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A NON-ABELIAN TENSOR PRODUCT OF LEIBNIZ ALGEBRAS

by **Allahtan V. GNEDBAYE**

Introduction.

Let $\mathfrak{g}$ be a Lie algebra and let M be a representation of $\mathfrak{g}$, seen as a right $\mathfrak{g}$-module. Given a $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{g}$, one can endow the $\mathbb{K}$-module M with a bracket ($[m, m'] := m^{\mu(m')}$) which is not skew-symmetric but satisfies the *Leibniz rule of derivations*:

$$[m, [m', m'']] = [[m, m'], m''] - [[m, m''], m'].$$

Such objects were baptized *Leibniz algebras* by Jean-Louis Loday and are studied as a non-commutative variation of Lie algebras (see [8]). One of the main examples of Lie algebras comes from the notion of *derivations*. For the Leibniz algebras, there is an analogue notion of *biderivations* (see [7]).

The aim of this article is to "integrate" the Leibniz algebra of biderivations by means of a non-abelian tensor product of Leibniz algebras as it is done for Lie algebras.

In the classical case, D. Guin (see [5]) has shown that, given crossed Lie $\mathfrak{g}$-algebras $\mathfrak{M}$ and $\mathfrak{N}$, the set of derivations $\mathrm{Der}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ has a structure of pre-crossed Lie $\mathfrak{g}$-algebra. Moreover the functor $\mathrm{Der}_{\mathfrak{g}}(\mathfrak{N}, -)$ is right adjoint to the functor $-\otimes_{\mathfrak{g}}\mathfrak{N}$ where $-\otimes_{\mathfrak{g}}-$ is the non-abelian tensor product of Lie algebras defined by G. J. Ellis (see [3]). D. Guin uses these objects

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to construct a non-abelian (co)homology theory for Lie algebras, which enables him to compare the $\mathbb{K}$-modules $\mathrm{HC}_1(A)$ and $\mathrm{K}_2^{M\mathrm{add}}(A)$ where A is an arbitrary associative algebra. We give a non-commutative version of his results, in the sense that Leibniz algebras play the role of Lie algebras, the additive Milnor K-theory $\mathrm{K}_*^{M\mathrm{add}}(A)$ (resp. the cyclic homology $\mathrm{HC}_*(A)$) being replaced by the Milnor-type Hochschild homology $\mathrm{HH}_*^M(A)$ (resp. the classical Hochschild homology $\mathrm{HH}_*(A)$).

To this end, we introduce the notion of *(pre)crossed* Leibniz $\mathfrak{g}$-algebra as a simultaneous generalization of notions of representation and two-sided ideal of the Leibniz algebra $\mathfrak{g}$. Given crossed Leibniz $\mathfrak{g}$-algebras $\mathfrak{M}$ and $\mathfrak{N}$, we equip the set $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ of biderivations with a structure of pre-crossed Leibniz $\mathfrak{g}$-algebra. On the other hand, we construct a *non-abelian tensor product* $\mathfrak{M}\star\mathfrak{N}$ of Leibniz algebras with mutual actions on one another. When $\mathfrak{M}$ and $\mathfrak{N}$ are crossed Leibniz $\mathfrak{g}$-algebras, this tensor product has also a structure of crossed Leibniz $\mathfrak{g}$-algebra. It turns out that the functor $-\star_{\mathfrak{g}}\mathfrak{N}$ is left adjoint to the functor $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},-)$. Another characterization of this tensor product is the following. If the Leibniz algebra $\mathfrak{g}$ is perfect (and free as a $\mathbb{K}$-module), then the Leibniz algebra $\mathfrak{g}\star\mathfrak{g}$ is the universal central extension of $\mathfrak{g}$ (see [4]). We give also low-degrees (co)homological interpretations of these objects, which yield an exact sequence of $\mathbb{K}$-modules

$$\begin{aligned} A/[A,A]\otimes\mathrm{HH}_1(A)\ \oplus\ \mathrm{HH}_1(A)\otimes A/[A,A] &\to \mathfrak{H}\mathfrak{L}_1(\mathfrak{A},\mathrm{L}(\mathfrak{A}))\\ &\to \mathfrak{H}\mathfrak{L}_1(\mathfrak{A},[\mathfrak{A},\mathfrak{A}]) \to\to \mathrm{HH}_1(\mathfrak{A}) \to \mathrm{HH}_1^{\mathfrak{M}}(\mathfrak{A}) \to [\mathfrak{A},\mathfrak{A}]/[\mathfrak{A},[\mathfrak{A},\mathfrak{A}]] \to 0\end{aligned}$$

where $\mathrm{L}(A)$ is the $\mathbb{K}$-module $A\otimes A/\operatorname{im}(b_3)$ equipped with a suitable Leibniz bracket (see section 1.2).

Throughout this paper the symbol $\mathbb{K}$ denotes a commutative ring with a unit element and $\otimes$ stands $\otimes_{\mathbb{K}}$.

Contents

1. Prerequisites on Leibniz algebras.

1.1. Leibniz algebras.

A *Leibniz algebra* is a $\mathbb{K}$-module $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called *bracket* and satisfying only the *Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for any $x, y, z \in \mathfrak{g}$. In the presence of the condition $[x, x] = 0$, the Leibniz identity is equivalent to the so-called *Jacobi identity*. Therefore Lie algebras are examples of Leibniz algebras.

A *morphism* of Leibniz algebras is a linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$f([x,y]) = [f(x), f(y)]$$

for any $x, y \in \mathfrak{g}_1$. It is clear that Leibniz algebras and their morphisms form a category that we denote by (**Leib**).

A *two-sided ideal* of a Leibniz algebra $\mathfrak{g}$ is a submodule $\mathfrak{h}$ such that $[x, y] \in \mathfrak{h}$ and $[y, x] \in \mathfrak{h}$ for any $x \in \mathfrak{h}$ and any $y \in \mathfrak{g}$. For any two-sided ideal $\mathfrak{h}$ in $\mathfrak{g}$, the quotient module $\mathfrak{g}/\mathfrak{h}$ inherits a structure of Leibniz algebra induced by the bracket of $\mathfrak{g}$. In particular, let $([x, x])$ be the two-sided ideal in $\mathfrak{g}$ generated by all brackets $[x, x]$. The Leibniz algebra $\mathfrak{g}/([\mathfrak{x},\mathfrak{x}])$ is in fact a Lie algebra, said *canonically associated* to $\mathfrak{g}$ and is denoted by $\mathfrak{g}_{\mathrm{Lie}}$.

Let $\mathfrak{g}$ be a Leibniz algebra. Denote by $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ the submodule generated by all brackets $[x, y]$. The Leibniz algebra $\mathfrak{g}$ is said to be *perfect* if $\mathfrak{g}' = \mathfrak{g}$. It is clear that any submodule of $\mathfrak{g}$ containing $\mathfrak{g}'$ is a two-sided ideal in $\mathfrak{g}$.

1.2. Examples.

Let M be a representation of a Lie algebra $\mathfrak{g}$ (the action of $\mathfrak{g}$ on M being denoted by m^g for $m \in M$ and $g \in \mathfrak{g}$). For any $\mathfrak{g}$-equivariant map $\mu : M \to \mathfrak{g}$, the bracket given by $[m, m'] := m^{\mu(m')}$ induces a structure of Leibniz (non-Lie) algebra on M. Observe that any Leibniz algebra $\mathfrak{g}$ can be obtained in such a way by taking the canonical projection $\mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$ (which is obviously $\mathfrak{g}_{\mathrm{Lie}}$-equivariant).

Let A be an associative algebra and let $b_3 : A^{\otimes 3} \to A^{\otimes 2}$ be the Hochschild boundary that is, the linear map defined by

$$b_3(a \otimes b \otimes c) := ab \otimes c - a \otimes bc + ca \otimes b, \;\; a, b, c \in A.$$

Then the bracket given by

$$[a \otimes b, c \otimes d] := (ab - ba) \otimes (cd - dc),\ a, b, c, d \in A,$$

defines a structure of Leibniz algebra on the $\mathbb{K}$-module $\mathrm{L}(A) := A^{\otimes 2}/\operatorname{im}(b_3)$. Moreover, we have an exact sequence of $\mathbb{K}$-modules

$$0 \to \mathrm{HH}_1(A) \to \mathrm{L}(A) \xrightarrow{b_2} A \to \mathrm{HH}_0(A)$$

where $\mathrm{HH}_*(A)$ denotes the Hochschild homology groups and $b_2(x, y) = [x, y] := xy - yx$ for any $x, y \in A$.

1.3. Free Leibniz algebra.

Let V be a $\mathbb{K}$-module and let $\overline{\mathrm{T}}(V) := \bigoplus_{n \geq 1} V^{\otimes n}$ be the reduced tensor module. The bracket defined inductively by

$$[x, v] = x \otimes v, \text{ if } x \in \overline{\mathrm{T}}(V) \text{ and } v \in V$$
$$[x, y \otimes v] = [x, y] \otimes v - [x \otimes v, y], \text{ if } x, y \in \overline{\mathrm{T}}(V) \text{ and } v \in V,$$

satisfies the Leibniz identity. The Leibniz algebra so defined is the *free Leibniz algebra* over V and is denoted by $\mathcal{F}(V)$ (see [8]). Observe that one has

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n = [\cdots [[v_1, v_2], v_3] \cdots v_n],\ \forall\, v_1, \cdots, v_n \in V.$$

Moreover, the *free Lie algebra* over V is nothing but the Lie algebra $\mathcal{F}(V)_{\mathrm{Lie}}$.

2. Crossed Leibniz algebras.

2.1. Leibniz action.

Let $\mathfrak{g}$ and $\mathfrak{M}$ be Leibniz algebras. A *Leibniz action* of $\mathfrak{g}$ on $\mathfrak{M}$ is a couple of bilinear maps

$$\mathfrak{g} \times \mathfrak{M} \to \mathfrak{M},\ (\mathfrak{g}, \mathfrak{m}) \mapsto {}^{\mathfrak{g}}\mathfrak{m} \quad \text{and} \quad \mathfrak{M} \times \mathfrak{g} \to \mathfrak{M}, (\mathfrak{m}, \mathfrak{g}) \mapsto \mathfrak{m}^{\mathfrak{g}}$$

satisfying the axioms

i) $m^{[g,g']} = (m^g)^{g'} - (m^{g'})^g$,

ii) ${}^{[g,g']}m = ({}^g m)^{g'} - {}^g(m^{g'})$,

iii) ${}^{g}({}^{g'}m) = -{}^{g}(m^{g'})$,

iv) ${}^{g}[m,m'] = [{}^{g}m,m'] - [{}^{g}m',m]$,

v) $[m,m']^{g} = [m^{g},m'] + [m,m'^{g}]$,

vi) $[m,{}^{g}m'] = -[m,m'^{g}]$

for any $m,m' \in \mathfrak{M}$ and $g,g' \in \mathfrak{g}$. We say that $\mathfrak{M}$ is a *Leibniz* $\mathfrak{g}$*-algebra.* Observe that the axiom i) applied to the triples $(m;g,g')$ and $(m;g',g)$ yields the relation

$$m^{[g,g']} = -m^{[g',g]}.$$

2.2. Examples.

Any two-sided ideal of a Leibniz algebra $\mathfrak{g}$ is a Leibniz $\mathfrak{g}$-algebra, the action being given by the initial bracket.

