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EDGE-DISJOINT ODD CYCLES IN GRAPHS WITH SMALL CHROMATIC NUMBER

by C. BERGE and B. REED

Dedicated to the memory of François Jaeger

1. A positional game.

An interesting positional game can be described as follows: two players, called respectively Red and Blue, play alternately by coloring with their color an edge in a graph G. The first one who achieves a monochromatic odd cycle wins and his opponent loses; if no monochromatic odd cycle is obtained, it is a draw. It would be interesting to know for which graphs one of the players can win.

A non-constructive proof for the existence of a winning strategy is:

THEOREM 1. — *If the players Red and Blue play the game with a graph G of chromatic number > 4, there exists a winning strategy for the first player.*

Proof. — Assume that Red is the first player and has no winning strategy with a graph G; we shall show that this leads to a contradiction.

Clearly, for his opponent, no strategy σ is winning, because otherwise Red could win by using the same strategy σ (determined for a fictitious game where Blue starts, his first move being an arbitrary chosen edge),

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which is a contradiction. Thus both Red and Blue can guarantee a draw. At the end of the game, all the edges of G have been colored; the red edges define a partial graph G_R with no odd cycles and the blue edges define a partial graph G_B with no odd cycles. Since the graph G_R is bipartite, its vertices can be colored with only two colors, say α and α', so that no two adjacent vertices have the same color; also, the vertices of G_B can be colored with two colors β and β' so that no two adjacent vertices of G_B have the same color. If we assign to each vertex a label (α, β), (α, β'), (α', β) or (α', β'), corresponding to the pair of colors used for this vertex in G_R and in G_B, no two adjacent vertices of G have the same label. This implies that the graph G is 4-colorable. A contradiction.

For 4-colorable graphs, and in particular for planar graphs, no simple structural property has been found so far to see that the game is unfair.

2. The König property.

Given a simple graph $G = (X, E)$, the *edge-transversal number* $\tau_e(G)$ is the least number of edges to remove from G in order to destroy all the odd cycles. The *edge-packing number* $\nu_e(G)$ is the maximum number of paiwise edge-disjoint odd cycles; if G has no odd cycles, we put $\nu_e(G) = 0$.

Clearly, we have always $\tau_e(G) \geq \nu_e(G)$, and if for some graph G this inequality holds with equality, it is easier to check that some edge-transversal is minimum (or that some edge-packing is maximum). It would be interesting to determine the graphs for which the values of these two coefficients can be obtained in polynomial time (see [2], [9] for some related results where the words "edge" and "vertex" are interchanged).

When these two coefficients are equal, we say that we have the *König property (for the odd cycles).*

We do not always have this König property, even for planar graphs; for instance, the complete graph K_4 is planar, but $\tau_e(K_4) = 2$ and $\nu_e(K_4) = 1$. However we have proved in [3] that for all the planar graphs having a fixed edge-packing number, their edge-transversal numbers are bounded.

Graphs for which each pair of odd cycles has at least one *vertex* in common and yet the number of *vertices* needed to hit them all is large, have been considered by Lovász [7]. Similarly, we have considered in [3] a family of graphs G for which $\nu_e(G) = 1$ and their edge-transversal numbers can be arbitrarily large.

We do not know any simple characterization for the graphs whose odd cycles satisfy the König property, but we have:

THEOREM 2. — *Let G be a graph such that $\tau_e(G') = \nu_e(G')$ for every partial graph G' of G. Then its chromatic number is < 4.*

Proof. — