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THE TUTTE POLYNOMIAL OF A MORPHISM OF MATROIDS I. SET-POINTED MATROIDS AND MATROID PERSPECTIVES

by Michel LAS VERGNAS *

1. Introduction.

The dichromatic polynomial of a graph - now currently called the *Tutte polynomial* - is a 2-variable polynomial introduced by Tutte in [28] as a self-dual generalization of the 1-variable chromatic polynomial considered by Birkhoff and Whitney. The extension of the dichromatic polynomial to matroids (combinatorial geometries) is due to Crapo [13]. The Tutte polynomial plays a central role in the theory of numerical invariants of matroids, and has numerous applications. We refer the reader to [8] for a recent survey on Tutte polynomials.

We have introduced in [19] a 3-variable polynomial associated with two matroids related by a strong map (or *matroid perspective* in our terminology) generalizing in several respects the Tutte polynomial of a matroid. The classical Tutte polynomial is equivalent up to a simple transformation to the generating function of cardinality and rank of subsets of elements. The 3-variable Tutte polynomial studied in the present paper is in a similar way equivalent to the the generating function of cardinality and ranks of subsets but in both matroids. The assumption that the matroids

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are related by a strong map is necessary for certain desirable properties to hold. Owing to the factorization theorem of strong maps, an alternative setting is constituted by matroids containing a subset of distinguished elements - we say, in the present paper, matroids *pointed by a subset of elements*, or *set-pointed* matroids. When the distinguished set is empty - or the two matroids of the strong map are equal - the polynomial reduces to the usual Tutte polynomial. When the distinguished subset is reduced to one element we recover a polynomial introduced by Brylawski in [3]. The homogenous 4-variable Tutte polynomial defined by Brylawski is equivalent to the present one in this special case (see the beginning of Section 3 below). A generalization of the Tutte polynomial to graphs pointed by a subset of distinguished vertices has been introduced by Tutte [29]. It turns out that this polynomial is the coefficient of the monomial of highest degree in the 3-variable of our polynomial (see the end of Section 8).

Several papers deal with applications of the Tutte polynomial of matroid perspectives.

• When $M \to M'$ is an oriented matroid perspective on a set E the evaluation $t(M, M'; 0, 0, 1)$ counts the number of subsets A of E such such that $-_A M$ is acyclic and $-_A M'$ is totally cyclic [20]. This result generalizes in a self-dual way previous results of the literature on acyclic orientations of graphs [27], orientations of regular matroids [7], regions of real hyperplane arrangements [31], non Radon partitions of real spaces [5], acyclic reorientations of oriented matroids (or, equivalently, regions of pseudohyperplane arrangements), [17] [21], and bounded regions of hyperplane arrangements [15]. It has applications to the counting of containments of a given flat in facets of acyclic reorientations of an oriented matroid (i.e. projective transforms in the real case): see [10], Section 4 (b). The evaluation $t(M, M'; 0, 0, 1)$ generalizes in a certain sense Crapo's β invariant of a matroid [13] [20]. More generally, the coefficients of $t(M, M'; x, y, 1)$ have a combinatorial interpretation in terms of reorientations [24].

• Let G and G^* be two graphs dually imbedded in a surface. By a result of J. Edmonds (1965) up to the natural bijection between edges induced by the duality, the bond matroid of G^* and the cycle matroid of G are related by a strong map. We have used this property in [23] in the case of 4-valent graphs imbedded in the projective plane and the torus to relate the 3-variable Tutte polynomial to Eulerian tours and cycle decompositions of the graph G.

• A matroid perspective is a special case of a q-matroid in the sense of [2] (see end of Section 6). The Martin polynomial of a q-matroid studied in [2] reduces in the case of a matroid perspective to its Tutte polynomial (specifically, if $Q = Q(M, M'^*)$ we have $m(Q; x) = t(M, M'; x, x, 1)$). A property of the Tutte poynomial of a matroid perspective dealing with connectivity (see [22] Theorem 8.3) is generalized in [2] to q-matroids (see Theorem 6.1 and Corollary 6.2).

• A (finite) vectorial matroid can be represented by a subspace $V \subseteq GF(q)^E$ for some finite field - in other words, a *linear code* in the context of coding theory. The matroids associated with two subspaces $V \subseteq V' \subseteq GF(q)^E$ are related by a strong map. In [14] we have unified, and generalized in terms of 3-variable Tutte polynomials, several classical results of the literature dealing with Tutte polynomials of vectorial matroids. In particular we have generalized a result of F. Jaeger on the generating function of weights of codewords of a linear code and its dual [16].

Our purpose in this paper is to present the basic algebraic properties of the 3-variable Tutte polynomial of matroid strong maps, or perspectives. The main content of the sections are as follows. Section 2 recalls the properties of strong maps useful for the sequel. In Section 3 we introduce the 3-variable Tutte polynomial of a matroid pointed by a subset, and describe its elementary properties. Section 4 shows how this polynomial can be used for the computation of the Tutte polynomial of a matroid with a *normal* subset. In section 5 we relate the (almost) equivalent points of view of Tutte polynomials of set-pointed matroids and matroid perspectives. We establish the linear relations between coefficients valid for all perspectives except a finite number (counterpart of a result of Brylawski for matroids [3]). Section 6 deals with the Higgs factorization of a strong map. Our main result here is that the Tutte polynomial of a strong map is equivalent to the collection of Tutte polynomials of factors of the Higgs factorization. As is well-known, the Tutte polynomial of a matroid can be expressed in terms of the Möbius function of its lattice of flats. We generalize this result to perspectives in Section 7, and also give a generalization of Stanley's factorization theorem [26]. In Section 8 we establish the expression of the Tutte polynomial of a matroid perspective in terms of activities of independent/spanning subsets.

A further generalization is introduced in [19] and [22]. A Tutte polynomial in $k + 1$ variables can be associated with any sequence of k matroids on the same set pairwise related by strong maps - a *matroid perspective sequence*. For the sake of simplicity we restrict ourself here to the case $k = 2$. Most results presented here generalize easily. We refer the reader to [9] for some developments on Tutte polynomials of matroid perspective sequences, with application to electrical network theory, and to [2] for a relation with q-matroids (see Section 7).

2. Notation and terminology.

All definitions and results of this section are classical and can be found in standard textbooks [1] [8] [30]. We recall some of them for the convenience of the reader.

Let M be a matroid on a (finite) set E.

We denote by $r_M(A)$ the rank in M of a subset $A \subseteq E$ and by $Cl_M(A)$ the closure of A in M. The rank of the matroid M is $r(M) = r_M(E)$.

We denote by $t(M)$ the *Tutte polynomial* of M, defined by

$$t(M;x,y) = \sum_{A\subseteq E} (x-1)^{r(M)-r_M(A)}(y-1)^{|A|-r_M(A)}. \tag{2.1}$$

The expressions $M\backslash A$, M/A denote the matroid obtained from M by deleting and contracting A respectively, and $M(A)$ denotes $M \setminus (E \setminus A)$. As usual, when $A = \{e\}$, we write $M\backslash e$ and M/e instead of $M\backslash\{e\}$ and $M/\{e\}$ respectively. For $X \subseteq E\backslash A$, we have

$$r_{M\backslash A}(X) = r_M(X) \text{ and } r_{M/A}(X) = r_M(X \cup A) - r_M(A). \tag{2.2}$$

We denote by M^* the *dual* (or *orthogonal*) matroid of M. The rank of $X \subseteq E$ in M^* is given by $r_M*(X) = |X| + r_M(E\backslash X) - r_M(E)$. We have $(M^*)^* = M$, and $(M\backslash A)^* = M/A$, $(M/A)^* = M\backslash A$ for $A \subseteq E$.

Let $E = E_1 + E_2$ be a partition of E. The following properties (i)-(ii) are equivalent:

(i) $r_M(E) = r_M(E_1) + r_M(E_2)$

(ii) $r_M(X \cup Y) = r_M(X) + r_M(Y)$ for all $X \subseteq E_1$ and $Y \subseteq E_2$.

Then the matroid M is said to *factorize* into $M(E_1)$ and $M(E_2)$. We write $M = M(E_1) \oplus M(E_2)$.

An *isthmus* of M is an element $e \in E$ such that $r(M\backslash e) = r(M) - 1$. A *loop* is an element $e \in E$ such that $r_M(\{e\}) = 0$. If $e \in E$ is an isthmus or a loop of M, we have $M = M(E\backslash e) \oplus M(e)$. Accordingly e is called a *factor of* M. If $e \in E$ is a factor of M, we have $M\backslash e = M/e$. Note that e is an isthmus of M if and only if e is a loop of M^*.

Let M, M' be two matroids on a set E. The following properties (i)-(iii) are equivalent :

(i) every flat of M' is a flat of M

(ii) every circuit of M is a union of circuits of M'

(iii) $r_{M'}(X) - r_{M'}(Y) \leqslant r_M(X) - r_M(Y)$ for all $Y \subseteq X \subseteq E$.

We write $M \to M'$ to indicate that these three equivalent statements are true. We may wish to give a distinctive name P (say) to a pair of matroids M, M' which are thus related, and we then say that $P : M \to M'$ constitutes a *matroid perspective*. We define the *degree* (or *rank drop*) of P by $d(P) = r(M) - r(M')$.

A matroid perspective in our sense is a particular case of *strong map* of matroids. No significant generality is lost, since it can easily be shown that any strong map is reducible to a perspective by adding loops and parallel elements to the matroids. We use the word *perspective* to emphasize that the matroids are on the same set, or more generally on two sets related by a bijection.

Let $P : M \to M'$ be a matroid perspective on a set E.

Given $A \subseteq E$, we have clearly $M\backslash A \to M'\backslash A$ and $M/A \to M'/A$. We say that $P\backslash A : M\backslash A \to M'\backslash A$ and $P/A : M/A \to M'/A$ are the matroid perspectives obtained from P by *deleting* and *contracting* A respectively.

We say that A is a *factor* of $P : M \to M'$ if A is a factor of both M and M'. We then write $P = P(A) \oplus P(E\backslash A)$, where $P(A) : M(A) \to M'(A)$.

The flats of M', together with the flats of M which have the same rank in M and in M', constitute the flats of a matroid L on E. If $r(M) - r(M') \leqslant 1$ we have $L = M$. If $r(M) - r(M') \geqslant 2$ then L is a new matroid of rank $r(M') + 1$, called the *Higgs lift of M' with respect to M*. The rank in L of $X \subseteq E$ is given by

$$r_L(X) = \min(r_M(X), r_{M'}(X) + 1). \tag{2.3}$$

We have $M \to L \to M'$. Iterating the lift construction, we get a sequence of $r(M) - r(M') + 1$ matroids on E such that $M_0 = M \to M_1 \to \ldots \to M_{r(M)-r(M')} = M'$ and $r(M_i) = r(M) - i$ for $i = 0, 1, \ldots, r(M) - r(M')$, called the *Higgs factorization* of $M \to M'$. The matroid M_i is the *$(r(M) - r(M') - i)$-th Higgs lift of M' with respect to M*. By (2.2), the rank in M_i of $X \subseteq E$ is given by

$$r_{M_i}(X) = \min(r_M(X), r_{M'}(X) + r(M) - r(M') - i). \tag{2.4}$$

Given a matroid N on a set E and $A \subseteq E$, we have $N\backslash A \to N/A$. Conversely, by a classical result of Edmonds and Higgs, if $M \to M'$ there is a matroid N on a set F containing E such that $M = N\backslash A$ and $M' = N/A$, where $A = F\backslash E$. A matroid N with these properties is called a *major* of the matroid perspective $M \to M'$. A canonical major, unique up to isomorphism, is given by a construction due to Higgs.

Let A be any set disjoint from E with $|A| = d = r(M) - r(M')$. We denote by $U_r(A)$ the uniform matroid of rank r on A. Observe that $M \oplus U_d(A) \to M' \oplus U_0(A)$ is a perspective of degree $2d$ on $E \cup A$. Then the d-th Higgs lift of $M' \oplus U_0(A)$ with respect to $M \oplus U_d(A)$ is a major of $M \to M'$. We will denote by $H(P) = H(M, M')$ this particular major, called the *Higgs major* of $P : M \to M'$. By (2.4) the rank in $H(M, M')$ of $X \subseteq E \cup A$ is given by

$$(2.5) \quad r_{H(M,M')}(X) \\ = \min(r_M(X \cap E) + |X \cap A|, r_{M'}(X \cap E) + r(M) - r(M')).$$

2. The Tutte polynomial of a set-pointed matroid.

Let M be a matroid on a set E and A be a subset of E. We define the *Tutte polynomial of M pointed by A*, denoted by $t(M; A)$, as the 3-variable polynomial given by

$$(3.1) \quad t(M; A; x, y, z) \\ = \sum_{X \subseteq E\backslash A} (x-1)^{r(M)-r_M(X \cup A)}(y-1)^{|X|-r_M(X)} z^{r_M(X \cup A)-r_M(X)}.$$

The cases $|A| = 0, 1$ reduce to known definitions.

Case $A = \varnothing$. We have $t(M; \varnothing) = t(M)$, the usual Tutte polynomial.

