ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 49, nº 3 (1999), p. 809-814
<http://www.numdam.org/item?id=AIF_1999__49_3_809_0>

Ann. Inst. Fourier, Grenoble
49, 3 (1999), 809–814

BIPARTITE GRAPHS THAT ARE NOT CIRCLE GRAPHS

by André BOUCHET

The graphs considered in this note are finite and simple. Let G be a graph defined on the vertex-set V, with edge-set E. An edge with ends x and y is denoted by xy. The *complement* of G is the graph $\overline{G}$, defined on the same vertex-set as G, such that xy is an edge of $\overline{G}$ if and only if xy is not an edge of G. We denote by $\overline{E}$ the edge-set of $\overline{G}$.

Set $n = |V|$. Naji [9] has defined a linear system of equations $\nu(G)$ with $n(n-1)$ unknowns α_{xy} in $\mathrm{GF}(2)$, defined for every ordered pair (x, y) of distinct vertices of G. The equations are of two types:

$$T(x,y,z) \,:\, \alpha_{xy} + \alpha_{xz} + \alpha_{yz} + \alpha_{zy} = 1$$

if either (i) xy and xz are edges of G and yz is an edge of $\overline{G}$ or (ii) xy and xz are edges of $\overline{G}$ and yz is an edge of G,

$$E(y,z) \,:\, \alpha_{yz} + \alpha_{zy} = 1$$

if yz is an edge of G.

Originally the first subsystem of equations was defined only in Case (i). In Case (ii) the equation $T(x, y, z)$ was replaced by $\alpha_{xy} + \alpha_{xz} = 0$. The two systems of equations are seen to be equivalent by making a linear combination of $T(x, y, z)$ and $E(y, z)$.

A *circle graph* is an intersection graph of finitely many chords of a circle. The following characterization of circle graphs is part of Naji's thesis. The original proof, which is long and difficult, has not been published. A

Keywords: Circle graph – Binary matroid.
Math. classification: 05C – 05C75.

short proof has been recently derived by Gasse [5] from a characterization of circle graphs by three obstructions, due to the author [3].

THEOREM 1. — *A graph G is a circle graph if and only if $\nu(G)$ has a solution.*

Here we focus on the *reduced system* of equations $\nu_R(G)$, which only contains the equations $T(x, y, z)$. If $\nu_R(G)$ and $\nu(G)$ were equivalent systems, then the complement of any circle graph would be a circle graph. However, it is easily checked that the 6-circuit is a circle graph, and that its complement is not. This example also shows that the converse of Corollary 3 fails.

THEOREM 2. — *If $\nu_R(G)$ has a solution for a bipartite graph G, then $\nu(G)$ also has a solution.*

Here we mention an interesting application to bipartite circle graphs. It has been proved by de Fraysseix [4] that a bipartite graph is a circle graph if and only if it is the fundamental graph of a planar graph. One of the nice corollaries of Naji's characterization is that it provides algebraic characterization of planarity. If you want to check whether a graph is planar you can construct Naji's system for one of its fundamental graphs. Theorem 2 is interesting as it removes some redundant constraints from the system.

COROLLARY 3. — *If a bipartite graph G is not a circle graph, then $\overline{G}$ also is not a circle graph.*

We first recall some definitions and results used in the sequel. The *symmetric difference* of two subsets of edges P and Q is $P\Delta Q := (P \setminus Q) \cup (Q \setminus P)$. If X is a subset of vertices of G, then we denote by $\nabla(X)$ the set of edges of G having one end in X and one end in $V \setminus X$. The set ∇X is a *cut*. If x is a vertex then we set $\nabla(x) = \nabla(\{x\})$, and we call $\nabla(x)$ a *vertex-cut*. Every cut can be expressed as a symmetric difference of vertex-cuts. A *circuit* of G is a subset C of edges such that the subgraph of G induced by C, denoted by $G[C]$, is connected and regular of degree 2. The circuit is *chordless* if the edge-set of the subgraph $G[V(C)]$ (induced by $V(C)$, the vertex-set of C) is equal to C. It is known that a subset Γ of edges is a cut if and only if $|\Gamma \cap C|$ is even, for every circuit C. It is also known that every circuit of a graph can be expressed as a symmetric difference of chordless circuits. (We refer the reader to [10] for a proof.) By combining the two preceding properties we obtain the following one.

PROPOSITION 4. — *A subset Γ of edges of a graph is a cut if and only if $|\Gamma \cap C|$ is even for every chordless circuit.*

If $\alpha = (\alpha_{xy} : x, y \in V, x \neq y)$ is a solution of $\nu_R(G)$, then we denote by E_α the subset of edges xy of G such that $\alpha_{xy} + \alpha_{yx} = 1$.

LEMMA 5. — *Let G be a graph and let α be a solution of $\nu_R(G)$. If C is a chordless even circuit of G, then $|E_\alpha \cap C|$ is even.*