A $\mathbb{K}$-module M equipped with two operations of a Leibniz algebra $\mathfrak{g}$ satisfying the axioms i), ii) and iii) is called a *representation of* $\mathfrak{g}$ (see [8]). Therefore representations of a Leibniz algebra $\mathfrak{g}$ are abelian Leibniz $\mathfrak{g}$-algebras.

2.3. Crossed Leibniz algebras.

Let $\mathfrak{g}$ be a Leibniz algebra. A *pre-crossed Leibniz* $\mathfrak{g}$*-algebra* is a Leibniz $\mathfrak{g}$-algebra $\mathfrak{M}$ equipped with a morphism of Leibniz algebras $\mu : \mathfrak{M} \to \mathfrak{g}$ such that

$$\mu({}^{g}m) = [g,\mu(m)] \quad \text{and} \quad \mu(m^{g}) = [\mu(m),g]$$

for any $g \in \mathfrak{g}$ and $m \in \mathfrak{M}$. Moreover if the relations

$${}^{\mu(m)}m' = [m,m'] \quad \text{and} \quad m^{\mu(m')} = [m,m'],\ \forall\, m,m' \in \mathfrak{M},$$

hold, then $(\mathfrak{M},\mu)$ is called a *crossed Leibniz* $\mathfrak{g}$*-algebra.*

2.4. Examples.

Any Leibniz algebra $\mathfrak{g}$, equipped with the identity map $\mathrm{id}_{\mathfrak{g}}$, is a crossed Leibniz $\mathfrak{g}$-algebra.

Any two-sided ideal $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{g}$, equipped with the inclusion map $\mathfrak{h} \hookrightarrow \mathfrak{g}$, is a crossed Leibniz $\mathfrak{g}$-algebra.

Let $\alpha : \mathfrak{c} \twoheadrightarrow \mathfrak{g}$ be a central extension of Leibniz algebras (i.e., a surjective morphism whose kernel is contained in the centre of $\mathfrak{c}$, see [4]). Define operations of $\mathfrak{g}$ on $\mathfrak{c}$ by

$$^{g}c := [\alpha^{-1}(g), c] \quad \text{and} \quad c^{g} := [c, \alpha^{-1}(g)]$$

where $\alpha^{-1}(g)$ is any pre-image of g in $\mathfrak{c}$. Then $(\mathfrak{c}, \alpha)$ is a crossed Leibniz $\mathfrak{g}$-algebra.

PROPOSITION 2.1. — *For any pre-crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M}, \mu)$, the image* $\mathrm{im}(\mu)$ *(resp. the kernel* $\ker(\mu)$*) is a two-sided ideal in $\mathfrak{g}$ (resp. $\mathfrak{M}$). Moreover, if $(\mathfrak{M}, \mu)$ is crossed, then* $\ker(\mu)$ *is contained in the centre of $\mathfrak{M}$.*

Proof. — Let m be an element of $\mathfrak{M}$. For any $g \in \mathfrak{g}$, we have

$$[\mu(m), g] = \mu(m^{g}) \in \mathrm{im}(\mu) \quad \text{and} \quad [g, \mu(m)] = \mu(^{g}m) \in \mathrm{im}(\mu).$$

Thus, $\mathrm{im}(\mu)$ is a two-sided ideal in $\mathfrak{g}$. Assume that $m \in \ker(\mu)$; then for any $m' \in \mathfrak{M}$, we have

$$\mu([m, m']) = [\mu(m), \mu(m')] = 0 = [\mu(m'), \mu(m)] = \mu([m', m]).$$

Therefore $\ker(\mu)$ is a two-sided ideal in $\mathfrak{M}$. Moreover if the Leibniz action of $\mathfrak{g}$ on $\mathfrak{M}$ is crossed, then we have

$$[m, m'] = {}^{\mu(m)}m' = 0 = m'^{\mu(m)} = [m', m]$$

for any $m \in \ker(\mu)$ and $m' \in \mathfrak{M}$. Thus $\ker(\mu)$ is contained in the centre of $\mathfrak{M}$. □

2.5. Morphism of pre-crossed Leibniz algebras.

Let $\mathfrak{g}$ be a Leibniz algebra and let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. A *morphism* from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ is a Leibniz algebra morphism $f : \mathfrak{M} \to \mathfrak{N}$ such that

$$f(^{g}m) = {}^{g}(f(m)), \; f(m^{g}) = (f(m))^{g} \text{ and } \mu = \nu f$$

for any $m \in \mathfrak{M}$ and $g \in \mathfrak{g}$. A *morphism of crossed Leibniz* $\mathfrak{g}$*-algebras* is the same as a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras. It is clear that pre-crossed (resp. crossed) Leibniz $\mathfrak{g}$-algebras and their morphisms form a category that we denote by $(\mathbf{pc\text{-}Leib}(\mathfrak{g}))$ (resp. $(\mathbf{c\text{-}Leib}(\mathfrak{g}))$).

PROPOSITION 2.2. — *Let $f : (\mathfrak{M}, \mu) \to (\mathfrak{N}, \nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra morphism. Then $(\mathfrak{M}, \mathfrak{f})$ is a crossed Leibniz $\mathfrak{N}$-algebra via the Leibniz action of $\mathfrak{N}$ on $\mathfrak{M}$ given by*

$$ {}^{n}m := {}^{\nu(n)}m \quad \text{and} \quad m^{n} := m^{\nu(n)}, \ \forall\, m \in \mathfrak{M}, \mathfrak{n} \in \mathfrak{N}. $$

Proof. — One easily checks that $\mathfrak{M}$ is a Leibniz $\mathfrak{N}$-algebra. For any $m, m' \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have

$$ f({}^{n}m) = f({}^{\nu(n)}m) = {}^{\nu(n)}f(m) = [n, f(m)], $$
$$ f(m^{n}) = f(m^{\nu(n)}) = f(m)^{\nu(n)} = [f(m), n]; $$

thus $(\mathfrak{M}, \mathfrak{f})$ is a pre-crossed Leibniz $\mathfrak{N}$-algebra. Moreover we have

$$ {}^{f(m)}m' = {}^{\nu(f(m))}m' = {}^{\mu(m)}m' = [m, m'], $$
$$ m^{f(m')} = m^{\nu(f(m'))} = m^{\mu(m')} = [m, m']; $$

thus $(\mathfrak{M}, \mathfrak{f})$ is a crossed Leibniz $\mathfrak{N}$-algebra. □

2.6. Exact sequences.

We say that a sequence

$$ (\mathfrak{L}, \lambda) \xrightarrow{\alpha} (\mathfrak{M}, \mu) \xrightarrow{\beta} (\mathfrak{N}, \nu) $$

is *exact* in the category $(\mathbf{pc\text{-}Leib}(\mathfrak{g}))$ (resp. $(\mathbf{c\text{-}Leib}(\mathfrak{g}))$ if the sequence

$$ \mathfrak{L} \xrightarrow{\alpha} \mathfrak{M} \xrightarrow{\beta} \mathfrak{N} $$

is exact as sequence of Leibniz algebras.

PROPOSITION 2.3. — *If the sequence*

$$ (\mathfrak{L}, \lambda) \xrightarrow{\alpha} (\mathfrak{M}, \mu) \xrightarrow{\beta} (\mathfrak{N}, \nu) $$

is exact in the category $(\mathbf{pc\text{-}Leib}(\mathfrak{g}))$ (resp. $(\mathbf{c\text{-}Leib}(\mathfrak{g}))$), then the map λ is zero. Moreover if the Leibniz $\mathfrak{g}$-algebra $(\mathfrak{L}, \lambda)$ is crossed, then the Leibniz algebra $\mathfrak{L}$ is abelian.

Proof. — Indeed, since $\beta\alpha = 0$, we have $\lambda = \nu\beta\alpha = 0$. From whence $\ker(\lambda) = \mathfrak{L}$, and by Proposition 2.1, it is clear that the Leibniz algebra $\mathfrak{L}$ is abelian. □

3. Biderivations of Leibniz algebras.

In this section, we fix a Leibniz algebra $\mathfrak{g}$.

3.1. Derivations and anti-derivations.

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. A *derivation* from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ is a linear map $d : \mathfrak{M} \to \mathfrak{N}$ such that

$$d([m, m']) = d(m)^{\mu(m')} + {}^{\mu(m)}d(m'),\ \forall\ m, m' \in \mathfrak{M}.$$

An *anti-derivation* from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ is a linear map $D : \mathfrak{M} \to \mathfrak{N}$ such that

$$D([m, m']) = D(m)^{\mu(m')} - D(m')^{\mu(m)},\ \forall\ m, m' \in \mathfrak{M}.$$

3.2. Examples.

Let $(\mathfrak{N}, \nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra and let n be any element of $\mathfrak{N}$. By the axiom iii) (resp. i)) of 2.1, the linear map

$$\mathfrak{g} \to \mathfrak{N},\ \mathfrak{g} \mapsto {}^{\mathfrak{g}}\mathfrak{n} \quad (\mathfrak{resp.}\ \mathfrak{g} \to \mathfrak{N},\ \mathfrak{g} \mapsto -\mathfrak{n}^{\mathfrak{g}})$$

is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$ to $(\mathfrak{N}, \nu)$.

3.3. Biderivations.

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. We denote by $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ the free $\mathbb{K}$-module generated by the triples (d, D, g), where d (resp. D) is a derivation (resp. an anti-derivation) from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ and g is an element of $\mathfrak{g}$ such that

$$\nu(d(m)) = \mu(m^g),\ \nu(D(m)) = -\mu({}^g m),$$
$${}^h d(m) = {}^h D(m),\ D(m^h) = -D({}^h m)$$

for any $h \in \mathfrak{g}$ and $m \in \mathfrak{M}$.