Case $|A| = 1$. Let $A = \{e\}$ for $e \in E$. The polynomial $t(M; \{e\})$ is equivalent to the Tutte polynomial of M pointed by e introduced by Brylawski in [3] as a T'-invariant of the category of pointed pregeometries (matroids). Let $t_B(M; e; z, x, z', x')$ denote this polynomial, which is in 4 variables. No closed formula for $t_B(M; e)$ is explicitly given in [3]. However from [3] Cor. 6.14 one can easily derive the identity $t_B(M; e; z, x, z', x') = x' t(M; \{e\}; z, x, z'/x')$ establishing the equivalence.

By a straightforward calculation it follows from (3.1) that

$$(3.2)\quad t(M;A;x,y,z) = z^{r(M)-r(M\backslash A)} \sum_{X\subseteq E\backslash A} (x-1)^{r(M/A)-r_{M/A}(X)} \times (y-1)^{|X|-r_{M\backslash A}(X)} z^{r(M\backslash A)-r(M/A)-(r_{M\backslash A}(X)-r_{M/A}(X))}.$$

Hence $t(M;A)$ is completely determined by $M\backslash A$, M/A and $r(M)$. This remark will be used below in Section 5. On the other hand $t(M;A)$ determines $t(M\backslash A)$ and $t(M/A)$, since the following proposition follows from (2.2):

PROPOSITION 3.1. — *Let M be a matroid on a set E and A be a subset of E. We have*

$$(3.3)\quad t(M\backslash A;x,y) = (x-1)^{r(M\backslash A)-r(M)} t(M;A;x,y,x-1)$$

$$(3.4)\quad t(M/A;x,y) = (y-1)^{r_M(A)} t(M;A;x,y,1/(y-1)).$$

□

By [3] Cor. 6.14, $t_B(M;e)$ and $t(M;\{e\})$ are completely determined by $t(M\backslash e)$ and $t(M/e)$. However for $|A|>1$, the two Tutte polynomials $t(M\backslash A)$ and $t(M/A)$ and $r_M(A)$ are not sufficient in general to determine $t(M;A)$. Actually $t(M;A)$, which is of degree $r_M(A)$ in z, is equivalent in a certain sense to $r_M(A)+1$ Tutte polynomials (see Theorem 6.1 and the counterexample of Figure 1).

In the remainder of this section we show that standard properties of Tutte polynomials with respects to deletion/contraction, direct sum and duality generalize to $t(M;A)$.

THEOREM 3.2 — *Let M be a matroid on a set E, and A be a subset of E. The following relations (i)-(iv) hold:*

(i) *if $e\in E\backslash A$ is neither an isthmus nor a loop of M*

$$t(M;A;x,y,z) = t(M\backslash e;A;x,y,z) + t(M/e;A;x,y,z);$$

(ii) *if $e\in E\backslash A$ is an isthmus of M*

$$t(M;A;x,y,z) = xt(M\backslash e;A;x,y,z);$$

(iii) *if $e\in E\backslash A$ is a loop of M*

$$t(M;A;x,y,z) = yt(M\backslash e;A;x,y,z);$$

(iv) *(case $A = E$)*

$$t(M; E; x, y, z) = z^{r_M(E)}.$$

Conversely the relations (i)-(iv) define $t(M; A)$ uniquely by induction on $|E \backslash A|$.

LEMMA 3.3. — *For any $e \in E\backslash A$ we have*

$$(3.5)\quad t(M; A; x, y, z) = (x-1)^{r(M)-r(M\backslash e)} t(M\backslash e; A; x, y, z) \\ +(y-1)^{1-r_M(e)} t(M/e; A; x, y, z).$$

Proof. — Let $X \subseteq E\backslash A$. Set

$$t(M; A; X; x, y, z) = (x-1)^{r(M)-r_M(X\cup A)}(y-1)^{|X|-r_M(X)} z^{r_M(X\cup A)-r_M(X)}.$$

We have

$$t(M; A) = \sum_{X\subseteq E\backslash A} t(M; A; X) = \sum_{e\notin X\subseteq E\backslash A} t(M; A; X) + \sum_{e\in X\subseteq E\backslash A} t(M; A; X).$$

It follows from (2.2) that

$$t(M; A; X; x, y, z) \\ = \begin{cases} (x-1)^{r(M)-r(M\backslash e)} t(M\backslash e; A; X; x, y, z) & \text{when } e \notin X \subseteq E\backslash A \\ (y-1)^{1-r_M(e)} t(M/e; A; X\backslash e; x, y, z) & \text{when } e \in X \subseteq E\backslash A \end{cases}$$

and this proves Lemma 3.3. □

Proof of Theorem 3.2. — (i) If $e \in E\backslash A$ is neither an isthmus nor a loop of M, we have $r(M) = r(M\backslash e)$ and $r_M(e) = 1$. Hence, by Lemma 3.3, $t(M; A) = t(M\backslash e; A) + t(M/e; A)$.

(ii) If $e \in E\backslash A$ is an isthmus of M, we have $M\backslash e = M/e$, $r(M) = r(M\backslash e) + 1$ and $r_M(e) = 1$. Hence, by Lemma 3.3, $t(M; A) = xt(M\backslash e; A)$.

(iii) If $e \in E\backslash A$ is a loop of M, we have $M\backslash e = M/e$, $r(M) = r(M\backslash e)$ and $r_M(e) = 0$. Hence, by Lemma 3.3, $t(M; A) = yt(M\backslash e; A)$.

The proof of (iv) is immediate, and it is also obvious that (i)-(iv) determine $t(M; A)$ uniquely by induction on $|E \setminus A|$. □

COROLLARY 3.4. — *The Tutte polynomial of a set-pointed matroid is a polynomial with non-negative integer coefficients.* □

Using straightforward substitutions we obtain

PROPOSITION 3.5. — *Let M be a matroid on a set E, and A be a subset of E. For any factorization $M = M(E_1) \oplus M(E_2)$ of M we have*

$$t(M;A) = t(M(E_1);A \cap E_1) \times t(M(E_2);A \cap E_2). \tag{3.6}$$

□

PROPOSITION 3.6. — *Let M be a matroid on a set E, and A be a subset of E. We have*

$$t(M^*;A;x,y,z) = z^{|A|}t(M;A;y,x,1/z). \tag{3.7}$$

□

By Theorem 3.2 the computation of $t(M;A)$ can be considered as the beginning of a computation of $t(M)$ when operations of deletion/contraction are first applied to all elements of $E\backslash A$. Then, when reduced to A, instead of going on with the computation of $t(M)$, we just keep track of the rank of the resulting matroid by means of the third variable. A finer analysis of this partial computation is obtained by introducing variables representing all possible matroids on A.

THEOREM 3.7. — *Let A be a set, $\mathcal{M}(A)$ be the set of all matroids on A, and let $(z_M)_{M\in\mathcal{M}(A)}$ be variables indexed by these matroids. Then by replacing relation (iv) in the statement of Theorem 3.2 by $t(M;E;x,y,z) = z_M$, we obtain inductive relations whose unique solution is the polynomial in $|\mathcal{M}(A)| + 2$ variables with non-negative integer coefficients given by*

$$\begin{aligned} &T(M;A;x,y,(z_N)_{N\in\mathcal{M}(A)}) \\ &\qquad = \sum_{X\subseteq E\backslash A} (x-1)^{r(M)-r_M(X\cup A)}(y-1)^{|X|-r_M(X)}z_{M(X\cup A)/X}. \end{aligned} \tag{3.8}$$

□

Properties of this 'big' set-pointed Tutte polynomial $T(M;A)$ with respect to direct sum and duality are analogous to those of $t(M)$ or $t(M;A)$.

We have

$$t(M;A;x,y,z) = T(M;A;x,y,(z^{r(N)})_{N\in\mathcal{M}(A)}) \tag{3.9}$$

$$t(M;x,y) = T(M;A;x,y,(t(N;x,y))_{N\in\mathcal{M}(A)}). \tag{3.10}$$

In the next section we will give an application of (3.9) and (3.10) to the computation of usual Tutte polynomials. We refer the reader to [9] for application of this polynomial to electrical network theory.

4. Normal subsets.

Tutte polynomials of set pointed matroids can be used in certain situations to simplify the calculation of Tutte polynomials of matroids. In application of Theorem 3.7, we show in this section that if A is a normal subset of a matroid M then the Tutte polynomial $t(M)$ can be computed from $t(M;A)$ and $t(M(A))$.

We recall a definition introduced in [19]. Let M be a matroid on a set E. A subset $A \subseteq E$ is *normal in* M if $X \cup Y$ spans A whenever $X \subseteq E \setminus A$, $Y \subseteq A$ and $X \cup Y_0$ spans A for some $Y_0 \subseteq A$ such that $r_M(Y_0) = r_M(Y)$. Roughly this definition means that subsets of A of equal rank have the same relative position with respect to $E \setminus A$ in M.

Example. — Let M_0 be the cycle-matroid of the graph with vertices a, b, c, d and edges $e_1 = ab$ $e_2 = ac$ $e_3 = bc$ $e_4 = ad$ $e_5 = bd$ $e_6 = ad$. Then $A = \{e_4, e_5, e_6\}$ is normal in M (but $A' = \{e_1, e_2, e_3\}$ is not).

PROPOSITION 2.1. — *Let M be a matroid on a set E, and A be a subset of E. The following properties (i)-(iii) are equivalent:*

(i) *A is normal in M*

(ii) *for all $X \subseteq E \setminus A$ and $Y \subseteq A$, we have*

$$r_M(X \cup Y) = \min(r_M(X) + r_M(Y), r_M(X \cup A)) \tag{4.1}$$

(iii) *M is the $r_M(A)$-th lift of $M/A \oplus U_0(A)$ with respect to $M \setminus A \oplus M(A)$.*

The equivalence of (i) and (ii) is Proposition 1.1 of [19]. We give a proof for completeness. We will often write $X + Y$ instead of $X \cup Y$ to denote the union of *disjoint* sets.

LEMMA 4.2. — *Let M be a matroid on a set E, A be a subset of E normal in M, and $X \subseteq E \setminus A$. If $Y \subset A$ is such that $Cl_M(X + Y) \cap (A \setminus Cl_M(Y)) \neq \varnothing$ then we have $Cl_M(X + Y) \supseteq A$.*

In particular if A contains no loops then for all $X \subseteq E \setminus A$ we have $Cl_M(X) \supseteq A$ or $Cl_M(X) \cap A = \varnothing$.

Proof. — We prove Lemma 4.2 by induction on $r_M(A) - r_M(Y)$. The lemma is trivial when $r_M(A) - r_M(Y) = 0$ or 1. Suppose $r_M(A) - r_M(Y) \geqslant 2$. Let $\{x_1, x_2, \ldots, x_k\}$ be a basis of $A \setminus Cl_M(Y)$ in $M(A)/(A \cap Cl_M(Y)$. We have $k \geqslant 2$. We may suppose that moreover $x_1 \in Cl_M(X+Y)$. We have $x_1 \in Cl_M(X + Y + x_2)$, and hence by the induction hypothesis $A \subseteq Cl_M(X+Y+x_2)$. Since $r_M(Y+x_1) = r_M(Y+x_2) = r_M(Y)+1$ and A is normal, we have $A \subseteq Cl_M(X+Y+x_1) = Cl_M(X+Y)$. □

Proof of Proposition 4.1. — (i) implies (ii). We have $r_M(X+Y) \leqslant \min(r_M(X)+r_M(Y), r_M(X+A))$. Suppose $r_M(X+Y) < r_M(X)+r_M(Y)$. Let Z be a basis of $M(X+Y)/X$. We have $|Z| = r_M(X+Y) - r_M(X) < r_M(Y)$, and hence $Y \not\subseteq Cl_M(Z)$. Since $Cl_M(X+Z) = Cl_M(X+Y)$ we have $Cl_M(X+Z) \cap (A \setminus Cl_M(Z)) \supseteq Y \setminus Cl_M(Z) \neq \varnothing$. If A is normal in M by Lemma 4.2 we have $A \subseteq Cl_M(X+Z)$, and hence $Cl_M(X+Y) = Cl_M(X+A)$. It follows that $r_M(X+Y) = r_M(X+A)$.

(ii) implies (i). Conversely suppose that $r_M(X+Y) = \min(r_M(X) + r_M(Y), r_M(X+A))$ for all $X \subseteq E \setminus A$ and $Y \subseteq A$. Let $X \subseteq E \setminus A$ and $Y, Y_0 \subseteq A$ be such that $A \subseteq Cl_M(X+Y_0)$ and $r_M(Y) = r_M(Y_0)$. Since $Cl_M(X+Y_0) = Cl_M(X+A)$ we have $r_M(X+Y_0) = r_M(X+A)$. Hence $r_M(X+A) = r_M(X+Y_0) \leqslant r_M(X)+r_M(Y_0) = r_M(X)+r_M(Y)$. Therefore, using the hypothesis, $r_M(X+Y) = r_M(X+A)$. Since $Y \subseteq A$, it follows that $Cl_M(X+Y) = Cl_M(X+A)$, and hence $A \subseteq Cl_M(X+Y)$.

We now prove the equivalence of (ii) and (iii). Set $N = M \setminus A \oplus M(A)$ and $N' = M/A \oplus U_0(A)$. Then $N \to N'$ is a matroid perspective on E. By (2.4), the matroid M is the $r_M(A)$-th lift of N' with respect to N if and only if for all $X \subseteq E \setminus A$ and $Y \subseteq A$ we have $r_M(X \cup Y) = \min(r_N(X \cup Y), r_{N'}(X \cup Y) + r_M(A))$.

