leibniz algebras. A Leibniz action of \(\mathfrak{g}\) on \(\mathfrak{m}\) is a couple of bilinear maps

\[\mathfrak{g} \times \mathfrak{m} \to \mathfrak{m}, (g, m) \mapsto {}^g m \quad \text{and} \quad \mathfrak{m} \times \mathfrak{g} \to \mathfrak{m}, (m, g) \mapsto m^g\]

satisfying the axioms

i) \(m^{[g \cdot g']} = (m^g)^{g'} - (m^{g'})^g,\)

ii) \([g \cdot g'] m = (g m)^{g'} - g (m^{g'}),\)
iii) \( g(g'm) = -g(m'g), \)
iv) \( g[m,m'] = [g_m,m'] - [g'm,m], \)
v) \( [m,m']^g = [m^g,m'] + [m,m'^g], \)
vi) \( [m, g'm'] = -[m, m'^g] \)

for any \( m, m' \in \mathcal{M} \) and \( g, g' \in \mathfrak{g}. \) We say that \( \mathcal{M} \) is a Leibniz \( \mathfrak{g} \)-algebra. Observe that the axiom i) applied to the triples \( (m; g, g') \) and \( (m; g', g) \) yields the relation

\[
m^{[g,g']} = -m^{[g',g]},
\]

2.2. Examples.

Any two-sided ideal of a Leibniz algebra \( \mathfrak{g} \) is a Leibniz \( \mathfrak{g} \)-algebra, the action being given by the initial bracket.

A \( \mathbb{K} \)-module \( M \) equipped with two operations of a Leibniz algebra \( \mathfrak{g} \) satisfying the axioms i), ii) and iii) is called a representation of \( \mathfrak{g} \) (see [8]). Therefore representations of a Leibniz algebra \( \mathfrak{g} \) are abelian Leibniz \( \mathfrak{g} \)-algebras.

2.3. Crossed Leibniz algebras.

Let \( \mathfrak{g} \) be a Leibniz algebra. A pre-crossed Leibniz \( \mathfrak{g} \)-algebra is a Leibniz \( \mathfrak{g} \)-algebra \( \mathcal{M} \) equipped with a morphism of Leibniz algebras \( \mu : \mathcal{M} \to \mathfrak{g} \) such that

\[
\mu(gm) = [g, \mu(m)] \quad \text{and} \quad \mu(g^g) = [\mu(m), g]
\]

for any \( g \in \mathfrak{g} \) and \( m \in \mathcal{M}. \) Moreover if the relations

\[
\mu(m)m' = [m, m'] \quad \text{and} \quad m\mu(m') = [m, m'], \quad \forall \ m, m' \in \mathcal{M},
\]

hold, then \( (\mathcal{M}, \mu) \) is called a crossed Leibniz \( \mathfrak{g} \)-algebra.
2.4. Examples.

Any Leibniz algebra $\mathfrak{g}$, equipped with the identity map $\text{id}_\mathfrak{g}$, is a crossed Leibniz $\mathfrak{g}$-algebra.

Any two-sided ideal $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{g}$, equipped with the inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$, is a crossed Leibniz $\mathfrak{g}$-algebra.

Let $\alpha : \mathfrak{c} \rightarrow \mathfrak{g}$ be a central extension of Leibniz algebras (i.e., a surjective morphism whose kernel is contained in the centre of $\mathfrak{c}$, see [4]). Define operations of $\mathfrak{g}$ on $\mathfrak{c}$ by
\[
\gamma^\mathfrak{g} := [\alpha^{-1}(g), \mathfrak{c}] \quad \text{and} \quad \mathfrak{c}^\mathfrak{g} := [\mathfrak{c}, \alpha^{-1}(g)]
\]
where $\alpha^{-1}(g)$ is any pre-image of $g$ in $\mathfrak{c}$. Then $(\mathfrak{c}, \alpha)$ is a crossed Leibniz $\mathfrak{g}$-algebra.

**Proposition 2.1.** — For any pre-crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{m}, \mu)$, the image $\text{im}(\mu)$ (resp. the kernel $\text{ker}(\mu)$) is a two-sided ideal in $\mathfrak{g}$ (resp. $\mathfrak{m}$). Moreover, if $(\mathfrak{m}, \mu)$ is crossed, then $\text{ker}(\mu)$ is contained in the centre of $\mathfrak{m}$.

**Proof.** — Let $m$ be an element of $\mathfrak{m}$. For any $g \in \mathfrak{g}$, we have
\[
[\mu(m), g] = \mu(m^g) \in \text{im}(\mu) \quad \text{and} \quad [g, \mu(m)] = \mu([g, m]) \in \text{im}(\mu).
\]
Thus, $\text{im}(\mu)$ is a two-sided ideal in $\mathfrak{g}$. Assume that $m \in \text{ker}(\mu)$; then for any $m' \in \mathfrak{m}$, we have
\[
\mu([m, m']) = [\mu(m), \mu(m')] = 0 = [\mu(m'), \mu(m)] = \mu([m', m]).
\]
Therefore $\text{ker}(\mu)$ is a two-sided ideal in $\mathfrak{m}$. Moreover if the Leibniz action of $\mathfrak{g}$ on $\mathfrak{m}$ is crossed, then we have
\[
[m, m'] = \mu(m)m' = 0 = m'\mu(m) = [m', m]
\]
for any $m, m' \in \text{ker}(\mu)$ and $m' \in \mathfrak{m}$. Thus $\text{ker}(\mu)$ is contained in the centre of $\mathfrak{m}$. \hfill \Box

2.5. Morphism of pre-crossed Leibniz algebras.

Let $\mathfrak{g}$ be a Leibniz algebra and let $(\mathfrak{m}, \mu)$ and $(\mathfrak{n}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. A *morphism* from $(\mathfrak{m}, \mu)$ to $(\mathfrak{n}, \nu)$ is a Leibniz algebra morphism $f : \mathfrak{m} \rightarrow \mathfrak{n}$ such that
\[
f(m^g) = ^g(f(m)), \quad f(m^\mathfrak{g}) = (f(m))^\mathfrak{g} \quad \text{and} \quad \mu = \nu f
\]
for any \( m \in \mathcal{M} \) and \( g \in \mathfrak{g} \). A morphism of crossed Leibniz \( \mathfrak{g} \)-algebras is the same as a morphism of pre-crossed Leibniz \( \mathfrak{g} \)-algebras. It is clear that pre-crossed (resp. crossed) Leibniz \( \mathfrak{g} \)-algebras and their morphisms form a category that we denote by \((\text{pc-Leib}(\mathfrak{g}))\) (resp. \((\text{c-Leib}(\mathfrak{g}))\)).

**Proposition 2.2.** — Let \( f : (\mathcal{M}, \mu) \to (\mathcal{N}, \nu) \) be a crossed Leibniz \( \mathfrak{g} \)-algebra morphism. Then \((\mathcal{M}, f)\) is a crossed Leibniz \( \mathfrak{N} \)-algebra via the Leibniz action of \( \mathfrak{N} \) on \( \mathcal{M} \) given by

\[
^n m := \nu(n)m \quad \text{and} \quad m^n := m^{\nu(n)}, \quad \forall m \in \mathcal{M}, n \in \mathfrak{N}.
\]

**Proof.** — One easily checks that \( \mathcal{M} \) is a Leibniz \( \mathfrak{N} \)-algebra. For any \( m, m' \in \mathcal{M} \) and \( n \in \mathfrak{N} \), we have

\[
f(nm) = f(\nu(n)m) = \nu(n)f(m) = [n, f(m)],
\]

\[
f(m^n) = f(m^{\nu(n)}) = f(m)^{\nu(n)} = [f(m), n];
\]

thus \((\mathcal{M}, f)\) is a pre-crossed Leibniz \( \mathfrak{N} \)-algebra. Moreover we have

\[
f(m)m' = \nu(f(m))m' = \mu(m)m' = [m, m'],
\]

\[
mf(m') = m^{\nu(f(m'))} = m^{\mu(m')} = [m, m'];
\]

thus \((\mathcal{M}, f)\) is a crossed Leibniz \( \mathfrak{N} \)-algebra. \( \square \)

2.6. Exact sequences.

We say that a sequence

\[
(\mathcal{L}, \lambda) \xrightarrow{\alpha} (\mathcal{M}, \mu) \xrightarrow{\beta} (\mathcal{N}, \nu)
\]

is exact in the category \((\text{pc-Leib}(\mathfrak{g}))\) (resp. \((\text{c-Leib}(\mathfrak{g}))\)) if the sequence

\[
\mathcal{L} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{N}
\]

is exact as sequence of Leibniz algebras.

**Proposition 2.3.** — If the sequence

\[
(\mathcal{L}, \lambda) \xrightarrow{\alpha} (\mathcal{M}, \mu) \xrightarrow{\beta} (\mathfrak{N}, \nu)
\]

is exact in the category \((\text{pc-Leib}(\mathfrak{g}))\) (resp. \((\text{c-Leib}(\mathfrak{g}))\)), then the map \( \lambda \) is zero. Moreover if the Leibniz \( \mathfrak{g} \)-algebra \((\mathcal{L}, \lambda)\) is crossed, then the Leibniz algebra \( \mathcal{L} \) is abelian.