PROPOSITION 3.1. — *If the Leibniz $\mathfrak{g}$-algebra $(\mathfrak{N}, \nu)$ is crossed, then there is a Leibniz algebra structure on the $\mathbb{K}$-module* $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ *for the bracket defined by*

$$[(d, D, g), (d', D', g')] := (\delta, \Delta, [g, g'])$$

where

$$\delta(m) := d'(m^g) - d(m^{g'}) \quad \text{and} \quad \Delta(m) = -D(m^{g'}) - d'({}^g m), \ \forall\ m \in \mathfrak{M}.$$

Proof. — Let us show that the maps δ and Δ are respectively a derivation and an anti-derivation. Indeed, for any $m, m' \in \mathfrak{M}$, we have

$$\begin{aligned}
\delta([m, m']) &= d'([m, m']^g) - d([m, m']^{g'}) \\
&= d'([m^g, m']) + d'([m, m'^g]) - d([m^{g'}, m']) - d([m, m'^{g'}]) \\
&= d'(m^g)^{\mu(m')} + {}^{\mu(m^g)}d'(m') + d'(m)^{\mu(m'^g)} + {}^{\mu(m)}d'(m'^g) \\
&\quad - d(m^{g'})^{\mu(m')} - {}^{\mu(m^{g'})}d(m') - d(m)^{\mu(m'^{g'})} \\
&\quad - {}^{\mu(m)}d(m'^{g'}) \\
&= (d'(m^g) - d(m^{g'}))^{\mu(m')} + {}^{\mu(m)}(d'(m'^g) - d(m'^{g'})) \\
&\quad + {}^{\nu(d(m))}d'(m') + d'(m)^{\nu(d(m'))} - {}^{\nu(d'(m))}d(m') \\
&\quad - d(m)^{\nu(d'(m'))} \\
&= \delta(m)^{\mu(m')} + {}^{\mu(m)}\delta(m') + [d(m), d'(m')] \\
&\quad + [d'(m), d(m')] - [d'(m), d(m')] - [d(m), d'(m')] \\
&= \delta(m)^{\mu(m')} + {}^{\mu(m)}\delta(m')
\end{aligned}$$

and

$$\begin{aligned}
\Delta([m, m']) &= -D([m, m']^{g'}) - d'({}^g[m, m'] \\
&= -D([m^{g'}, m']) - D([m, m'^{g'}]) - d'([{}^g m, m']) + d'([{}^g m', m]) \\
&= -D(m^{g'})^{\mu(m')} + D(m')^{\mu(m^{g'})} - D(m)^{\mu(m'^{g'})} + D(m'^{g'})^{\mu(m)} \\
&\quad - d'({}^g m)^{\mu(m')} - {}^{\mu({}^g m)}d'(m') + d'({}^g m')^{\mu(m)} + {}^{\mu({}^g m')}d'(m) \\
&= (-D(m^{g'}) - d'({}^g m))^{\mu(m')} - (-D(m'^{g'}) - d'({}^g m'))^{\mu(m)} \\
&\quad + D(m')^{\nu(d'(m))} - D(m)^{\nu(d'(m'))} + {}^{\nu(D(m))}d'(m') \\
&\quad - {}^{\nu(D(m')))}d'(m) \\
&= \Delta(m)^{\mu(m')} - \Delta(m')^{\mu(m)} + [D(m'), d'(m)] \\
&\quad - [D(m), d'(m')] + [D(m), d'(m')] - [D(m'), d'(m)] \\
&= \Delta(m)^{\mu(m')} - \Delta(m')^{\mu(m)}.
\end{aligned}$$

On the other hand, we have

$$\nu(\delta(m)) = \nu(d'(m^g)) - \nu(d(m^{g'})) = \mu((m^g)^{g'}) - \mu((m^{g'})^g) = \mu(m^{[g,g']}),$$

$$\nu(\Delta(m)) = -\nu(D(m^{g'})) - \nu(d'({}^g m)) = \mu({}^g(m^{g'})) - \mu(({}^g m)^{g'}) = -\mu({}^{[g,g']}m),$$

$$\begin{aligned} {}^h\delta(m) &= {}^h d'(m^g) - {}^h d(m^{g'}) = {}^h D'(m^g) - {}^h D(m^{g'}) \\ &= -\,{}^h D'({}^g m) - {}^h D(m^{g'}) = -{}^h d'({}^g m) - {}^h D(m^{g'}) \\ &= {}^h\Delta(m), \end{aligned}$$

$$\begin{aligned} \Delta({}^h m) &= -\,D(({}^h m)^{g'}) - d'({}^g({}^h m)) \\ &= -\,D({}^{[h,g']}m) - D({}^h(m^{g'})) + d'({}^g(m^h)) \\ &= D((m^h)^{g'}) + d'({}^g(m^h)) = -\Delta(m^h). \end{aligned}$$

Therefore the triple $(\delta, \Delta, [g, g'])$ is a biderivation from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$. Moreover, let (d, D, g), (d', D', g') and (d'', D'', g'') be biderivations from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$. We set

$$\begin{aligned} (\delta, \Delta, [g', g'']) &:= [(d', D', g'), (d'', D'', g'')], \\ &\qquad (\delta_0, \Delta_0, g_0) := [(d, D, g), (\delta, \Delta, [g', g''])], \\ (\delta', \Delta', [g, g']) &:= [(d, D, g), (d', D', g')], \\ &\qquad (\delta_1, \Delta_1, g_1) := [(\delta', \Delta', [g, g']), (d'', D'', g'')], \\ (\delta'', \Delta'', [g, g'']) &:= [(d, D, g), (d'', D'', g'')], \\ &\qquad (\delta_2, \Delta_2, g_2) := [(\delta'', \Delta'', [g, g'']), (d', D', g')]. \end{aligned}$$

It is clear that $g_0 = g_1 - g_2$. For any $m \in \mathfrak{M}$, we have

$$\begin{aligned} (\delta_1 - \delta_2)(m) &= d''(m^{[g,g']}) - \delta'(m^{g''}) - d'(m^{[g,g'']}) + \delta''(m^{g'}) \\ &= d''((m^g)^{g'}) - d''((m^{g'})^g) - d'((m^{g''})^g) + d((m^{g''})^{g'}) \\ &\quad - d'((m^g)^{g''}) + d'((m^{g''})^g) + d''((m^{g'})^g) - d((m^{g'})^{g''}) \\ &= d''((m^g)^{g'}) - d'((m^g)^{g''}) - d(m^{[g',g'']}) \\ &= \delta(m^g) - d(m^{[g',g'']}) = \delta_0(m) \end{aligned}$$

and

$$\begin{aligned} (\Delta_1 - \Delta_2)(m) &= -\,\Delta'(m^{g''}) - d''({}^{[g,g']}m) + \Delta''(m^{g'}) + d'({}^{[g,g'']}m) \\ &= D((m^{g''})^{g'}) + d'({}^g(m^{g''})) - d''(({}^g m)^{g'}) + d''({}^g(m^{g'})) \\ &\quad - D((m^{g'})^{g''}) - d''({}^g(m^{g'})) + d'(({}^g m)^{g''}) - d'({}^g(m^{g''})) \\ &= -\,D(m^{[g',g'']}) - d''(({}^g m)^{g'}) + d'(({}^g m)^{g''}) \\ &= -\,D(m^{[g',g'']}) - \delta({}^g m) = \Delta_0(m). \end{aligned}$$

Therefore the $\mathbb{K}$-module $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ is a Leibniz algebra. □

Let us equip the set $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ with a Leibniz action of $\mathfrak{g}$.

PROPOSITION 3.2. — *Let $(\mathfrak{M},\mu)$ (resp. $(\mathfrak{N},\nu)$) be a pre-crossed (resp. crossed) Leibniz $\mathfrak{g}$-algebra. The set $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M},\mathfrak{N})$ is a pre-crossed Leibniz $\mathfrak{g}$-algebra for the operations defined by*

$$^{h}(d,D,g) := ({}^{h}d, {}^{h}D, [h,g]) \quad \text{and} \quad (d,D,g)^{h} := (d^{h}, D^{h}, [g,h])$$

where

$$\begin{aligned} ({}^{h}d)(m) = d(m^{h}) - d(m)^{h},\ ({}^{h}D)(m) := {}^{h}d(m) - d({}^{h}m), \\ (d^{h})(m) := d(m)^{h} - d(m^{h}),\ (D^{h})(m) := D(m)^{h} - D(m^{h}). \end{aligned}$$

Proof. — Everything can be smoothly checked and we merely give an example of these verifications. By definition we have

$$\begin{aligned} {}^{h}[(d,D,g),(d',D',g')] &= ({}^{h}\delta, {}^{h}\Delta, [h,[g,g']]), \\ [{}^{h}(d,D,g),(d',D',g')] &= (\delta_1, \Delta_1, [[h,g],g']), \\ [{}^{h}(d',D',g'),(d,D,g)] &= (\delta_2, \Delta_2, [[h,g'],g]). \end{aligned}$$

For any $m \in \mathfrak{M}$ we have

$$\begin{aligned} (\delta_1 - \delta_2)(m) &= d'(m^{[h,g]}) - ({}^{h}d)(m^{g'}) - d(m^{[h,g']}) + ({}^{h}d')(m^{g}) \\ &= d'((m^{h})^{g}) - d'((m^{g})^{h}) - d((m^{g'})^{h}) + d(m^{g'})^{h} \\ &\quad - d((m^{h})^{g'}) + d((m^{g'})^{h}) + d'((m^{g})^{h}) - d'(m^{g})^{h} \\ &= (d'((m^{h})^{g}) - d((m^{h})^{g'})) - (d'(m^{g}) - d(m^{g'}))^{h} \\ &= \delta(m^{h}) - \delta(m)^{h} = ({}^{h}\delta)(m) \end{aligned}$$

and

$$\begin{aligned} (\Delta_1 - \Delta_2)(m) &= -({}^{h}D)(m^{g'}) - d'({}^{[h,g]}m) + ({}^{h}D')(m^{g}) + d({}^{[h,g']}m) \\ &= -{}^{h}D(m^{g'}) + d({}^{h}(m^{g'})) - d'(({}^{h}m)^{g}) + d'({}^{h}(m^{g})) \\ &\quad + {}^{h}D'(m^{g}) - d'({}^{h}(m^{g})) + d(({}^{h}m)^{g'}) - d({}^{h}(m^{g'})) \\ &= {}^{h}(D'(m^{g}) - D(m^{g'})) - (d'(({}^{h}m)^{g}) - d(({}^{h}m)^{g'})) \\ &= {}^{h}\delta(m) - \delta({}^{h}m) = ({}^{h}\Delta)(m). \end{aligned}$$

Thus we get

$${}^{h}[(d,D,g),(d',D',g')] = [{}^{h}(d,D,g),(d',D',g')] - [{}^{h}(d',D',g'),(d,D,g)].$$

□

Now we can state the fundamental result which is a consequence of Propositions 3.1 and 3.2.

THEOREM 3.3. — *For any pre-crossed (resp. crossed) Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M}, \mu)$ (resp. $(\mathfrak{N}, \nu)$), the Leibniz $\mathfrak{g}$-algebra* $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ *is pre-crossed for the morphism*

$$\rho : \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N}) \to \mathfrak{g},\ (\mathfrak{d}, \mathfrak{D}, \mathfrak{g}) \mapsto \mathfrak{g}. \qquad \square$$

3.4. Remarks.

For any element g of $\mathfrak{g}$, the linear map $\mathrm{ad}_g : h \mapsto [h, g]$ (resp. $\mathrm{Ad}_g : h \mapsto -[g, h]$) is a derivation (resp. an anti-derivation) of the Leibniz algebra $\mathfrak{g}$. In the classical sense (i.e., without "crossing", see [7]) the couple $(\mathrm{ad}_g, \mathrm{Ad}_g)$ is called *inner biderivation* of $\mathfrak{g}$. Therefore the pre-crossed Leibniz $\mathfrak{g}$-algebra $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, \mathfrak{N})$ can be seen as the set of biderivations from $(\mathfrak{M}, \mu)$ to $(\mathfrak{N}, \nu)$ over inner biderivations of $\mathfrak{g}$.