Since $r_N(X \cup Y) = r_M(X) + r_M(Y)$ and $r_{N'}(X \cup Y) = r_{M/A}(X) = r_M(X \cup A) - r_M(A)$, the above condition is equivalent to $r_M(X \cup Y) = \min(r_M(X) + r_M(Y), r_M(X \cup A))$ for all $X \subseteq E \setminus A$ and $Y \subseteq A$. □

The Higgs major of a matroid perspective $M \to M'$ is a special case of a matroid with a normal subset. It is easy to prove that a matroid M on E is the Higgs major of $M \setminus A \to M/A$ if and only if $A \subseteq E$ is independent and normal in M, and $r(M \setminus A) = r(M)$.

We recall that the rank function of the *truncation* $\mathrm{Tr}(M)$ of a matroid M is given by $r_{\mathrm{Tr}(M)}(X) = \min(r_M(X), r(M) - 1)$ for $X \subseteq E(M)$. More generally the *k-truncation* $\mathrm{Tr}_k(M)$ of M is defined by $r_{\mathrm{Tr}_k(M)}(X) =$

$\min(r_M(X), k)$ for $X \subseteq E(M)$. The matroids $\mathrm{Tr}_k(M)$ are called *iterated truncations* of M.

THEOREM 4.3. — *Let M be a matroid on a set E and $A \subseteq E$ be normal in M. We have*

$$t(M) = \sum_{i=0}^{i=r_M(A)} t_i(M;A)t(\mathrm{Tr}_i(M(A))) \tag{4.2}$$

where $t_i(M;A;x,y)$ is the coefficient of z^i in $t(M;A;x,y,z)$ and $\mathrm{Tr}_i(M(A))$ denotes the i-truncation of $M(A)$.

In Theorem 4.3 the key property is that if A is normal in M then any minor of M on A is an iterated truncation of $M(A)$, and hence is determined by its rank.

LEMMA 4.4. — *Let M be a matroid on a set E and $A \subseteq E$ be normal in M. Then any minor N of M on A is an iterated truncation of $M(A)$. We have $N = \mathrm{Tr}_{r(N)}(M(A))$.*

Proof. — A minor of M on A is of the form $(M/B)(A)$ for some $B \subseteq E \setminus A$. Let $X \subseteq A$. By (4.1) we have

$$\begin{aligned} r_{(M/B)(A)}(X) &= r_{M/B}(X) = r_M(X \cup B) - r_M(B) \\ &= \min(r_M(X) + r_M(B), r_M(A \cup B)) - r_M(B) \\ &= \min(r_M(X), r_M(A \cup B) - r_M(B)) \\ &= \min(r_{M(A)}(X), r_M(A \cup B) - r_M(B)). \end{aligned}$$

Hence $(M/B)(A)$ is the $(r_M(A \cup B) - r_M(B))$-truncation of $M(A)$.

□

Proof of Theorem 4.3. — Let $\mathcal{M}(A)$ be the set of matroids of M on A. Set

$$T(M;A;x,y,(z_N)_{N \in \mathcal{M}(A)}) = \sum_{N \in \mathcal{M}(A)} T_N(M;A;x,y)z_N$$

where $T(M;A)$ is the 'big' Tutte polynomial of M pointed by A. By Theorem 3.7 $T_N(M;A) \equiv 0$ if N is not a minor of M on A. By Lemma 4.4, for N a minor of M on A we have $N = \mathrm{Tr}_i(M(A))$ with $i = r(N)$. Hence, setting $d = r(M(A))$ and denoting by $\mathcal{M}(M;A)$ the set of minors of M on

A, by (3.9) and (3.10) we have

$$\begin{aligned}
t(M) &= \sum_{N\in\mathcal{M}(M;A)} T_N(M;A).t(N) \\
&= \sum_{i=0}^{i=d} \sum_{\substack{N\in\mathcal{M}(M;A)\\ r(N)=i}} T_N(M;A).t(N) \\
&= \sum_{i=0}^{i=d} \Big[\sum_{\substack{N\in\mathcal{M}(M;A)\\ r(N)=i}} T_N(M;A)\Big] t(\mathrm{Tr}_i(M(A))) \\
&= \sum_{i=0}^{i=d} t_i(M;A)t(\mathrm{Tr}_i(M(A))).
\end{aligned}$$

□

Example (continued). — The Tutte polynomial of M_0 is given by

$$t(M_0) = x^3 + x^2y + xy^2 + y^3 + 2x^2 + 3xy + 2y^2 + x + y.$$

We have

$$t(M_0; A) = (x+1)z^2 + (x+y+1)z$$

and hence

$$\begin{aligned}
t_0(M_0;A) &= 0 \\
t_1(M_0;A) &= x+y+1 \\
t_2(M_0;A) &= x+1.
\end{aligned}$$

On the other hand

$$\begin{aligned}
t(\mathrm{Tr}_2(M_0(A))) &= t(M_0(A)) = x^2+xy \\
t(\mathrm{Tr}_1(M_0(A))) &= y^2+x+y \\
t(\mathrm{Tr}_0(M_0(A))) &= y^3.
\end{aligned}$$

By Theorem 4.3 we have

$$t(M_0) = 0.y^3 + (x+y+1)(y^2+x+y) + (x+1)(x^2+xy).$$

We supplement Theorem 4.3 by observing that the Tutte polynomial $t(\mathrm{Tr}(M))$ of the truncation of a matroid M can be calculated from the Tutte polynomial $t(M)$. This result is stated in [3] (cf Proposition 4.11), but the explicit expression is not given.

Proposition 4.5. — *Let M be a rank $r \geqslant 1$ matroid with Tutte polynomial given by $t(M;x,y) = \sum_{i=0}^{i=r} u_i x^i$ where $u_i \in \mathbf{Z}[y]$ for $i = 0,1,\ldots,r$. Then the Tutte polynomial of the truncation of M is given by*

$$t(\mathrm{Tr}(M);x,y) = \sum_{i=1}^{i=r-1} \Big[\sum_{j=i+1}^{j=r} u_j \Big] x^i + (y-1)u_0 + y \sum_{j=1}^{j=r} u_j. \tag{4.3}$$

More generally, for any integer k with $1 \leqslant k \leqslant r-1$, the Tutte polynomial of the k-th iterated truncation, or $(r-k)$-truncation, is given by

$$\begin{aligned} t(\mathrm{Tr}_{r-k}(M);x,y) = {} & \sum_{i=1}^{i=r-k} \Big(\sum_{j=k+i}^{j=r} \binom{j-i-1}{k-1} u_j \Big) x^i + \\ & + \sum_{j=0}^{j=k} y^j (y-1)^{k-j} u_j + \sum_{j=k+1}^{j=r} \Big(\sum_{i=1}^{i=k} \binom{j-i-1}{k-i} y^i \Big) u_j. \end{aligned} \tag{4.4}$$

By (2.4) it follows from the expression of its rank function that the truncation $\mathrm{Tr}(M)$ is the $(r(M)-1)$-th lift of the rank zero matroid with respect to M. We postpone the proof of Proposition 4.5 until Section 6 where it will be given in terms of lifts.

5. The Tutte polynomial of a matroid perspective.

Let $P : M \to M'$ be a matroid perspective on a set E, and let N be a major of P on a set F containing E. Set $A = F\backslash E$. By (3.2) $z^{-(r(N)-r(M))} t(N;A)$ is a polynomial which depends only on $M = N\backslash A$ and $M' = N/A$.

We define the *Tutte polynomial of* $P : M \to M'$, denoted by $t(P)$ or $t(M,M')$, by

$$t(P;x,y,z) = t(M,M';x,y,z) = z^{-(r(N)-r(M))} t(N;F\backslash E;x,y,z) \tag{5.1}$$

where N is any major of $M \to M'$ on a set F containing E. By Corollary 3.4, $t(M,M')$ is a polynomial with non-negative integer coefficients. From (3.2) we obtain

$$\begin{aligned} & t(M,M';x,y,z) \\ & = \sum_{X \subseteq E} (x-1)^{r(M')-r_{M'}(X)} (y-1)^{|X|-r_M(X)} z^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}. \end{aligned} \tag{5.2}$$

By the Edmonds-Higgs theorem, set-pointed matroids and matroid perspectives are almost equivalent objects in the present context. Parallel theories can be developed in either language. In the remainder of the paper, theorems will be stated in terms of matroid perspectives, which are more natural in applications [20] [23] [24]. For the convenience of the reader we restate below theorems of Section 3 in terms of perspectives. Besides basic properties of Tutte polynomials of matroid perspectives, the main result of this section is a theorem giving all linear relations between coefficients valid for all perspectives except a finite number.

Straightforward substitutions yield the following simple but useful relations between Tutte polynomials of matroids and Tutte polynomials of matroid perspectives.

PROPOSITION 5.1. — *Let M be a matroid on a set E. We have*

$$t(M,M;x,y,z) = t(M;x,y), \tag{5.3}$$

$$t(M,U_0(E);x,y,z) = t(M;z+1,y), \tag{5.4}$$

$$t(U_{|E|}(E),M;x,y,z) = z^{|E|-r(M)}t\left(M;x,\frac{1}{z}+1\right). \tag{5.5}$$

□

PROPOSITION 5.2. — *Let $M \to M'$ be a matroid perspective on a set E. We have*

$$t(M;x,y) = t(M,M';x,y,x-1) \tag{5.6}$$

$$t(M';x,y) = (y-1)^{r(M)-r(M')}t\left(M,M';x,y,\frac{1}{y-1}\right). \tag{5.7}$$

□

The polynomial $t(M,M';x,y,z)$ is of degree $r(M)-r(M')$ in z. By [3] Cor. 6.14 and (5.1), when $r(M)-r(M')=1$, we have the following converse to (5.6),(5.7) :

$$t(M,M';x,y,z) = \frac{z(y-1)-1}{xy-x-y}t(M;x,y) + \frac{-z+x-1}{xy-x-y}t(M';x,y). \tag{5.8}$$

A generalization of (5.8) when $r(M)-r(M')>1$ will be given in Section 6 (Theorem 6.1). We point out that $t(M)$ and $t(M')$ do not determine $t(M,M')$ in general.

By Propositions 5.1 and 5.2, a theorem on Tutte polynomials of matroids is also, in a trivial way, a theorem on Tutte polynomials of matroid perspectives. Properties reducible in this way to properties of Tutte polynomials of matroids will generally be omitted here.

The following evaluations of $t(M, M')$ are not reducible to evaluations of $t(M)$ or $t(M')$:

$t(M, M'; 1, 1, 1)$ counts the number of independent sets of M which are spanning in M' (proof by inspection of (5.2))

$t(M, M'; 0, 0, 0)$ is related to Möbius functions of the lattices of flats of M and M' (see Corollary 7.4 below)

We have studied in different papers several remarkable evaluations of $t(M, M')$ (see Section 1).

THEOREM 5.3. — *Let $M \to M'$ be a matroid perspective on a set E. The following relations (i)-(v) hold:*

(i) *if $e \in E$ is neither an isthmus nor a loop of M*

$$t(M, M'; x, y, z) = t(M\backslash e, M'\backslash e; x, y, z) + t(M/e, M'/e; x, y, z);$$

(ii) *if $e \in E$ is an isthmus of M', and hence also an isthmus of M*

$$t(M, M'; x, y, z) = xt(M\backslash e, M'\backslash e; x, y, z);$$

(iii) *if $e \in E$ is a loop of M, and hence also a loop of M'*

$$t(M, M'; x, y, z) = yt(M\backslash e, M'\backslash e; x, y, z);$$

(iv) *if $e \in E$ is an isthmus of M and is not an isthmus of M'*

$$t(M, M'; x, y, z) = zt(M\backslash e, M'\backslash e; x, y, z) + t(M/e, M'/e; x, y, z);$$

(v) *(case $E = \varnothing$)*

$$t(\varnothing, \varnothing; x, y, z) = 1.$$

Conversely these relations define $t(M, M')$ uniquely by induction on $|E|$.

Proof. — Theorem 5.3 can be either derived from Theorem 3.2 using (5.1), or proved directly using the relation

$$(5.9)\quad t(M, M'); x, y, z) = (x-1)^{r(M)-r(M')} z^{r(M)-r(M')-(r(M\backslash e)-r(M'\backslash e))} \times t(M\backslash e, M'\backslash e; x, y, z) + (y-1)^{1-r_M(e)} t(M/e, M'/e; x, y, z)$$

valid for any $e \in E$. We leave details to the reader. □

PROPOSITION 5.4. — *Let P be a matroid perspective on a set E, and $E = E_1 + E_2$ be a partition of E such that $P = P(E_1) \oplus P(E_2)$. We have*

$$t(P) = t(P(E_1)) \times t(P(E_2)). \tag{5.10}$$

□

PROPOSITION 5.5. — *Let $M \to M'$ be a matroid perspective. We have*

$$t(M'^*, M^*; x, y, z) = z^{r(M)-r(M')} t(M, M'; y, x, 1/z). \tag{5.11}$$

□

Coefficients of Tutte polynomials are not linearly independent. Brylawski has established all linear relations between coefficients of Tutte polynomials valid for all matroids except a finite number [3]. These relations arise from the identity $t(M; x, x/(x-1)) \equiv x^{|E|}(x-1)^{-|E|+r(M)}$. The first ones are the well-known properties $t_{0,0} = 0$ i.e. the constant term of the Tutte polynomial of a non-empty matroid is zero, $t_{1,0} = t_{0,1}$ i.e. the coefficients of x and y are equal in $t(M; x, y)$ if M has at least two elements, etc. For Tutte polynomials of matroid perspectives the following generalization holds.

