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METRIC COSET SCHEMES REVISITED

by P. CAMION, B. COURTEAU(*) and A. MONTPETIT

1. Introduction.

Let the finite set X be endowed with the structure of an Abelian group and let (X, R), $R = \{R_i\}_{0 \leqslant i \leqslant n}$ be an association scheme such that, to the partition $\{R_i\}_{0 \leqslant i \leqslant n}$ of $X \times X$ corresponds a partition $\{X_0 = \{0\}, X_1, \ldots, X_n\}$ of X so that $(x, y) \in R_i$ if and only if $x - y \in X_i$. We then call (X, R) an Abelian scheme. Notice that what we here call an Abelian scheme corresponds to a special instance of what is usually named a **Schur-ring**. Let Y be a subgroup of X that is called an additive code of (X, R). P. Delsarte introduces a partition $\bar{R}$ on the set $\bar{X} \times \bar{X}$, where $\bar{X} = X/Y$, with the help of what he calls the **restricted distribution matrix** of Y in (X, R). Let us call in general $(\bar{X}, \bar{R})$ a **coset configuration**. He then characterizes the coset configurations $(\bar{X}, \bar{R})$ which are association schemes, more precisely Abelian schemes, then called **coset schemes**.

As pointed out by P. Solé [38], Delsarte observed [13] that in the linear case the coset configuration of a completely regular code is a P-polynomial association scheme whose dual is a scheme on the words of the dual code called the distance scheme [5] and in the unrestricted case an s'-partition design is obtained [8]. We here confine ourselves to studying additive completely regular codes in Abelian groups in relation with partition

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designs. In this way we obtain a construction of metric Abelian schemes and an algorithm to compute their intersection matrices.

1.1. Some references.

Theorem 3.5 of Godsil and Martin [18] generalizes both Delsarte's result and Brouwer et al. [3, Theorem 11.1.6] of which an application is the characterization of **metric coset schemes**. The paper of Godsil and Martin brings a unifying viewpoint on the whole subject.

We here confine ourselves to particular instances of those general results. We recall all needed basic facts and we give elementary proofs providing tools for computations. We emphazise through examples that completely regular codes (Definition 2.12) in metric Abelian schemes is not all we need to characterize metric coset schemes. In such examples the coset scheme of an additive code which is not completely regular can be made metric for a suitable ordering of its relations.

Finally we introduce a new construction of Abelian metric schemes which are obtained as coset schemes.

The notion of partition design is taken from [10] [8] but the same notion is known as coherent partition [20] or equitable partition [16] [37]. Equitable partitions of graphs are the same thing as the graph divisors introduced by Sachs and his co-workers, and an exposition of their point of view appears in [33, Chap. 4].

The concept of completely regular codes in those particular distance-regular graphs which are known as Hamming graphs, was introduced by N.V. Semakov, V.A. Zinoviev and C.V. Zaitsev in [34] where uniformly packed codes, which form a particular class of completely regular codes were not only introduced but deeply investigated. In [21] the authors show that completely regular codes share properties of perfect codes. Those concepts need that of distance distribution matrix which is a particular case of that of **outer distribution matrix** of a subset Y, called a code, in an association scheme (X, R) (which is not necessarily metric) introduced by Delsarte [13]. The combinatorial matrix is taken from [10] and has been studied in [10] [8] and [9] in the case where (X, R) is a Hamming scheme. All these concepts have been generalized by Montpetit in [28] [29] for regular graphs which include the particular case of distance-regular graphs (see also [3]).

We assume in this paper that all basic concepts of association schemes are known to the reader. If the reader needs the necessary notions about association schemes, he may find those in [13] or [1] or [7]. An association scheme (X, R) is defined by a partition of $X \times X$ into relations $R_0, R_1, \ldots, R_n$ that satisfy a series of conditions. We recall those conditions.

Let X be a finite set and let $R = \{R_0, R_1, \ldots, R_n\}$ be a family of $n+1$ relations on X; in other words, R_i is a subset of $X \times X$, $i = 0, \ldots, n$. The **point** $y \in X$ is said to be the **i-th associate** of the point x if the couple (x, y) belongs to R_i. The configuration (X, R) is called a **commutative association scheme with** n **classes** if the following conditions are satisfied:

A_1 The diagonal relation $\{(x, x) | x \in X\}$ is the relation R_0 of R.

A_2 The family R forms a partition of $X \times X$ i.e.,

$$X \times X = R_0 \cup \ldots \cup R_n, \ R_i \cap R_j = \varnothing \quad \text{if } i \neq j.$$

A_3 The reciprocal relation R_i^T of R_i, $R_i^T = \{(x, y) | (y, x) \in R_i\}$ also belongs to $R, i = 0, \ldots, n$. We thus have $R_i^T = R_{i'}$, for some $i' \in \{0, 1, \ldots, n\}$.

A_4 For any triple of integers $i, j, k \in \{0, 1, \ldots, n\}$ the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ does not depend on the choice of (x, y) in R_k. This constant number is denoted by p_{ij}^k.

A_5 We have that $p_{ij}^k = p_{ji}^k$ for all i, j, k.

The adjacency matrix of R_i is denoted by D_i, $i = 0, \ldots, n$. Note that the set of indices is not always explicitly given when it is obvious, for instance when it is the set $\{0, 1, \ldots, n\}$. The relations

$$D_i D_j = \sum_{k=0}^{n} p_{ij}^k D_k \tag{1}$$

yield the integers p_{ij}^k which are the **intersection numbers** of the scheme. In this notation k is a superscript used to avoid three subscripts.

Definition 1.1. — *The* **intersection matrix relative to** R_i *is the matrix* $[L_i(k, j) = p_{ij}^k]$.

Let us tell a further word on the general outer distribution matrix B of a subset Y of X in (X, R). All following notions have been introduced in [13] [14].

Definition 1.2. — *The* **outer distribution matrix** *of a subset* Y *in* (X, R) *in an association scheme* (X, R) *is the matrix* $B = [B(x, j)]$ *whose*

(x,j)-entry is

$$B_j(x) = B(x,j) = |R_j \cap (x \times Y)|, x \in X, \ i = 0, \dots, n.$$

Thus $B_j(x)$ is the number of j-th associates of x in Y.

DEFINITION 1.3. — *A code Y in (X,R) is* **regular** *when the row-vector $B(x) = [B_0(x), \dots, B_n(x)]$ is independent of the choice of x in Y.*

DEFINITION 1.4. — *Let $s' + 1$ be the rank of the outer distribution matrix (or of its combinatorial matrix, Definition 2.11, if the scheme is metric) of a code Y. Then the* **external degree** *of Y is s'.*

The reason for that denomination is as follows.

PROPOSITION 1.1 [13]. — *If X is endowed with the structure of an Abelian group and if a code Y is a subgroup of X, then the external degree s' of the code Y is the number of nonzero weights of its dual Y° (Definition 2.13).*

DEFINITION 1.5. — *Let $t' + 1$ be the number of distinct rows of the outer distribution matrix (or combinatorial matrix, Definition 2.11, if the scheme is metric) of a code Y. Then the* **combinatorial number** *of Y is t'.*

An analog of Lloyd's theorem for completely regular codes is obtained in [21]. In Section 4 of [28] properties of perfect codes in t-regular graphs, (which were introduced by A. Neumaier [31]) including Lloyd's theorem are derived from properties of completely regular codes in t-regular graphs. See also [32]. Finally, weakly metric association schemes are introduced in [38] where the author obtains a Lloyd theorem in those non symmetric association schemes.

We have included several references which are not quoted in the text but are however closely related to the topic.

1.2. Our concern.

Our investigation concerns **codes in the theory of association schemes**, also called **Algebraic Combinatorics** [1]. Let (X,R), $R = \{R_i\}_{0 \leqslant i \leqslant n}$ be the (metric or P-polynomial) association scheme of a distance-regular graph (X,Γ). A subset Y of vertices of (X,Γ) is called a **code** of (X,Γ).

Let the set X be endowed with the structure of an Abelian group and let (X, R) have the property that, to the partition $\{R_i\}_{0 \leqslant i \leqslant n}$ of $X \times X$ corresponds a partition $\{X_i\}_{0 \leqslant i \leqslant n}$ of X so that x and y are at distance i apart in (X, Γ) (or $(x, y) \in R_i$) if and only if $x - y \in X_i$. In particular (x, y) is an edge in (X, Γ) when $x - y \in X_1$. We say that Y is an **additive code** if it is a subgroup of X. We then consider the graph $(X/Y, \Delta)$ on the set of classes of the quotient group X/Y defined by: $(\bar{x}, \bar{y})$ is an edge if and only if $(\bar{x} - \bar{y}) \cap X_1 \neq \varnothing$. We here focus on the following theorem, consequence of Theorem 11.1.6 in Brouwer et al. [3]):

A necessary and sufficient condition for an additive code Y of a distance-regular graph (X, Γ) to be completely regular is that $(X/Y, \Delta)$ is a distance-regular graph.

We reconsider its proof in giving one's mind to work out tools for constructing instances of distance-regular graphs.

Therefore we give two distinct proofs of that result essentially based upon the fact that the distance partition of a completely regular code is a partition design of which the associate matrix, in the case under consideration is essentially the first intersection matrix of a metric scheme. The striking fact is that an additive code can exist which admits a partition design of which the associate matrix is an intersection matrix of the metric coset scheme of that code (actually a Hamming scheme), but not the first (and that code is thus not completely regular).