On the other hand, given a pre-crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M}, \mu)$, one easily checks that the map $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{M}, -)$ is a functor from the category of crossed Leibniz $\mathfrak{g}$-algebras to the category of pre-crossed Leibniz $\mathfrak{g}$-algebras.

4. Non-abelian tensor product of Leibniz algebras.

4.1. Leibniz pairings.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. A *Leibniz pairing* of $\mathfrak{M}$ and $\mathfrak{N}$ is a triple $(\mathfrak{P}, \mathfrak{h}_1, \mathfrak{h}_2)$ where $\mathfrak{P}$ is a Leibniz algebra and $h_1 : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{P}$ (resp. $h_2 : \mathfrak{N} \times \mathfrak{M} \to \mathfrak{P}$) is a bilinear map such that

$$\begin{gathered}
h_1(m, [n, n']) = h_1(m^n, n') - h_1(m^{n'}, n), \\
h_2(n, [m, m']) = h_2(n^m, m') - h_2(n^{m'}, m), \\
h_1([m, m'], n) = h_2({}^m n, m') - h_1(m, n^{m'}), \\
h_2([n, n'], m) = h_1({}^n m, n') - h_2(n, m^{n'}), \\
h_1(m, {}^{m'}n) = -h_1(m, n^{m'}),\ h_2(n, {}^{n'}m) = -h_2(n, m^{n'}), \\
h_1(m^n, {}^{m'}n') = [h_1(m, n), h_1(m', n')] = h_2({}^m n, m'^{n'}), \\
h_1({}^n m, n'^{m'}) = [h_2(n, m), h_2(n', m')] = h_2(n^m, {}^{n'}m'), \\
h_1(m^n, n'^{m'}) = [h_1(m, n), h_2(n', m')] = h_2({}^m n, {}^{n'}m'), \\
h_1({}^n m, {}^{m'}n') = [h_2(n, m), h_1(m', n')] = h_2(n^m, m'^{n'})
\end{gathered}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$.

4.2. Example.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two-sided ideals of a same Leibniz algebra $\mathfrak{g}$. Take $\mathfrak{P} := \mathfrak{M} \cap \mathfrak{N}$ and define

$$h_1(m,n) := [m,n] \quad \text{and} \quad h_2(n,m) := [n,m].$$

Then the triple $(\mathfrak{P}, \mathfrak{h}_1, \mathfrak{h}_2)$ is a Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$.

4.3. Non-abelian tensor product.

A Leibniz pairing $(\mathfrak{P}, \mathfrak{h}_1, \mathfrak{h}_2)$ of $\mathfrak{M}$ and $\mathfrak{N}$ is said to be *universal* if for any other Leibniz pairing $(\mathfrak{P}', \mathfrak{h}_1', \mathfrak{h}_2')$ of $\mathfrak{M}$ and $\mathfrak{N}$ there exists a unique Leibniz algebra morphism $\theta : \mathfrak{P} \to \mathfrak{P}'$ such that

$$\theta h_1 = h_1' \quad \text{and} \quad \theta h_2 = h_2'.$$

It is clear that a universal pairing, when it exists, is unique up to a unique isomorphism. Here is a construction of the universal pairing as a *non-abelian tensor product.*

DEFINITION-THEOREM 4.1. — *Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. Let V be the free $\mathbb{K}$-module generated by the symbols $m * n$ and $n * m$ where $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. Let $\mathfrak{M} \star \mathfrak{N}$ be the Leibniz algebra quotient of the free Leibniz algebra generated by V by the two-sided ideal defined by the relations*

i) $\lambda(m * n) = \lambda m * n = m * \lambda n,\ \lambda(n * m) = \lambda n * m = n * \lambda m,$

ii) $(m + m') * n = m * n + m' * n,\ (n + n') * m = n * m + n' * m,$
$m * (n + n') = m * n + m * n',\ n * (m + m') = n * m + n * m',$

iii) $m * [n, n'] = m^n * n' - m^{n'} * n,\ n * [m, m'] = n^m * m' - n^{m'} * m,$
$[m, m'] * n = {}^m n * m' - m * n^{m'},\ [n, n'] * m = {}^n m * n' - n * m^{n'},$

iv) $m * {}^{m'}n = -m * n^{m'},\ n * {}^{n'}m = -n * m^{n'},$

v) $m^n * {}^{m'}n' = [m * n, m' * n'] = {}^m n * m'^{n'},$
$m^n * n'^{m'} = [m * n, n' * m'] = {}^m n * {}^{n'}m',$
${}^n m * n'^{m'} = [n * m, n' * m'] = n^m * {}^{n'}m'$
${}^n m * {}^{m'}n' = [n * m, m' * n'] = n^m * m'^{n'}$
for any $\lambda \in \mathbb{K}$, $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$. Define maps

$$h_1 : \mathfrak{M} \times \mathfrak{N} \to \mathfrak{M} \star \mathfrak{N},\ \mathfrak{h}_1(\mathfrak{m}, \mathfrak{n}) := \mathfrak{m} * \mathfrak{n}$$

and

$$h_2 : \mathfrak{N} \times \mathfrak{M} \to \mathfrak{M} \star \mathfrak{N}, \ \mathfrak{h}_2(\mathfrak{n}, \mathfrak{m}) := \mathfrak{n} * \mathfrak{m}.$$

Then the triple $(\mathfrak{M} \star \mathfrak{N}, \mathfrak{h}_1, \mathfrak{h}_2)$ *is the universal Leibniz pairing of* $\mathfrak{M}$ *and* $\mathfrak{N}$ *and called the non-abelian tensor product (or tensor product for short) of* $\mathfrak{M}$ *and* $\mathfrak{N}$.

Proof. — It is straightforward to see that the triple $(\mathfrak{M} \star \mathfrak{N}, \mathfrak{h}_1, \mathfrak{h}_2)$ so-defined is a Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$. For the universality, notice that if $(\mathfrak{P}, \mathfrak{h}_1', \mathfrak{h}_2')$ is another Leibniz pairing of $\mathfrak{M}$ and $\mathfrak{N}$, then the map θ is necessarily given on generators by

$$\theta(m * n) = h_1'(m, n) \quad \text{and} \quad \theta(n * m) = h_2'(n, m)$$

for any $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$. □

As an illustration of this construction, we give now a description of the non-abelian tensor product when the actions are trivial.

PROPOSITION 4.2. — *If the Leibniz algebras* $\mathfrak{M}$ *and* $\mathfrak{N}$ *act trivially on each other, then there is an isomorphism of abelian Leibniz algebras*

$$\mathfrak{M} \star \mathfrak{N} \cong \mathfrak{M}_{\mathfrak{ab}} \otimes \mathfrak{N}_{\mathfrak{ab}} \ \oplus \ \mathfrak{N}_{\mathfrak{ab}} \otimes \mathfrak{M}_{\mathfrak{ab}}$$

where $\mathfrak{M}_{\mathfrak{ab}} := \mathfrak{M}/[\mathfrak{M}, \mathfrak{M}]$ *and* $\mathfrak{N}_{\mathfrak{ab}} := \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$.

Proof. — Recall that the underlying $\mathbb{K}$-module of the free Leibniz algebra generated by V is

$$\overline{\mathrm{T}}(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

Since the actions are trivial, the definition of the bracket on $\overline{\mathrm{T}}(V)$ and the relations v) enable us to see that $\mathfrak{M} \star \mathfrak{N}$ is an abelian Leibniz algebra and that the summands $V^{\otimes n}$ (for $n \geq 2$) are killed. Relations i) and ii) of 4.1 say that the $\mathbb{K}$-module $\mathfrak{M} \star \mathfrak{N}$ is the quotient of $\mathfrak{M} \otimes \mathfrak{N} \ \oplus \ \mathfrak{N} \otimes \mathfrak{M}$ by the relations iii). These later imply that $\mathfrak{M} \star \mathfrak{N}$ is the abelian Leibniz algebra $\mathfrak{M}_{\mathfrak{ab}} \otimes \mathfrak{N}_{\mathfrak{ab}} \ \oplus \ \mathfrak{N}_{\mathfrak{ab}} \otimes \mathfrak{M}_{\mathfrak{ab}}$. □

4.4. Compatible Leibniz actions.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual Leibniz actions on one another. We say that these actions are *compatible* if we have

$$\begin{aligned}
{}^{({}^m n)}m' &= [m^n, m'], & {}^{({}^n m)}n' &= [n^m, n'], \\
{}^{(n^m)}m' &= [{}^n m, m'], & {}^{(m^n)}n' &= [{}^m n, n'], \\
m^{({}^{m'} n)} &= [m, m'^n], & n^{({}^{n'} m)} &= [n, n'^m], \\
m^{(n^{m'})} &= [m, {}^n m'], & n^{(m^{n'})} &= [n, {}^m n']
\end{aligned}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$.

4.5. Examples.

If $\mathfrak{M}$ and $\mathfrak{N}$ are two-sided ideals of a same Leibniz algebra, then the actions (given by the initial bracket) are compatible.

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. Then one can define a Leibniz action of $\mathfrak{M}$ on $\mathfrak{N}$ (resp. of $\mathfrak{N}$ on $\mathfrak{M}$) by setting

$$^{m}n := {}^{\mu(m)}n \quad \text{and} \quad n^{m} := n^{\mu(m)}$$
$$(\text{resp. } {}^{n}m := {}^{\nu(n)}m \quad \text{and} \quad m^{n} := m^{\nu(n)}).$$

If the Leibniz $\mathfrak{g}$-algebras $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ are crossed, then these Leibniz actions are compatible.

4.6. First crossed structure.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be Leibniz algebras with mutual compatible actions on one another. Consider the operations of $\mathfrak{M}$ on $\mathfrak{M} \star \mathfrak{N}$ given by

$$^{m}(m' * n') := [m, m'] * n' - {}^{m}n' * m', \; {}^{m}(n' * m') := {}^{m}n' * m' - [m, m'] * n',$$
$$(m * n)^{m'} := [m, m'] * n + m * n^{m'}, \; (n * m)^{m'} := n^{m'} * m + n * [m, m']$$

and those of $\mathfrak{N}$ on $\mathfrak{M} \star \mathfrak{N}$ given by

$$^{n}(m' * n') := {}^{n}m' * n' - [n, n'] * m', \; {}^{n}(n' * m') := [n, n'] * m' - {}^{n}m' * n',$$
$$(m * n)^{n'} := m^{n'} * n + m * [n, n'], \; (n * m)^{n'} := [n, n'] * m + n * m^{n'}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$. Then we have

PROPOSITION 4.3. — *With the above operations, the map*

$$\mu : \mathfrak{M} \star \mathfrak{N} \to \mathfrak{M}, \; \mathrm{m} * \mathrm{n} \mapsto \mathrm{m}^{\mathrm{n}}, \; \mathrm{n} * \mathrm{m} \mapsto {}^{\mathrm{n}}\mathrm{m}$$
$$(\textit{resp. } \nu : \mathfrak{M} \star \mathfrak{N} \to \mathfrak{N}, \; \mathrm{m} * \mathrm{n} \mapsto {}^{\mathrm{m}}\mathrm{n}, \; \mathrm{n} * \mathrm{m} \mapsto \mathrm{n}^{\mathrm{m}})$$

induces on $\mathfrak{M} \star \mathfrak{N}$ a structure of crossed Leibniz $\mathfrak{M}$-algebra (resp. $\mathfrak{N}$-algebra).