THEOREM 5.6. — *Let t_{ijk} denotes the coefficient of $x^i y^j z^k$ in the Tutte polynomial $t(M, M'; x, y, z)$ of a matroid perspective $M \to M'$. For non-negative integers n, d let*

$$f_n \equiv \sum_{0 \leqslant i+j \leqslant n,\ k \geqslant 0} (-1)^{i+k} \binom{n-j+k}{i+k} t_{ijk}$$

$$f_{n,d} \equiv \sum_{0 \leqslant i+j \leqslant n,\ 0 \leqslant k \leqslant d} (-1)^{i+k} \binom{n-j+k}{i+k} t_{ijk}.$$

If the number of elements of M (or M') is at least $n+1$ then the coefficients in $t(M, M')$ satisfy the relation

$$f_n \equiv \sum_{0 \leqslant i+j \leqslant n,\ k \geqslant 0} (-1)^{i+k} \binom{n-j+k}{i+k} t_{ijk} = 0. \tag{5.12}$$

Conversely $f_{n,d}$ for $n = 0, 1, \ldots$ constitute a basis of the vector space of linear forms f in finitely many variables t_{ijk} with $i, j = 0, 1, \ldots$ $0 \leqslant k \leqslant d$ such that $f = 0$ is satisfied by coefficients of Tutte polynomials of all but a finite number of matroid perspectives $M \to M'$ with $r(M) - r(M') = d$.

The case $d=0$ of Theorem 5.6, i.e. the case of matroids, is Theorem 6.6 of Brylawski [3]. The relation $f_n = 0$ can be derived from Brylawski's theorem $f_{n,0} = 0$ by means of (5.6). Obviously f_n is not a finite sum. In order to have a basis theorem similar to Brylawski's, we have to restrict the class of perspectives so that only finitely many terms of f_n are 'active'. In Theorem 5.6 we consider perspectives of fixed degree d, so that $t_{ijk} = 0$ for $k > d$. Slightly different versions of the converse part of Theorem 5.6 can be obtained by proofs similar to the following one. For instance, it can be shown that the linear forms f_n restricted to variables i, j, k such that $i+k \leqslant r$ resp. $i \leqslant r'$ and $k \leqslant r-r'$ constitute a basis of the vector space of linear forms f in these variables such that $f=0$ is satisfied by coefficients of Tutte polynomials of all but a finite number of matroid perspectives $M \to M'$ with $r(M)=r$ resp. $r(M)=r$ and $r(M')=r'$.

LEMMA 5.7. — *Let n, r, r' be integers such that $0 \leqslant r' < r \leqslant n$. We have*

$$(5.13)\quad t(U_{r,n}, U_{r',n}; x, y, z) = \Big[\sum_{i=0}^{i=r'} \binom{n-i-1}{r'-i} x^i\Big] z^{r-r'} + \sum_{i=1}^{i=r-r'-1} \binom{n}{r-i} z^i + \sum_{i=0}^{i=n-r} \binom{n-i-1}{n-r-i} y^i.$$

Proof. — Let E be a set with n elements. The rank function of $U_r(E)$ is given by $r_{U_r(E)}(X) = \min(|X|, r)$ for $X \subseteq E$. Using this expression, we obtain (5.13) from (5.2) by a straightforward computation. □

Proof of Theorem 5.6. — We could use (5.6) to derive (5.12) from Brylawski's result. However the following direct proof is in fact simpler. The identity

$$t\left(M, M'; x, \frac{x}{x-1}, x-1\right) \equiv x^{|E|}(x-1)^{-|E|+r(M)}$$

is an immediate consequence from (5.2).

Since obviously $t_{ijk} = 0$ for $j > |E|$, we get the *polynomial* identity

$$\sum_{i,j,k \geqslant 0} t_{ijk} x^{i+j} (x-1)^{|E|-j+k} \equiv x^{|E|}(x-1)^{r(M)}.$$

Let n be an integer such that $n < |E|$. We rewrite this polynomial identity as

$$(x-1)^{|E|-n} \sum_{0\leqslant i+j\leqslant n,\ k\geqslant 0} t_{ijk} x^{i+j}(x-1)^{n-j+k} \equiv x^{|E|}(x-1)^{r(M)} - \sum_{i+j>n,\ k\geqslant 0} t_{ijk} x^{i+j}(x-1)^{|E|-j+k}.$$

The right hand side is a polynomial of degree $> n$. Since the constant term of $(x-1)^{|E|-n}$ is not 0, the polynomial $\sum_{0\leqslant i+j\leqslant n,\ k\geqslant 0} t_{ijk}x^{i+j}$-$(x-1)^{n-j+k}$ has smallest degree $> n$. Since the coefficient of x^n in this polynomial is 0 we obtain (5.12).

We now prove that $f_{n,d}$ for $n = 0, 1, \ldots$ constitute a basis. The independence is clear, by checking which variables have non-zero coefficient in $f_{n,d}$. Let $f \equiv \sum_{i,j,k} a_{ijk}t_{ijk}$ be a linear form in finitely many variables t_{ijk} for $i, j = 0, 1, \ldots$ and $k = 0, 1, \ldots, d$, such that $f = 0$ for all but a finite number of matroid perspectives M, M' with $r(M) - r(M') = d$. We show that f is a finite linear combination of forms $f_{n,d}$ with $n = 0, 1, \ldots$.

Let n be the greatest value of $i + j$ such that $a_{ijk} \neq 0$ for some k. Consider integers i, k, m such that $0 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant d \leqslant m$. Set

$$\begin{aligned} M &= U_{i,i} \oplus U_{0,n-i} \oplus U_{d-k,d-k} \oplus U_{k,m} \\ M' &= U_{i,i} \oplus U_{0,n-i} \oplus U_{0,d-k} \oplus U_{0,m} \end{aligned}$$

with element sets of M and M' chosen so that M, M' is a matroid perspective. Then M and M' have $n + d - k + m$ elements and $r(M) - r(M') = d$.

By Proposition 5.4 and Lemma 5.7 we have

$$t(M, M'; x, y, z) = x^i y^{n-i}(z+1)^{d-k}\Big[\sum_{1\leqslant \ell\leqslant k} \binom{m}{k-\ell} z^\ell + \binom{m-1}{\ell-1} + y(\ldots)\Big].$$

Therefore, by the choice of n, for $m \geqslant d$ we have

$$\sum_{0\leqslant j\leqslant d-k} \binom{d-k}{j}\Big[\sum_{1\leqslant \ell\leqslant k} \binom{m}{k-\ell} a_{i,n-i,j+\ell} + \binom{m-1}{k-1} a_{i,n-i,j}\Big] = 0.$$

The left hand side of this equality is a polynomial in m. This polynomial having value zero for infinitely many values of m is identically zero. Setting $m = 1$ we get

$$\sum_{0 \leqslant j \leqslant d-k} \binom{d-k}{j} (a_{i,n-i,k+j} + a_{i,n-i,k+j-1}) = 0$$

or, equivalently,

$$\sum_{0 \leqslant j \leqslant d-k+1} \binom{d-k+1}{j} a_{i,n-i,k+j-1} = 0.$$

This last equality holding for $k = 1, 2, \ldots, d$, it follows that $a_{i,n-i,k} = (-1)^k a_{i,n-i,0}$ for $k = 1, 2, \ldots, d$.

Consider now integers i, m such that $1 \leqslant i \leqslant n$ and $m \geqslant d+1$. Set

$$\begin{aligned} M &= U_{i-1,i-1} \oplus U_{0,n-i} \oplus U_{d+1,m} \\ M' &= U_{i-1,i-1} \oplus U_{0,n-i} \oplus U_{1,m} \end{aligned}$$

with element sets of M and M' chosen so that M, M' is a matroid perspective. Then M and M' have $n+m$ elements and $r(M) - r(M') = d$. We have

$$\begin{aligned} t(M, M'; x, y, z) = x^{i-1} y^{n-i} \Big[(x+m-1) z^d &+ \sum_{1 \leqslant \ell \leqslant d-1} \binom{m}{d+1-\ell} z^\ell + \\ &+ \binom{m-1}{d} + \binom{m-2}{d} y + y^2(\ldots) \Big]. \end{aligned}$$

Hence for infinitely many values of m

$$\begin{aligned} a_{i,n-i,d} + (m-1) a_{i-1,n-i,d} &+ \sum_{1 \leqslant \ell \leqslant d-1} \binom{m}{d+1-\ell} a_{i-1,n-i,\ell} \\ &+ \binom{m-1}{d} a_{i-1,n-i,0} + \binom{m-2}{d} a_{i-1,n-i+1,0} = 0. \end{aligned}$$

The left hand side is an identically zero polynomial in m. Setting $m = 1$, we get

$$a_{i,n-i,d} + (-1)^d a_{i-1,n-i+1,0} = 0.$$

Combining this relation with $a_{i,n-i,k} = (-1)^k a_{i,n-i,0}$ we obtain $a_{i,n-i,k} = (-1)^{i+k} a_{0,n,0}$ for $i = 0, 1, \ldots, n$ and $k = 1, 2, \ldots, d$. Set

$$f' \equiv f - a_{0,n,0} f_{n,d} \equiv \sum_{i,j,k} a'_{ijk} t_{ijk}.$$

In view of (5.12), the greatest value of $i+j$ such that $a'_{ijk} \neq 0$ for some k is now $< n$. It follows by induction that f is a finite linear combination of the $f_{n,d}$ as required. □

6. The Higgs factorization.

The main result of this section is that the Tutte polynomial of a matroid perspective of degree d is computationally equivalent to the $d+1$ Tutte polynomials of the matroids of its Higgs factorization.

Let $P : M \to M'$ be a matroid perspective of *degree* $d = d(P) = r(M) - r(M')$. For $k = 0, 1, \ldots, r(M) - r(M')$ we denote by $t_k(P) = t_k(M, M')$ the polynomial in two variables defined by

$$t(M, M'; x, y, z) = \sum_{k=0}^{k=d} t_k(M, M'; x, y) z^k. \tag{6.1}$$

We have

$$t_k(M, M'; x, y) = \sum_{\substack{X \subseteq E \\ r_M(X) - r_{M'}(X) = d-k}} (x-1)^{r(M') - r_{M'}(X)} (y-1)^{|X| - r_M(X)}. \tag{6.2}$$

THEOREM 6.1. — *Let $M \to M'$ be a matroid perspective of degree $d = r(M) - r(M')$ and $M_0 = M, M_1, \ldots, M_d = M'$ be its Higgs factorization. Then for $i = 0, 1, \ldots, d$ we have*

$$t(M_i; x, y) = \sum_{k=0}^{k=i} (y-1)^{i-k} t_k(M, M'; x, y) + \sum_{k=i+1}^{k=d} (x-1)^{k-i} t_k(M, M'; x, y). \tag{6.3}$$

Conversely

$$t_0(M, M', x, y) = \frac{1}{xy - x - y} \left[-t(M_0; x, y) + (x-1) t(M_1; x, y) \right] \tag{6.4}$$

$$t_k(M, M'; x, y) = \frac{1}{xy - x - y} \big[(y-1) t(M_{k-1}; x, y) + (-xy + x + y - 2) t(M_k; x, y) + (x-1) t(M_{k+1}; x, y) \big] \tag{6.5}$$

for $k = 1, 2, \dots, d-1$

$$(6.6)\quad t_d(M, M'; x, y) = \frac{1}{xy - x - y}\big[(y-1)t(M_{d-1}; x, y) - t(M_d; x, y)\big]$$

or, equivalently,

$$(6.7)\quad t(M, M'; x, y, z) = \frac{1}{xy - x - y}\big[(yz - z - 1)t(M; x, y) + (-z + x - 1)z^{d-1} \times t(M'; x, y) + (z - x + 1)(yz - z - 1)\sum_{i=1}^{i=d-1} z^{i-1} t(M_i; x, y)\big].$$

Proof. — For $X \subseteq E$ and $i = 0, 1, \dots, d$ set

$$t(M_i; X; x, y) = (x-1)^{r(M_i) - r_{M_i}(X)}(y-1)^{|X| - r_{M_i}(X)}.$$

We have

$$t(M_i; x, y) = \sum_{k=0}^{k=d} \sum_{\substack{X \subseteq E \\ r_M(X) - r_{M'}(X) = d-k}} t(M_i; X; x, y).$$

By (2.4) we have $r_{M_i} = \min(r_M(X), r_{M'}(X) + d - i)$. If $r_M(X) - r_{M'}(X) = d - k$, we have $r_{M_i}(X) = r_{M'}(X) + d - i = r_M(X) + k - i$ for $0 \leqslant k \leqslant i$ and $r_{M_i}(X) = r_M(X) = r_{M'}(X) + d - k$ for $i + 1 \leqslant k \leqslant d$. By straightforward substitution we obtain

$$\sum_{\substack{X \subseteq E \\ r_M(X) - r_{M'}(X) = d-k}} t(M_i; X; x, y) = \begin{cases} (y-1)^{i-k} t_k(M, M'; x, y) & \text{if } 0 \leqslant k \leqslant i \\ (x-1)^{k-i} t_k(M, M'; x, y) & \text{if } i + 1 \leqslant k \leqslant d. \end{cases}$$

The first part of Theorem 6.1 follows.