2. Partition designs in metric schemes and coset configurations in Abelian schemes.

2.1. Distance-regular graphs.

Let $R = \{R_0, R_1, \ldots, R_n\}$ be a family of $n + 1$ relations R_i on X such that (X, R) is a symmetric association scheme. Thus (X, R) is an association scheme in which every relation R_i is symmetric, $i = 0, \ldots, n$. From now on, when the range of the index i clearly is $\{0, \ldots, n\}$, we will omit: "$i = 0, \ldots, n$". Relation R_1 defines an undirected graph (X, Γ) on X. A **path** of length j fom x to y is a sequence of vertices $x_0 = x, x_1, \ldots, x_j = y$ such that (x_i, x_{i+1}) is an edge of Γ. Two vertices x and y are at distance $d(x, y) = i$ apart if the shortest path between them in (X, Γ) has i edges. The mapping $(x, y) \mapsto d(x, y)$ is known to satisfy the axioms of a distance.

DEFINITION 2.1. — *An undirected graph* (X, Γ) *is* **distance-regular** *if and only if there exists an association scheme* (X, R) *such that*

$$(x, y) \in R_i \Longleftrightarrow d(x, y) = i,$$

where $d()$ *is the distance relation in* (X, Γ).

In such a situation, then the associaton scheme (X, R) is said to be **metric** or **P-polynomial** [13, Section 4.2]. Properties of distance-regular graphs and P-polynomial association scheme are discussed in [1, Chapter 3].

Remark 2.1. — First notice that a distance-regular graph is connected. Indeed one of the conditions for (X, R) to be an association scheme [13] is that $R = \{R_0, R_1, \ldots, R_n\}$ is a partition of $X \times X$.

It actually is enough for a connected undirected graph of diameter n to be distance-regular that for any triple of integers i, j, k in the range $[0, n]$, then for any pair $\{x, y\}$ of vertices at distance k apart, the number of vertices z such that $d(z, x) = i,\ d(z, y) = j$ is a constant number p_{ij}^k. Those numbers are the intersection numbers of the association scheme (X, R). Notice that the number of triples $\{x, y, z\}$ with $d(y, x) = k, d(z, y) = j,\ d(z, x) = i$ also is p_{ji}^k and thus the condition: $p_{ij}^k = p_{ji}^k$ is satisfied.

We take from [1, Chap. III] the following proposition.

PROPOSITION 2.1. — *Let* (X, R) *be a symmetric association scheme. Then* (X, R) *is a metric scheme if and only if the first intersection matrix* $L_1 = [p_{1j}^k]$ *is a tridiagonal matrix with nonzero off-diagonal entries of the form:*

$$L_1 = \begin{bmatrix} 0 & v_1 & 0 & \ldots & 0 & 0 \\ 1 & a_1 & b_1 & \ldots & 0 & 0 \\ 0 & c_2 & a_2 & \ldots & 0 & 0 \\ 0 & 0 & c_3 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & b_{n-2} & 0 \\ 0 & 0 & 0 & \ldots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \ldots & c_n & a_n \end{bmatrix} ; \; p_{1j}^{j-1} = b_{j-1}, p_{1j}^{j} = a_j, p_{1j}^{j+1} = c_{j+1},$$

where v_1 *is the* **valence** *of* (X, R), *i.e., the number of points which are 1-associated with any* $x \in X$.

2.2. Abelian schemes.

What we will define as an Abelian scheme is usually known as a Schur ring (more precisely the Bose-Mesner algebra associated with an

Abelian scheme is called an S-ring), since Schur first defined that structure in general finite groups [34] [35]. An account on S-rings is given in [1, pages 104-113]. We found it convenient to just use the words Abelian schemes because only finite Abelian groups are considered here.

DEFINITION 2.2. — *Let X be a finite Abelian group and let $R = \{R_0, R_1, \dots, R_n\}$ be a family of $n+1$ relations R_i on X such that (X, R) is an association scheme. Then (X, R) is an* **association scheme invariant under translation** *if*

$$(x, y) \in R_i \Rightarrow (x+z, y+z) \in R_i,$$

for each $z \in X$ and $i = 0, 1, \dots, n$. Since the structure of an Abelian group is essential for that additional axiom, such an association scheme is simply called an **Abelian scheme** *in the sequel and (X, R) will always denote an Abelian scheme.*

The Abelian scheme (X, R) may be given by the family $\{X_0 = \{0\}, X_1, \cdots, X_n\}$ of subsets of X defined by

$$X_i = \{x - y | (x, y) \in R_i\}.$$

We cannot have $(x, y) \in R_i$ and $(x', y') \in R_j$ with $i \neq j$, and $x-y = x'-y'$. Indeed $x' - x = y' - y \implies (x + (x' - x), y + (y' - y)) \in R_i$. Thus $X_i \cap X_j = \varnothing$ for $i \neq j$. But $\{R_0, R_1, \dots, R_n\}$ is a partition of $X \times X$. Hence $\{X_0 = \{0\}, X_1, \cdots, X_n\}$ is a partition of X and

$$(x, y) \in R_i \Leftrightarrow x - y \in X_i.$$

The classes $\{X_1, \dots, X_n\}$ containing all nonzero elements are called the **Abelian classes** of (X, R). The class X_i will denote the set of **weight** i elements in X. However if the Abelian scheme is not metric, the weight is only a label. If the Abelian scheme is metric it is given by an ordered partition $[X_0 = \{0\}, X_1, \dots, X_n]$, because distances are ordered. Since a metric scheme is a symmetric association scheme then $X_i = -X_i$.

2.3. Codes in Abelian schemes and in metric schemes.

Let x and x' be in a same coset of an additive code Y in an Abelian scheme (X, R). The fact that the rows of the outer distribution matrix B of Y in (X, R), indexed by x and x', are identical follows invariance under translation. Indeed if $(x, y) \in R_i$ $y \in Y$, then $(x', y + x' - x) \in$

R_i, where $y + x' - x \in Y$, and thus the number of i-th associates of x in Y is equal to that of x' in Y. This leads to the following definition.

DEFINITION 2.3. — *Let Y be an additive code in an Abelian scheme. The rows, denoted by $\bar{B}(\bar{x})$, $\bar{x} \in \bar{X}$, of the* **restricted distribution matrix** *$\bar{B}$ of $\bar{X}$ are numbered by the cosets $\bar{x}$ of $\bar{X} = X/Y$,*

$$\bar{B}_j(\bar{x}) = B_j(x), \ x \in \bar{x}, \ j = 0, \dots, n,$$

where B is the outer distribution matrix of Y in (X, R).

DEFINITION 2.4. — *For an additive code Y of an Abelian scheme (X, R),* **the distribution partition** *for Y in (X, R) is the partition of $\bar{X} = X/Y$ given by the equivalence relation*

$$\bar{x} \equiv \bar{y} \Longleftrightarrow \bar{B}(\bar{x}) = \bar{B}(\bar{y}),$$

where $\bar{B}$ is the restricted distribution matrix of Y in (X, R). The classes of the distribution partition will be denoted by $\{\bar{X}_0 = Y, \bar{X}_1, \dots, \bar{X}_{t'}\}$.

Notice that t' is the combinatorial number of Y (Definition 1.5). The next example is taken from [11].

Example 2.1. — Consider the 2-error-correcting code Y of length $n = 15$ over $\mathbb{F}_2$. It has 128 elements called **codewords**. The Abelian group X is $(\mathbb{F}_2^{15}, +)$. Then the Abelian scheme (X, R), $R = \{R_i\}_{0 \leqslant i \leqslant n}$ considered here has for Abelian classes $X_1, \dots, X_{15}$ where X_i is the set of codewords of weight i. Thus $(x, y) \in R_i \Longleftrightarrow x - y \in X_i$. It is a Hamming scheme, thus a metric scheme denoted by $H(15, 2)$. The corresponding Hamming graph has 2^{15} vertices (which are the vectors of $\mathbb{F}_2^{15}$ and x, y is an egdge if $x - y$ is a vector with exactly one nonzero component). The coset group X/Y is denoted by $\bar{X}$. The **restricted distribution matrix** $\bar{B}$ of $\bar{X}$ is given by the first 8 columns of $\bar{B}$, since here $\bar{B}_i(\bar{x}) = \bar{B}_i(\bar{x}, n - i))$.

Distances to Y	Coset Weight Distributions							
0	1	0	0	0	0	18	30	15
1	0	1	0	0	6	12	19	26
2	0	0	1	0	6	15	16	26
2	0	0	1	2	4	11	20	26
3	0	0	0	1	3	18	30	12
3	0	0	0	1	7	14	18	24
3	0	0	0	3	5	10	22	24
3	0	0	0	3	9	6	10	36

DEFINITION 2.5. — *The* **coset configuration** *of an additive code* Y *in an Abelian scheme* (X, R) *is the partition of* $\bar{X} \times \bar{X}$ *into the relations* $\bar{R}_i$, $i = 0, \dots, t'$ *defined by*

$$(\bar{x}, \bar{y}) \in \bar{R}_i \iff \bar{x} - \bar{y} \in \bar{X}_i,$$

where $\{\bar{X}_0 = Y, \bar{X}_1, \dots, \bar{X}_{t'}\}$ *is the distribution partition of* $\bar{X}$.