**Proof.** — Indeed, since \( \beta\alpha = 0 \), we have \( \lambda = \nu\beta\alpha = 0 \). From whence \( \ker(\lambda) = \mathcal{L} \), and by Proposition 2.1, it is clear that the Leibniz algebra \( \mathcal{L} \) is abelian. \( \square \)

In this section, we fix a Leibniz algebra $\mathfrak{g}$.

3.1. Derivations and anti-derivations.

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. A derivation from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ is a linear map $d: \mathfrak{M} \to \mathfrak{N}$ such that

$$d([m, m']) = d(m)\mu(m') + \mu(m)d(m'), \forall m, m' \in \mathfrak{M}.$$ 

An anti-derivation from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ is a linear map $D: \mathfrak{M} \to \mathfrak{N}$ such that

$$D([m, m']) = D(m)\mu(m') - \mu(m')D(m), \forall m, m' \in \mathfrak{M}.$$

3.2. Examples.

Let $(\mathfrak{N}, \nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra and let $n$ be any element of $\mathfrak{N}$. By the axiom iii) (resp. i)) of 2.1, the linear map

$$g \to \mathfrak{N}, \ g \mapsto \mathfrak{N} \ (\text{resp. } g \to \mathfrak{N}, \ g \mapsto -\mathfrak{N})$$

is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \text{id}_\mathfrak{g})$ to $(\mathfrak{N}, \nu)$.

3.3. Biderivations.

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. We denote by $\text{Bider}_\mathfrak{g}(\mathfrak{M}, \mathfrak{N})$ the free $\mathbb{K}$-module generated by the triples $(d, D, g)$, where $d$ (resp. $D$) is a derivation (resp. an anti-derivation) from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ and $g$ is an element of $\mathfrak{g}$ such that

$$\nu(d(m)) = \mu(gm), \ \nu(D(m)) = -\mu(gm),$$

$$^h d(m) = ^h D(m), \ D(m^h) = -D(^h m)$$

for any $h \in \mathfrak{g}$ and $m \in \mathfrak{M}$. 
PROPOSITION 3.1. — If the Leibniz g-algebra \((\mathfrak{g}, \nu)\) is crossed, then there is a Leibniz algebra structure on the \(K\)-module \(\text{Bider}_g(\mathfrak{g}, \mathfrak{g})\) for the bracket defined by

\[
[(d, D, g), (d', D', g')] := (\delta, \Delta, [g, g'])
\]

where

\[
\delta(m) := d'(m^g) - d(m^g') \quad \text{and} \quad \Delta(m) = -D(m^g') - d'(q_m), \, \forall \, m \in \mathfrak{g}.
\]

Proof. — Let us show that the maps \(\delta\) and \(\Delta\) are respectively a derivation and an anti-derivation. Indeed, for any \(m, m' \in \mathfrak{g}\), we have

\[
\delta([m, m']) = d'(\{m, m'[g] - d([m, m']^g])
\]

\[
= d'(\{m^g, m'\}) + d'(\{m, m'^g\}) - d([m^g, m']) + d([m, m'^g])
\]

\[
= d'(m^g)\mu(m') + \mu(m^g)d'(m') + d'(m)\mu(m'^g) + \mu(m)d'(m'^g)
\]

\[
- d(m^g)\mu(m') - \mu(m'^g)d(m') - d(m)\mu(m'^g)
\]

\[
- \mu(m)d(m'^g)
\]

\[
= (d'(m^g) - d(m^g'))\mu(m') + \mu(m)(d'(m^g) - d(m^g'))
\]

\[
+ \nu(d(m))d'(m') + d'(m)\nu(d(m')) - \nu(d'(m))d(m')
\]

\[
- d(m)\nu(d'(m'))
\]

\[
= \delta(m)\mu(m') + \mu(m)\delta(m') + [d(m), d'(m')]
\]

\[
+ [d'(m), d(m')] - [d(m'), d'(m')] - [d(m), d'(m')]
\]

\[
= \delta(m)\mu(m') + \mu(m)\delta(m').
\]

and

\[
\Delta([m, m']) = -D([m, m']^g) - d'(q_{m, m'})
\]

\[
= -D([m^g, m']) - D([m, m'^g]) - d'(\{q_m, m'\}) + d'(\{q_m, m'^g\})
\]

\[
= -D(m^g')\mu(m') + D(m')\mu(m'^g) - D(m)\mu(m'^g) + D(m'g')\mu(m)
\]

\[
- d'(q_m)\mu(m') - \mu(q_m)d'(m') + d'(q_m')\mu(m) + \mu(q_m')d'(m)
\]

\[
= (-D(m^g') - d'(q_m))\mu(m') - (-D(m'^g) - d'(q_m'))\mu(m)
\]

\[
+ D(m')\nu(d(m')) - D(m)\nu(d'(m')) + \nu(D(m))d'(m') - \nu(D(m'))d'(m)
\]

\[
= \Delta(m)\mu(m') - \Delta(m')\mu(m) + [D(m'), d'(m')]
\]

\[
- [D(m), d'(m')] + [D(m), d'(m')] - [D(m'), d'(m')]
\]

\[
= \Delta(m)\mu(m') - \Delta(m')\mu(m).
\]
On the other hand, we have

\[ \nu(\delta(m)) = \nu(d'(m^g)) - \nu(d(m^g')) = \mu((m^g)g') - \mu((m^g')g) = \mu(m^{[g,g']}) , \]

\[ \nu(\Delta(m)) = -\nu(D(m^g)) - \nu(d'(m^g')) = \mu(q(m^g')) - \mu((q(m)g') = -\mu([g,g']_m) , \]

\[ h_\delta(m) = h_\delta'(m^g) - h_\delta(d(m^g')) = hD'(m^g) - hD(m^g') = -hD'(q_m) - hD(m^g') = -hD'(q_m) - hD(m^g') \]

\[ = h\Delta(m) , \]

\[ \Delta^h(m) = -D(h(m)^g') - d'(q(h_m)) = -D(h(m)^g') - D(h(m^g')) + d'(q(m^h)) \]

\[ = D((m^h)^g') + d'(q(m^h)) = -\Delta(m^h) . \]

Therefore the triple \((\delta, \Delta, [g,g'])\) is a biderivation from \((\mathcal{M}, \mu)\) to \((\mathcal{N}, \nu)\).

Moreover, let \((d, D, g), (d', D', g')\) and \((d'', D'', g'')\) be biderivations from \((\mathcal{M}, \mu)\) to \((\mathcal{N}, \nu)\). We set

\[ (\delta_0, \Delta_0, g_0) := [(d, D, g), (\delta, \Delta, [g,g'])] , \]

\[ (\delta', \Delta', [g,g']) := [(d, D, g), (d', D', g')] , \]

\[ (\delta_1, \Delta_1, g_1) := [(\delta, \Delta', [g,g']), (d'', D'', g'')] , \]

\[ (\delta_2, \Delta_2, g_2) := [(\delta'', \Delta'', [g,g''], (d', D', g')] . \]

It is clear that \(g_0 = g_1 - g_2\). For any \(m \in \mathcal{M}\), we have

\[ (\delta_1 - \delta_2)(m) = d''(m^g g') - d'(m^g') + \delta''(m^g') \]

\[ = d''((m^g)g') - d''((m^g')g) - d'((m^g')g) + d'((m^g')g') \]

\[ = d''((m^g)g') + d'((m^g')g) + d'((m^g')g) - d'((m^g')g') \]

\[ = \delta(m^g) - d(m^{[g,g']}) = \delta_0(m) \]

and

\[ (\Delta_1 - \Delta_2)(m) = -\Delta'(m^g') - d''(m^g g') + \Delta''(m^g) + d'([g,g']_m) \]

\[ = D((m^g')g') + d'((m^g')g) - d''(m^g) + d'((m^g)g') - d'((m^g)g') \]

\[ = -D(m^{[g,g']}) - d''((m^g)g') + d'((m^g)g') \]

\[ = -D(m^{[g,g']}) - \delta(m) = \Delta_0(m) . \]

Therefore the \(K\)-module \(\text{Bider}_g(\mathcal{M}, \mathcal{N})\) is a Leibniz algebra. \(\square\)
Let us equip the set $Bideg(\mathcal{M}, \mathcal{N})$ with a Leibniz action of $g$.