By (6.3) the polynomials $t(M_i)$ for $i = 0, 1, \dots, d$ are linear combinations with coefficients in $\mathbb{Q}(x, y)$ of the polynomials $t_k(M, M')$ with $k = 0, 1, \dots, d$. The second part of Theorem 6.1 amounts to inverting the corresponding $(d+1) \times (d+1)$ matrix (a particular instance of Toeplitz matrix). Surprisingly enough this computation can be completely carried out, yielding the simple formulas (6.4)-(6.6). Expanding the determinant $\Delta_d(x, y)$ of the matrix along its last column we find it satisfies the relation $\Delta_d = (xy - x - y)\Delta_{d-1}$. Hence $\Delta_d = (xy - x - y)^d$. Similar inductive relations are satisfied by determinants obtained by deleting one row and one column.

Alternatively, given (6.4)-(6.6) we can check their validity directly. Using (2.4), for any $X \subseteq E$ we have

$$\frac{1}{xy-x-y}\left[-t(M_0;x,y)+(x-1)t(M_1;x,y)\right]$$
$$=\begin{cases} 0 & \text{if } r_M(X)-r_{M'}(X)<d \\ t_0(M,M';X;x,y) & \text{if } r_M(X)-r_{M'}(X)=d. \end{cases}$$

Analogous identities hold in the two other cases. □

The particular case $r(M) - r(M') = 1$ of (6.7) is equivalent to Corollary 6.14 of [3] - see (5.8).

When $r(M) - r(M') \geqslant 2$, the polynomials $t(M)$ and $t(M')$ are not sufficient in general to determine $t(M, M')$, as shown by the following counterexample. Let M, L, M', N' be the cycle-matroids of the graphs with edge-sets $\{e_1 = ab, e_2 = ac, e_3 = bc, e_4 = ad, e_5 = ae\}$, $\{e_1 = ab, e_2 = ac, e_3 = bc, e_4 = ad, e_5 = bd\}$, $\{e_1 = ab, e_2 = ac, e_3 = bc, e_4 = ab, e_5 = ab\}$, $\{e_1 = ab, e_2 = ab, e_3 = ab, e_4 = ac, e_5 = bc\}$ respectively. These matroids are displayed in Figure 1 as matroids of affine dependencies of set of points in dimensions 3,2,1,1 respectively. We have $M \to M'$ and $M \to N'$. Obviously $M' \approx N'$, and hence $t(M') = t(N')$. However $t(M, M') = (x^2 + 3x + y + 2)z^2 + (3x + 2y + 3)z + x + y + 1 \neq t(M, N') = (x^2 + 3x + 3)z^2 + (xy + 2x + y + 5)z + y + 2$.

We observe moreover that $M \to L \to M'$ and $M \to L \to N'$ factorize $M \to M'$ and $M \to N'$ respectively. Since the Tutte polynomials of the matroids composing these two factorizations are equal, it follows that Theorem 6.1 cannot be extended to general factorizations (actually $M \to L \to M'$ is a Higgs factorization, whereas $M \to L \to N'$ is not).

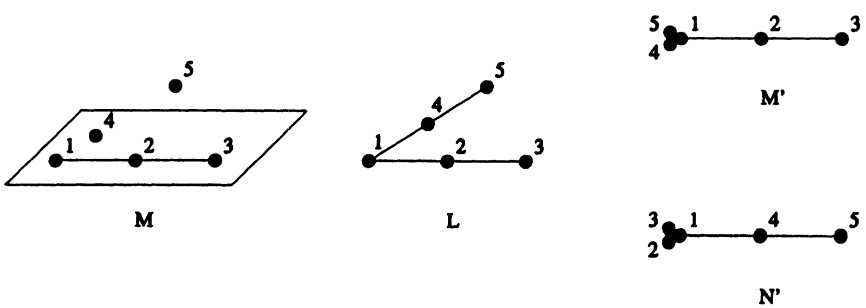

Figure 1

By Theorem 6.1, for numbers x, y, z such that $xy - x - y \neq 0$, the evaluation $t(M, M'; x, y, z)$ can be expressed as a linear combination of $t(M_i; x, y)$ with $i = 0, 1, \dots, d$. The situation is different when $xy - x - y = 0$. Proposition 6.2 relates $t(M, M'; 0, 0, z)$ to the β invariants of $M_0, M_1, \dots, M_d$. We recall that $\beta(M)$ is the coefficient of x in $t(M; x, y)$, or alternatively, since $t(M; 0, 0) = 0$, the limit of $t(M; x, 0)/x$ when $x \to 0$.

PROPOSITION 6.2. — *Let $M \to M'$ be a matroid perspective with $d = r(M) - r(M')$, and $M_0 = M, M_1, \dots, M_d = M'$ be its Higgs factorization. We have*

$$t_0(M, M'; 0, 0) = \beta(M_0) + \beta(M_1) \tag{6.8}$$

$$t_i(M, M'; 0, 0) = \beta(M_{i-1}) + 2\beta(M_i) + \beta(M_{i+1}) \tag{6.9}$$

for $i = 1, 2, \dots, d-1$

$$t_d(M, M'; 0, 0) = \beta(M_{d-1}) + \beta(M_d). \tag{6.10}$$

Proof. — By (6.4) and the continuity of the polynomial $t_0(M, M')$ we have

$$\begin{aligned} t_0(M, M'; 0, 0) &= \lim_{x \to 0} t_0(M, M'; x, 0) \\ &= \lim_{x \to 0} \frac{1}{-x}\left[-t(M_0; x, 0) + (x-1)t(M_1; x, 0)\right] \\ &= \beta(M_0) + \beta(M_1). \end{aligned}$$

The two other cases are similar. □

COROLLARY 6.3.

$$t(M, M'; 0, 0, 0) = \beta(M) + \beta(M_1) \tag{6.11}$$

$$t(M, M'; 0, 0, 1) = 2\beta(M) + 2\beta(M') + 4 \sum_{k=1}^{k=d-1} \beta(M_k). \tag{6.12}$$

□

Observe that Proposition 6.2 implies that $t(M, M'; 0, 0, -1) = 0$. However this equality is trivial since by (5.6) we have $t(M, M'; 0, 0, -1) = t(M; 0, 0) = 0$.

PROPOSITION 6.4. — *Let $M \to M'$ be a matroid perspective with $d = r(M) - r(M')$, and $M_0 = M, M_1, \ldots, M_d = M'$ be its Higgs factorization. Let i,j be two integers such that $0 \leqslant i < j \leqslant d$. We have*

$$(6.13)\quad t(M_i, M_j; x, y, z) = \sum_{k=0}^{k=i} (y-1)^{i-k} t_k(M, M'; x, y) + \sum_{k=i+1}^{k=j-1} z^{k-i} t_k(M, M'; x, y) + z^{j-i} \sum_{k=j}^{k=d} (x-1)^{k-j} t_k(M, M'; x, y).$$

Proof. — Proposition 6.4 is an immediate corollary of Theorem 6.1. Alternatively a direct proof is as follows. For $X \subseteq E$ and $i = 0, 1, \ldots, d$ set

$$t(M_i, M_j; X; x, y, z) = (x-1)^{r(M_j) - r_{M_j}(X)} (y-1)^{|X| - r_{M_i}(X)} z^{r(M_i) - r(M_j) - (r_{M_i}(X) - r_{M_j}(X))}.$$

By (5.2) we have

$$t(M_i, M_j) = \sum_{X \subseteq E} t(M_i, M_j; X) = \sum_{k=0}^{k=d} \sum_{\substack{X \subseteq E \\ r_M(X) - r_{M'}(X) = d-k}} t(M_i, M_j; X).$$

By (2.4) we have

$$\sum_{\substack{X \subseteq E \\ r_M(X) - r_{M'}(X) = d-k}} t(M_i, M_j; X; x, y, z) = \begin{cases} (y-1)^{i-k} t_k(M, M'; x, y) & \text{if } 0 \leqslant k \leqslant i \\ z^{k-i} t_k(M, M'; x, y) & \text{if } i \leqslant k \leqslant j \\ z^{j-i} (x-1)^{k-j} t_k(M, M'; x, y) & \text{if } j \leqslant k \leqslant d. \end{cases}$$

□

COROLLARY 6.5. — *The Tutte polynomial of the Higgs major $H(M, M')$ of a matroid perspective $M \to M'$ can be calculated from $t(M, M')$. Set $d = r(M) - r(M')$. We have*

$$(6.14)\qquad t(H(M, M'); x, y) = \sum_{k=0}^{k=d} t_k(M, M'; x, y) t(U_{k,d}; x, y).$$

Proof. — By (2.5), we get Corollary 6.5 from Theorem 6.1, Proposition 4.1 and Proposition 4.4. Alternatively by (5.1) Formula (6.14) is a particular case of (4.1). □

We end this section by a proof of Proposition 4.5.

Lemma 6.6. — *Let a, b, c be three non-negative integers such that $b \leqslant a$. We have*

(6.15)

$$\sum_{k=0}^{k=\min(a-b,c)} (-1)^k \binom{a-k}{b}\binom{c}{k} = \begin{cases} (-1)^{a-b}\binom{c-b-1}{a-b} & \text{if } c \geqslant a \\ 0 & \text{if } b < c < a \\ \binom{a-c}{a-b} & \text{if } c \leqslant b. \end{cases}$$

Proof. — We first prove (6.15) when $c \geqslant a$ by induction on $a+b$.

For $b = 0$ (6.15) reduces to $\sum_{k=0}^{k=a}(-1)^k\binom{c}{k} = (-1)^a\binom{c-1}{a}$ which can easily be proved by induction on a.

For $b = a$ (6.15) is trivially true.

Suppose $0 < b < a$. For $k = 0, 1, \ldots, a-b-1$ we have $\binom{a-k}{b} = \binom{a-k-1}{b} + \binom{a-k-1}{b-1}$. Hence

$$\begin{aligned}
&\sum_{k=0}^{k=a-b} (-1)^k \binom{a-k}{b}\binom{c}{k} \\
&\quad = (-1)^{a-b}\binom{c}{a-b} + \sum_{k=0}^{k=a-b} (-1)^k \binom{a-k}{b}\binom{c}{k} \\
&\quad = \sum_{k=0}^{k=a-b-1} (-1)^k \binom{a-k-1}{b}\binom{c}{k} + (-1)^{a-b}\binom{c}{a-b} \\
&\qquad\qquad + \sum_{k=0}^{k=a-b-1} (-1)^k \binom{a-k-1}{b-1}\binom{c}{k} \\
&\quad = \sum_{k=0}^{k=a-b-1} (-1)^k \binom{a-k-1}{b}\binom{c}{k} + \sum_{k=0}^{k=a-b} (-1)^k \binom{a-k-1}{b-1}\binom{c}{k} \\
&\quad = (-1)^{a-b-1}\binom{c-b-1}{a-b-1} + (-1)^{a-b}\binom{c-b}{a-b} \text{(by induction)} \\
&\quad = (-1)^{a-b}\binom{c-b-1}{a-b}.
\end{aligned}$$

We have $\sum_{k=0}^{k=a-b}(-1)^k\binom{a-k}{b}\binom{c}{k} = (-1)^{a-b}\binom{c-b-1}{a-b}$ for all integers $c \geqslant a$. Both terms of this equality can be considered as polynomials in c. Since equality holds for infinitely many values of c, the corresponding polynomial identity holds. As easily checked, this identity reduces to (6.15) when evaluated at non negative integer values of c. □

Proof of Proposition 4.5. — Set $r = r(M)$ and let $M_0 = M, M_1, \dots, M_r = U_0(E)$ be the Higgs factorization of the matroid perspective $M \to U_0(E)$. We have $\mathrm{Tr}_{r-k}(M) = M_k$. Hence by Theorem 6.1

$$\begin{aligned} t(\mathrm{Tr}_{r-k}(M);x,y) &= t(M_k;x,y) \\ &= \sum_{\ell=0}^{\ell=k-1} (y-1)^{k-\ell} t_\ell(M,U_0(E)) + \sum_{\ell=k}^{\ell=r} (x-1)^{\ell-k} t_\ell(M,U_0(E)). \end{aligned}$$

By Proposition 4.1 we have

$$\begin{aligned} &t(M,U_0(E);x,y,z) \\ &= t(M;z+1,y) = \sum_{j=0}^{j=r} u_j(z+1)^j = \sum_{j=0}^{j=r}\Big(\sum_{\ell=0}^{\ell=j}\tbinom{j}{\ell}z^\ell\Big)u_j = \sum_{\ell=0}^{\ell=r}\Big(\sum_{j=\ell}^{j=r}\tbinom{j}{\ell}u_j\Big)z^\ell. \end{aligned}$$