DEFINITION 2.6. — *The partition* $\{Y = E_0, E_1, \cdots, E_\rho\}$ *of* X *in a metric scheme* (X, R), *where* E_i *is the set of points at distance* i *apart from the code* Y, $i = 0, \dots, \rho$ *is the* **distance partition** *of* X *with respect to* Y.

DEFINITION 2.7. — *The integer* ρ *introduced in Definition 2.6 is the* **covering radius** *of the code* Y. *It is the largest distance of a point to* Y *in* X.

DEFINITION 2.8. — *Let* (X, R) *be a metric scheme. A partition* $\pi = \{E_0, E_1, \cdots, E_r\}$ *of* X *will be called a* **partition design** *[10], [9], if for any* $i, j \in \{0, \cdots, r\}$ *and any* $x \in E_i$ *the number* $M(i,j) = \mid \{y \in E_j : d(x,y) = 1\} \mid$ *does not depend of the choice of* x *in* E_i. *The matrix* $M = [M(i,j)]$ *is called the* **associate matrix** *of the partition design* π.

DEFINITION 2.9. — *A code* Y *is said to* **admit** *the* r*-partition design* $\pi = \{E_0, E_1, \cdots, E_r\}$ *of* X *if* $Y = \bigcup_{v \in S} E_v$ *where* S *is a nonempty subset of* $\{0, 1, \dots, r\}$.

DEFINITION 2.10. — *Let* $Y \subset X$ *be a code in the metric scheme* (X, R). *The* **distance distribution matrix** *[13] [14] of* Y *in* X *is the matrix* $B = [B(x,j)]$ *whose* (x,j)*-entry, is*

$$B_j(x) = B(x,j) = \mid \{y \in Y \mid d(x,y) = j\} \mid,$$

where $x \in X$, *and* $j \in \{0, 1, \cdots, n\}$.

The reader is invited to compare this definition with that of the outer distribution matrix (Definition 1.2).

DEFINITION 2.11. — *The* **combinatorial matrix** *of a code* Y *in a metric scheme* (X, R) *is the matrix* $A = [A(x,j)]$ *where the element in position* $(x,j), x \in X, j \geqslant 0$ *is* $A(x,j)$ = *number of paths of length* j *joining* x *to an element* $y \in Y$ *in the distance-regular graph* (X, Γ).

DEFINITION 2.12. — *A code Y in a metric scheme (X, R) is* **completely regular** *when the row-vector $B(x) = [B_0(x), \ldots, B_n(x)]$ only depends on the distance $d(x, Y) = i$ between x and Y.*

This definition can be usefully confronted with Definition 1.3.

Let us now recall some results essentially given by Theorem 3.2 and Corollary 3.1 of [9] or Proposition 3.14 and Corollary 3.15 of [8] and whose proofs have been extended to distance-regular graphs by A. Montpetit [28, Proposition 3.2.10 and 4.2.2], see also [3]. In fact Proposition 4.2.2 of [28] generalizes Proposition 2.3 below to codes in t-regular graphs.

PROPOSITION 2.2. — *For a code Y in a distance-regular graph, the covering radius, the external degree and the combinatorial number satisfy*

$$\rho \leqslant s' \leqslant t'.$$

THEOREM 2.1. — *If a code Y in a distance-regular graph admits a r-partition design, then we have*

$$\rho \leqslant s' \leqslant t' \leqslant r.$$

The code Y with external degree s' admits an s'-partition design if and only if the distinct rows of the combinatorial matrix A of Y are linearly independent, i.e., if and only if the equality $t' = s'$ is satisfied.

PROPOSITION 2.3. — *A code Y in a distance-regular graph is completely regular (Definition 2.12) if and only if the distance partition (Definition 2.6) $\{Y = E_0, E_1, \ldots, E_\rho\}$ is a partition design (Definition 2.8) or equivalently if Y admits a partition design whose number of classes is equal to the covering radius ρ of Y.*

Remark 2.2. — It follows from [9, Proposition 2.2] that the matrices A and B are *equivalent* in the sense that $\operatorname{rank}(A) = \operatorname{rank}(B)$ and the number of distinct rows is the same in A and B. As pointed out in [9] the combinatorial matrix A is especially suited to the study of partition designs. The present remark will be emphasized in Section 4.4 where a construction of metric schemes is introduced.

2.4. New Abelian schemes from additive codes.

2.4.1. Coset schemes.

One of the remarkable results of Delsarte [13, Theorem 6.10, Theorem 6.11 and the remark page 88] is the following.

THEOREM 2.2. — *In an Abelian scheme (X, R) the coset configuration of an additive code Y is an Abelian scheme in its turn if and only if the combinatorial number of Y is equal to its external degree or equivalently if the distinct rows of the restricted distribution matrix are linearly independant.*

Theorem 2.2 is generalized as Theorem 3.5 of [18]. A complete exposition of topics related to Theorem 2.2 can be found in [7, Section 5]. We summarize some results after the following proposition.

LEMMA 2.1. — *Let Y be an additive completely regular code in a metric Abelian scheme with Abelian classes $X_1, \dots, X_n$ and let $\pi = \{Y = E_0, \cdots, E_\rho\}$ be the distance partition of X with respect to Y. Then E_i is a union of cosets of Y, $i = 0, \dots, \rho$. The coset configuration $(\bar{X} = X/Y, \bar{R})$ of such a completely regular code is a symmetric Abelian scheme. A coset $\bar{x}$ belongs to $\bar{E}_i$ if and only if i is the smallest integer such that $\bar{x} \cap X_i \neq 0$.*

Proof. — We first prove that $(\bar{X}, \bar{R})$ is an Abelian scheme. The metric of the scheme (X, R) being invariant under translation, each class E_i is an union of cosets of Y. We denote by $\bar{E}_i$ the image $\varphi(E_i)$ of E_i under the homomorphism φ from X onto the quotient group X/Y. We first notice that, Y being completely regular (Definition 2.12) then $\bar{\pi} = \{\varphi(E_0), \cdots, \varphi(E_\rho)\}$ also is the distribution partition (Definition 2.4), of X/Y. The distribution matrix having $\rho + 1$ distinct rows, the rank $s' + 1$ of that matrix is at most $\rho + 1$ and since $\rho \leqslant s'$, by Proposition 2.2, then the $\rho + 1$ distinct rows are linearly independent. By Delsarte's Theorem 2.2 we know that the coset configuration $(\bar{X}, \bar{R})$ is thus an Abelian scheme. By hypothesis all rows indexed in E_i of the distance distribution matrix B of Y in X are equal. This means that the coset configuration $(\bar{X} = X/Y, \bar{R})$ is given by: $(\bar{x}, \bar{y}) \in \bar{R}_i \Longleftrightarrow \bar{x} - \bar{y} \in \bar{E}_i = \varphi(E_i)$.

Since $d(E_i, Y) = i$, then for every $\bar{x}$ in $\bar{E}_i$ and for every x in $\bar{x}$, we have a couple $(x, y) \in E_i \times Y$ with $d(x, y) = i$. Thus $d(x' = x - y, 0) = i$ and $x' \in \bar{x} \cap X_i$. There doesn't exist an $x \in E_i \cap X_j$ for $j < i$, since for all $x \in E_i$, $d(x, 0) \geqslant i$. Now we have seen in Section 2.2 that in a metric Abelian scheme $X_i = -X_i$. Thus the permutation $x \mapsto -x$ of X maps

$\bar{x} \cap X_i$ onto $-\bar{x} \cap X_i$ which shows that

$$\bar{x} \in \bar{E}_i \Longrightarrow -\bar{x} \in \bar{E}_i;\ \bar{E}_i = -\bar{E}_i.$$

Thus $(\bar{X}, \bar{R})$ is symmetric. □

2.4.2. The dual of an Abelian scheme.

NOTATION 2.1. — *We denote by X' the character group of a finite Abelian group X. The image under the character $x' \in X'$ of an element $x \in X$ is the complex number denoted by $< x, x' >$.*

We know that X' is isomorphic to X and that X is the character group of X'. The dual of a subgroup Y of X is then defined [13].

DEFINITION 2.13. — *We observe that the subset Y° of X' defined by*

$$Y^\circ = \{x' \in X' | < x, x' >= 1, \forall x \in Y\} \tag{2}$$

is itself a group. It is called **the dual** *of Y in X'.*

PROPOSITION 2.4. — *Let Y be an subgroup of a finite Abelian group X. Then the dual of Y° is Y.*

NOTATION 2.2. — *We used the notations ρ, s' and t' for the covering radius, the external degree and the combinatorial number of Y, respectively. The corresponding values for Y° will be denoted by ρ', s and t respectively, in the sequel.*

We do not have room enough here to precisely define the **dual scheme** of an Abelian scheme. We refer the reader to [7, Section 4.7.2] in which the presentation is very close to that of [19]. We only need to know that given an Abelian scheme (X, R), with Abelian classes $X_1, \ldots, X_n$, and a matrix $[< x, x' >, x \in X, x' \in X']$ of characters then the theorem of Tamaschke-Delsarte asserts that an Abelian scheme (X', R') is automatically defined on the character group X' of X. It also has n Abelian classes $X'_1, \ldots, X'_n$. It is called the dual scheme of (X, R). Conversely the dual of (X', R') is (X, R).