Proof. — Once again everything can be readily checked thanks to the compatibility conditions. For example we have

$$\begin{aligned} {}^{\mu(m*n)}(m' * n') &= {}^{m^{n}}(m' * n') = [m^{n}, m'] * n' - {}^{(m^{n})}n' * m' \\ &= {}^{(m^{n})}n' * m' - m^{n} * n'^{m'} - {}^{(m^{n})}n' * m' \\ &= m^{n} * {}^{m'}n = [m * n, m' * n'] \end{aligned}$$

for any $m, m' \in \mathfrak{M}$ and $n, n' \in \mathfrak{N}$. □

4.7. Second crossed structure.

Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras, equipped with the mutual Leibniz actions given in Examples 4.5. One easily checks that the operations given by

$$
\begin{aligned}
&{}^{g}(m * n) := {}^{g}m * n - {}^{g}n * m, \ {}^{g}(n * m) := {}^{g}n * m - {}^{g}m * n, \\
&(m * n)^{g} := m^{g} * n + m * n^{g}, \ (n * m)^{g} := n^{g} * m + n * m^{g},
\end{aligned}
$$

define a Leibniz action of $\mathfrak{g}$ on $\mathfrak{M} \star \mathfrak{N}$.

PROPOSITION 4.4. — *Let $(\mathfrak{M}, \mu)$ and $(\mathfrak{N}, \nu)$ be pre-crossed Leibniz $\mathfrak{g}$-algebras. Then the map $\eta : \mathfrak{M} \star \mathfrak{N} \to \mathfrak{g}$ defined on generators by*

$$
\eta(m * n) := [\mu(m), \nu(n)] \quad \text{and} \quad \eta(n * m) := [\nu(n), \mu(m)],
$$

confers to $\mathfrak{M} \star \mathfrak{N}$ a structure of pre-crossed Leibniz $\mathfrak{g}$-algebra. Moreover, if one of the Leibniz $\mathfrak{g}$-algebras $\mathfrak{M}$ or $\mathfrak{N}$ is crossed, then the Leibniz $\mathfrak{g}$-algebra $\mathfrak{M} \star \mathfrak{N}$ is crossed.

Proof. — It is immediate to check that the map η passes to the quotient and defines a Leibniz algebra morphism. Moreover we have

$$
\begin{aligned}
\eta({}^{g}(m * n)) &= [\mu({}^{g}m), \nu(n)] - [\nu({}^{g}n), \mu(m)] \\
&= [[g, \mu(m)], \nu(n)] - [[g, \nu(n)], \mu(m)] \\
&= [g, [\mu(m), \nu(n)]] = [g, \eta(m * n)]; \\
\eta({}^{g}(n * m)) &= -\eta({}^{g}(m * n)) = -[g, \eta(m * n)] \\
&= -[g, [\mu(m), \nu(n)]] = [g, [\nu(n), \mu(m)]] = [g, \eta(n * m)]; \\
\eta((m * n)^{g}) &= [\mu(m^{g}), \nu(n)] + [\mu(m), \nu(n^{g})] \\
&= [[\mu(m), g], \nu(n)] + [\mu(m), [\nu(n), g]] \\
&= [[\mu(m), \nu(n)], g] = [\eta(m * n), g]; \\
\eta((n * m)^{g}) &= [\nu(n^{g}), \mu(m)] + [\nu(n), \mu(m^{g})] \\
&= [[\nu(n), g], \mu(m)] + [\nu(n), [\mu(m), g]] \\
&= [[\nu(n), \mu(m)], g] = [\eta(n * m), g];
\end{aligned}
$$

thus $(\mathfrak{M} \star \mathfrak{N}, \eta)$ is a pre-crossed Leibniz $\mathfrak{g}$-algebra. Assume that, for instance, the Leibniz $\mathfrak{g}$-algebra $\mathfrak{M}$ is crossed. Then we have

$$\begin{aligned}
{}^{\eta(m*n)}(m' * n') &= {}^{[\mu(m),\nu(n)]}(m' * n') = {}^{\mu(m^{\nu(n)})}(m' * n') \\
&= {}^{\mu(m^{\nu(n)})}m' * n' - {}^{\mu(m^{\nu(n)})}n' * m' \\
&= [m^{\nu(n)}, m'] * n' - {}^{\mu(m^{\nu(n)})}n' * m' \\
&= {}^{\mu(m^{\nu(n)})}n' * m' - m^{\nu(n)} * n'^{\mu(m')} - {}^{\mu(m^{\nu(n)})}n' * m' \\
&= m^{\nu(n)} * {}^{\mu(m')}n' = [m * n, m' * n']
\end{aligned}$$

and

$$\begin{aligned}
(m * n)^{\eta(m' * n')} &= (m * n)^{[\mu(m'),\nu(n')]} = (m * n)^{\mu(m'^{\nu(n')})} \\
&= m^{\mu(m'^{\nu(n')})} * n + m * n^{\mu(m'^{\nu(n')})} \\
&= [m, m'^{\nu(n')}] * n + m * n^{\mu(m'^{\nu(n')})} \\
&= {}^{\mu(m)}n * m'^{\nu(n')} - m * n^{\mu(m'^{\nu(n')})} + m * n^{\mu(m'^{\nu(n')})} \\
&= [m * n, m' * n'].
\end{aligned}$$

By the same way, one easily gets

$$\begin{aligned}
{}^{\eta(m*n)}(n' * m') &= [m * n, n' * m'],\ (m * n)^{\eta(n' * m')} = [m * n, n' * m'], \\
{}^{\eta(n*m)}(n' * m') &= [n * m, n' * m'],\ (n * m)^{\eta(n' * m')} = [n * m, n' * m'], \\
{}^{\eta(n*m)}(m' * n') &= [n * m, m' * n'],\ (n * m)^{\eta(m' * n')} = [n * m, m' * n'].
\end{aligned}$$

So we have proved that the Leibniz $\mathfrak{g}$-algebra $\mathfrak{M} \star \mathfrak{N}$ is crossed. □

4.8. Remark.

It is clear that if $(\mathfrak{M}, \mu)$ (resp. $(\mathfrak{N}, \nu)$) is a crossed Leibniz $\mathfrak{g}$-algebra, then the map $\mathfrak{M} \star -$ (resp. $- \star \mathfrak{N}$) is a functor from the category of pre-crossed Leibniz $\mathfrak{g}$-algebras to the category of crossed Leibniz $\mathfrak{g}$-algebras.

PROPOSITION 4.5. — *Let $(\mathfrak{N}, \nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra. The functor $F(-) := - \star \mathfrak{N}$ is a right exact functor from the category of pre-crossed Leibniz $\mathfrak{g}$-algebras to the category of crossed Leibniz $\mathfrak{g}$-algebras.*

Proof. — Taking into account Proposition 2.3, let

$$0 \to (\mathfrak{P}, o) \xrightarrow{\mathfrak{f}} (\mathfrak{Q}, \lambda) \xrightarrow{\mathfrak{g}} (\mathfrak{R}, \gamma) \to o$$

be an exact sequence of pre-crossed Leibniz $\mathfrak{g}$-algebras. Consider the sequence of Leibniz algebras

$$F(\mathfrak{P}) \xrightarrow{\mathfrak{F}(\mathfrak{f})} \mathfrak{F}(\mathfrak{Q}) \xrightarrow{\mathfrak{F}(\mathfrak{g})} \mathfrak{F}(\mathfrak{R}) \to o.$$

It is clear that the morphism $F(g)$ is surjective. Since the map $F(f)$ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras, by Proposition 2.2, $(F(\mathfrak{P}), \mathfrak{F}(\mathfrak{f}))$ is a crossed Leibniz $F(\mathfrak{Q})$-algebra; and by Proposition 2.1, the image $\operatorname{im} F(f)$ is a two-sided ideal in $F(\mathfrak{Q})$. By composition we have $F(g)F(f) = F(gf) = 0$, which yields a factorisation

$$\overline{F(g)} : F(\mathfrak{Q})/\operatorname{im}\mathfrak{F}(\mathfrak{f}) \to \mathfrak{F}(\mathfrak{R}).$$

In fact, the morphism $\overline{F(g)}$ is an isomorphism. To see it, let us consider the map

$$\Gamma : F(\mathfrak{R}) \to \mathfrak{F}(\mathfrak{Q})/\operatorname{im}\mathfrak{F}(\mathfrak{f})$$

given on generators by

$$\Gamma(r * n) := g^{-1}(r) * n \bmod \operatorname{im} F(f) \text{ and } \Gamma(n * r) := n * g^{-1}(r) \bmod \operatorname{im} F(f)$$

where $g^{-1}(r)$ is any pre-image of r in $\mathfrak{Q}$. Indeed, if q and q' are two pre-images of r, then $q - q' = f(p)$ for some p in $\mathfrak{P}$. Therefore we have

$$\begin{aligned} q * m - q' * n &= (q - q') * n = f(p) * n = F(f)(p * n) \in \operatorname{im} F(f), \\ n * q - n * q' &= n * (q - q') = n * f(p) = F(f)(n * p) \in \operatorname{im} F(f); \end{aligned}$$

thus the map Γ is well-defined. One easily checks that Γ is a morphism of Leibniz algebras and inverse to $\overline{F(g)}$. □

5. Adjunction theorem.

In this section we show that, for any crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{N}, \nu)$, the functor $- \star \mathfrak{N}$ is left adjoint to the functor $\operatorname{Bider}_{\mathfrak{g}}(\mathfrak{N}, -)$. For technical reasons, we assume that the relations

$$iv) \qquad m * {}^{\mu(m')}n = -m * n^{\mu(m')}, \; n * {}^{\nu(n')}m = -n * m^{\nu(n')}$$

defining the tensor product $\mathfrak{M} \star \mathfrak{N}$ are extended to the relations

$$iv)' \qquad m * {}^{g}n = -m * n^{g}, \; n * {}^{g}m = -n * m^{g}$$

for any $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$ and $g \in \mathfrak{g}$. To avoid confusion, we denote this later tensor product by $\mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N}$. For instance, the Leibniz $\mathfrak{g}$-algebras $\mathfrak{M} \star \mathfrak{N}$ and $\mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N}$ coincide if the maps μ and ν are surjective.