Hence $t_\ell(M,U_0(E);x,y) = \sum_{j=\ell}^{j=r}\binom{j}{\ell}u_j$ and therefore

$$\begin{aligned} &t(Tr_{r-k}(M);x,y) \\ &\quad = \sum_{\ell=0}^{\ell=k-1}\Big[\sum_{j=\ell}^{j=r}\binom{j}{\ell}u_j\Big](y-1)^{k-\ell} + \sum_{\ell=k}^{\ell=r}\Big[\sum_{j=\ell}^{j=r}\binom{j}{\ell}u_j\Big](x-1)^{\ell-k}. \end{aligned}$$

We have

$$\begin{aligned} &\sum_{\ell=0}^{\ell=k-1}\Big[\sum_{j=\ell}^{j=r}\binom{j}{\ell}u_j\Big](y-1)^{k-\ell} \\ &\quad = \sum_{\ell=0}^{\ell=k-1}(-1)^{k-\ell}\Big[\sum_{j=\ell}^{j=r}\binom{j}{\ell}u_j\Big]\Big[\sum_{i=0}^{i=k-\ell}(-1)^i\binom{k-\ell}{i}y^i\Big] \\ &\quad = \sum_{i=0}^{i=k}\Big[\sum_{j=0}^{j=r}(-1)^{k-i}\Big[\sum_{\ell=0}^{\ell=\min(j,k-1,k-i)}(-1)^\ell\binom{k-\ell}{i}\binom{j}{\ell}\Big]u_j\Big]y^i \\ &\quad = \sum_{j=0}^{j=k} y^j(y-1)^{k-j}u_j + \sum_{j=k+1}^{j=r}\Big[\sum_{i=1}^{i=k}\binom{j-i-1}{k-i}y^i\Big]u_j \end{aligned}$$

by using Lemma 6.6. Similarly

$$\sum_{\ell=k}^{\ell=r}\Big[\sum_{j=\ell}^{j=r}\binom{j}{\ell}u_j\Big](x-1)^{\ell-k}$$
$$=\sum_{\ell=k}^{\ell=r}(-1)^{\ell-k}\Big[\sum_{j=\ell}^{j=r}\binom{j}{\ell}u_j\Big]\Big[\sum_{i=0}^{i=\ell-k}(-1)^i\binom{\ell-k}{i}x^i\Big]$$
$$=\sum_{i=0}^{i=r-k}\Big[\sum_{j=k+i}^{j=r}(-1)^{k+i}\Big[\sum_{\ell=k+i}^{\ell=j}(-1)^{\ell}\binom{\ell-k}{i}\binom{j}{\ell}\Big]u_j\Big]x^i$$
$$=\sum_{i=0}^{i=r-k}\Big[\sum_{j=k+i}^{j=r}\binom{j-i-1}{k-1}u_j\Big]x^i$$

again by Lemma 6.6. □

7. Tutte polynomials in terms of lattice of flats.

Let M be a matroid on a set E. We denote by $\mathcal{F}(M)$ the lattice of flats of M and by $\mu_{\mathcal{F}(M)}$ the Möbius function of $\mathcal{F}(M)$. We shall sometimes use the standard lattice-theoretic notations 0, 1, $A \wedge B$, $A \vee B$ for the elements $Cl_M(\varnothing)$, E, $A \cap B$, $Cl_M(A \cup B)$ respectively of the lattice $\mathcal{F}(M)$, where $A, B \in \mathcal{F}(M)$ and $Cl_M(\varnothing)$ is the set of loops of M.

As well-known, the Tutte polynomial of a matroid M can be expressed in terms of $\mathcal{F}(M)$ and its Möbius function. This result generalizes to Tutte polynomials of matroid perspectives. We also give a generalization of Stanley's factorization theorem.

THEOREM 7.1. — *Let $M \to M'$ be a matroid perspective on a set E. We have*

$$(7.1)\quad t(M,M';x,y,z) = \sum_{X,Y\in\mathcal{F}(M)} \mu_{\mathcal{F}(M)}(Y,X)(x-1)^{r(M')-r_{M'}(X)} \times y^{|Y|}(y-1)^{-r_M(X)}z^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}.$$

The following lemma is due to Rota.

LEMMA 7.2 [25] (see also [1] Theorem 4.27). — *Let P be a poset and $\varphi : P \to P$ be a closure on P. Let $x \in P$ and $y \in \varphi(P)$. We have*

$$\sum_{t\in P,\varphi(t)=y} \mu_P(x,t) = \begin{cases} 0 & \text{if } x \notin \varphi(P) \\ \mu_{\varphi(P)}(x,y) & \text{if } x \in \varphi(P). \end{cases}$$

□

LEMMA 7.3. — *Let $\varphi : 2^E \to 2^E$ be a closure on a set E and $\mathcal{F}$ be the set of closed subsets of E. Let $X \in \mathcal{F}$. We have*

$$\sum_{\substack{Y\subseteq X\\ \varphi(Y)=X}} (y-1)^{|Y|} = \sum_{Z\in\mathcal{F}} \mu_{\mathcal{F}}(Z,X)y^{|Z|}.$$

Proof. — The Möbius function of the Boolean lattice $\mathcal{B}$ of subsets of E is given by $\mu_{\mathcal{B}}(A,B) = (-1)^{|B\setminus A|}$ if $A \subseteq B$ and 0 otherwise. Applying Lemma 7.2 we have

$$\begin{aligned}
\sum_{\substack{Y\subseteq X\\ \varphi(Y)=X}} (y-1)^{|Y|} &= \sum_{\substack{Y\subseteq X\\ \varphi(Y)=X}} \sum_{Z\subseteq Y} (-1)^{|Y\setminus Z|} y^{|Z|} \\
&= \sum_{Z\subseteq X} \Big(\sum_{\substack{Z\subseteq Y\\ \varphi(Y)=X}} (-1)^{|Y\setminus Z|} \Big) y^{|Z|} \\
&= \sum_{Z\in\mathcal{F}} \mu_{\mathcal{F}}(Z,X)y^{|Z|}.
\end{aligned}$$

□

Proof of Theorem 7.1. — Set $f(k,k') = (x-1)^{r(M')-k'}(y-1)^{-k}z^{r(M)-r(M')-(k-k')}$. Then

$$\begin{aligned}
t(M,M';x,y,z) &= \sum_{X\subseteq E} f(r_M(X), r_{M'}(X))(y-1)^{|X|} \\
&= \sum_{X\in\mathcal{F}(M)} \sum_{\substack{Y\subseteq X\\ Cl_M(Y)=X}} f(r_M(Y), r_{M'}(Y))(y-1)^{|Y|} \\
&= \sum_{X\in\mathcal{F}(M)} f(r_M(X), r_{M'}(X)) \sum_{\substack{Y\subseteq X\\ Cl_M(Y)=X}} (y-1)^{|Y|} \\
&= \sum_{X\in\mathcal{F}(M)} f(r_M(X), r_{M'}(X)) \sum_{Y\in\mathcal{F}(M)} \mu_{\mathcal{F}(M)}(Y,X)y^{|Y|}.
\end{aligned}$$

The last equality is Lemma 7.3. We have used the fact that $r_M(Y) = r_M(X)$ and $r_{M'}(Y) = r_{M'}(X)$ when $Cl_M(Y) = X$. The first equality holds by definition of closure and rank in a matroid. The second uses crucially the fact that $Cl_M(Y) \subseteq Cl_{M'}(Y)$ for all $Y \subseteq E$, which holds if and only if $M \to M'$ is a perspective. □

COROLLARY 7.4. — *Let $M \to M'$ be a matroid perspective. If M has no loops then*

$$(7.2)\quad t(M,M';0,0,0) = (-1)^{r(M)} \sum_{\substack{X\in\mathcal{F}(M)\\ r_M(X)-r_{M'}(X)=r(M)-r(M')}} \mu_{\mathcal{F}(M)}(0,X)$$

$$(7.3)\quad t(M,M';0,0,1) = (-1)^{r(M')} \sum_{X\in\mathcal{F}(M)} \mu_{\mathcal{F}(M)}(0,X)(-1)^{r_M(X)-r_{M'}(X)}.$$

□

The proof of Corollary 7.4 is immediate by inspection of (7.1). Let M be a loopless matroid on a set E. From (7.2) with $M' = U_0(E)$ and (5.2) we obtain $t(M;1,0) = t(M,U_0(E);0,0,0) = (-1)^{r(M)}\mu_{\mathcal{F}(M)}(0,1)$, a classical evaluation of the Tutte polynomial.

Let $M \to M'$ be a matroid perspective on E. We define

$$(7.4)\quad p(M,M';u,w) = \sum_{X\in\mathcal{F}(M)} \mu_{\mathcal{F}(M)}(0,X)u^{r(M')-r_{M'}(X)}w^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}$$

$$(7.5)\quad q(M,M';u,v,w) = \sum_{Y\in\mathcal{F}(M)} v^{|Y|}p(M/Y,M'/Y;u,w).$$

When M has no loops, we have clearly

$$(7.6)\quad q(M,M';u,0,w) = p(M,M';u,w).$$

When $M = M'$ the polynomials $p(M,M')$ and $q(M,M')$ reduce respectively to the Poincaré polynomial and the coboundary polynomial of M (see [12] [13]).

In [13] a different generalization of the coboundary polynomial is introduced, namely the polynomial $\sum_{X,Y\in\mathcal{F}(M)} \mu_{\mathcal{F}(M)}(Y,X)u^{r(M')-r_{M'}(X)}v^{r_M(Y)}$ for a matroid perspective $M \to M'$ (cf [13] Thm.9).

LEMMA 7.5. — *Let $M \to M'$ be a matroid perspective, and $Y \in \mathcal{F}(M)$. We have*

$$(7.7)\quad p(M/Y,M'/Y;u,w) = \sum_{X\in\mathcal{F}(M)} \mu_{\mathcal{F}(M)}(Y,X)u^{r(M')-r_{M'}(X)}w^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}.$$

Proof. — We have

$$p(M/Y, M'/Y; u, w) = \sum_{Z \in \mathcal{F}(M/Y)} \mu_{\mathcal{F}(M/Y)}(0, Z) \times u^{r(M'/Y) - r_{M'/Y}(Z)} w^{r(M/Y) - r(M'/Y) - (r_{M/Y}(Z) - r_{M'/Y}(Z))}.$$

The mapping $Z \mapsto X = Cl_M(Y \cup Z)$ is an isomorphism from $\mathcal{F}(M/Y)$ onto the interval $[Y, 1]$ of $\mathcal{F}(M)$. Hence $\mu_{\mathcal{F}(M/Y)}(0, Z) = \mu_{\mathcal{F}(M)}(Y, X)$.

We have $r(M/Y) = r(M) - r_M(Y)$, $r(M'/Y) = r(M') - r_{M'}(Y)$, $r_{M/Y}(Z) = r_M(Y \cup Z) - r_M(Y) = r_M(X) - r_M(Y)$, $r_{M'/Y}(Z) = r_{M'}(Y \cup Z) - r_{M'}(Y) = r_{M'}(X) - r_{M'}(Y)$. We point out that the equality $r_{M'}(Y \cup Z) = r_{M'}(X)$ holds because $M \to M'$ is a perspective. Substituting these different equalities in the above expression for $p(M/Y, M'/Y; u, w)$, we obtain Lemma 7.5. □

PROPOSITION 7.6.

$$q(M, M'; u, v, w) = (v-1)^{r(M)} t\left(M, M'; \frac{u}{v-1} + 1, v, \frac{w}{v-1}\right). \tag{7.8}$$

When M has no loops

$$p(M, M'; u, w) = (-1)^{r(M)} t(M, M'; 1-u, 0, -w). \tag{7.9}$$

Proof. — From Lemma 7.5 and (7.5) we obtain

$$q(M, M'; u, v, w) = \sum_{X, Y \in \mathcal{F}(M)} \mu_{\mathcal{F}(M)}(Y, X) u^{r(M') - r_{M'}(X)} v^{|Y|} w^{r(M) - r(M') - (r_M(X) - r_{M'}(X))}.$$

Formula (7.8) follows then from Theorem 7.1. We obtain (7.9) from (7.6). □

Stanley's factorization theorem for modular flats [26] generalizes to $p(M, M')$. We recall that a flat A is *modular in* a matroid M if $r_M(A \cup X) + r_M(A \cap X) = r_M(A) + r_M(X)$ for all flats X of M.

THEOREM 7.7. — *Let $M \to M'$ be a matroid perspective and A be a modular flat of M such that $r_M(A) - r_{M'}(A) = r(M) - r(M')$. Then $p(M(A), M'(A))$ divides $p(M, M')$. We have*

$$\text{(7.10)}\quad p(M,M';u,w) = p(M(A),M'(A);u,w)\Bigg(\sum_{\substack{X\in\mathcal{F}(M)\\ X\wedge A=0}} \mu_{\mathcal{F}(M)}(0,X)u^{r(M)-r_M(A)-r_M(X)}\Bigg).$$

Stanley's theorem is obtained when $M=M'$.