DEFINITION 2.14. — *The* **restriction** (Y, R^Y) *of an Abelian scheme (X, R) to an additive code Y with weights $i_0 = 0, i_1, \ldots, i_s$ is defined by the partition $\{Y_0, Y_{i_1}, \ldots, Y_{i_s}\}$ of the Abelian group Y, where $Y_{i_\nu} = Y \cap X_{i_\nu}$ as follows:*

$$(x, y) \in R^Y_\nu \ if\ x - y \in Y_{i_\nu},\ \nu = 0, 1, \ldots, s.$$

Theorem 6.10 and the remark page 88 of Delsarte [13] can be stated as follows.

THEOREM 2.3. — *The restriction (Y, R^Y) of an Abelian scheme (X, R) to an additive s-weight code Y is an Abelian subscheme if and only if the restricted distribution matrix of the dual Y° of Y (which has rank $s+1$) in (X', R') has exactly $s+1$ distinct rows, or equivalently if $s = t$, where t is the combinatorial number of Y°.*

Theorem 2.3 with Theorem 2.2 give the following.

THEOREM 2.4. — *In an Abelian scheme (X, R) the coset configuration $(\bar{X}, \bar{R})$ of a code Y is an Abelian scheme in its turn if and only the restriction (Y°, R^{Y°) of the Abelian scheme (X', R') (dual to (X, R)) to Y° is an Abelian scheme. That Abelian scheme is the dual scheme of $(\bar{X}, \bar{R})$.*

3. Coset configurations which are metric Abelian schemes.

3.1. The group algebra of an Abelian scheme.

DEFINITION 3.1. — *The* **group algebra** *$\mathbb{C}X$ of the Abelian group X over the field $\mathbb{C}$ consists of all formal sums*

$$\mathbf{a} = \sum_{x \in X} a_x Z^x, a_x \in \mathbb{C}, x \in X.$$

Addition and multiplication of elements of $\mathbb{C}X$ are defined in a natural way by

$$\sum_{x\in X} a_x Z^x + \sum_{x\in X} b_x Z^x = \sum_{x\in X} (a_x + b_x) Z^x,\ r\sum_{x\in X} a_x Z^x = \sum_{x\in X} r a_x Z^x, r \in \mathbb{C}$$

and

$$\sum_{x\in X} a_x Z^x \sum_{x\in X} b_x Z^x = \sum_{z\in X} Z^z \sum_{x+y=z} a_x b_y.$$

Clearly Z^0 is the unity of $\mathbb{C}X$ and can be written 1. If $S \subseteq X$ is a subset of X, then we shall write

$$Z^S = \sum_{x\in S} X^x.$$

3.2. Algebraic tools.

The following two theorems and two lemmas give reformulations of Abelian schemes and partition designs in terms of the group algebra $\mathbb{C}X$ of X. The first one is stated as Theorem 4.30 in [7].

THEOREM 3.1. — *Let X be an Abelian group and let $\{X_0 = \{0\}, \ldots, X_n\}$ be a partition of X. That partition defines an Abelian scheme on X, i.e., $X_1, \ldots, X_n$ are the Abelian classes of a scheme, if and only if the subalgebra of $\mathbb{C}X$ generated by $\{Z^{X_i} | i = 0, \ldots, n\}$ has dimension $n+1$. In that case we have*

$$Z^{X_i} Z^{X_j} = \sum_{k=0}^{n} p_{ij}^{k} Z^{X_k}$$

where the coefficients p_{ij}^k are the intersection numbers of the scheme.

The theorem is simply, in the particular case where X is an Abelian group, a reformulation of [13, Theorem 2.1] characterizing an association scheme by its Bose-Mesner algebra. It clearly shows how an Abelian scheme is a particular instance of an S-ring. The second theorem characterizes in algebraic terms metric Abelian schemes. Delsarte proved [13, Section 5.2] that an association scheme is metric (Definition 2.1) if and only if it is P-polynomial. The reader desiderous to see the necessary proofs is refered to [1, Section 3.1]. We only here need to reformulate the P-polynomial property for Abelian schemes. The next theorem follows from using the representation of $\mathbb{C}X$ as used in [7, Section 4].

THEOREM 3.2. — *An Abelian scheme is metric with respect to the ordered set of Abelian classes $[X_1, \ldots, X_n]$ if and only if there exists a polynomial $v_i(z)$ of degree i, $i = 0, \ldots, n$ such that*

$$Z^{X_i} = v_i(Z^{X_1}), \ i = 0, \ldots, n.$$

We now have to prove two lemmas.

LEMMA 3.1. — *Let (X, R) be a metric Abelian scheme with the ordering $[X_1, \ldots, X_n]$ of its Abelian classes. Let $\pi = \{E_0, \cdots, E_r\}$ be a partition of X. Then π is a partition design (Definition 2.8) if and only if in the group algebra $\mathbb{C}X$ we have*

$$Z^{E_i} Z^{X_1} = \sum_{j=0}^{r} M(j,i) Z^{E_j} \tag{3}$$

for $i, j = 0, \cdots, r$. The matrix $M = [M(i,j)]$ is then the associate matrix of π.

Proof. — In the group algebra $\mathbb{C}X$, we have

$$Z^{E_i}Z^{X_1} = \sum_{x\in E_i} X^x \sum_{h\in X_1} Z^h = \sum_{x\in E_i}\sum_{h\in X_1} Z^{x+h},$$

$$Z^{E_i}Z^{X_1} = \sum_{j=0}^{r}\sum_{y\in E_j} M(y,i)Z^y \tag{4}$$

where $M(y,i) =| \{x \in E_i \mid y - x = h \in X_1\} |=| \{x \in E_i \mid d(y,x) = 1\} |$, since $X_1 = -X_1$. Thus $M(y,i)$ is the number of points in E_i adjacent to y.

If $\pi = \{E_0, \cdots, E_r\}$ is a partition design, then $M(y,i) = M(j,i)$ for any $y \in E_j$. Then (4) becomes

$$Z^{E_i}Z^{X_1} = \sum_{j=0}^{r} M(j,i) \sum_{y\in E_j} Z^y = \sum_{j=0}^{r} M(j,i)Z^{E_j}.$$

Conversely, if this last equality is valid, then (4) gives $M(y,i) = M(j,i)$ for any $y \in E_j$ because in the group algebra the set $\{Z^y \mid y \in X\}$ is linearly independent. □

Let $Y \subset X$ be an additive code and let $\varphi : X \to \bar{X} = X/Y$ be the canonical homomorphism defined by $\varphi(x) = x + Y = \bar{x}$. Let $\bar{\varphi} : \mathbb{C}X \to \mathbb{C}\bar{X}$ be the algebra homomorphism defined on $\mathbb{C}X$ by extending φ by linearity:

$$\bar{\varphi}\Big(\sum_{x\in X} a_x Z^x\Big) = \sum_{x\in X} a_x \bar{\varphi}(Z^x) = \sum_{x\in X} a_x Z^{\varphi(x)}.$$

By applying the algebra homomorphism $\bar{\varphi}$ to the relation (3), we obtain the following lemma.

LEMMA 3.2. — *Let Y be an additive code of X and $\pi = \{E_i \mid i = 0, \cdots, r\}$ be a partition design of X such that $Y = E_0$, and satisfying the property $x \in E_i \Rightarrow x + Y \subset E_i$ for $i = 0, \cdots, r$. Set $\bar{E}_i = \{\varphi(x) \mid x \in E_i\} = \varphi(E_i)$. Then the image $\bar{\pi} = \varphi(\pi) = \{\bar{E}_i \mid i = 0, \cdots, r\}$ of π is a partition of $\bar{X} = X/Y$ such that*

$$Z^{\bar{E}_i}\alpha(Z) = \sum_{j=0}^{r} M(j,i)Z^{\bar{E}_j} \tag{5}$$

where $\alpha(Z) = \bar{\varphi}(Z^{X_1}) = \sum_{\bar{x}\in\bar{X}} \alpha_{\bar{x}} Z^{\bar{x}}$ with $\alpha_{\bar{x}} =| X_1 \cap \bar{x} |$.

Proof. — We have

$$\bar\varphi(Z^{E_j}) = \bar\varphi \sum_{x\in E_j} Z^x = \sum_{x\in E_j} Z^{\varphi(x)} = \mid Y \mid \sum_{\bar x\in \bar E_j} Z^{\bar x} = \mid Y \mid Z^{\bar E_j}$$

and

$$\bar\varphi(Z^{X_1}) = \bar\varphi \sum_{x\in X_1} Z^x = \sum_{x\in X_1} Z^{\varphi(x)} = \sum_{\bar x\in \bar X} \alpha_{\bar x} Z^{\bar x}$$

where $\alpha_{\bar x} = \mid X_1 \cap \bar x \mid$ is the number of weight one elements in the coset $\bar x$.

Relations (5) are then obtained by applying $\bar\varphi$ to relations (3). □

Remark 3.1. — If Y is 1-error-correcting, then by definition there is at most one x in the coset $\bar u$ such that $w(x) = 1$. In this case, $\alpha_{\bar u} = 1$ for $\bar u \in \varphi(X_1)$ and $\alpha_{\bar u} = 0$ otherwise, so $\alpha(Z) = \sum_{\bar u\in\varphi(X_1)} Z^{\bar u} = Z^{\bar X_1}$ where $\bar X_1 = \varphi(X_1)$. If Y is completely regular, then by definition $\alpha(Z) = cZ^{\bar X_1}$ where c is the number of weight one elements in any given coset of Y at distance one from Y. In fact, Y may be define to be 1-regular if the coset weight distributions of Y are the same for any coset at distance one from Y. In this case we also have by definition that $\alpha(Z) = cZ^{\bar X_1}$.