THEOREM 5.1. — *Let $(\mathfrak{M}, \mu)$ be a pre-crossed Leibniz $\mathfrak{g}$-algebra and let $(\mathfrak{N}, \nu)$ and $(\mathfrak{P}, \lambda)$ be crossed Leibniz $\mathfrak{g}$-algebras. There is an isomorphism of $\mathbb{K}$-modules*

$$\operatorname{Hom}_{(\mathbf{pc\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M}, \operatorname{Bider}_{\mathfrak{g}}(\mathfrak{N}, \mathfrak{P})) \cong \operatorname{Hom}_{(\mathbf{c\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N}, \mathfrak{P}).$$

Proof. — Let $\phi \in \mathrm{Hom}_{(\mathbf{pc\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M}, \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},\mathfrak{P}))$ and put $(d_m, D_m, g_m) := \phi(m)$ for $m \in \mathfrak{M}$. Notice that we have $g_m = \mu(m)$ thanks to the relation $\rho\phi = \mu$, where $\rho : \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},\mathfrak{P}) \to \mathfrak{g}$ is the crossing morphism. We associate to ϕ the map $\Phi : \mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N} \to \mathfrak{P}$ defined on generators by

$$\Phi(m * n) := -D_m(n) \quad \text{and} \quad \Phi(n * m) := d_m(n), \ \forall\, m \in \mathfrak{M}, \mathfrak{n} \in \mathfrak{N}.$$

LEMMA 5.2. — *The map Φ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras.*

Conversely, given an element $\sigma \in \mathrm{Hom}_{(\mathbf{c\text{-}Leib}(\mathfrak{g}))}(\mathfrak{M} \star_{\mathfrak{g}} \mathfrak{N}, \mathfrak{P})$, we associate the map $\Sigma : \mathfrak{M} \to \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{N},\mathfrak{P})$ defined by

$$\Sigma(m) := (\delta_m, \Delta_m, \mu(m)), \ \forall\, m \in \mathfrak{M},$$

where

$$\delta_m(n) := \sigma(n * m) \quad \text{and} \quad \Delta_m(n) := -\sigma(m * n), \ \forall\, n \in \mathfrak{N}.$$

LEMMA 5.3. — *The map Σ is a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras.*

It is clear that the maps $\phi \mapsto \Phi$ and $\sigma \mapsto \Sigma$ are inverse to each other, which proves the adjunction theorem. $\square$

Proof of Lemma 5.2. — There is a lot of things to check in order to show that the map Φ is well-defined. Let us give some examples of these verifications. For any $m, m' \in \mathfrak{M}$, $n, n' \in \mathfrak{N}$ and $h \in \mathfrak{g}$, we have

$$\begin{aligned}
\Phi({}^{n}m * n' - n * m^{n'}) &= -\, D_{{}^{\nu(n)}m}(n') - d_{m^{\nu(n')}}(n) \\
&= -\, ({}^{\nu(n)}D_m)(n') - ((d_m)^{\nu(n')})(n) \\
&= -{}^{\nu(n)}D_m(n') + d_m({}^{\nu(n)}n') - d_m(n)^{\nu(n')} + d_m(n^{\nu(n')}) \\
&= -{}^{\nu(n)}d_m(n') + d_m([n,n']) - d_m(n)^{\nu(n')} + d_m([n,n']) \\
&= d_m([n,n']) = \Phi([n,n'] * m).
\end{aligned}$$

We also compute

$$\Phi(m * {}^{h}n) = -D_m({}^{h}n) = D_m(n^h) = -\Phi(m * n^h),$$
$$\Phi(n * {}^{h}m) = d_{{}^{h}m}(n) = ({}^{h}d_m)(n) = -((d_m)^h)(n) = -d_{m^h}(n) = -\Phi(n * m^h)$$

and

$$\begin{aligned}\Phi(m^n * {}^{m'}n') &= -D_{m^{\nu(n)}}({}^{\mu(m')}n') = -((D_m)^{\nu(n)})({}^{\mu(m')}n')\\ &= -D_m({}^{\mu(m')}n')^{\nu(n)} + D_m(({}^{\mu(m')}n')^{\nu(n)})\\ &= -D_m({}^{\mu(m')}n')^{\nu(n)} + D_m([{}^{\mu(m')}n', n])\\ &= -D_m(n)^{\nu({}^{\mu(m')}n')} = D_m(n)^{\lambda(D_{m'}(n'))}\\ &= [D_m(n), D_{m'}(n')] = [\Phi(m * n), \Phi(m' * n')]\\ &= \Phi([m * n, m' * n']).\end{aligned}$$

Now let $m \in \mathfrak{M}$, $n \in \mathfrak{N}$ and $g \in \mathfrak{g}$. One has successively

$$\begin{aligned}\Phi({}^g(m * n)) &= \Phi({}^gm * n) - \Phi({}^gn * m) = -D_{{}^gm}(n) - d_m({}^gn)\\ &= ({}^gD_m)(n) - d_m({}^gn) = -{}^gD_m(n) = {}^g\Phi(m * n),\end{aligned}$$

$$\Phi({}^g(n*m)) = -\Phi({}^g(m*n)) = -{}^g\Phi(m*n) = {}^gD_m(n) = {}^gd_m(n) = {}^g\Phi(n*m),$$

$$\begin{aligned}\Phi((m * n)^g) &= \Phi(m^g * n) + \Phi(m * n^g) = -D_{m^g}(n) - D_m(n^g)\\ &= -((D_m)^g)(n) - D_m(n^g) = -D_m(n)^g = \Phi(m * n)^g,\\ \Phi((n * m)^g) &= \Phi(n^g * m) + \Phi(n * m^g) = d_m(n^g) + d_{m^g}(n)\\ &= d_m(n^g) + ((d_m)^g)(n) = d_m(n)^g = \Phi(n * m)^g;\end{aligned}$$

$$\begin{aligned}\lambda\Phi(m * n) &= -\lambda(D_m(n)) = \nu({}^{\mu(m)}n) = [\mu(m), \nu(n)] = \eta(m * n),\\ \lambda\Phi(n * m) &= \lambda(d_m(n)) = \nu(n^{\mu(m)}) = [\nu(n), \mu(m)] = \eta(n * m).\end{aligned}$$

Therefore the map Φ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. □

Proof of Lemma 5.3. — Let us first show that $\Sigma(m)$ is a well-defined biderivation. For any $n, n' \in \mathfrak{N}$, we have

$$\begin{aligned}&\delta_m(n)^{\nu(n')} + {}^{\nu(n)}\delta_m(n')\\ &= \sigma(n * m)^{\nu(n')} + {}^{\nu(n)}\sigma(n' * m) = \sigma((n * m)^{\nu(n')}) + \sigma({}^{\nu(n)}(n' * m))\\ &= \sigma(n^{\nu(n')} * m) + \sigma(n * m^{\nu(n')}) + \sigma({}^{\nu(n)}n' * m) - \sigma({}^{\nu(n')}m * n')\\ &= 2\sigma([n, n'] * m) - \sigma({}^{\nu(n)}m * n' - n * m^{\nu(n')})\\ &= 2\sigma([n, n'] * m) - \sigma([n, n'] * m) = \sigma([n, n'] * m) = \delta_m([n, n']),\end{aligned}$$

thus δ_m is a derivation. Moreover, we have

$$\begin{aligned}&\Delta_m(n)^{\nu(n')} - \Delta_m(n')^{\nu(n)}\\ &= -\sigma(m * n)^{\nu(n')} + \sigma(m * n')^{\nu(n)} = \sigma((m * n')^{\nu(n)}) - \sigma((m * n)^{\nu(n')})\\ &= \sigma(m^{\nu(n)} * n') + \sigma(m * n'^{\nu(n)}) - \sigma(m^{\nu(n')} * n) - \sigma(m * n^{\nu(n')})\\ &= \sigma(m^{\nu(n)} * n' - m^{\nu(n')} * n) - \sigma(m * {}^{\nu(n)}n') - \sigma(m * n^{\nu(n')})\\ &= \sigma(m * [n, n']) - \sigma(m * [n, n']) - \sigma(m * [n, n'])\\ &= -\sigma(m * [n, n']) = \Delta_m([n, n']),\end{aligned}$$

thus Δ_m is an anti-derivation. We have also

$$\lambda(\delta_m(n)) = \lambda(\sigma(n * m)) = \eta(n * m) = [\nu(n), \mu(m)] = \nu(n^{\mu(m)}),$$
$$\lambda(\Delta_m(n)) = -\lambda(\sigma(m * n)) = -\eta(m * n) = -[\mu(m), \nu(n)] = -\nu({}^{\mu(m)}n),$$
$${}^h\delta_m(n) = {}^h\sigma(n{*}m) = \sigma({}^h(n{*}m)) = -\sigma({}^h(m{*}n)) = -{}^h\sigma(m{*}n) = -{}^h\Delta_m(n),$$
$$\Delta_m({}^hn) = -\sigma(m * {}^hn) = \sigma(m * n^h) = -\Delta_m(n^h).$$

Therefore $\Sigma(m) = (\delta_m, \Delta_m, \mu(m))$ is a biderivation from $(\mathfrak{N}, \nu)$ to $(\mathfrak{P}, \lambda)$.

For any $h \in \mathfrak{g}$, $m \in \mathfrak{M}$ and $n \in \mathfrak{N}$, we have

$$\begin{aligned}({}^h(\delta_m))(n) &= \delta_m(n^h) - \delta_m(n)^h = \sigma(n^h * m) - \sigma(n * m)^h \\ &= -\sigma(n * m^h) = \sigma(n * {}^hm) = \delta_{{}^hm}(n),\end{aligned}$$

$$\begin{aligned}({}^h(\Delta_m))(n) &= {}^h\Delta_m(n) - \delta_m({}^hn) = {}^h\sigma(m * n) - \sigma({}^hn * m) \\ &= \sigma({}^hm * n) = \Delta_{{}^hm}(n);\end{aligned}$$

and obviously $[h, \mu(m)] = \mu({}^hm)$, thus we have $\Sigma({}^hm) = {}^h\Sigma(m)$. On the other side, we have

$$\begin{aligned}((\delta_m)^h)(n) &= \delta_m(n)^h - \delta_m(n^h) = \sigma(n * m)^h - \sigma(n^h * m) \\ &= \sigma(n * m^h) = \delta_{m^h}(n)\end{aligned}$$

and

$$\begin{aligned}((\Delta_m)^h)(n) &= \Delta_m(n)^h - \delta_m(n^h) = -\sigma(m * n)^h + \sigma(m * n^h) \\ &= -\sigma(m^h * n) = \Delta_{m^h}(n).\end{aligned}$$

Since $[\mu(m), h] = \mu(m^h)$, we get $\Sigma(m^h) = \Sigma(m)^h$. By definition of the map Σ, we have $\rho\Sigma(m) = \mu(m)$. Therefore the map Σ is a morphism of pre-crossed Leibniz $\mathfrak{g}$-algebras. □

6. Cohomological characterizations.

6.1. Non-abelian Leibniz cohomology.

Let $\mathfrak{g}$ be a Leibniz algebra viewed as the crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{g}, \mathrm{id}_\mathfrak{g})$, and let $(\mathfrak{M}, \mu)$ be a crossed Leibniz $\mathfrak{g}$-algebra. Given an element $m \in \mathfrak{M}$, we denote by d_m (resp. D_m) the derivation (resp. anti-derivation) $g \mapsto {}^gm$ (resp. $g \mapsto -m^g$) from $(\mathfrak{g}, \mathrm{id}_\mathfrak{g})$ to $(\mathfrak{M}, \mu)$, and by $\overline{\mu(m)} := \mu(m) \bmod Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the centre of $\mathfrak{g}$. One easily checks that the triple $(d_m, D_m, \overline{\mu(m)})$ is a well-defined element of $\mathrm{Bider}_\mathfrak{g}(\mathfrak{g}, \mathfrak{M})$.