LEMMA 7.8 (see Proof of Theorem 2 of [26]). — *Let M be a matroid and A be a modular flat of M. For any $X\in\mathcal{F}(M)$*

$$\text{(7.11)}\qquad \mu_{\mathcal{F}(M)}(0,X) = \sum_{\substack{Y\in\mathcal{F}(M)\\ Y\wedge A=0,\, Y\leqslant X\leqslant Y\vee A}} \mu_{\mathcal{F}(M)}(0,Y)\mu_{\mathcal{F}(M)}(Y,X).$$

For any $X,Y\in\mathcal{F}(M)$ such that $Y\wedge A=0$ and $Y\leqslant X\leqslant Y\vee A$

$$\text{(7.12)}\qquad r_M(X) = r_M(Y) + r_M(X\wedge A).$$

LEMMA 7.9. — *Let $M\to M'$ be a matroid perspective and A be a modular flat of M such that $r_M(A)-r_{M'}(A) = r(M)-r(M')$. Then A is a modular flat of M' and for any flat X of M we have $r_M(X)-r_{M'}(X) = r_M(X\wedge A)-r_{M'}(X\wedge A)$.*

Proof. — Since $M\to M'$ is a perspective, we have $r(M)-r(M') = r_M(A)-r_{M'}(A)\leqslant r_M(Cl_{M'}(A))-r_{M'}(Cl_{M'}(A))\leqslant r(M)-r(M')$. Hence $r_M(A) = r_M(Cl_{M'}(A))$. Therefore since A is a flat of M by hypothesis, and $Cl_{M'}(A)$ is a flat of M since $M\to M'$ is a perspective, we have $A = Cl_{M'}(A)$ i.e. A is a flat of M'.

Since A is modular in M, for any flat X of M we have $r_M(X\cup A)+r_M(X\cap A) = r_M(A)+r_M(X)$. By submodularity $r_{M'}(X\cup A)+r_{M'}(X\cap A)\leqslant r_{M'}(A)+r_{M'}(X)$. On the other hand, $M\to M'$ being a perspective, we have $r_M(A)-r_{M'}(A)\leqslant r_M(X\cup A)-r_{M'}(X\cup A)\leqslant r(M)-r(M')$. Therefore since $r_M(A)-r_{M'}(A) = r(M)-r(M')$ we have $r_M(A)-r_{M'}(A) = r_M(X\cup A)-r_{M'}(X\cup A) = r(M)-r(M')$. It follows that $r_M(X\cap A)-r_{M'}(X\cap A)\geqslant r_M(X)-r_{M'}(X)$. Since $M\to M'$ is a perspective the reverse inequality holds, and we have the equality $r_M(X\cap A)-r_{M'}(X\cap A) = r_M(X)-r_{M'}(X)$. The modularity of A in M' follows. □

Proof of Theorem 7.7. — By (7.11) we have

$$\begin{aligned}
&p(M,M';u,w)\\
&\quad= \sum_{X\in\mathcal{F}(M)} \mu_{\mathcal{F}(M)}(0,X)u^{r(M')-r_{M'}(X)}w^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}\\
&\quad= \sum_{X\in\mathcal{F}(M)} \Bigg(\sum_{\substack{Y\in\mathcal{F}(M)\\ Y\wedge A=0, Y\leqslant X\leqslant Y\vee A}} \mu_{\mathcal{F}(M)}(0,Y)\mu_{\mathcal{F}(M)}(Y,X)\Bigg) u^{r(M')-r_{M'}(X)}\\
&\qquad\qquad \times w^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}\\
&\quad= \sum_{\substack{Y\in\mathcal{F}(M)\\ Y\wedge A=0}} \mu_{\mathcal{F}(M)}(0,Y)\Bigg(\sum_{\substack{X\in\mathcal{F}(M)\\ Y\leqslant X\leqslant Y\vee A}} \mu_{\mathcal{F}(M)}(Y,X)u^{r(M')-r_{M'}(X)}\\
&\qquad\qquad \times w^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}\Bigg).
\end{aligned}$$

Let $X, Y \in \mathcal{F}(M)$ be such that $Y \wedge A = 0$ and $Y \leqslant X \leqslant Y \vee A$. By (7.12) we have $r_M(X) = r_M(Y) + r_M(X \wedge A)$. By Lemma 7.9 we have $r_M(X) - r_{M'}(X) = r_M(X \wedge A) - r_{M'}(X \wedge A)$ and $r_M(Y) = r_{M'}(Y)$. It follows that $r_{M'}(X) = r_{M'}(Y) + r_{M'}(X \wedge A)$.

The mapping $X \mapsto Z = X \wedge A$ is an order isomorphism of the interval $[Y, Y \vee A]$ of $\mathcal{F}(M)$ onto the interval $[0, A]$. Hence $\mu_{\mathcal{F}(M)}(Y,X) = \mu_{\mathcal{F}(M)}(0,Z)$. Using this mapping and the above relations we obtain

$$\begin{aligned}
&\sum_{\substack{X\in\mathcal{F}(M)\\ Y\leqslant X\leqslant Y\vee A}} \mu_{\mathcal{F}(M)}(Y,X)u^{r(M')-r_{M'}(X)}w^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}\\
&\quad= u^{r(M')-r_{M'}(A)-r_{M'}(Y)} \sum_{\substack{Z\in\mathcal{F}(M)\\ Z\leqslant A}} \mu_{\mathcal{F}(M)}(0,Z)u^{r_{M'}(A)-r_{M'}(Z)}\\
&\qquad\qquad \times w^{r_M(A)-r_{M'}(A)-(r_M(Z)-r_{M'}(Z))}\\
&\quad= u^{r(M')-r_{M'}(A)-r_{M'}(Y)}p(M(A), M'(A); u, w)\\
&\quad= u^{r(M)-r_M(A)-r_M(Y)}p(M(A), M'(A); u, w).
\end{aligned}$$

□

Remark 7.10 — Since $p(M, M'; u, u) = p(M; u)$, the Poincaré polynomial of the matroid M, it follows that in the formula (7.10) the quotient $p(M, M')/p(M(A), M'(A))$ is equal to $p(M)/p(M(A))$. This last quotient has been identified by Brylawski as the Poincaré polynomial of the complete Brown truncation of M relative to A divided by $u - 1$ ([4] Cor.7.4).

We recall that the *principal extension of a matroid M with respect to a subset of elements A* is the matroid whose flats are all flats of M not containing A and all sets of the form $X \cup \{p\}$ for X flat of M containing A, where p is a new element. The *Brown truncation* of a matroid M relative to a subset of elements A is obtained by first performing a principal extension of M with respect to A, and then by contracting the new element.

Let us denote by $\mathrm{Tr}(M; A)$ the result of this operation. Note that $M \to \mathrm{Tr}(M; A)$ is a perspective. The usual truncation is $\mathrm{Tr}(M; E)$, when A is the whole set E of elements. The *complete Brown truncation of M relative to A* in the sense of Brylawski is obtained by iterating $r_M(A) - 1$ times the Brown truncation relative to A. We denote it here by $\mathrm{Br}(M; A)$.

COROLLARY 7.11. — *Under the hypothesis of Theorem 7.7 we have*

$$p(M, M'; u, w) = \frac{1}{u-1} p(M(A), M'(A); u, w).p(\mathrm{Br}(M; A; u)). \tag{7.13}$$

□

The above proof of Theorem 7.7 follows Stanley's proof. Using Brylawski's result, Theorem 7.7 can be derived from Stanley's Factorization Theorem by means of Theorem 6.1.

LEMMA 7.12. — *Let $M \to M'$ be a matroid perspective on a set E and $A \subseteq E$. Then* $\mathrm{Tr}(M; A) \to \mathrm{Tr}(M'; A)$.

Proof. — Let N, N' be the principal extensions of M, M' respectively relative to A. By [18], $N \to N'$ is a matroid perspective. Contracting the new element, we deduce that $\mathrm{Tr}(M; A) \to \mathrm{Tr}(M'; A)$ is a matroid perspective. □

PROPOSITION 7.13. — *Let $M \to M'$ be a matroid perspective and A be a modular flat of M such that $r_M(A) - r_{M'}(A) = d = r(M) - r(M')$. Let $M_0 = M \to M_1 \to \ldots \to M_d = M'$ be the Higgs factorization of $M \to M'$. Then if $M'(A)$ is loopless, the complete Brown truncations of $M_0, M_1, \ldots, M_d$ relative to A are all equal.*

Proof. — Suppose first $r_{M'}(A) = 1$. The condition $r_M(A) - r_{M'}(A) = d = r(M) - r(M')$, implies $r_{M_{d-1}}(A) = 2$ if $d \geqslant 1$ (otherwise Lemma 7.13 is trivial). Hence the (unique) major of $M' \to M_{d-1}$ is an extension of M_{d-1} by an element $\{e\}$ spanned by A. Since M' is loopless, the element e is in general position in A. Suppose e is spanned by a flat F of M_{d-1}. Since A is modular in M_{d-1} by Lemma 7.9, it follows that $e \in A \cap F$. If $A \not\leqslant F$ the intersection $A \cap F$ is of rank 1, and hence e is parallel to some element in

A, a contradiction. Therefore $A \subseteq F$, and by definition $\{e\}$ is the principal extension of M_{d-1} relative to A. It follows that M' is the complete Brown truncation of M_{d-1}.

Suppose now $r_{M'}(A) \geqslant 2$. Then by Lemma 7.12 we have $\mathrm{Tr}(M;A) \to \mathrm{Tr}(M';A)$. The Higgs factorization of $\mathrm{Tr}(M;A) \to \mathrm{Tr}(M';A)$ is given by $\mathrm{Tr}(M_0) \to \mathrm{Tr}(M_1) \to \ldots \to \mathrm{Tr}(M_d)$. Observe that for $i = 1,2,\ldots,d$ the subset A is a modular flat of $\mathrm{Tr}(M_i;A)$ and that the complete Brown truncation of $\mathrm{Tr}(M_i;A)$ relative to A is equal to the complete Brown truncation of M_i relative to A. Using induction and the above case $r_{M'}(A) = 1$, Lemma 7.13 follows. □

N.B. If loops are permitted, the matroids $\mathrm{Br}(M_i;A)$ may not be equal, but they are equal as geometries for all i such that $r_{M_i}(A)$ is not 0.

Alternative proof of Theorem 7.7. — Let $M_0 = M \to M_1 \to \ldots \to M_d = M'$ be the Higgs factorization of $M \to M'$. By Lemma 7.9 the subset A is a modular flat of M_i for $i = 1,2,\ldots,d$. Hence by the theorems of Stanley and Brylawski applied to M_i

$$p(M_i;u) = \frac{1}{u-1} p(M_i(A);u) p(\mathrm{Br}(M_i;A;u).$$

By Lemma 7.13, if $M_i(A)$ is loopless, we have $\mathrm{Br}(M_i(A)) = \mathrm{Br}(M;A)$ hence $p(\mathrm{Br}(M_i(A))) = p(\mathrm{Br}(M;A))$. Otherwise $p(M_i(A)) \equiv 0$. Using formula (6.7) with $x = u-1$ $y = 0$ and $z = -w$ in view of (7.9), we obtain (7.13) directly by factorizing $p(\mathrm{Br}(M;A))$. Details are left to the reader. □

8. Matroid perspectives on ordered sets. Activities.

The original definition by Tutte of the dichromatic polynomial of a graph in [28] is not by a closed formula as in Section 2, but uses activities of spanning trees. Tutte's definition was extended to general matroids by Crapo [12].

Specifically, given a matroid M, we have

$$t(M;x,y) = \sum_{B \text{ basis of } M} x^{\iota(B)} y^{\epsilon(B)} \tag{8.1}$$

where $\iota(B)$, $\epsilon(B)$ denote the internal and external activities respectively of B with respect to a given ordering of the set of elements of M.

We give in this section the extension of this formula to matroid perspectives.

Let E be an ordered set and M be a matroid on E.

Let $X \subseteq E$ be independent in M. For any $e \in E \backslash X$, there is at most one circuit of M contained in $X \cup \{e\}$, and if such a circuit exists then it contains e. We denote by $\epsilon_M(X)$ the number of elements $e \in E \backslash X$ such that $X \cup \{e\}$ contains a circuit with smallest element e.

Dually, for any $X \subseteq E$ spanning in M and any $e \in X$, we denote by $\iota_M(X)$ the number of elements $e \in X$ such that $(E \backslash X) \cup \{e\}$ contains a cocircuit with smallest element e.

Note that X being spanning in M if and only if $E \backslash X$ is independent in M^*, we have $\iota_M(X) = \epsilon_{M^*}(E \backslash X)$.

We say that $\epsilon_M(X)$ and $\iota_M(X)$ are the *external* respectively *internal activities* of X. In the matroid case, when X is a base - i.e. X is both independent and spanning - these definitions reduce to the usual ones.

THEOREM 8.1. — *Let E be an ordered set and $M \to M'$ be a matroid perspective on E. We have*

$$(8.2) \quad t(M, M'; x, y, z) = \sum_{\substack{X \subseteq E \\ X \text{ independent in } M \\ \text{and spanning in } M'}} x^{\iota_{M'}(X)} y^{\epsilon_M(X)} z^{r(M) - r(M') - (r_M(X) - r_{M'}(X))}.$$

The proof of Theorem 8.1 is by deletion/contraction of the greatest element. It generalizes the proof given by Brylawski in the matroid case [6].