3.3. Tools for further constructions.

3.3.1. Characterisation of metric coset schemes.

We now come back to Section 1.2 in the introduction and we arrive at the theorem under investigation.

THEOREM 3.3. — *Let Y be an additive code in a metric Abelian scheme (X, R) with the ordering $[X_1, \ldots, X_n]$ of its Abelian classes such that $Y \cap X_1 = \varnothing$. A necessary and sufficient condition for its coset configuration $(\bar X = X/Y, \bar R)$ to be the metric scheme of the graph $(\bar X, \Delta)$ in which $(\bar x, \bar y)$ is an edge if and only if $(\bar x - \bar y) \cap X_1 \neq \varnothing$ is that Y is a completely regular code.*

Proof. — The condition is necessary.

Since the coset configuration $(\bar X, \bar R)$ on the quotient group X/Y is the Abelian scheme defined by the distance-regular graph $(X/Y, \Delta)$, its Abelian classes $\{\bar X_1, \ldots, \bar X_{t'}\}$ are given by: $(\bar x - \bar y) \in \bar X_i \Leftrightarrow \delta(\bar x, \bar y) = i$, where δ is the distance relation in $(X/Y, \Delta)$. Let us see that $\bar X_i$ is a set of cosets in which every point is at distance exactly i from Y in (X, R). First notice that all points in a same coset $\bar{x}$ are at equal distance from Y, the metric of (X,R) being invariant under translation. We thus show that the distance to Y is precisely i for a point in $\bar{x}$ and for all $\bar{x}\in\bar{X}_i$. For in the distance-regular graph $(X/Y,\Delta)$ of $(\bar{X},\bar{R})$, a coset $\bar{x}\in\bar{X}_i$ is at distance i from $\bar{X}_0=Y:\delta(\bar{x},Y)=i$. This means that such a coset, say $\bar{x}$, is a sum of i cosets contained in $\bar{X}_1$ and not less. Since a coset in $\bar{X}_1$ is, by hypothesis, translated from Y by an element u of weight one (i.e. $u\in X_1$), then $\bar{x}$ is translated fom Y by an element x which is the sum of i elements of weight 1. Thus in the distance-regular graph (X,Γ) of $(X,R), d(x,0)=j\leqslant i$. The weight of x is j in (X,R) and if j were smaller than i, then $\bar{x}$ being a sum of j cosets contained in $\bar{X}_1$ would be in $\bar{X}_{j'}\neq\bar{X}_i$. Denoting by ρ the covering radius of Y (Definition 2.7), we thus have $t'=\rho$. Since by hypothesis any two cosets in $\bar{X}_i$ have the same distance distribution, then Y is completely regular.

The condition is sufficient.

First proof (Relying on Theorem 2.2).

We aim to show that $(\bar{X}=X/Y,\bar{R})$ is a metric Abelian scheme with respect to the ordered set of Abelian classes $E_1,\cdots,E_\rho$ where $\pi=\{E_1,\cdots,E_\rho\}$ is the distance partition of X with respect to Y. Since by hypothesis Y is completely regular, then π is a partition design, by Proposition 2.3. Let M be the associate matrix of the partition design π.

By Lemma 2.1 the coset configuration $(\bar{X},\bar{R})$ is a symmetric Abelian scheme with classes $\bar{X}_i=\bar{E}_i$ $i=0,\ldots,\rho$. By Proposition 2.1 we now have to verify that the first intersection matrix $\bar{L}_1=[\bar{p}^i_{1j}]$ has the desired property for a suitable choice of the class $\bar{X}_1$. We first notice that if $M(i,j)\neq 0$ then $|i-j|\leqslant 1$ and M is thus tridiagonal. The hypotheses of Lemma 3.2 are satisfied and we may write

$$Z^{\bar{E}_i}\alpha(Z)=\sum_{k=0}^{\rho}M(k,i)Z^{\bar{E}_k}, \tag{6}$$

where by Remark 3.1 $\alpha(Z)=cZ^{\bar{X}_1}$. By definition of the intersection numbers $\bar{p}^i_{1j}$ we have that

$$Z^{\bar{X}_1}Z^{\bar{E}_i}=c^{-1}Z^{\bar{E}_1}Z^{\bar{E}_i}=\sum_{k=0}^{\rho}\bar{p}^k_{1i}Z^{\bar{E}_k}. \tag{7}$$

Thus from (6) and (7) $\bar{p}^k_{1i}=c^{-1}M(k,i)$. We have in particular that $\bar{p}^0_{11}=c^{-1}M(0,1)$. Since $\bar{p}^0_{11}$ is the valence v_1 of $\bar{R}_1$ in the symmetric association scheme $(\bar{X}, \bar{R})$, then $\bar{p}^0_{11} = \bar{E}_1$, the number of cosets at distance 1 from Y. We can alternatively see that $|Y|M(0,1) = |E_1|c = |Y||\bar{E}_1|c$. By relations (17) of [9] we have $|E_0|M(0,1) = |E_1|M(1,0)$, then $M(1,0) = c$ and $\bar{p}^1_{10} = 1$ as required by Proposition 2.1. Since by definition $Y \cap X_1 = \varnothing$ then $M(0,0) = 0$ as required. Now let y be any point in E_i. By definition of E_i there exists a shorstest path $x = x_0, x_1, \ldots, x_{i-1}, x_i = y$ joining some point $x \in Y$ to y. But then $x_{i-1} \in E_{i-1}$ and $d(x_{i-1}, x_i) = 1$. This shows that $M(i-1, i) \neq 0$ for $0 < i \leqslant \rho$. Considering a point $y \in E_{i+1}$, $i < \rho$, we similarly see that $M(i, i+1) \neq 0$ for $0 \leqslant i < \rho$.

Second proof (ignoring Theorem 2.2).

In matrix notation, relations (6) give

$$\begin{aligned} &\alpha(Z)[Z^{\bar{E}_0}, \cdots, Z^{\bar{E}_\rho}] = [Z^{\bar{E}_0}, \cdots, Z^{\bar{E}_\rho}]M, \\ &\alpha(Z)^s[Z^{\bar{E}_0}, \cdots, Z^{\bar{E}_\rho}] = [Z^{\bar{E}_0}, \cdots, Z^{\bar{E}_\rho}]M^s \end{aligned} \tag{8}$$

for $s = 0, \cdots, \rho$. Finally, multiplying both sides to the right by the column vector $e_1 = [1, 0, \cdots, 0]^t$, we obtain

$$[1, \alpha(Z), \alpha(Z)^2, \cdots, \alpha(Z)^\rho] = [Z^{\bar{E}_0}, \cdots, Z^{\bar{E}_\rho}]A^{(\rho)} \tag{9}$$

because by Theorem 3.1 of [9], whose proof is easily extended to the case at hand (see [28, Prop.3.2.8]), the column number s of the restricted $(r+1) \times (r+1)$ combinatorial matrix $A^{(r)}$ of a code Y admitting a r-partition design is simply $M^s e_1$.

Since Y is completely regular, then $A^{(\rho)}$ is upper triangular and invertible [9, Theorem 3.1], we may write by inverting $A^{(\rho)}$,

$$Z^{\bar{E}_i} = \sum_{s=0}^{\rho} g_{i,s}(\alpha(Z))^s = g_i(\alpha(Z)) = g_i(cZ^{\bar{X}_1})) \tag{10}$$

where $[g_{i,0}, \cdots, g_{i,\rho}]^t$ is the column number i in $[A^{(\rho)}]^{-1}$.

This last matrix being upper triangular in its turn, $g_{i,i+1} = \cdots = g_{i,\rho} = 0$ and so degree $(g_i) = i$ for $i = 0, \cdots, \rho$.

A Schur-ring has been brought to the fore, which implies that $(\bar{X}, \bar{R})$ is an Abelian scheme, by Theorem 3.1. Theorem 3.2 achieves the proof. □

3.3.2. Direct computation of the intersection numbers and eigenvalues of the coset scheme when the code is 1-error-correcting.

Let us first observe that from [1, Chap.III] polynomials v_i satisfy the recurrence

$$xv_i(x) = b_{i-1}v_{i-1}(x) + a_i v_i(x) + c_{i+1}v_{i+1}(x), \ 0 \leqslant i < \rho, \tag{11}$$

which gives an easy way of computing the polynomials v_i when $[M(i,j] = [\bar{p}^i_{1j}]$ is known. Next we have that (Definition 1.1) $\bar{L}_\ell = v_\ell(\bar{L}_1) = [\bar{p}^i_{\ell j}], \ell = 2, \ldots, \rho$. The eigenvalues $\bar{P}_k(i)$ of $(\bar{X}, \bar{R})$ can be computed similarly.

We now use the argument of the second proof to obtain those numbers.