DEFINITION-PROPOSITION 6.1. — *Let $\mathfrak{J}$ be the $\mathbb{K}$-module freely generated by the biderivations $(d_m, D_m, \overline{\mu(m)}), m \in \mathfrak{M}$. Then $\mathfrak{J}$ is a two-sided ideal of* $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$. *The Leibniz algebra* $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})/\mathfrak{J}$ *is denoted by* $\mathfrak{H}\mathfrak{L}^1(\mathfrak{g}, \mathfrak{M})$.

Proof. — For any $m \in \mathfrak{M}$ and $(d, D, g) \in \mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$, we have

$$[(d, D, g), (d_m, D_m, \overline{\mu(m)})] = (\delta_m, \Delta_m, [g, \overline{\mu(m)}])$$

with

$$\begin{aligned}\delta_m(x) &= d_m([x, g]) - d([x, \overline{\mu(m)}) = {}^{[x,g]}m - d([x, \mu(m)])\\ &= {}^{\mu(d(x))}m - d(x)^{\mu(m)} - {}^{x}d(\mu(m))\\ &= [d(x), m] - [d(x), m] - {}^{x}D(\mu(m))\\ &= d_{m_1}(x)\end{aligned}$$

where $m_1 := -D(\mu(m))$,

$$\begin{aligned}\Delta_m(x) &= -\,D([x, \overline{\mu(m)}]) - d_m([g, x]) = -D([x, \mu(m)]) - {}^{[g,x]}m\\ &= -\,D(x)^{\mu(m)} - D(\mu(m))^x + {}^{\mu(D(x))}m\\ &= -\,[D(x), m] + D(\mu(m))^x + [D(x), m]\\ &= D_{m_1}(x),\end{aligned}$$

$$\mu(m_1) = -\mu(D(\mu(m))) = [g, \mu(m)] = [g, \overline{\mu(m)}];$$

thus we have $[(d, D, g), (d_m, D_m, \overline{\mu(m)})] \in \mathfrak{J}$. On the other side, we have

$$[(d_m, D_m, \overline{\mu(m)}), (d, D, g)] = (\delta'_m, \Delta'_m, [\overline{\mu(m)}, g])$$

with

$$\begin{aligned}\delta'_m(x) &= d([x, \overline{\mu(m)}]) - d_m([x, g]) = d([x, \mu(m)]) - {}^{[x,g]}m\\ &= d(x)^{\mu(m)} + {}^{x}d(\mu(m)) - {}^{\mu(d(x))}m\\ &= [d(x), m] + {}^{x}d(\mu(m)) - [d(x), m]\\ &= d_{m_2}(x)\end{aligned}$$

where $m_2 := d(\mu(m))$,

$$\begin{aligned}\Delta'_m(x) &= -\,D_m([x, g]) - d([\overline{\mu(m)}, x]) = m^{[x,g]} - d([\mu(m), x])\\ &= m^{\mu(d(x))} - d(\mu(m))^x - {}^{\mu(m)}d(x)\\ &= [m, d(x)] - d(\mu(m))^x - [m, d(x)]\\ &= D_{m_2}(x),\end{aligned}$$

$$\mu(m_2) = \mu(d(\mu(m))) = [\mu(m), g] = [\overline{\mu(m)}, g];$$

thus we have $[(d_m, D_m, \overline{\mu(m)}), (d, D, g)] \in \mathfrak{J}$. Therefore the set $\mathfrak{J}$ is a two-sided ideal of $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{M})$. $\square$

Similarly, given a crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{M}, \mu)$, one defines

$$\mathfrak{HL}^{\mathrm{o}}(\mathfrak{g}, \mathfrak{M}) := \{\mathfrak{m} \in \mathfrak{M} : {}^{\mathfrak{g}}\mathfrak{m} = \mathfrak{m}^{\mathfrak{g}} = \mathrm{o}, \ \forall\ \mathfrak{g} \in \mathfrak{g}\}$$

that is, the set of invariant elements of $\mathfrak{M}$. From the relations

$$[m, m'] = m^{\mu(m')} = 0 = {}^{\mu(m')}m = [m', m], \ m \in \mathfrak{HL}^{\mathrm{o}}(\mathfrak{g}, \mathfrak{M}), \ \mathfrak{m}' \in \mathfrak{M},$$

it is clear that $\mathfrak{HL}^{\mathrm{o}}(\mathfrak{g}, \mathfrak{M})$ is contained in the centre of the Leibniz algebra $\mathfrak{M}$.

PROPOSITION 6.2. — *For any exact sequence of crossed Leibniz $\mathfrak{g}$-algebras*

$$0 \to (\mathfrak{A}, \mathrm{o}) \xrightarrow{\alpha} (\mathfrak{B}, \lambda) \xrightarrow{\beta} (\mathfrak{C}, \mu) \to \mathrm{o},$$

there exists an exact sequence of $\mathbb{K}$-modules

$$0 \to \mathfrak{HL}^{\mathrm{o}}(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{HL}^{\mathrm{o}}(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{HL}^{\mathrm{o}}(\mathfrak{g}, \mathfrak{C}) \xrightarrow{\partial} \mathfrak{HL}^{1}(\mathfrak{g}, \mathfrak{A})$$
$$\to \mathfrak{HL}^{1}(\mathfrak{g}, \mathfrak{B}) \xrightarrow{\beta^{1}} \mathfrak{HL}^{1}(\mathfrak{g}, \mathfrak{C})$$

where β^1 is a Leibniz algebra morphism.

Proof. — Everything goes smoothly except the definition of the connecting homomorphism ∂. Given an element $c \in \mathfrak{HL}^{\mathrm{o}}(\mathfrak{g}, \mathfrak{C})$, let $b \in \mathfrak{B}$ be any pre-image of c in $\mathfrak{B}$. For any $x \in \mathfrak{g}$, we have

$$\beta({}^{x}b) = {}^{x}c = 0 = c^{x} = \beta(b^{x}).$$

Thus the element ${}^{x}b$ (resp. b^x) is in $\ker(\beta) = \mathrm{im}(\alpha)$. Since the morphism α is injective, the map $d^c : x \mapsto \alpha^{-1}({}^{x}b)$ (resp. $D^c : x \mapsto \alpha^{-1}(b^x)$) is a derivation (resp. an anti-derivation) from $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$ to $(\mathfrak{A}, \mathrm{o})$. One easily checks that the triple $(d^c, D^c, 0)$ is a well-defined element of $\mathrm{Bider}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{A})$ whose class in $\mathfrak{HL}^{1}(\mathfrak{g}, \mathfrak{A})$ does not depend on the choice of the pre-image b. We put

$$\partial(c) := \mathrm{class}(d^c, D^c, 0).$$ $\square$

6.2. Non-abelian Leibniz homology.

Let $\mathfrak{g}$ be a Leibniz algebra viewed as the crossed Leibniz $\mathfrak{g}$-algebra $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}})$, and let $(\mathfrak{N}, \nu)$ be a crossed Leibniz $\mathfrak{g}$-algebra.

Definition-Proposition 6.3. — *The map $\Psi_{\mathfrak{N}} : \mathfrak{N} \star \mathfrak{g} \to \mathfrak{N}$ given on generators by*

$$\Psi_{\mathfrak{N}}(n * g) := n^g \quad \textit{and} \quad \Psi_{\mathfrak{N}}(g * n) := {}^g n, \; g \in \mathfrak{g}, \; \mathfrak{n} \in \mathfrak{N},$$

is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. We define the low-degrees non-abelian homology of $\mathfrak{g}$ with coefficients in $\mathfrak{N}$ to be

$$\mathfrak{HL}_0(\mathfrak{g}, \mathfrak{N}) := \operatorname{coker}_{\mathfrak{N}} \quad \mathfrak{and} \quad \mathfrak{HL}_1(\mathfrak{g}, \mathfrak{N}) := \ker_{\mathfrak{N}}.$$

Proof. — To see that the map $\Psi_{\mathfrak{N}}$ is a Leibniz algebra morphism is equivalent to the fact that the Leibniz action of $\mathfrak{N}$ on $\mathfrak{g}$ is well-defined. The definition of the crossing homomorphism $\eta_{\mathfrak{N}} : \mathfrak{N} \star \mathfrak{g} \to \mathfrak{g}$ implies that $\Psi_{\mathfrak{N}}$ is a morphism of crossed Leibniz $\mathfrak{g}$-algebras. □

Proposition 6.4. — *For any exact sequence of crossed Leibniz $\mathfrak{g}$-algebras*

$$0 \to (\mathfrak{A}, o) \xrightarrow{\alpha} (\mathfrak{B}, \lambda) \xrightarrow{\beta} (\mathfrak{C}, \mu) \to o,$$

there exists an exact sequence of $\mathbb{K}$-modules

$$\mathfrak{HL}_1(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{HL}_1(\mathfrak{g}, \mathfrak{B}) \to \mathfrak{HL}_1(\mathfrak{g}, \mathfrak{C}) \xrightarrow{\partial} \mathfrak{HL}_0(\mathfrak{g}, \mathfrak{A}) \to \mathfrak{HL}_0(\mathfrak{g}, \mathfrak{B}) \\ \to \mathfrak{HL}_0(\mathfrak{g}, \mathfrak{C}) \to o.$$

Proof. — We know that the functor $- \star \mathfrak{g}$ is right exact (Proposition 4.5). Therefore Proposition 6.4 is nothing but the "snake-lemma" applied to diagram

$$\begin{tikzcd} & \mathfrak{A} \star \mathfrak{g} \arrow[r] \arrow[d, "\Psi_{\mathfrak{A}}"] & \mathfrak{B} \star \mathfrak{g} \arrow[r] \arrow[d, "\Psi_{\mathfrak{B}}"] & \mathfrak{C} \star \mathfrak{g} \arrow[r] \arrow[d, "\Psi_{\mathfrak{C}}"] & 0 \\ 0 \arrow[r] & \mathfrak{A} \arrow[r] & \mathfrak{B} \arrow[r] & \mathfrak{C} \arrow[r] & 0 \end{tikzcd}$$

which is obviously commutative. □

6.3. Universal central extension.

Let $\mathfrak{g}$ be a Leibniz algebra and let $\Psi := \Psi_{\mathfrak{g}}$ be the morphism defining the homolgy $\mathfrak{HL}_*(\mathfrak{g}, \mathfrak{g})$. From the relations $v)$ of Definition-Theorem 4.1, it is clear that $\Psi : \mathfrak{g} \star \mathfrak{g} \twoheadrightarrow [\mathfrak{g}, \mathfrak{g}]$ is a central extension of Leibniz algebras (see [4]).