**Proposition 3.2.** — Let $(\mathcal{M}, \mu)$ (resp. $(\mathcal{N}, \nu)$) be a pre-crossed (resp. crossed) Leibniz $g$-algebra. The set $Bideg(\mathcal{M}, \mathcal{N})$ is a pre-crossed Leibniz $g$-algebra for the operations defined by

$$(d', D', g)(h, g') := (d^h, D^h, [g, h])$$

where

$$(^h d)(m) = d(m^h) - d(m)^h, \quad (^h D)(m) := ^h d(m) - d(^h m),$$

$$(d^h)(m) := d(m)^h - d(m^h), \quad (D^h)(m) := D(m)^h - D(m^h).$$

**Proof.** — Everything can be smoothly checked and we merely give an example of these verifications. By definition we have

$$[(d, D, g), (d', D', g')] = ([^h \delta, ^h \Delta, [h, [g, g']]],$$

$$[h(d, D, g), (d', D', g')] = ([\delta_1, \Delta_1, [[h, g], g']],$$

$$[h(d', D', g'), (d, D, g)] = ([\delta_2, \Delta_2, [h, g'], g']).$$

For any $m \in \mathcal{M}$ we have

$$(\delta_1 - \delta_2)(m) = d'(m^{[h, g]}) - (d^h)(m^g) - d((m)^g)^h - (d'(m^g)^h)$$

$$= d'((m^h)^g) - d'((m^g)^h) - d((m^g)^h) + d((m^g)^h)$$

$$= (d'((m^h)^g) - d((m)^g)^h) - (d'(m^g) - d((m^g)^h)$$

$$= \delta(m^h) - \delta(m)^h = (\delta^h)(m)$$

and

$$(\Delta_1 - \Delta_2)(m) = - hD)(m^g) - d'((m^{[h, g]})m + (hD')(m^g) + d((m^g)^h)$$

$$hD)(m^g) + d((m^g)^h)$$

$$hD'(m^g) - d'((m^g)^h)$$

$$= hD'(m^g) - D(m^g) - (d'((m^g)^h) - d((m^g)^h))$$

Thus we get

$$[(d, D, g), (d', D', g')] = [(d, D, g), (d', D', g')] - [(h(d', D', g'), (d, D, g)].$$

Now we can state the fundamental result which is a consequence of Propositions 3.1 and 3.2.
THEOREM 3.3. — For any pre-crossed (resp. crossed) Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M}, \mu)$ (resp. $(\mathfrak{N}, \nu)$), the Leibniz $\mathfrak{g}$-algebra $\text{Bider}_\mathfrak{g}(\mathfrak{M}, \mathfrak{N})$ is pre-crossed for the morphism $\rho : \text{Bider}_\mathfrak{g}(\mathfrak{M}, \mathfrak{N}) \to \mathfrak{g}$, $(\mathfrak{O}, \mathfrak{D}, \mathfrak{g}) \mapsto \mathfrak{g}$. □

3.4. Remarks.

For any element $g$ of $\mathfrak{g}$, the linear map $\text{ad}_g : h \mapsto [h, g]$ (resp. $\text{Ad}_g : h \mapsto -[g, h]$) is a derivation (resp. an anti-derivation) of the Leibniz algebra $\mathfrak{g}$. In the classical sense (i.e., without “crossing”, see [7]) the couple $(\text{ad}_g, \text{Ad}_g)$ is called inner biderivation of $\mathfrak{g}$. Therefore the pre-crossed Leibniz $\mathfrak{g}$-algebra $\text{Bider}_\mathfrak{g}(\mathfrak{M}, \mathfrak{N})$ can be seen as the set of biderivations from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ over inner biderivations of $\mathfrak{g}$.

On the other hand, given a pre-crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M}, \mu)$, one easily checks that the map $\text{Bider}_\mathfrak{g}(\mathfrak{M}, -)$ is a functor from the category of crossed Leibniz $\mathfrak{g}$-algebras to the category of pre-crossed Leibniz $\mathfrak{g}$-algebras.


4.1. Leibniz pairings.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. A Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$ is a triple $(\mathfrak{P}, \eta_1, \eta_2)$ where $\mathfrak{P}$ is a Leibniz algebra and $\eta_1 : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{P}$ (resp. $\eta_2 : \mathfrak{N} \times \mathfrak{M} \to \mathfrak{P}$) is a bilinear map such that

\[
\eta_1(m, [n, n']) = \eta_1(m^n, n') - \eta_1(m^{n'}, n),
\eta_2(n, [m, m']) = \eta_2(n^m, m') - \eta_2(n^{m'}, m),
\eta_1([m, m'], n) = \eta_2(m^n, m') - \eta_1(m, n^{m'}),
\eta_2([n, n'], m) = \eta_1(m^n, n') - \eta_2(n, m^{n'}),
\eta_1(m, m'n) = -\eta_1(m, n^{m'}), \quad \eta_2(n, n'm) = -\eta_2(n, m^{n'}),
\eta_1(m^n, m'n') = [\eta_1(m, n), \eta_1(m', n')] = \eta_2(m^n, m'n'),
\eta_1(m'n, m') = [\eta_2(n, m), \eta_2(n', m')] = \eta_2(m'n, m'),
\eta_1(m'n, m) = [\eta_1(m, n), \eta_2(n', m')] = \eta_2(m'n, m'),
\eta_1(m'n', n) = [\eta_2(n, m), \eta_1(m', n')] = \eta_2(m'n', m')
\]

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$.
4.2. Example.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two-sided ideals of a same Leibniz algebra $\mathfrak{g}$. Take $\mathfrak{P} := \mathfrak{M} \cap \mathfrak{N}$ and define

$$h_1(m,n) := [m,n] \quad \text{and} \quad h_2(n,m) := [n,m].$$

Then the triple $(\mathfrak{P}, h_1, h_2)$ is a Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$.

4.3. Non-abelian tensor product.

A Leibniz pairing $(\mathfrak{P}, h_1, h_2)$ of $\mathfrak{M}$ and $\mathfrak{N}$ is said to be universal if for any other Leibniz pairing $(\mathfrak{P}', h_1', h_2')$ of $\mathfrak{M}$ and $\mathfrak{N}$ there exists a unique Leibniz algebra morphism $\theta : \mathfrak{P} \to \mathfrak{P}'$ such that

$$\theta h_1 = h_1' \quad \text{and} \quad \theta h_2 = h_2'.$$

It is clear that a universal pairing, when it exists, is unique up to a unique isomorphism. Here is a construction of the universal pairing as a non-abelian tensor product.

**Definition-Theorem 4.1.** — Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. Let $V$ be the free $K$-module generated by the symbols $m \ast n$ and $n \ast m$ where $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. Let $\mathfrak{M} \ast \mathfrak{N}$ be the Leibniz algebra quotient of the free Leibniz algebra generated by $V$ by the two-sided ideal defined by the relations

i) $\lambda (m \ast n) = \lambda m \ast n = m \ast \lambda n$, $\lambda (n \ast m) = \lambda n \ast m = n \ast \lambda m$,

ii) $(m + m') \ast n = m \ast n + m' \ast n$, $(n + n') \ast m = n \ast m + n' \ast m$,

$$(m + m') \ast n = m \ast n + m' \ast n', \quad n \ast (m + m') = n \ast m + n \ast m',$$

iii) $m \ast [n, n'] = m^n \ast n' - m^{n'} \ast n$, $n \ast [m, m'] = n^m \ast m' - n^{m'} \ast m$,

$[m, m'] \ast n = m^n \ast m' - m^m \ast n'$, $[n, n'] \ast m = n^n \ast m' - n^m \ast n'$,

iv) $m \ast m' = - m \ast n^{m'}$, $n \ast n' = - n \ast m^{n'}$,

v) $m^n \ast m'^n = [m \ast n, m' \ast n'] = m^n \ast m'^n$,

$m^n \ast n'^m = [m \ast n, n' \ast m'] = m^n \ast n'^m$,

$m \ast n'^m = [n \ast m, n' \ast m'] = n^m \ast n'^m$,

$m \ast m' = [n \ast m, m' \ast n'] = n^m \ast m'^n$,

for any $\lambda \in K$, $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$. Define maps

$$h_1 : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{M} \ast \mathfrak{N}, \quad h_1(m,n) := m \ast n.$$
and
\[ h_2 : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \ast \mathcal{N}, \quad h_2(n, m) := n \ast m. \]
Then the triple \((\mathcal{M} \ast \mathcal{N}, h_1, h_2)\) is the universal Leibniz pairing of \(\mathcal{M}\) and \(\mathcal{N}\) and called the non-abelian tensor product (or tensor product for short) of \(\mathcal{M}\) and \(\mathcal{N}\).

Proof. — It is straightforward to see that the triple \((\mathcal{M} \ast \mathcal{N}, h_1, h_2)\) so-defined is a Leibniz pairing of \(\mathcal{M}\) and \(\mathcal{N}\). For the universality, notice that if \((\mathcal{P}, h'_1, h'_2)\) is another Leibniz pairing of \(\mathcal{M}\) and \(\mathcal{N}\), then the map \(\theta\) is necessarily given on generators by
\[ \theta(m \ast n) = h'_1(m, n) \quad \text{and} \quad \theta(n \ast m) = h'_2(n, m) \]
for any \(m \in \mathcal{M}\) and \(n \in \mathcal{N}\). \(\Box\)

As an illustration of this construction, we give now a description of the non-abelian tensor product when the actions are trivial.

Proposition 4.2. — If the Leibniz algebras \(\mathcal{M}\) and \(\mathcal{N}\) act trivially on each other, then there is an isomorphism of abelian Leibniz algebras
\[ \mathcal{M} \ast \mathcal{N} \cong \mathcal{M}_{\text{ab}} \otimes \mathcal{N}_{\text{ab}} \oplus \mathcal{N}_{\text{ab}} \otimes \mathcal{M}_{\text{ab}} \]
where \(\mathcal{M}_{\text{ab}} := \mathcal{M}/[\mathcal{M}, \mathcal{M}]\) and \(\mathcal{N}_{\text{ab}} := \mathcal{N}/[\mathcal{N}, \mathcal{N}]\).