Invoking (10) and (8) we can write

$$Z^{\bar{E}_i} Z^{\bar{E}_j} = g_i(\alpha(Z)) Z^{\bar{E}_j} = \sum_{s=0}^{i} g_{i,s}(\alpha(Z))^s Z^{\bar{E}_j}$$

$$= \sum_{s=0}^{i} g_{i,s}(\alpha(Z))^s [Z^{\bar{E}_0}, \cdots, Z^{\bar{E}_\rho}] e_j = \sum_{s=0}^{i} g_{i,s}[Z^{\bar{E}_0}, \cdots, Z^{\bar{E}_\rho}](M^s e_j)$$

$$= \sum_{k=0}^{\rho} \sum_{s=0}^{i} g_{i,s}(M^s e_j)_k Z^{\bar{E}_k} = \sum_{k=0}^{\rho} \bar{p}^k_{ij} Z^{\bar{E}_k}, \tag{12}$$

where $M^s e_j$ is the column number j of M^s. So

$$Z^{\bar{E}_i} Z^{\bar{E}_j} = \sum_{k=0}^{\rho} \bar{p}^k_{ij} Z^{\bar{E}_k} \tag{13}$$

and by Theorem 3.1 the partition $\bar{\pi} = \{\bar{E}_0 = \{0\}, \bar{E}_1, \cdots, \bar{E}_\rho\}$ determines on $X/Y = \bar{X}$ the Abelian scheme $(\bar{X}, \bar{R})$ where the coefficients $\bar{p}^k_{ij}$ are its intersection numbers.

Let us defined what was called in [7] the **Hecke representation** $\bar{D}_{\bar{z}}$ of $Z^{\bar{z}} \in \mathbb{C}(\bar{X})$.

The entries $\bar{D}_{\bar{z}}(x,y)$ of the matrix $\bar{D}_{\bar{z}}$ are

$$\bar{D}_{\bar{z}}(\bar{x}, \bar{y}) = \begin{cases} 1 & \text{if } \bar{x} - \bar{y} = \bar{z}, \\ 0 & \text{otherwise.} \end{cases}$$

The image of $\alpha(Z)$ by the Hecke representation of $\mathbb{C}(\bar{X})$ is

$$\bar{D} = \sum_{\bar{x} \in \bar{E}_1} \alpha_{\bar{x}} \bar{D}_{\bar{x}} = \bar{D}_1, \tag{14}$$

where $\alpha_{\bar{x}} =\mid X_1 \cap \bar{x} \mid= 1$. The image of (10) then gives

$$\bar{D}_i = g_i(\bar{D}_1) \tag{15}$$

where g_i is the polynomial of degree i whose coefficients are given by column number i of $(A^{(\rho)})^{-1}$.

This implies that the scheme $(\bar{X}, \bar{R})$ is P-polynomial with respect to the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_\rho$ of $\bar{D}_1$ [13, Section 5.1], that is the eigenvalues $\bar{P}_k(i)$ of $\bar{D}_k$ which define the first eigenmatrix $\bar{P} = [\bar{P}(i,k) = \bar{P}_k(i)]$ of the coset scheme $(X/Y, \bar{\pi}) = (\bar{X}, \bar{R})$ satisfy

$$\bar{P}_k(i) = g_k(\lambda_i) \tag{16}$$

where $\lambda_0, \cdots, \lambda_\rho$ are the eigenvalues of $\bar{D}_1$.

Remark 3.2. — For the determination of the eigenmatrix $\bar{P}$ of the coset scheme $(\bar{X}, \bar{R})$ we may use the definition of the polynomials g_k given by (10), to express (16) in matrix form as follows:

$$\begin{bmatrix} 1 & \lambda_0 \cdots (\lambda_0)^\rho \\ \vdots & \\ 1 & \lambda_\rho \cdots (\lambda_\rho)^\rho \end{bmatrix} = \begin{bmatrix} 1 & \bar{P}_1(0) \cdots \bar{P}_\rho(0) \\ \vdots & \\ 1 & \bar{P}_1(\rho) \cdots \bar{P}_\rho(\rho) \end{bmatrix} A^{(\rho)} = \bar{P}A^{(\rho)}. \tag{17}$$

3.3.3. From coset schemes to restrictions.

For the following statement, we only need to know that an association scheme is Q-polynomial when its dual is metric. From Theorem 3.3 and Theorem 2.4 we can state :

THEOREM 3.4. — *Let (X, R) be a Q-polynomial Abelian scheme. Let Y be an additive code of (X, R). Then he dual Y^0 of Y is completely regular in the metric Abelian scheme (X', R') dual to (X, R) if and only if the restriction (Y, R^Y) is a Q-polynomial Abelian subscheme of (X, R).*

4. Concrete examples.

4.1. The metric coset scheme of the binary Golay code.

It was observed in [8] that the binary (23,12,7) binary Golay code $Y = \mathcal{G}_{23}$, as a perfect code admits a partition design $\pi = \{E_0 = Y, E_1, \ldots, E_3\}$

which is the distance partition of $X = (\mathbb{F}_2^{23}, +)$ with respect to Y. The associate matrix M of π is

$$M = \begin{bmatrix} 0 & 23 & 0 & 0 \\ 1 & 0 & 22 & 0 \\ 0 & 2 & 0 & 21 \\ 0 & 0 & 3 & 20 \end{bmatrix}.$$

By Proposition 2.3, Y is completely regular and Theorem 3.3 applies. Thus the coset configuration of Y in the Hamming scheme (X, R) is an Abelian scheme. This was shown in [7, Example 4.46] by using the fact that the Mathieu group acting on a 23-set may be represented in $GL(11, 2)$. Exercise 4.40 in [7] would show that this Abelian scheme is metric. The intersection numbers of the scheme are computed in [7, Example 4.46] through combinatorial enumeration. We find again those values by matrix computation following Section 3.3.2. We here have $\rho = 3$. Then the combinatorial matrix (Definition 2.11) of Y in the Hamming scheme (X, R) is computed, through the powers of M.

$$A^{(\rho)} = \begin{bmatrix} 1 & 0 & 23 & 0 \\ 0 & 1 & 0 & 67 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}; \ [g_{i,s}]^T = [A^{(\rho)}]^{-1} = \begin{bmatrix} 1 & 0 & \frac{23}{2} & 0 \\ 0 & 1 & 0 & \frac{-67}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix}.$$

Let us denote the matrix $[(M^s e_j)_k]$ by [h(s,k,j)]. We have that $[h(0, k, j)]$ is the identity matrix and $[h(1, k, j)] = M$. Next

$$[h(2,k,j)] = \begin{bmatrix} 23 & 0 & 506 & 0 \\ 0 & 67 & 0 & 462 \\ 2 & 0 & 107 & 420 \\ 0 & 6 & 60 & 463 \end{bmatrix}; [h(3,k,j)] = \begin{bmatrix} 0 & 1541 & 0 & 10626 \\ 67 & 0 & 2860 & 9240 \\ 0 & 260 & 1260 & 10647 \\ 6 & 120 & 1521 & 10520 \end{bmatrix}.$$

Finally as for the intersection matrices (Definition 1.1) we have that $\bar{L}_0$ is the identity matrix and $\bar{L}_1 = M$. Next, using relations (12), we obtain

$$\bar{L}_2 = \begin{bmatrix} 0 & 0 & 253 & 0 \\ 0 & 22 & 0 & 231 \\ 1 & 0 & 42 & 210 \\ 0 & 3 & 30 & 220 \end{bmatrix}; \ \bar{L}_3 = \begin{bmatrix} 0 & 0 & 0 & 1771 \\ 0 & 0 & 231 & 1540 \\ 0 & 21 & 210 & 1540 \\ 1 & 20 & 220 & 1530 \end{bmatrix}.$$

Remark 4.1. — Notice that the Abelian scheme just considered is not isomorphic with a Hamming scheme. For it has 2^{11} elements and as a Hamming scheme, since 11 is prime it would have 11 classes instead of 3.

4.2. Metric coset schemes whose codes are not completely regular.

DEFINITION 4.1. — *We denote by $\mathbb{F}_q^*$ the set of nonzero elements of $\mathbb{F}_q$. We consider in the Abelian group $X = (\mathbb{F}_q^m, +)$ a subset E of nonzero elements such that $\mathbb{F}_q^* E = E$ and a subset E^* obtained by choosing a nonzero vector in each one dimensional subspace contained in $E \cup \{0\}$. For any subset F of X, then $\mathcal{C}(F)$ denotes the subspace generated by the rows of the matrix whose column set is F. It is an additive code in an Abelian scheme on the Abelian group $(\mathbb{F}_q, +)^F$. The main Abelian scheme that we have in mind is the classical Hamming scheme whose classes are all subsets of $(\mathbb{F}_q, +)^F$ with constant Hamming weight, the Hamming weight of a vector being the number of its nonzero coordinates. The set F is refered to as the* **set of coordinate forms** *of $\mathcal{C}(F)$. The subspace $\mathcal{C}(E^*)$ is called the* **projective code** *associated to E. The code $\mathcal{C}(E^*)$ is only defined up to equivalence.*

We here refer to [9, Section 4] and [7, Example 5.17]. Let X_k be the set of vectors of weight k in $\mathbb{F}_q^m$. The set X_k may be considered the set of columns of a generator matrix that is denoted by G_k. Thus the linear code Y given by that generator matrix is denoted by $\mathcal{C}(X_k)$. The generator matrix of $\mathcal{C}(X_k^*)$ is denoted by G_k^*. The code spanned by the transposed of G_k, i.e., by the columns in X_k is denoted by T. To shed light on the example dealt with below, we first establish a theorem which is an easy consequence of the discussion in [7, Example 5.17]. It also is very much related with the results in Section 7 of [38].