Proof. — Let $v_0, v_1, \cdots, v_{2l-1}$ be the successive vertices of C. For each triple $\{x, y, z\}$ giving rise to an equation $T(x, y, z)$ set $\tau(x, y, z) = \alpha_{xy} + \alpha_{xz} + \alpha_{yz} + \alpha_{zy}$. Set

$$X = \sum_{i=0}^{2l-1} \tau(v_{i-1}, v_i, v_{i+1}) + \sum_{i=0}^{2l-1} \sum_{j \notin \{i-1,i,i+1\}} \tau(v_i, v_j, v_{j+1}),$$

where the arithmetic on the indices is mod. $2l$. The reader will verify that

$$X = \sum_{i=0}^{2l-1} (\alpha_{v_i v_{i+1}} + \alpha_{v_{i+1} v_i}),$$

which implies

$$X = |E_\alpha \cap C| \pmod 2. \tag{1}$$

The number of terms of the form $\tau(v_p, v_q, v_r)$ in the expression of X is even. Since each of these terms has value 1 in $\nu_R(G)$, it follows

$$X = 0 \pmod 2. \tag{2}$$

The equalities (1) and (2) imply $|E_\alpha \cap C| = 0 \pmod 2$. □

LEMMA 6. — *Let α be a solution of $\nu_R(G)$ and let Γ be a cut of G. There exists a solution α' of $\nu_R(G)$ such that*

$$E_{\alpha'} = E_\alpha \Delta \Gamma.$$

Proof. — Since Γ is a symmetric difference of vertex-cuts, it is sufficient to prove the lemma when $\Gamma = \nabla(v)$, for some vertex v. Change α_{vx} into $1 + \alpha_{vx}$, for every vertex x different from v, and change α_{xv} into $1 + \alpha_{xv}$, for every vertex x joined to v by an edge of $\overline{G}$. We easily check that we obtain a new solution α' of $\nu_R(G)$. For every edge xy, the values

$\alpha_{xy} + \alpha_{yx}$ and $\alpha'_{xy} + \alpha'_{yx}$ are distinct if and only if e belongs to $\nabla(v)$. It follows $E_{\alpha'} = E_\alpha \Delta \nabla v$. □

Proof of Theorem 2. — Let α be a solution of $\nu_R(G)$. Lemma 5 and Proposition 4 imply that E_α is a cut of G. Since G is bipartite the edge-set of G is also a cut. Therefore $\Gamma = E \Delta E_\alpha$ is a cut. Lemma 6 implies the existence of a solution α' of $\nu_R(G)$ such that $E_{\alpha'} = E_\alpha \Delta \Gamma = E$. Hence every edge xy of G satisfies $\alpha'_{xy} + \alpha'_{yx} = 1$, and α' is a solution of $\nu(G)$.

□

Proof of Corollary 3. — Assume indirectly that $\overline{G}$ is a circle graph. Then $\nu(\overline{G})$ has a solution by Naji's theorem. Hence $\nu_R(G)$ has a solution. The systems $\nu_R(G)$ and $\nu_R(\overline{G})$ are equal. Hence $\nu_R(G)$ has a solution. Theorem 2 implies that $\nu(G)$ has a solution. Naji's theorem implies that G is a circle graph, a contradiction. □

QUESTION. — Naji's result is difficult to prove. Is there an elementary direct proof of Corollary 3?

The influence of François Jaeger in the study of circle graphs.

In the 70's the main problem was to find a good characterization of circle graphs, which was expected by many researchers, and to devise a recognition algorithm of polynomial complexity. An approach was to consider the binary matroids generated by the neighborhoods of a circle graph.

If A is a collection of subsets of a finite set E, then the collection of minimal nonempty subsets of E that can be expressed as symmetric differences of subsets in A is the collection of circuits of a binary matroid, which we call the *binary matroid generated by A*. If x is a vertex of a simple graph $G = (V, E)$, then $N(x) = \{y \in V : xy \in E\}$ is the *neighborhood* of x.

If G is a bipartite graph with classes X and Y, then de Fraysseix [4] proved that G is a circle graph if and only if the binary matroids generated by $\{N(x) : x \in X\}$ and $\{N(x) : x \in Y\}$ are graphic (which is equivalent to say that G is a fundamental graph of a planar graph).

This is a good characterization of bipartite circle graphs because graphic matroids are characterized by excluding a finite number of excluded minors, by a theorem of Tutte [11].

A main result of François Jaeger [6], [7], [8], when G is a simple graph, says that, if P is a subset of V, then the binary matroid generated by $\{N(x) \cup P \cap \{x\} : x \in V\}$ is graphic. If G is a circle graph and x is any vertex of G, then the graph obtained from G by replacing the induced subgraph $G[N(x)]$ by the complementary subgraph is also a circle graph. A graph derived from G by performing a sequence of such transformations is said to be *locally equivalent* to G. If we apply Jaeger's result to the graphs locally equivalent to G, one proves that G is a circle graph if and only if, for every pair of subsets P and Q of V, the binary matroid generated by $\{Q \cap (N(x) \cup P \cap \{x\}) : x \in V\}$ is graphic [1]. Unfortunately the converse does not hold; the 5-wheel is not a circle graph and satisfies the preceding condition. It would be interesting to compare the class of circle graphs with the class of graphs satisfying Jaeger's conditions. A good characterization of circle graphs can be found in [3].

Let $A(G)$ be the adjacency matrix of a simple graph, with coefficients in the field GF(2). François proved that, if $A(G)$ has an inverse, then it is the adjacency matrix of a graph G' locally equivalent to G. Accordingly G' is a circle graph if G is a circle graph.

François was also the origin of fruitful ideas concerning circle graphs. Naji prepared his thesis with him and François foreseen the importance of circle graph orientations, which led to the equations studied in that note. The matrix $A(G)$ considered above has an interesting property, when G is any circle graph; it can be transformed into an antisymmetric matrix with integral coefficients, by changing into -1 some values equal to 1, in such a way that every principal minor has a determinant equal to 0 or 1. This property, which is now referred to *principal unimodularity*, is related to Poincaré duality. I gave a simple direct proof of the principal unimodularity property by using an idea of François [2].

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