Proof. — Recall that the underlying \(\mathbb{K}\)-module of the free Leibniz algebra generated by \(V\) is
\[ \overline{T}(V) = V \oplus V \otimes^2 V \oplus \cdots \oplus V \otimes^n V \oplus \cdots \]
Since the actions are trivial, the definition of the bracket on \(\overline{T}(V)\) and the relations v) enable us to see that \(\mathcal{M} \ast \mathcal{N}\) is an abelian Leibniz algebra and that the summands \(V \otimes^n V\) (for \(n \geq 2\)) are killed. Relations i) and ii) of 4.1 say that the \(\mathbb{K}\)-module \(\mathcal{M} \ast \mathcal{N}\) is the quotient of \(\mathcal{M} \otimes \mathcal{N} \oplus \mathcal{N} \otimes \mathcal{M}\) by the relations iii). These later imply that \(\mathcal{M} \ast \mathcal{N}\) is the abelian Leibniz algebra \(\mathcal{M}_{\text{ab}} \otimes \mathcal{N}_{\text{ab}} \oplus \mathcal{N}_{\text{ab}} \otimes \mathcal{M}_{\text{ab}}\). \(\Box\)

4.4. Compatible Leibniz actions.

Let \(\mathcal{M}\) and \(\mathcal{N}\) be Leibniz algebras with mutual Leibniz actions on one another. We say that these actions are compatible if we have
\[
\begin{align*}
(m^n) m' &= [m^n, m'], \quad (m^m) n' = [m^m, n'], \\
(n^m) m' &= [m, m'], \quad (m^n) n' = [m^n, n'], \\
m(m^n) &= [m, m^n], \quad n(m^m) = [n, m^m], \\
m(n^m) &= [m^n, m'], \quad n(m^n) = [n, m^n].
\end{align*}
\]
for any $m, m' \in \mathcal{M}$ and $n, n' \in \mathcal{N}$.

4.5. Examples.

If $\mathcal{M}$ and $\mathcal{N}$ are two-sided ideals of a same Leibniz algebra, then the actions (given by the initial bracket) are compatible.

Let $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ be pre-crossed Leibniz $g$-algebras. Then one can define a Leibniz action of $\mathcal{M}$ on $\mathcal{N}$ (resp. of $\mathcal{N}$ on $\mathcal{M}$) by setting

$m_n := \mu(m)n$ and $n^m := \nu(n)m$

(resp. $m^w := \nu(n)m$ and $m^n := m^\nu(n)$).

If the Leibniz $g$-algebras $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ are crossed, then these Leibniz actions are compatible.

4.6. First crossed structure.

Let $\mathcal{M}$ and $\mathcal{N}$ be Leibniz algebras with mutual compatible actions on one another. Consider the operations of $\mathcal{M}$ on $\mathcal{M} \ast \mathcal{N}$ given by

$m'(m' \ast n') := [m, m'] \ast n' - m' \ast m', \quad m'(n' \ast m') := m' \ast m' - [m, m'] \ast n'$

and those of $\mathcal{N}$ on $\mathcal{M} \ast \mathcal{N}$ given by

$n'(m' \ast n') := [n, n'] \ast m' - n' \ast m', \quad n'(n' \ast m') := n' \ast m' - [n, n'] \ast n'$

for any $m, m' \in \mathcal{M}$ and $n, n' \in \mathcal{N}$. Then we have

**Proposition 4.3.** — With the above operations, the map

$\mu : \mathcal{M} \ast \mathcal{N} \to \mathcal{M}, \quad m \ast n \mapsto m^n, \quad n \ast m \mapsto n^m$

(resp. $\nu : \mathcal{M} \ast \mathcal{N} \to \mathcal{N}, \quad m \ast n \mapsto n^m, \quad n \ast m \mapsto n^m$)

induces on $\mathcal{M} \ast \mathcal{N}$ a structure of crossed Leibniz $\mathcal{M}$-algebra (resp. $\mathcal{N}$-algebra).

**Proof.** — Once again everything can be readily checked thanks to the compatibility conditions. For example we have

$\mu(m \ast n)(m' \ast n') = m^n(m' \ast n') = [m^n, m'] \ast n' - (m^n)n' \ast m' = (m^n)n' \ast m' - m^n \ast n'm' - (m^n)n' \ast m' = m^n \ast m'n = [m \ast n, m' \ast n']$
for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$.

**4.7. Second crossed structure.**

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $g$-algebras, equipped with the mutual Leibniz actions given in Examples 4.5. One easily checks that the operations given by

\[
\begin{align*}
\circ (m \ast n) & := \circ m \ast n - \circ n \ast m, \quad \circ (n \ast m) := \circ n \ast m - \circ m \ast n, \\
(m \ast n)^g & := m^g \ast n + m \ast n^g, \quad (n \ast m)^g := n^g \ast m + n \ast m^g,
\end{align*}
\]

define a Leibniz action of $g$ on $\mathfrak{M} \ast \mathfrak{N}$.

**Proposition 4.4.** — Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $g$-algebras. Then the map $\eta : \mathfrak{M} \ast \mathfrak{N} \rightarrow g$ defined on generators by

\[
\begin{align*}
\eta(m \ast n) & := [\mu(m), \nu(n)] \quad \text{and} \quad \eta(n \ast m) := [\nu(n), \mu(m)],
\end{align*}
\]

confers to $\mathfrak{M} \ast \mathfrak{N}$ a structure of pre-crossed Leibniz $g$-algebra. Moreover, if one of the Leibniz $g$-algebras $\mathfrak{M}$ or $\mathfrak{N}$ is crossed, then the Leibniz $g$-algebra $\mathfrak{M} \ast \mathfrak{N}$ is crossed.

**Proof.** — It is immediate to check that the map $\eta$ passes to the quotient and defines a Leibniz algebra morphism. Moreover we have

\[
\begin{align*}
\eta(\circ (m \ast n)) & = [\mu(\circ m), \nu(n)] - [\nu(\circ n), \mu(m)] \\
& = [[g, \mu(m)], \nu(n)] - [[g, \nu(n)], \mu(m)] \\
& = [g, [\mu(m), \nu(n)]]; \\
\eta(\circ (n \ast m)) & = - \eta(g (m \ast n)) = -[g, \eta(m \ast n)] \\
& = - [g, [\mu(m), \nu(n)]]; \\
\eta((m \ast n)^g) & = [\mu(m^g), \nu(n)] + [\mu(m), \nu(n^g)] \\
& = [[\mu(m), g], \nu(n)] + [\mu(m), [\nu(n), g]] \\
& = [[\mu(m), \nu(n)], g] = [\eta(m \ast n), g]; \\
\eta((n \ast m)^g) & = [\nu(n^g), \mu(m)] + [\nu(n), \mu(m^g)] \\
& = [[\nu(n), g], \mu(m)] + [\nu(n), [\mu(m), g]] \\
& = [[\nu(n), \mu(m)], g] = [\eta(n \ast m), g];
\end{align*}
\]

thus $(\mathfrak{M} \ast \mathfrak{N}, \eta)$ is a pre-crossed Leibniz $g$-algebra. Assume that, for instance,
the Leibniz \( g \)-algebra \( \mathcal{M} \) is crossed. Then we have
\[
\eta(m \ast n)(m' \ast n') = [\mu(m), \nu(n)](m' \ast n') = \mu(m \nu(n))(m' \ast n')
\]
\[
= \mu(m \nu(n))m' \ast n' - \nu(m \nu(n))m' \ast n
\]
\[
= [m \nu(n), m'] \ast n' - \mu(m \nu(n))m' \ast n
\]
\[
= \mu(m \nu(n))n' \ast m' - m \nu(n) * n' \mu(m) - \mu(m \nu(n))m' \ast n
\]
\[
= m \nu(n) \ast m \nu(n') = [m \ast n, m' \ast n']
\]
and
\[
(m \ast n)\eta(m' \ast n') = (m \ast n)[\mu(m'), \nu(n')] = (m \ast n)\mu(m' \nu(n'))
\]
\[
= m \mu(m' \nu(n')) \ast n + m \ast n \mu(m' \nu(n'))
\]
\[
= [m, m \nu(n')] \ast n + m \ast n \mu(m' \nu(n'))
\]
\[
= \mu(m \nu(n') \ast m' - m \ast n \mu(m' \nu(n')) + m \ast n \mu(m' \nu(n'))
\]
\[
= [m \ast n, m' \ast n'].
\]
By the same way, one easily gets
\[
\eta(m \ast n)(n' \ast m') = [m \ast n, n' \ast m'], (m \ast n)\eta(n' \ast m') = [m \ast n, n' \ast m'],
\]
\[
\eta(n \ast m)(n' \ast m') = [n \ast m, n' \ast m'], (n \ast m)\eta(n' \ast m') = [n \ast m, n' \ast m'],
\]
\[
\eta(n \ast m)(m' \ast n') = [n \ast m, m' \ast n'], (n \ast m)\eta(m' \ast n') = [n \ast m, m' \ast n'].
\]
So we have proved that the Leibniz \( g \)-algebra \( \mathcal{M} \ast \mathcal{N} \) is crossed. \( \square \)

4.8. Remark.

It is clear that if \( (\mathcal{M}, \mu) \) (resp. \( (\mathcal{N}, \nu) \)) is a crossed Leibniz \( g \)-algebra, then the map \( \mathcal{M} \ast - \) (resp. \( - \ast \mathcal{N} \)) is a functor from the category of pre-crossed Leibniz \( g \)-algebras to the category of crossed Leibniz \( g \)-algebras.

**Proposition 4.5.** — Let \( (\mathcal{N}, \nu) \) be a crossed Leibniz \( g \)-algebra. The functor \( F(-) := - \ast \mathcal{N} \) is a right exact functor from the category of pre-crossed Leibniz \( g \)-algebras to the category of crossed Leibniz \( g \)-algebras.

**Proof.** — Taking into account Proposition 2.3, let
\[
0 \rightarrow (\mathcal{F}, \omega) \xrightarrow{f} (\Omega, \lambda) \xrightarrow{g} (\mathcal{R}, \gamma) \rightarrow 0
\]
be an exact sequence of pre-crossed Leibniz \( g \)-algebras. Consider the sequence of Leibniz algebras
\[
F(\mathcal{F}) \xrightarrow{\mathcal{F}(f)} F(\Omega) \xrightarrow{\mathcal{F}(g)} F(\mathcal{R}) \rightarrow 0.
\]
It is clear that the morphism \( F(g) \) is surjective. Since the map \( F(f) \) is a morphism of crossed Leibniz \( g \)-algebras, by Proposition 2.2, \((F(\mathfrak{P}), \mathfrak{F}(f))\) is a crossed Leibniz \( F(\Omega) \)-algebra; and by Proposition 2.1, the image \( \text{im} F(f) \) is a two-sided ideal in \( F(\Omega) \). By composition we have \( F(g) F(f) = F(gf) = 0 \), which yields a factorisation

\[
\overline{F(g)} : F(\Omega)/\text{im} \mathfrak{F}(f) \rightarrow \mathfrak{F}(\mathfrak{A}).
\]

In fact, the morphism \( \overline{F(g)} \) is an isomorphism. To see it, let us consider the map

\[
\Gamma : F(\mathfrak{A}) \rightarrow \mathfrak{F}(\Omega)/\text{im} \mathfrak{F}(f)
\]
given on generators by

\[
\Gamma(r \ast n) := g^{-1}(r) \ast n \mod \text{im} F(f) \quad \text{and} \quad \Gamma(n \ast r) := n \ast g^{-1}(r) \mod \text{im} F(f)
\]
where \( g^{-1}(r) \) is any pre-image of \( r \) in \( \Omega \). Indeed, if \( q \) and \( q' \) are two pre-images of \( r \), then \( q - q' = f(p) \) for some \( p \) in \( \mathfrak{P} \). Therefore we have

\[
q \ast m - q' \ast n = (q - q') \ast n = f(p) \ast n = F(f)(p \ast n) \in \text{im} F(f),
\]
\[
n \ast q - n \ast q' = n \ast (q - q') = n \ast f(p) = F(f)(n \ast p) \in \text{im} F(f);
\]
thus the map \( \Gamma \) is well-defined. One easily checks that \( \Gamma \) is a morphism of Leibniz algebras and inverse to \( \overline{F(g)} \).

5. Adjunction theorem.

In this section we show that, for any crossed Leibniz \( g \)-algebra \((\mathfrak{M}, \nu)\), the functor \(- \ast \mathfrak{M}\) is left adjoint to the functor \( \text{Bider}_g(\mathfrak{M}, -) \). For technical reasons, we assume that the relations

\[
iv) \quad m \ast \mu(m')n = -m \ast n \mu(m'), \quad n \ast \nu(n')m = -n \ast m \nu(n')
\]
defining the tensor product \( \mathfrak{M} \ast \mathfrak{M} \) are extended to the relations

\[
iv)' \quad m \ast qn = -m \ast n q, \quad n \ast qm = -n \ast m q
\]
for any \( m, m' \in \mathfrak{M}, n, n' \in \mathfrak{M} \) and \( g \in g \). To avoid confusion, we denote this later tensor product by \( \mathfrak{M} \ast_g \mathfrak{M} \). For instance, the Leibniz \( g \)-algebras \( \mathfrak{M} \ast \mathfrak{M} \) and \( \mathfrak{M} \ast_g \mathfrak{M} \) coincide if the maps \( \mu \) and \( \nu \) are surjective.

**Theorem 5.1.** — Let \((\mathfrak{M}, \mu)\) be a pre-crossed Leibniz \( g \)-algebra and let \((\mathfrak{N}, \nu)\) and \((\mathfrak{P}, \lambda)\) be crossed Leibniz \( g \)-algebras. There is an isomorphism of \( \mathbb{K} \)-modules

\[
\text{Hom}(\text{pc-Leib}(g))(\mathfrak{M}, \text{Bider}_g(\mathfrak{N}, \mathfrak{P})) \cong \text{Hom}(\text{c-Leib}(g))(\mathfrak{M} \ast_g \mathfrak{N}, \mathfrak{P}).
\]
Proof. — Let \( \phi \in \text{Hom}(\text{pc-Leib}(g))(\mathcal{M}, \text{Bider}_g(\mathcal{N}, \mathcal{P})) \) and put \((d_m, D_m, g_m) := \phi(m)\) for \(m \in \mathcal{M}\). Notice that we have \(g_m = \mu(m)\) thanks to the relation \(\rho \phi = \mu\), where \(\rho : \text{Bider}_g(\mathcal{N}, \mathcal{P}) \to g\) is the crossing morphism. We associate to \(\phi\) the map \(\Phi : \mathcal{M} \star_g \mathcal{N} \to \mathcal{P}\) defined on generators by
\[
\Phi(m \star n) := -D_m(n) \quad \text{and} \quad \Phi(n \star m) := d_m(n), \ \forall \ m \in \mathcal{M}, n \in \mathcal{N}.
\]

**Lemma 5.2.** — The map \(\Phi\) is a morphism of crossed Leibniz \(g\)-algebras.

Conversely, given an element \(\sigma \in \text{Hom}(\text{c-Leib}(g))(\mathcal{M} \star_g \mathcal{N}, \mathcal{P})\), we associate the map \(\Sigma : \mathcal{M} \to \text{Bider}_g(\mathcal{N}, \mathcal{P})\) defined by
\[
\Sigma(m) := (\delta_m, \Delta_m, \mu(m)), \ \forall \ m \in \mathcal{M},
\]
where
\[
\delta_m(n) := \sigma(n \star m) \quad \text{and} \quad \Delta_m(n) := -\sigma(m \star n), \ \forall \ n \in \mathcal{N}.
\]

**Lemma 5.3.** — The map \(\Sigma\) is a morphism of pre-crossed Leibniz \(g\)-algebras.

It is clear that the maps \(\phi \mapsto \Phi\) and \(\sigma \mapsto \Sigma\) are inverse to each other, which proves the adjunction theorem. \(\square\)

**Proof of Lemma 5.2.** — There is a lot of things to check in order to show that the map \(\Phi\) is well-defined. Let us give some examples of these verifications. For any \(m, m' \in \mathcal{M}, n, n' \in \mathcal{N}\) and \(h \in g\), we have
\[
\Phi([m \star n'] - n \star m' = -D_{\nu(n)m}(n') - d_{m',\nu(n}')(n) = -((\nu(n)D_m)(n') - ((d_m)\nu(n'))(n)
\]
\[
= -\nu(n)d_m(n') + d_m((\nu(n)n') - d_m(n')\nu(n') + d_m(n') = -\nu(n)[m_n, n'] = m([n, n']).
\]

We also compute
\[
\Phi(m \star h_n) = -D_m(h) = D_m(n^h) = -\Phi(m \star n^h), \quad \Phi(n \star h_m) = d_m(n) = (h_d_m)(n) = -((d_m^h)(n) = -d_m^h(n) = -\Phi(n \star m^h).
\]
and
\[ \Phi(m^n \ast m'n') = -D_{m^\nu(n)}(\mu(m')n') = -(\nu(m')n')(\mu(m')n') \]
\[ = -D_m(\mu(m')n')^\nu(n) + D_m(\mu(m')n')^\nu(n) \]
\[ = -D_m(\mu(m')n')^\nu(n) + D_m([\mu(m')n', n]) \]
\[ = -D_m(n)^\nu(m'n') = D_m(n)^\nu(D_m(n')) \]
\[ = [D_m(n), D_{m'}(n')] = [\Phi(m \ast n), \Phi(m' \ast n')] \]
\[ = \Phi([m \ast n, m' \ast n']). \]

Now let \( m \in \mathcal{M}, n \in \mathfrak{N} \) and \( g \in \mathfrak{g} \). One has successively
\[ \Phi(\theta(m \ast n)) = \Phi(\theta(m \ast n)) = -D_{\theta(n)}(n) - d_m(\theta(n)) \]
\[ = (D_m)(n) - d_m(\theta(n)) = -\theta(D_m(n)) = \theta(\Phi(m \ast n)), \]
\[ \Phi(\theta(n \ast m)) = -\Phi(\theta(m \ast n)) = -\Phi(m \ast n) = \theta(D_m(n)) = \theta(d_m(n)) = \theta(\Phi(n \ast m)), \]
\[ \Phi((m \ast n)^g) = \Phi(m^g \ast n) + \Phi(m \ast n^g) = -D_{m^g}(n) - D_{m}(n^g) \]
\[ = -((D_m)^g)(n) - d_m(\theta(n)) = -D_{m^g}(n) = \Phi(m \ast n)^g, \]
\[ \Phi((n \ast m)^g) = \Phi(n^g \ast m) + \Phi(n \ast m^g) = d_m(n^g) + d_m(n^g) \]
\[ = d_m(n^g) + (d_m)^g(n) = d_m(n^g)^g = \Phi(n \ast m)^g; \]
\[ \lambda_\Phi(m \ast n) - \lambda(D_m(n)) = \nu(\mu(m), n) = [\mu(m), \nu(n)] = \eta(m \ast n), \]
\[ \lambda_\Phi(n \ast m) = \lambda(d_m(n)) = \nu(n^\mu(m)) = [\nu(n), \mu(m)] = \eta(n \ast m). \]

Therefore the map \( \Phi \) is a morphism of crossed Leibniz \( \mathfrak{g} \)-algebras. \( \square \)

Proof of Lemma 5.3. — Let us first show that \( \Sigma(m) \) is a well-defined biderivation. For any \( n, n' \in \mathfrak{N} \), we have
\[
\delta_m(n)^\nu(n') + \nu(n)\delta_m(n')
\]
\[ = \sigma(n \ast m)^\nu(n') + \nu(n)\sigma(n' \ast m) = \sigma((n \ast m)^\nu(n')) + \sigma(\nu(n')(n' \ast m)) \]
\[ = \sigma(n^\nu(n') \ast m) + \sigma(n \ast m^\nu(n')) + \sigma(\nu(n')n' \ast m) - \sigma(\nu(n')m \ast n') \]
\[ = 2\sigma([n, n'] \ast m) - \sigma(\nu(n)m \ast n' - n \ast m^\nu(n')) \]
\[ = 2\sigma([n, n'] \ast m) - \sigma([n, n'] \ast m) = \sigma([n, n'] \ast m) = \delta_m([n, n']), \]
thus \( \delta_m \) is a derivation. Moreover, we have
\[
\Delta_m(n)^\nu(n') - \Delta_m(n')^\nu(n)
\]
\[ = -\sigma(m \ast n)^\nu(n') + \sigma(m \ast n')^\nu(n) = \sigma((m \ast n')^\nu(n)) - \sigma((m \ast n)^\nu(n')) \]
\[ = \sigma(m^\nu(n) \ast n') + \sigma(m \ast n^\nu(n)) - \sigma(m^\nu(n') \ast n) - \sigma(m \ast n^\nu(n')) \]
\[ = \sigma(m^\nu(n) \ast n' - m^\nu(n') \ast n) - \sigma(m^\nu(n) \ast n') - \sigma(m \ast n^\nu(n')) \]
\[ = \sigma(m \ast [n, n']) - \sigma(m \ast [n, n']) - \sigma(m \ast [n, n']) \]
\[ = -\sigma(m \ast [n, n']) = \Delta_m([n, n']), \]
thus $\Delta_m$ is an anti-derivation. We have also
\[
\lambda(\delta_m(n)) = \lambda(\sigma(n \ast m)) = \eta(n \ast m) = [\nu(n), \mu(m)] = \nu(n^\mu(m)),
\]
\[
\lambda(\Delta_m(n)) = -\lambda(\sigma(m \ast n)) = -[\mu(n), \nu(m)] = -\nu(\mu(m)^n),
\]
\[
h \delta_m(n) = h \sigma(n \ast m) = \sigma(h(n \ast m)) = -h \sigma(h(m \ast n)) = -h \Delta_m(n),
\]
\[
\Delta_m(hn) = -\sigma(m \ast hn) = \sigma(m \ast n^h) = -\Delta_m(n^h).
\]
Therefore $\Sigma(m) = (\delta_m, \Delta_m, \mu(m))$ is a biderivation from $(\mathfrak{N}, \nu)$ to $(\mathfrak{P}, \lambda)$.

For any $h \in \mathfrak{g}$, $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have
\[
(h \delta_m)(n) = \delta_m(n^h) - \delta_m(n)^h = \sigma(n^h \ast m) - \sigma(n \ast m)^h
\]
\[
= -\sigma(n \ast m^h) = \sigma(n \ast h m) = \delta_m(n),
\]
\[
(h \Delta_m)(n) = h \Delta_m(n) - \delta_m(hn) = h \sigma(m \ast n) - \sigma(hn \ast m)
\]
\[
= \sigma(hm \ast n) = \delta_{hn}(n);
\]
and obviously $[h, \mu(m)] = \mu(h m)$, thus we have $\Sigma(hm) = h \Sigma(m)$. On the other side, we have
\[
((\delta_m)^h)(n) = \delta_m(n)^h - \delta_m(n^h) = \sigma(n \ast m)^h - \sigma(n^h \ast m)
\]
\[
= \sigma(n \ast m^h) = \delta_{nm}(n)
\]
and
\[
((\Delta_m)^h)(n) = \Delta_m(n)^h - \delta_m(n^h) = -\sigma(m \ast n)^h + \sigma(m \ast n^h)
\]
\[
= -\sigma(m^h \ast n) = \Delta_{hm}(n).
\]
Since $[\mu(m), h] = \mu(m^h)$, we get $\Sigma(m^h) = \Sigma(m)^h$. By definition of the map $\Sigma$, we have $\rho \Sigma(m) = \mu(m)$. Therefore the map $\Sigma$ is a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras.

\[\square\]

6. Cohomological characterizations.

6.1. Non-abelian Leibniz cohomology.

Let $\mathfrak{g}$ be a Leibniz algebra viewed as the crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{g}, \text{id}_\mathfrak{g})$, and let $(\mathfrak{M}, \mu)$ be a crossed Leibniz $\mathfrak{g}$-algebra. Given an element $m \in \mathfrak{M}$, we denote by $d_m$ (resp. $D_m$) the derivation (resp. anti-derivation) $g \mapsto gm$ (resp. $g \mapsto -gm$) from $(\mathfrak{g}, \text{id}_\mathfrak{g})$ to $(\mathfrak{M}, \mu)$, and by $\mu(m) := \mu(m) \mod Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$. One easily checks that the triple $(d_m, D_m, \mu(m))$ is a well-defined element of $\text{Bider}_\mathfrak{g}(\mathfrak{g}, \mathfrak{M})$. 

DEFINITION-PROPOSITION 6.1. — Let \( \mathfrak{J} \) be the \( \mathbb{K} \)-module freely generated by the biderivations \( (d_m, D_m, \mu(m)) \), \( m \in \mathcal{M} \). Then \( \mathfrak{J} \) is a two-sided ideal of \( \text{Bider}_\mathcal{M}(\mathfrak{g}, \mathcal{M}) \). The Leibniz algebra \( \text{Bider}_\mathcal{M}(\mathfrak{g}, \mathcal{M})/\mathfrak{J} \) is denoted by \( \mathfrak{L}^\text{L}(\mathfrak{g}, \mathcal{M}) \).

Proof. — For any \( m \in \mathcal{M} \) and \( (d, D, g) \in \text{Bider}_\mathcal{M}(\mathfrak{g}, \mathcal{M}) \), we have

\[
[(d, D, g), (d_m, D_m, \mu(m))] = (\delta_m, \Delta_m, [g, \mu(m)])
\]

with

\[
\delta_m(x) = d_m([x, g]) - d([x, \mu(m))] = [x, g]m - d([x, \mu(m)])
\]

\[
= \mu(d(x))m - d(x)\mu(m) - \varepsilon d(\mu(m))
\]

\[
= [d(x), m] - [d(x), m] - \varepsilon D(\mu(m))
\]

\[
= d_m(x)
\]

where \( m_1 := -D(\mu(m)) \),

\[
\Delta_m(x) = -D([x, \mu(m)]) - d_m([g, x]) = -D([x, \mu(m)]) - [g, x]m
\]

\[
= -D(x)\mu(m) - D(\mu(m))x + \mu(D(x))m
\]

\[
= -[D(x), m] + D(\mu(m))x + [D(x), m]
\]

\[
= D_m(x),
\]

\[
\mu(m_1) = -\mu(D(\mu(m))) = [g, \mu(m)] = [g, \mu(m)];
\]

thus we have \( [(d, D, g), (d_m, D_m, \mu(m))] \in \mathfrak{J} \). On the other side, we have

\[
[(d_m, D_m, \mu(m)), (d, D, g)] = (\delta'_m, \Delta'_m, [\mu(m), g])
\]

with

\[
\delta'_m(x) = d([x, \mu(m)]) - d_m([x, g]) = d([x, \mu(m)]) - [x, g]m
\]

\[
= d(x)\mu(m) + \varepsilon d(\mu(m)) - \mu(d(x))m
\]

\[
= [d(x), m] + \varepsilon d(\mu(m)) - [d(x), m]
\]

\[
= d_{m_2}(x)
\]

where \( m_2 := d(\mu(m)) \),

\[
\Delta'_m(x) = -D_m([x, g]) - d([\mu(m), x]) = m[x, g] - d([\mu(m), x])
\]

\[
= m\mu(d(x)) - d(\mu(m))x - \mu(m)d(x)
\]

\[
= [m, d(x)] - d(\mu(m))x - [m, d(x)]
\]

\[
= D_{m_2}(x),
\]

\[
\mu(m_2) = \mu(d(\mu(m))) = [\mu(m), g] = [\mu(m), g];
\]
thus we have $[(d_m, D_m, \mu(m)), (d, D, g)] \in \mathcal{J}$. Therefore the set $\mathcal{J}$ is a two-sided ideal of $\text{Bider}_g(g, \mathcal{M})$.

Similarly, given a crossed Leibniz $g$-algebra $(\mathcal{M}, \mu)$, one defines
\[ \mathcal{H}\mathcal{L}^0(g, \mathcal{M}) := \{ m \in \mathcal{M} : \mu(m) = m^g = 0, \forall g \in g \} \]
that is, the set of invariant elements of $\mathcal{M}$. From the relations
\[ [m, m'] = m^{\mu(m')} = 0 = \mu(m')m = [m', m], \ m \in \mathcal{H}\mathcal{L}^0(g, \mathcal{M}), \ m' \in \mathcal{M}, \]
it is clear that $\mathcal{H}\mathcal{L}^0(g, \mathcal{M})$ is contained in the centre of the Leibniz algebra $\mathcal{M}$.

**PROPOSITION 6.2.** — For any exact sequence of crossed Leibniz $g$-algebras
\[ 0 \rightarrow (\mathfrak{A}, 0) \overset{\alpha}{\rightarrow} (\mathfrak{B}, \lambda) \overset{\beta}{\rightarrow} (\mathfrak{C}, \mu) \rightarrow 0, \]
there exists an exact sequence of $K$-modules
\[ 0 \rightarrow \mathcal{H}\mathcal{L}^0(g, \mathfrak{A}) \rightarrow \mathcal{H}\mathcal{L}^0(g, \mathfrak{B}) \rightarrow \mathcal{H}\mathcal{L}^0(g, \mathfrak{C}) \overset{\partial}{\rightarrow} \mathcal{H}\mathcal{L}^1(g, \mathfrak{A}) \rightarrow \mathcal{H}\mathcal{L}^1(g, \mathfrak{B}) \overset{\beta^1}{\rightarrow} \mathcal{H}\mathcal{L}^1(g, \mathfrak{C}) \]
where $\beta^1$ is a Leibniz algebra morphism.

**Proof.** — Everything goes smoothly except the definition of the connecting homomorphism $\partial$. Given an element $c \in \mathcal{H}\mathcal{L}^0(g, \mathfrak{C})$, let $b \in \mathfrak{B}$ be any pre-image of $c$ in $\mathfrak{B}$. For any $x \in g$, we have
\[ \beta^{x}(b) = \mu^{x}(c) = 0 = c^{x} = \beta^{x}(b^{x}). \]
Thus the element $\beta^{x}(b)$ (resp. $b^{x}$) is in $\ker(\beta) = \text{im}(\alpha)$. Since the morphism $\alpha$ is injective, the map $d^{c} : x \mapsto \alpha^{-1}(\mu^{x}(c))$ (resp. $D^{c} : x \mapsto \alpha^{-1}(\mu^{x}(b^{x}))$) is a derivation (resp. an anti-derivation) from $(g, \text{id}_g)$ to $(\mathfrak{A}, 0)$. One easily checks that the triple $(d^{c}, D^{c}, 0)$ is a well-defined element of $\text{Bider}_g(g, \mathfrak{A})$ whose class in $\mathcal{H}\mathcal{L}^1(g, \mathfrak{A})$ does not depend on the choice of the pre-image $b$. We put
\[ \partial(c) := \text{class}(d^{c}, D^{c}, 0). \]

**6.2. Non-abelian Leibniz homology.**

Let $g$ be a Leibniz algebra viewed as the crossed Leibniz $g$-algebra $(g, \text{id}_g)$, and let $(\mathcal{M}, \nu)$ be a crossed Leibniz $g$-algebra.
DEFINITION-PROPOSITION 6.3. — The map $\Psi_\mathfrak{N} : \mathfrak{N} \ast \mathfrak{g} \to \mathfrak{N}$ given on generators by

$$
\Psi_\mathfrak{N}(n \ast g) := n^g \quad \text{and} \quad \Psi_\mathfrak{N}(g \ast n) := g_n, \quad g \in \mathfrak{g}, \quad n \in \mathfrak{N},
$$
is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. We define the low-degrees non-abelian homology of $\mathfrak{g}$ with coefficients in $\mathfrak{N}$ to be

$$
\mathfrak{H}_0(\mathfrak{g}, \mathfrak{N}) := \text{coker} \Psi_\mathfrak{N} \quad \text{and} \quad \mathfrak{H}_1(\mathfrak{g}, \mathfrak{N}) := \ker \Psi_\mathfrak{N}.
$$

Proof. — To see that the map $\Psi_\mathfrak{N}$ is a Leibniz algebra morphism is equivalent to the fact that the Leibniz action of $\mathfrak{N}$ on $\mathfrak{g}$ is well-defined. The definition of the crossing homomorphism $\eta_\mathfrak{N} : \mathfrak{N} \ast \mathfrak{g} \to \mathfrak{g}$ implies that $\Psi_\mathfrak{N}$ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. \qed

PROPOSITION 6.4. — For any exact sequence of crossed Leibniz $\mathfrak{g}$-algebras

$$
0 \to (\mathfrak{A}, \rho) \to (\mathfrak{B}, \lambda) \to (\mathfrak{C}, \mu) \to 0,
$$

there exists an exact sequence of $\mathbb{K}$-modules

$$
\mathfrak{H}_1(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{H}_1(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{H}_1(\mathfrak{g}, \mathfrak{C}) \to \mathfrak{H}_0(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{H}_0(\mathfrak{g}, \mathfrak{B})
$$

$$
\to \mathfrak{H}_0(\mathfrak{g}, \mathfrak{C}) \to 0.
$$

Proof. — We know that the functor $- \ast \mathfrak{g}$ is right exact (Proposition 4.5). Therefore Proposition 6.4 is nothing but the “snake-lemma” applied to diagram

$$
\begin{array}{ccccccccc}
\mathfrak{A} \ast \mathfrak{g} & \to & \mathfrak{B} \ast \mathfrak{g} & \to & \mathfrak{C} \ast \mathfrak{g} & \to & 0 \\
\downarrow \Psi_\mathfrak{A} & & \downarrow \Psi_\mathfrak{B} & & \downarrow \Psi_\mathfrak{C} & & \\
0 & \to & \mathfrak{A} & \to & \mathfrak{B} & \to & \mathfrak{C} & \to & 0
\end{array}
$$

which is obviously commutative. \qed

6.3. Universal central extension.

Let $\mathfrak{g}$ be a Leibniz algebra and let $\Psi := \Psi_\mathfrak{g}$ be the morphism defining the homology $\mathfrak{H}_*(\mathfrak{g}, \mathfrak{g})$. From the relations $v)$ of Definition-Theorem 4.1, it is clear that $\Psi : \mathfrak{g} \ast \mathfrak{g} \to [\mathfrak{g}, \mathfrak{g}]$ is a central extension of Leibniz algebras (see [4]).
THEOREM 6.5. — If the Leibniz algebra \( g \) is perfect and free as a \( K \)-module, then the morphism \( \Psi : g \times g \to [g, g] = g \) is the universal central extension of \( g \). Moreover, we have an isomorphism of \( K \)-modules
\[
\mathfrak{H}_1(g, g) \cong \mathfrak{H}_2(g).
\]

Proof. — It is enough to prove the universality of the central extension \( \Psi : g \times g \to [g, g] = g \). Let \( \alpha : C \to g \) be a central extension of \( g \). Since \( \ker(\alpha) \) is central in \( C \), the quantity \( [\alpha^{-1}(x), \alpha^{-1}(y)] \) does not depend on the choice of the pre-images \( \alpha^{-1}(x) \) and \( \alpha^{-1}(y) \) where \( x, y \in g \). One easily checks that the map \( \phi : g \times g \to C \) given on generators by
\[
\phi(x \star y) := [\alpha^{-1}(x), \alpha^{-1}(y)]
\]
is a well-defined Leibniz algebra morphism such that \( \alpha \phi = \Psi \). The uniqueness of the map \( \phi \) follows from Lemma 2.4 of [4] since the perfectness of \( g \) implies that of \( g \star g \):
\[
x \star y = \left( \sum_i [x_i, x'_i] \right) \star \left( \sum_j [y_j, y'_j] \right) = \sum_{i,j} [x_i \star x'_i, y_j \star y'_j].
\]