Remark 4.2. — The codes of Definition 4.1 were introduced in [6] and were investigated more deeply in [2]. The weight enumeration obtained in [6] for the binary field $\mathbb{F}_2$ was there obtained for any finite field $\mathbb{F}_q$. Remark 4.6 below also gives an easy way of calculating the weight distribution.

THEOREM 4.1. — *If the number of nonzero weights of Y equals the number of nonzero weights of T, then the restriction (Y, R^Y) to Y of the Hamming scheme $H(|X_k|, q)$ defined on the set of all codewords over $\mathbb{F}_q$ of length $|X_k|$ is an association scheme.*

We first show that the number ℓ of nonzero weigths of T is at least t, the combinatorial number of Y°. For we notice that a codeword of T is a syndrom $G_k x^T = u$ of the code Y°, where $x \in \mathbb{F}_q^{|X_k|}$. Thus x is in a coset of Y°. Let v be any other syndrom of Y° with the same weight as u. There exists an $m \times m$ monomial matrix σ over $\mathbb{F}_q$ (i.e., a matrix with exactly one nonzero element in each row and in each column) such that $\sigma u = v$. Thus $\sigma G_k x^T = v$. This means that there exists a vector y obtained by permuting the positions of x such that $G_k y^T = v$. Thus the coset corresponding to the syndrom v has the same weight distribution as that corresponding to u. Now the number of distinct weight distributions of cosets of Y°, is by definition $t+1$ where t is the combinatorial number of Y° (Definition 1.5 and Proposition 2.4) and we have that $t \leqslant \ell$. By assumption $\ell = s$ (Proposition 1.1) and by Proposition 2.2, $s \leqslant t$. Hence $s = t$: Theorem 2.2 and Theorem 2.3 apply. □

Remark 4.3. — The cosets of Y° consist in a group isomorphic with its group of syndroms. That group is a subgroup of $(\mathbb{F}_q^m, +)$ in general. But if $\operatorname{rank}(X_k) = m$ then the coset scheme obtained under the assumption of Theorem 4.1 is isomorphic with a Hamming scheme. The code Y always has at most m nonzero weights since $\ell \leqslant m$ and $s \leqslant t \leqslant \ell$. This also can be seen by observing as in [7, Example 5.17] that all linear combinations of the rows of G_k with the same number of nonzero components yield codewords of Y with equal weight. If Y has exactly m nonzero weights then the restriction (Y, R^Y) to Y of the hamming scheme $H(|X_k|, q)$ defined on the set of all codewords over $\mathbb{F}_q$ which, as a consequence of Theorem 4.1 is a subscheme is thus isomorphic with the Hamming scheme $H(m,q)$. That particular situation was considered by T. Bier [2] where this author proves that if the code with generator matrix G_k has m distinct weights then the restriction (Y, R^Y) is essentially the Hamming scheme $H(m,q)$.

Corollary 4.1. — *Let $q = k = 2$ in the statement of Theorem 4.1. Then Y° is completely regular (and the coset configuration of Y° always is a metric Abelian scheme). The valence of the corresponding distance-regular graph is $\binom{m}{2}$, the number of its vertices is 2^{m-1} and its diameter is $\left\lfloor \frac{m}{2} \right\rfloor$.*

Here G_2 has rank $m-1$ over $\mathbb{F}_2$ and each syndrom of Y° has even weight. Thus $t \leqslant \left\lfloor \frac{m}{2} \right\rfloor$, by the argument above. But every syndrom of weight $2w$ is obtained as a sum of w columns of G_2 and not less. Thus the corresponding coset is at distance w from Y°, $w = 0, \dots, \left\lfloor \frac{m}{2} \right\rfloor$. Hence $t = \left\lfloor \frac{m}{2} \right\rfloor$ and Y° is completely regular. Theorem 3.3 applies.

Remark 4.4. — The distance-regular graph here has all vectors of even weight as vertices and (x,y) is an edge whenever the Hamming weight $W_H(x-y)$ is 2. That distance-regular graph is given in Example 1, Section

5.3.3 of [13] in which it is shown that we have a tight design if and only if m is odd. That graph is also considered in [3, p.114] as the *half-cube* (bipartite half of the Hamming graph $H(m,2)$).

COROLLARY 4.2. — *Let Y be the additive group of $\mathbb{F}_2^\ell$, $\ell = \binom{m}{2}$ spanned by the characteristic vectors of cycles of the complete undirected graph of m vertices. Then Y is completely regular in the Hamming scheme $H(\ell,2)$.*

Remark 4.5. — The author of [2] also considers the case $q = 2$, k even. He essentially asserts that if Y has exactly $\left\lfloor \frac{m}{2} \right\rfloor$ distinct weights in that case, then the coset configuration of Y° is a coset sheme. This here appears as a corollary of Theorem 4.1.

On the other hand let Y^* be the projective code associated to X_k, i.e. the code $\mathcal{C}(X_k^*)$ (Definition 4.1). If the restriction (Y, R^Y) of the Hamming scheme $H(|X_k|, q)$ to Y is an Abelian scheme with Abelian classes $Y_{i_1}, \ldots, Y_{i_m}$ (Definition 2.14) (thus isomorphic with $H(m,q)$) then the restriction (Y^*, R^{*Y^*}) of the Hamming scheme $H(|X_k^*|, q)$ to Y^* clearly is an Abelian scheme isomorphic with (Y, R^Y), with Abelian classes $Y_{i_1}^*, \ldots, Y_{i_m}^*$. Since a Hamming scheme is selfdual, then the coset configuration of $Y^{*\circ}$ not only is an association scheme, by Theorem 2.4 but it is a metric scheme. It clearly is the Hamming scheme $H(m,q)$ endowing the group of syndroms of Y°. The Abelian class $Y_{i_1}^*$ defining the edges of the distance-regular graph of (Y^*, R^{*Y}) is the set of $(q-1)m$ nonzero codewords which are scalar multiples of the rows of G_k^*. We see next that the dual $Y^{*\circ}$ of Y^*, which is a 1-error-correcting code is however not completely regular if $k > 1$, $q > 2$.

THEOREM 4.2. — *Let X_k be the set of vectors of weight k in $\mathbb{F}_q^m$. Let the generator matrix of the linear code Y^* have as set of columns X_k^*. If Y^* has exactly m nonzero weights, then the restriction (Y^*, R^{*Y^*}) of the Hamming scheme $H(|X_k^*|, q)$ is a Hamming scheme as well as the coset configuration of the dual $Y^{*\circ}$ of Y^* and $Y^{*\circ}$ is not completely regular if $k > 1, q > 2$.*

We only need to show that the covering radius ρ' of $Y^{*\circ}$ is smaller than m for $k > 1$ (Proposition 2.3 and Theorem 2.1). It is readily seen that a syndrom of $Y^{*\circ}$ is a linear combination of at most $\left\lceil \frac{w}{k} \right\rceil$ columns of G_k^* if $w > k$ and is less than 3 if $w \leqslant k$.

Example 4.1. — Here $q = 3$ and $k = 2$ The code Y^* spanned over $\mathbb{F}_3$ by the following generator matrix has three nonzero weights. For every linear combination of 2 (resp. 3) rows has weight 5 (resp. 3). The code $Y^{*\circ}$ is seen to have as generator matrix G_3°.

$$G_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix} G_3^\circ = \begin{bmatrix} 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}.$$

The argument used for Y^* applies and thus both coset configurations of Y^* and $Y^{*\circ}$ are Abelian metric schemes and however none of those codes is completely regular.

Remark 4.6. — In general the weights of the code $\mathcal{C}(X_k^*)$ (Definition 4.1) are given by the Krawtchouk polynomials (see [27, Chap. 5, §7]). The m possible weights of Y are obtained from Delsarte results [14] [15] (see [7, Example 5.17])

$$w_{m,i} = (|X_k| - P_k(i))(q-1)/q, \ i = 0, \dots, m.$$

The hypothesis of Theorem 4.2 are satisfied for $q = 3$ with $k = 2$, $m = 3, 5, 6, 8, 9, 11, \dots$ and for $q = 2$ with $k = 3$, $m = 8, 12, 14, \dots$

4.3. Partition designs admitted by the dual codes of $\mathcal{C}(X_k^*)$.

After [9, Section 4] every code $Y^{*\circ}$ dual to $\mathcal{C}(X_k^*)$ admits a partition design. The associate matrix $M(u, v)$ can be computed with the help of [9, Proposition 4.2] as follows. To the partition $\{E_0, E_1, \dots, E_t\}$ corresponds a partition $\{\Delta_0, \Delta_1, \dots, \Delta_t\}$ of syndroms, where Δ_i is the set of syndroms of weight i. For any choice of an element $a \in \Delta_u$, the number $M(u, v)$ counts the pairs $(b, h) \in \Delta_v \times \mathbb{F}_q^* X_k^*$ satisfying $a = b + h$.

Considering a sum $a = b + h$, let α be the number of nonzero components h_i of h such that $a_i = 0$. Let β be the number of nonzero components h_i of h such that $b_i = 0$. Next let γ_0 be the number of nonzero components b_i of b with $h_i = 0$ and finally let γ_1 be the number of nonzero components b_i of b such that $h_i \neq 0$ and $b_i + h_i \neq 0$. We have that

$$v - \alpha + \beta = u; \ \alpha + \beta + \gamma_1 = k; \ \alpha + \gamma_0 + \gamma_1 = v \tag{18}$$

or

$$\alpha = k - u + \gamma_0; \ \beta = k - v + \gamma_0; \ \gamma_1 = v - k + u - 2\gamma_0, \tag{19}$$

where α, β, γ_1 should be non-negative integers and $k \geqslant |u-v|, 0 \leqslant \gamma_0 \leqslant m-k$, else $M(u,v) = 0$. we have

$$(20) \qquad M_k(u,v) = \sum_{\gamma_0=0}^{m-k} \binom{u}{\beta}\binom{u-\beta}{\gamma_1}(q-2)^{\gamma_1}\binom{m-u}{\alpha}(q-1)^{\alpha}$$

where α, β, γ_1 take their values from (19).

We also have that $M_k = L_k$ where L_k is the k-th intersection matrix of $H(m,q)$ (Definition 1.1). It is furthermore known that $L_k = v_k(L_1)$ where $v_k(x)$ is the polynomial that is computed with recurrence (11).

Remark 4.7. — We will introduce in the sequel a new construction of metric Abelian schemes. Relations (20) are the basic data for those computations.

Example 4.2. — For $q = 3, m = 3, k = 2$ then the associate matrix of the partition design admitted by $Y^{*\circ}$ is the matrix M_2 below.

We have seen that if the code $Y^{*\circ}$ were completely regular, then that associate matrix would be the first intersection matrix of a metric scheme. Here M_2 is the second intersection matrix of $H(3,3)$ whose first intersection matrix is L_1.

$$M_2 = \begin{bmatrix} 0 & 0 & 12 & 0 \\ 0 & 4 & 4 & 4 \\ 1 & 2 & 5 & 4 \\ 0 & 3 & 6 & 3 \end{bmatrix}; \; L_1 = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}.$$

We check that the recurrence for Hamming schemes

$$L_1 L_1 = v_1 L_0 + a_1 L_1 + c_2 L_2 = m(q-1)L_0 + L_1 + 2L_2$$

gives $L_2 = M_2$ as expected.

4.4. Completely regular codes admitted by the considered partition design.

To end this investigation we develop a construction of metric coset schemes in which is emphasized how easily examples are worked out with the help of the associate matrix of a partition design and the combinatorial matrix of a code admitted by that partition design.

4.4.1. $Y = \mathcal{C}(X_k)$ for $q = 2$ $m = 5$ and $k = 3$.

That code is admitted by a partition design whose associate matrix M_3 is given by relations (20). By relations (23) of [9] or (11) of [8] we can compute the combinatorial matrix of Y (Definition 2.11). If e_1 is the first unit column-vector, corresponding to the class $E_0 = Y^\circ$ of the partition design, then we simply have that

$$A = [e_1, Me_1, M^2e_1, \ldots] \tag{21}$$

$$M_3 = \begin{bmatrix} 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 3 & 0 & 6 & 0 & 1 \\ 1 & 0 & 6 & 0 & 3 & 0 \\ 0 & 4 & 0 & 6 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \end{bmatrix}; \ A = \begin{bmatrix} 1 & 0 & 10 & 0 & 640 & \\ 0 & 0 & 0 & 60 & 0 & \\ 0 & 0 & 6 & 0 & 624 & \\ 0 & 1 & 0 & 64 & 0 & \vdots \\ 0 & 0 & 6 & 0 & 624 & \\ 0 & 0 & 0 & 60 & 0 & \end{bmatrix}.$$

Since the columns of the matrix are related by a recurrence of degree at most equal to that of the minimal polynomial of M, we only need to compute the first m columns of A. The distinct number of rows of the $m + 1 \times m$ matrix obtained is $t + 1$ where t is the combinatorial number of Y°. Its rank is $s + 1$ where s is the external degree of Y°, which is the number of nonzero weights of Y. Here we have $s = 3$ (the nonzero weights of Y are 4, 6 and 10) and $t = 3$. From Theorem 2.1 the covering radius ρ' of Y° is at most $s = 3$ and it is then readily checked that the weight-one syndrom of Y° cannot be obtained from a sum of 2 columns of the matrix G_3. Thus $\rho' = s = t$ and by Proposition 2.3 Y° is completely regular. Since $\operatorname{rank}(G_3) = 5$ over $\mathbb{F}_2$, the corresponding coset scheme has 2^5 elements and it has diameter 3. It is not a Hamming scheme.

We have just checked for clarity on the parity-check matrix of Y° that the covering radius of Y° is 3. It is however important to notice that considering matrix A is sufficient to see that $\rho' = s = t$. Here the three distinct rows of A can be rearranged to obtain the upper triangular 4×4 matrix $A^{(\rho')}$. Indeed by Proposition 2.2 of [9] we see from the distance distribution matrix of Y° (Remark 2.2) that a row $\bar{B}(\bar{x})$ only depends on the distance from $x \in \bar{x}$ to Y°. The associate matrix of the distance partition (which is a partition design, by Theorem 2.1) is computed from the matrix $A^{(\rho')}$. Since $MA^{(\rho')}_{j-1} = A^{(\rho')}_j$ and $A^{(\rho')}$ is upper triangular, relations

$$A(i,j+1) = M(i,0)A^{(\rho')}(0,j) + \ldots + M(i,j)A^{(\rho')}(j,j), \ i = 0, \ldots \rho'. \tag{22}$$

give $M(i,j)$ when $M(i,\ell)$ is known, $\ell = 0, \ldots, j-1$.

4.4.2. A code consisting of two classes of the partition design.

Next we consider the union of E_0 and E_5 in the considered partition design. The same construction as above where $e_1 + e_5$, where e_5 is the fifth unit vector, replaces e_1 yields the following combinatorial matrix A below. That new code is obtained from Y° by joining to it the coset whose syndrom has weight 5. We have here for that code that $\rho' = s = t = 2$. That code is completely regular and the corresponding coset scheme is a metric Abelian scheme with 2^4 element and diameter 2. The intersection matrices of the metric schemes obtained in Sections 4.4.1 and 4.4.2 are given here next to A.

$$A = \begin{bmatrix} 1 & 0 & 10 & 60 & 640 \\ 0 & 0 & 6 & 60 & 624 \\ 0 & 1 & 6 & 64 & 624 \\ \vdots & & & & \\ 0 & 1 & 6 & 64 & 624 \\ 0 & 0 & 6 & 60 & 624 \\ 1 & 0 & 10 & 60 & 640 \end{bmatrix}; \quad \begin{bmatrix} 0 & 10 & 0 & 0 \\ 1 & 0 & 9 & 0 \\ 0 & 6 & 0 & 4 \\ 0 & 0 & 10 & 0 \end{bmatrix}; \quad \begin{bmatrix} 0 & 10 & 0 \\ 1 & 6 & 3 \\ 0 & 6 & 4 \end{bmatrix}.$$

4.4.3. $Y = \mathcal{C}(X_k)$ for $q = 2$ $m = 6$, $k = 3$.

We first compute the combinatorial matrix of $Y^\circ = E_0$ to see that Y° is not completely regular. We observe that $\mathrm{rank}(G_6) = 6$ over $\mathbb{F}_2$. The code considered will consist of two classes of the partition design, $E_0 \cup E_6$. Thus the coset scheme that we obtain has 2^5 elements. We have

$$A^{(\rho')} = \begin{bmatrix} 1 & 0 & 40 & 0 \\ 0 & 2 & 0 & 512 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 480 \end{bmatrix}; \; M = \begin{bmatrix} 0 & 20 & 0 & 0 \\ 2 & 0 & 18 & 0 \\ 0 & 12 & 0 & 8 \\ 0 & 0 & 20 & 0 \end{bmatrix}.$$

We thus have an example where the constant c in the proof of Theorem 3.3 is 2. This construction leads to the same scheme as that of Section 4.4.1.

4.4.4. $Y = \mathcal{C}(X_k)$ for $q = 2, m = 7, k = 3$.

Here again the code consists in the union of E_0 and E_7. The associate matrix of the distance partition is the first intersection matrix

$$L_1 = \begin{bmatrix} 0 & 35 & 0 \\ 1 & 18 & 16 \\ 0 & 20 & 15 \end{bmatrix}.$$

The distance-regular graph has 64 vertices, valence 35 and diameter 2.

4.4.5. The case $q = k = 2$.

The parameters of those metric Abelian schemes are given in Corollary 4.1 and are not listed in the array below. The first intersection matrix can be computed for every scheme by using relations (20), (21) and (22). All other intersection matrices are then computed with the help of relations (11).

4.4.6. Table of results.

m	k	Classes	v	Diameter	$\lvert\bar{X}\rvert$	c
5	3	E_0	10	3	2^5	1
5	3	$E_0 \cup E_5$	10	2	2^4	1
6	3	$E_0 \cup E_6$	10	3	2^5	2
7	3	$E_0 \cup E_7$	35	2	2^6	1
8	4	$E_0 \cup E_8$	35	2	2^6	2
10	3	$E_0 \cup E_{10}$	120	3	2^9	1

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Added in Proof.

J.A. Bondy observes that the graph given in the second row of the array above is known as the **Clebsch graph** introduced by Seidel (see [30, Section 2.4]).

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