THEOREM 6.5. — *If the Leibniz algebra $\mathfrak{g}$ is perfect and free as a $\mathbb{K}$-module, then the morphism $\Psi : \mathfrak{g}\star\mathfrak{g} \twoheadrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ is the universal central extension of $\mathfrak{g}$. Moreover, we have an isomorphism of $\mathbb{K}$-modules*

$$\mathfrak{H}\mathfrak{L}_1(\mathfrak{g},\mathfrak{g}) \cong \mathrm{HL}_2(\mathfrak{g}).$$

Proof. — It is enough to prove the universality of the central extension $\Psi : \mathfrak{g}\star\mathfrak{g} \twoheadrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Let $\alpha : \mathfrak{C} \twoheadrightarrow \mathfrak{g}$ be a central extension of $\mathfrak{g}$. Since $\ker(\alpha)$ is central in $\mathfrak{C}$, the quantity $[\alpha^{-1}(x), \alpha^{-1}(y)]$ does not depend on the choice of the pre-images $\alpha^{-1}(x)$ and $\alpha^{-1}(y)$ where $x, y \in \mathfrak{g}$. One easily checks that the map $\phi : \mathfrak{g}\star\mathfrak{g} \to \mathfrak{C}$ given on generators by

$$\phi(x * y) := [\alpha^{-1}(x), \alpha^{-1}(y)]$$

is a well-defined Leibniz algebra morphism such that $\alpha\phi = \Psi$. The uniqueness of the map ϕ follows from Lemma 2.4 of [4] since the perfectness of $\mathfrak{g}$ implies that of $\mathfrak{g}\star\mathfrak{g}$:

$$x * y = \Big(\sum_i [x_i, x_i']\Big) * \Big(\sum_j [y_j, y_j']\Big) = \sum_{i,j} [x_i * x_i', y_j * y_j'].$$

By definition we have $\mathfrak{H}\mathfrak{L}_1(\mathfrak{g},\mathfrak{g}) = \ker()$. After [4] the kernel of the universal central extension of a Leibniz algebra $\mathfrak{g}$ is canonically isomorphic to $\mathrm{HL}_2(\mathfrak{g})$. Therefore we have

$$\mathfrak{H}\mathfrak{L}_1(\mathfrak{g},\mathfrak{g}) \cong \mathrm{HL}_2(\mathfrak{g}).$$ □

7. The Milnor-type Hochschild homology.

Let A be an associative algebra viewed as a Leibniz (in fact Lie) algebra for the bracket given by $[a,b] := ab - ba, a, b \in A$. Recall that the $\mathbb{K}$-module $\mathrm{L}(A) := A^{\otimes 2}/\operatorname{im}(b_3)$ is a Leibniz (non-Lie) algebra for the bracket defined by

$$[x\otimes y, x'\otimes y'] := (xy - yx)\otimes(x'y' - y'x'),\ \forall\, x, y, x', y' \in A.$$

PROPOSITION 7.1. — *The operations given by*

$$A \times \mathrm{L}(A) \to \mathrm{L}(A),\ {}^{a}(x\otimes y) := [a,x]\otimes y - [a,y]\otimes x,$$
$$\mathrm{L}(A) \times A \to \mathrm{L}(A),\ (x\otimes y)^{a} := [x,a]\otimes y + x\otimes[y,a]$$

confer to $\mathrm{L}(A)$ a structure of Leibniz A-algebra. Moreover the map

$$\mu_A : \mathrm{L}(A) \to A,\ x\otimes y \mapsto [x,y] = xy - yx$$

equips L(A) *with a structure of crossed Leibniz* A*-algebra.*

Proof. — The operations are well-defined since we have

$$\begin{aligned}{}^{a}(b_3(x\otimes y\otimes z)) =& b_3(ax\otimes y\otimes z - a\otimes z\otimes xy - za\otimes x\otimes y\\ &+ a\otimes yz\otimes x + a\otimes zx\otimes y - a\otimes y\otimes zx)\end{aligned}$$

and

$$\begin{aligned}(b_3(x\otimes y\otimes z))^{a} =& b_3(-ax\otimes y\otimes z + xy\otimes a\otimes z + x\otimes y\otimes za\\ &- x\otimes a\otimes yz - zx\otimes a\otimes y - zx\otimes y\otimes a).\end{aligned}$$

One easily checks that the couple $(\mathrm{L}(A),\mu_A)$ is a pre-crossed Leibniz A-algebra. Moreover we have

$$\begin{aligned}&{}^{\mu_A(x\otimes y)}(x'\otimes y') - [x\otimes y, x'\otimes y'] = b_3([x,y]\otimes x'\otimes y' - [x,y]\otimes y'\otimes x')\\ &(x\otimes y)^{\mu_A(x\otimes y)} - [x\otimes y, x'\otimes y'] = b_3(x\otimes [x',y']\otimes y - x\otimes y\otimes [x',y']).\end{aligned}$$

Thus the Leibniz A-algebra $(\mathrm{L}(A),\mu_A)$ is crossed. □

It is clear that the inclusion map $[A,A]\hookrightarrow A$ induces a structure of crossed Leibniz A-algebra on the two-sided ideal $[A,A]$, and that the map $\mu_A : \mathrm{L}(A)\to [A,A]$ is a morphism of crossed Leibniz A-algebras. Moreover we have an exact sequence of $\mathbb{K}$-modules

$$0\to \mathrm{HH}_1(A)\to \mathrm{L}(A)\xrightarrow{\mu_A} [A,A]\to 0.$$

LEMMA 7.2. — *The Leibniz algebra A acts trivially on* $\mathrm{HH}_1(A)$.

Proof. — One easily checks that

$${}^{a}(x\otimes y) = a\otimes [x,y] + b_3(a\otimes x\otimes y - a\otimes y\otimes x) \equiv a\otimes [x,y] \text{ in } \mathrm{L}(A)$$

and

$$(x\otimes y)^{a} = [x,y]\otimes a + b_3(x\otimes a\otimes y - x\otimes y\otimes a) \equiv [x,y]\otimes a \text{ in } \mathrm{L}(A).$$

Therefore, if $\omega = \sum\lambda_i(x_i\otimes y_i)\in \mathrm{HH}_1(A)$, that is $\sum\lambda_i[x_i,y_i] = 0$, then we have

$${}^{a}\omega = \sum\lambda_i{}^{a}(x_i\otimes y_i) \equiv \sum\lambda_i(a\otimes [x_i,y_i]) \equiv a\otimes\sum\lambda_i[x_i,y_i] = 0$$

and

$$\omega^{a} = \sum\lambda_i(x_i\otimes y_i)^{a} \equiv \sum\lambda_i([x_i,y_i]\otimes a) \equiv (\sum\lambda_i[x_i,y_i])\otimes a = 0$$

for any $a\in A$. □

As an immediate consequence, we get the following

COROLLARY 7.3. — *The sequence*

$$0 \to \mathrm{HH}_1(A) \to \mathrm{L}(A) \xrightarrow{\mu_A} [A,A] \to 0$$

is an exact sequence of crossed Leibniz A-algebras. □

We deduce from Proposition 6.4 an exact sequence of $\mathbb{K}$-modules

$$\begin{gathered}\mathfrak{HL}_1(\mathfrak{A},\mathrm{HH}_1(\mathfrak{A})) \to \mathfrak{HL}_1(\mathfrak{A},\mathrm{L}(\mathfrak{A})) \to \mathfrak{HL}_1(\mathfrak{A},[\mathfrak{A},\mathfrak{A}]) \to \\ \to \mathfrak{HL}_0(\mathfrak{A},\mathrm{HH}_1(\mathfrak{A})) \to \mathfrak{HL}_0(\mathfrak{A},\mathrm{L}(\mathfrak{A})) \to \mathfrak{HL}_0(\mathfrak{A},[\mathfrak{A},\mathfrak{A}]) \to 0.\end{gathered}$$

Since A and $\mathrm{HH}_1(A)$ act trivially on each other, we have

$$\mathfrak{HL}_0(\mathfrak{A},\mathrm{HH}_1(\mathfrak{A})) = \mathrm{HH}_1(\mathfrak{A})$$

and

$$\mathfrak{HL}_1(\mathfrak{A},\mathrm{HH}_1(\mathfrak{A})) = \mathfrak{A}\star\mathrm{HH}_1(\mathfrak{A}) \cong \mathfrak{A}/[\mathfrak{A},\mathfrak{A}]\otimes\mathrm{HH}_1(\mathfrak{A}) \oplus \mathrm{HH}_1(\mathfrak{A})\otimes\mathfrak{A}/[\mathfrak{A},\mathfrak{A}].$$

On the other hand, it is clear that

$$\mathfrak{HL}_1(\mathfrak{A},[\mathfrak{A},\mathfrak{A}]) \cong [\mathfrak{A},\mathfrak{A}]/[\mathfrak{A},[\mathfrak{A},\mathfrak{A}]].$$

Therefore we can state

THEOREM 7.4. — *For any associative algebra A with unit, there exists an exact sequence of $\mathbb{K}$-modules*

$$\begin{gathered}A/[A,A]\otimes\mathrm{HH}_1(A)\oplus\mathrm{HH}_1(A)\otimes A/[A,A] \to \mathfrak{HL}_1(\mathfrak{A},\mathrm{L}(\mathfrak{A})) \to \mathfrak{HL}_1(\mathfrak{A},[\mathfrak{A},\mathfrak{A}]) \\ \to \mathrm{HH}_1(A) \to \mathrm{HH}_1^M(A) \to [A,A]/[A,[A,A]] \to 0\end{gathered}$$

where $\mathrm{HH}_1^M(A)$ denotes the Milnor-type Hochschild homology of A.

Proof. — Recall that $\mathrm{HH}_1^M(A)$ is defined to be the quotient of $A\otimes A$ by the relations

$$a\otimes[b,c]=0,\ [a,b]\otimes c=0,\ b_3(a\otimes b\otimes c)=0$$

for any $a,b,c\in A$ (see [6, 10.6.19]). By definition $\mathrm{L}(A) = A\otimes A/\operatorname{im}(b_3)$ and from the proof of Lemma 7.2, we get

$$\Psi_{L(A)}(a*(x\otimes y)) = {}^a(x\otimes y) \equiv a\otimes[x,y]$$

and

$$\Psi_{L(A)}((x\otimes y)*a) = (x\otimes y)^a \equiv [x,y]\otimes a.$$

Therefore it is clear that $\mathfrak{HL}_0(\mathfrak{A},\mathrm{L}(\mathfrak{A})) = \operatorname{coker}(_{\mathfrak{L}(\mathfrak{A})})$ is isomorphic to $\mathrm{HH}_1^M(A)$. □

Remark. — The $\mathbb{K}$-modules $\mathrm{HH}_1(A)$ and $\mathrm{HH}_1^M(A)$ coincide when the associative algebra A is *superperfect* as a Leibniz algebra that is, $A = [A, A]$ and $\mathrm{HL}_2(A) = 0$. Also, if the associative algebra A is commutative, then we have

$$\mathrm{HH}_1(A) \cong \mathrm{HH}_1^M(A) \cong \Omega^1_{A|\mathbb{K}}.$$

Let us also mention that the Milnor-type Hochschild homology appears in the description of the obstruction to the stability

$$\mathrm{HL}_n(gl_{n-1}(A)) \to \mathrm{HL}_n(gl_n(A)) \to \mathrm{HH}_{n-1}^M(A) \to 0$$

where $gl_n(A)$ is the Lie algebra of matrices with entries in the associative algebra A (see [2], [6, 10.6.20]).

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