By definition we have \( \mathfrak{H}_1(g, g) = \ker() \). After [4] the kernel of the universal central extension of a Leibniz algebra \( g \) is canonically isomorphic to \( \mathfrak{H}_2(g) \). Therefore we have
\[
\mathfrak{H}_1(g, g) \cong \mathfrak{H}_2(g).
\]

7. The Milnor-type Hochschild homology.

Let \( A \) be an associative algebra viewed as a Leibniz (in fact Lie) algebra for the bracket given by \([a, b] := ab - ba, a, b \in A\). Recall that the \( K \)-module \( L(A) := A^\otimes 2 / \text{im}(b_3) \) is a Leibniz (non-Lie) algebra for the bracket defined by
\[
[x \otimes y, x' \otimes y'] := (xy - yx) \otimes (x'y' - y'x'), \forall x, y, x', y' \in A.
\]

PROPOSITION 7.1. — The operations given by
\[
A \times L(A) \to L(A), \quad (x \otimes y) := [a, x] \otimes y - [a, y] \otimes x,
\]
\[
L(A) \times A \to L(A), \quad (x \otimes y)^a := [x, a] \otimes y + x \otimes [y, a]
\]
confer to \( L(A) \) a structure of Leibniz \( A \)-algebra. Moreover the map
\[
\mu_A : L(A) \to A, \ x \otimes y \mapsto [x, y] = xy - yx
\]
equips \(L(A)\) with a structure of crossed Leibniz \(A\)-algebra.

**Proof.** — The operations are well-defined since we have

\[
\begin{align*}
q(b_3(x \otimes y \otimes z)) &= b_3(ax \otimes y \otimes z - a \otimes x \otimes xy - za \otimes x \otimes y
\quad + a \otimes xz \otimes x + a \otimes xz \otimes y - a \otimes y \otimes zx) 
\end{align*}
\]

and

\[
(b_3(x \otimes y \otimes z))^a = b_3(-ax \otimes y \otimes z + xy \otimes a \otimes z + x \otimes y \otimes za
\quad - x \otimes a \otimes yz - xz \otimes a \otimes y - xz \otimes y \otimes a).
\]

One easily checks that the couple \((L(A), \mu_A)\) is a pre-crossed Leibniz \(A\)-algebra. Moreover we have

\[
\begin{align*}
\mu_A^a(x \otimes y)(x' \otimes y') - [x \otimes y, x' \otimes y'] &= b_3([x, y] \otimes x' \otimes y' - [x, y] \otimes y' \otimes x') \\
(x \otimes y)\mu_A^a(x \otimes y) - [x \otimes y, x' \otimes y'] &= b_3(x \otimes [x', y'] \otimes y - x \otimes y \otimes [x', y']).
\end{align*}
\]

Thus the Leibniz \(A\)-algebra \((L(A), \mu_A)\) is crossed. \(\square\)

It is clear that the inclusion map \([A, A] \hookrightarrow A\) induces a structure of crossed Leibniz \(A\)-algebra on the two-sided ideal \([A, A]\), and that the map \(\mu_A : L(A) \rightarrow [A, A]\) is a morphism of crossed Leibniz \(A\)-algebras. Moreover we have an exact sequence of \(\mathbb{K}\)-modules

\[
0 \rightarrow \text{HH}_1(A) \rightarrow L(A) \xrightarrow{\mu_A} [A, A] \rightarrow 0.
\]

**Lemma 7.2.** — The Leibniz algebra \(A\) acts trivially on \(\text{HH}_1(A)\).

**Proof.** — One easily checks that

\[
\begin{align*}
\alpha(x \otimes y) &= a \otimes [x, y] + b_3(a \otimes x \otimes y - a \otimes y \otimes x) \equiv a \otimes [x, y] \text{ in } L(A)
\end{align*}
\]

and

\[
\begin{align*}
(x \otimes y)^a &= [x, y] \otimes a + b_3(x \otimes a \otimes y - x \otimes y \otimes a) \equiv [x, y] \otimes a \text{ in } L(A).
\end{align*}
\]

Therefore, if \(\omega = \sum \lambda_i(x_i \otimes y_i) \in \text{HH}_1(A)\), that is \(\sum \lambda_i[x_i, y_i] = 0\), then we have

\[
\begin{align*}
\alpha \omega &= \sum \lambda_i \alpha(x_i \otimes y_i) \equiv \sum \lambda_i(a \otimes [x_i, y_i]) \equiv a \otimes \sum \lambda_i[x_i, y_i] = 0
\end{align*}
\]

and

\[
\begin{align*}
\omega^a &= \sum \lambda_i(x_i \otimes y_i)^a \equiv \sum \lambda_i([x_i, y_i] \otimes a) \equiv (\sum \lambda_i[x_i, y_i]) \otimes a = 0
\end{align*}
\]

for any \(a \in A\). \(\square\)

As an immediate consequence, we get the following
COROLLARY 7.3. — The sequence

\[ 0 \to \text{HH}_1(A) \to L(A) \xrightarrow{\mu A} [A, A] \to 0 \]

is an exact sequence of crossed Leibniz $A$-algebras. \[\square\]

We deduce from Proposition 6.4 an exact sequence of $K$-modules

\[ \begin{align*} 
\text{HH}_1(A, \text{HH}_1(A)) &\to \text{HH}_1(A, L(A)) \to \text{HH}_1(A, [A, A]) \\
&\to \text{HH}_0(A, \text{HH}_1(A)) \to \text{HH}_0(A, L(A)) \to \text{HH}_0(A, [A, A]) \to 0.
\end{align*} \]

Since $A$ and $\text{HH}_1(A)$ act trivially on each other, we have

\[ \text{HH}_0(A, \text{HH}_1(A)) = \text{HH}_1(A) \]

and

\[ \text{HH}_1(A, \text{HH}_1(A)) = \mathbb{A} \ast \text{HH}_1(A) \cong \mathbb{A}/[\mathbb{A}, \mathbb{A}] \otimes \text{HH}_1(A) \cong \text{HH}_1(A) \otimes \mathbb{A}/[\mathbb{A}, \mathbb{A}]. \]

On the other hand, it is clear that

\[ \text{HH}_1(A, [A, A]) \cong [A, A]/[A, [A, A]]. \]

Therefore we can state

THEOREM 7.4. — For any associative algebra $A$ with unit, there exists an exact sequence of $K$-modules

\[ \begin{align*} 
A/[A, A] \otimes \text{HH}_1(A) \oplus \text{HH}_1(A) \oplus A/[A, A] &\to \text{HH}_1(A, L(A)) \to \text{HH}_1(A, [A, A]) \\
&\to \text{HH}_1(A) \to \text{HH}_1^M(A) \to [A, A]/[A, [A, A]] \to 0
\end{align*} \]

where $\text{HH}_1^M(A)$ denotes the Milnor-type Hochschild homology of $A$.

Proof. — Recall that $\text{HH}_1^M(A)$ is defined to be the quotient of $A \otimes A$ by the relations

\[ a \otimes [b, c] = 0, \quad [a, b] \otimes c = 0, \quad b_3(a \otimes b \otimes c) = 0 \]

for any $a, b, c \in A$ (see [6, 10.6.19]). By definition $L(A) = A \otimes A/\text{im}(b_3)$ and from the proof of Lemma 7.2, we get

\[ \Psi_{L(A)}(a \ast (x \otimes y)) = a^* (x \otimes y) \equiv a \otimes [x, y] \]

and

\[ \Psi_{L(A)}((x \otimes y) \ast a) = (x \otimes y)^a \equiv [x, y] \otimes a. \]

Therefore it is clear that $\text{HH}_0(A, L(A)) = \text{coker}(\xi_0(A))$ is isomorphic to $\text{HH}_1^M(A)$. \[\square\]
Remark. — The $\mathbb{K}$-modules $HH_1(A)$ and $HH^M_1(A)$ coincide when the associative algebra $A$ is superperfect as a Leibniz algebra that is, $A = [A, A]$ and $HL_2(A) = 0$. Also, if the associative algebra $A$ is commutative, then we have

$$HH_1(A) \cong HH^M_1(A) \cong \Omega^1_{A|\mathbb{K}}.$$

Let us also mention that the Milnor-type Hochschild homology appears in the description of the obstruction to the stability

$$HL_n(gl_{n-1}(A)) \to HL_n(gl_n(A)) \to HH^M_{n-1}(A) \to 0$$

where $gl_n(A)$ is the Lie algebra of matrices with entries in the associative algebra $A$ (see [2], [6, 10.6.20]).

Acknowledgements. It is a pleasure to warmly thank E. Graham, D. Guin, A. Kuku, M. Livernet, J.-L. Loday and M. Wambst for pertinent comments and suggestions improving this text. Also, I am grateful to UNESCO and the Abdus Salam ICTP (Trieste, Italy) for support and hospitality. Particular thoughts to Mara Chiandotto for her medical advices.

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