THEOREM 8.2. — *Let E be a set with a total ordering and let M be a matroid on E. Let e be the greatest element of E with respect to the given ordering.*
If $X \subseteq E$ is independent in M, then

$$(8.3) \qquad \epsilon_{M\backslash e}(X) = \epsilon_M(X) + r_M(\{e\}) - 1 \text{ if } e \notin X$$

$$(8.4) \qquad \epsilon_{M/e}(X) = \epsilon_M(X) \text{ if } e \in X.$$

If $X \subseteq E$ is spanning in M, then

$$\iota_{M\setminus e}(X) = \iota_M(X) \text{ if } e \notin X \tag{8.5}$$

$$\iota_{M/e}(X) = \iota_M(X) - (r(M) - r(M \setminus e)) \text{ if } e \in X. \tag{8.6}$$

Proof. — For $x \in E \setminus X$ set $\epsilon_M(X;x) = 1$ if there exists a circuit of M contained in X with smallest element x, $\epsilon_M(X;x) = 0$ otherwise. We have $\epsilon_M(X) = \sum_{x\in E\setminus X} \epsilon_M(X;x)$.

(8.3) Suppose $e \notin X$. Clearly for all $x \in E \setminus (X \cup \{e\})$ we have $\epsilon_M(X;x) = \epsilon_{M\setminus e}(X;x)$. On the other hand since e is the greatest element of E for the ordering we have $\epsilon_M(X;e) = 1$ if and only if e is a loop of M. Hence

$$\begin{aligned}
\epsilon_M(X) &= \sum_{x\in E\setminus X} \epsilon_M(X;x) \\
&= \sum_{x\in E\setminus(X\cup\{e\})} \epsilon_M(X;x) + \epsilon_M(X;e) \\
&= \sum_{x\in(E\setminus\{e\})\setminus X} \epsilon_{M\setminus e}(X;x) + \epsilon_M(X;e) \\
&= \epsilon_{M\setminus e}(X) - r_M(\{e\}) + 1.
\end{aligned}$$

(8.4) Suppose $e \in X$. Let $x \in E\setminus X$ and C be a circuit of M contained in $X \cup \{x\}$. Then $C \setminus \{e\}$ is a circuit of M/e contained in $(X \setminus \{e\}) \cup \{x\}$. This is clear when $e \in C$, and follows from the fact that X is independent when $e \notin C$. Conversely if C' is a circuit of M/e contained in $(X\setminus\{e\})\cup\{x\}$ then C' or $C' \cup \{e\}$ is a circuit of M contained in $X \cup \{x\}$. Hence e being the greatest element of E for the ordering we have $\epsilon_M(X;x) = 1$ if and only if $\epsilon_{M/e}(X \setminus \{e\};x) = 1$.

(8.5) and (8.6) follow from (8.3) and (8.4) by duality. □.

Proof of Theorem 8.1. — For $X \subset E$ we set

$$f(M,M';X;x,y,z) = x^{\iota_{M'}(X)} y^{\epsilon_M(X)} z^{r(M)-r(M')-(r_M(X)-r_{M'}(X))}.$$

Let e be the greatest element of E for the ordering. We show that for this choice of e the function $f(M,M')$ defined by

$$f(M,M';x,y,z) = \sum_{\substack{X\subseteq E \\ X \text{ independent in } M \\ \text{spanning in } M'}} f(M,M';X;x,y,z)$$

satisfies the inductive relations of Theorem 5.3. It follows that $f(M,M') = t(M,M')$ proving Theorem 8.1.

We have

$$f(M,M') = \sum_{\substack{X\subseteq E \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M,M';X)$$

$$= \sum_{\substack{X\subseteq E\setminus\{e\} \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M,M';X) + \sum_{\substack{e\in X\subseteq E \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M,M';X).$$

In each of the cases (i)-(v) below, which refer to the corresponding cases of Theorem 5.3, we evaluate the two terms of the above sum. We denote by the letter X a subset of E independent in M and spanning in M'. If $e \in X$ we set $Y = X \setminus \{e\}$.

(i) *e is neither an isthmus nor a loop of M*

(i.a) If $e \notin X$ by Lemma 8.2 we have $\iota_{M'}(X) = \iota_{M'\setminus\{e\}}(X)$ and $\epsilon_{M'}(X) = \epsilon_{M'\setminus\{e\}}(X)$. Since M and M' are in perspective, a non-isthmus of M is also a non isthmus of M', and hence $r(M) = r(M \setminus e)$ and $r(M') = r(M' \setminus e)$. We have thus $r(M) - r(M') - (r_M(X) - r_{M'}(X)) = r(M\setminus e) - r(M'\setminus e) - (r_{M\setminus e}(X) - r_{M'\setminus e}(X))$. It follows that $f(M,M';X) = f(M \setminus e, M' \setminus e; X)$.

Now a subset X of $E \setminus \{e\}$ is independent in M if and only if X is independent in $M \setminus e$, and since e is not an isthmus of M' we have X spanning in M' if and only if X is spanning in $M' \setminus e$. Therefore

$$\sum_{\substack{X\subseteq E\setminus\{e\} \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M,M';X) \qquad \sum_{\substack{X\subseteq E\setminus\{e\} \\ X \text{ indep. in } M\setminus e \\ \text{span. in } M'\setminus e}} f(M\setminus e, M'\setminus e; X) = f(M\setminus e, M'\setminus e).$$

(i.b) If $e \in X$ by Lemma 8.2 we have $\iota_{M'}(X) = \iota_{M'/e}(Y)$ and $\epsilon_M(X) = \epsilon_{M/e}(Y)$. On the other hand $r(M) - r_M(X) = r(M/e) - r_{M/e}(Y)$ and $r(M') - r_{M'}(X) = r(M'/e) - r_{M'/e}(Y)$, and hence $r(M) - r(M') - ((r_M(X) - r_{M'}(X)) = r(M/e) - r(M'/e) - (r_{M/e}(Y) - r_{M'/e}(Y)$. Therefore $f(M,M';X) = f(M/e, M'/e; Y)$.

Since e is not a loop of M, we have X independent in M if and only if Y is independent in M/e, and X is spanning in M' if and only if Y is spanning in M'. It follows that

$$\sum_{\substack{e \in X \subseteq E \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M, M'; X) = \sum_{\substack{Y \subseteq E \setminus \{e\} \\ Y \text{ indep. in } M/e \\ \text{span. in } M'/e}} f(M/e, M'/e; Y) = f(M/e, M'/e).$$

(ii) *e is an isthmus of M'* (and hence also an isthmus of M since M and M' are in perspective).

Every subset X of E spanning in M' contains e. Hence

$$\sum_{\substack{X \subseteq E \setminus \{e\} \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M, M'; X) = 0.$$

If $e \in X$ by Lemma 8.2 we have $\iota_{M'}(X) = \iota_{M' \setminus e}(Y) + 1$ and $\epsilon_M(X) = \epsilon_{M \setminus e}(Y)$. On the other hand $r(M) = r(M \setminus e) + 1$, $r(M') = r(M' \setminus e) + 1$, $r_M(X) = r_{M \setminus e}(Y) + 1$ and $r_{M'}(X) = r_{M' \setminus e}(Y) + 1$. Hence $r(M) - r(M') - ((r_M(X) - r_{M'}(X)) = r(M/e) - r(M'/e) - (r_{M/e}(Y) - r_{M'/e}(Y)$. It follows that $f(M, M'; X; x, y, z) = xf(M \setminus e, M' \setminus e; Y; x, y, z)$. Since a subset X of E containing e is such that X is independent in M and spanning in M' if and only if $Y = X \setminus \{e\}$ is independent in $M \setminus e$ and spanning in $M' \setminus e$ we have

$$\begin{aligned} f(M, M') &= \sum_{\substack{e \in X \subseteq E \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M, M'; X) \\ &= x \sum_{\substack{Y \subseteq E \setminus \{e\} \\ Y \text{ indep. in } M \setminus e \\ \text{span. in } M' \setminus e}} f(M \setminus e, M' \setminus e; Y) = xf(M \setminus e, M' \setminus e). \end{aligned}$$

(iii) *e is a loop of M* (and hence also a loop of M' since M and M' are in perspective)

Every subset X of E independent in M is contained in $E \setminus \{e\}$. Hence

$$\sum_{\substack{e \in X \subseteq E \\ X \text{ indep. in } M \\ \text{span. in } M'}} f(M, M'; X) = 0.$$

If $e \notin X$ by Lemma 8.2 we have $\iota_{M'}(X) = \iota_{M' \setminus e}(X)$ and $\epsilon_M(X) = \epsilon_{M \setminus e}(X) + 1$. On the other hand $r(M) = r(M \setminus e)$ and $r(M') = r(M' \setminus e)$.

It follows that $f(M,M';X;x,y,z) = yf(M\setminus e, M'\setminus e; X;x,y,z)$. Since a subset X of $E\setminus\{e\}$ is independent in M and spanning in M' if and only if X is independent in $M\setminus e$ and spanning in $M\setminus e$ we have

$$f(M,M') = \sum_{\substack{X\subseteq E\setminus\{e\}\\ X \text{ indep. in } M\\ \text{span. in } M'}} f(M,M';X)$$

$$= y \sum_{\substack{X\subseteq E\setminus\{e\}\\ X \text{ indep. in } M\setminus e\\ \text{span. in } M'\setminus e}} f(M\setminus e, M'\setminus e; X) = yf(M\setminus e, M'\setminus e).$$

(iv) *e is an isthmus of M but is not an isthmus of M'*

(iv.a) If $e\notin X$ by Lemma 8.2 we have $\iota_{M'}(X) = \iota_{M'\setminus e}(X)$ and $\epsilon_M(X) = \epsilon_{M\setminus e}(X)$. On the other hand $r(M) = r(M\setminus e)+1$ and $r(M') = r(M'\setminus e)$, and hence $r(M)-r(M')-(r_M(X)-r_{M'}(X)) = r(M\setminus e)-r(M\setminus e')-(r_{M\setminus e}(X)-r_{M'\setminus e}(X))+1$. It follows that $(f(M,M';X;x,y,z) = zf(M\setminus e, M'\setminus e; X;x,y,z)$. A subset X of $E\setminus\{e\}$ is independent in M if and only if X is independent in $M\setminus e$, and, since e is not an isthmus of M', the subset X is spanning in M' if and only if X is spanning in M'. Therefore

$$\sum_{\substack{X\subseteq E\setminus\{e\}\\ X \text{ indep. in } M\\ \text{span. in } M'}} f(M,M';X) = z \sum_{\substack{X\subseteq E\setminus\{e\}\\ X \text{ indep. in } M\setminus e\\ \text{span. in } M'\setminus e}} f(M\setminus e, M'\setminus e; X) = zf(M\setminus e, M'\setminus e).$$

(iv.b) If $e\in X$, since e is neither a loop of M nor an isthmus of M', the proof of (i.b) holds.

(v) The case $E=\varnothing$ is immediate. □

When $M=M'$ Theorem 8.1 reduces to the classical expression of the Tutte polynomial of a matroid in terms of activities of bases (see [28] [13]). Theorem 8.1 generalizes also the following less well-known result of Tutte. Let G be a graph with a distinguished set of vertices S and a totally ordered set of edges. In [29] Tutte introduces a polynomial $W(G,S)$ obtained as the generating function of activities of S-trees of G, where an S-tree is a spanning forest of G with exactly $|S|$ components, each containing exactly one vertex of S. The main result, [29] Theorem 6.2, is that $W(G,S)$ is independent of the ordering used to define it.

For the sake of brevity we do not recall here Tutte's definitions of activities of S-trees (the reader is referred to [29] Section 6) but give equivalent definitions. Let G' be the graph obtained from G by identifying all vertices in S to one vertex, keeping all edges. An S-tree of G is a spanning tree of G'. Let $C(G)$ and $C(G')$ denote the cycle matroids of the graphs G and G' respectively. The internal activity of an S-tree T in the sense of [29] is equal to $\iota_{C(G')}(T)$ and its external activity is equal to $\epsilon_{C(G)}(T)$. Hence

$$W(G,S;x,y) = \sum_{T \text{ spanning tree of } G} x^{\iota_{C(G')}(T)} y^{\epsilon_{C(G)}(T)}.$$

Observe that $C(G) \to C(G')$ is a matroid perspective of degree $|S| - 1$. With notations of Section 6 we have

$$W(G,S) = t_{|S|-1}(C(G), C(G')). \tag{8.7}$$

Therefore Tutte's Theorem 6.2 is contained in Theorem 8.1.

The coefficients of the chromatic polynomial of a graph have been expressed in terms of broken circuits by H. Whitney. This result has been generalized to matroids by Brylawski [6]. A similar result holds for matroid perspectives.

A *broken circuit* of a matroid M on a totally ordered set E is a set of the form $C \setminus \{e\}$ where C is a circuit of M with smallest element e.

THEOREM 8.3. — *Let $M \to M'$ be a matroid perspective on a set E with a total ordering. We have*

$$\text{(8.8)}\quad p(M, M(;u,w) = \sum_{\substack{X \subseteq E \\ X \text{ contains no} \\ \text{broken circuit of } M}} (-1)^{r_M(X)} u^{r(M') - r_{M'}(X)} w^{r(M) - r(M') - (r_M(X) - r_{M'}(X))}.$$

□

The proof of Theorem 8.3 by deletion/contraction of the greatest element is analogous to the proof of Theorem 8.1 (see also [6]). We omit it.

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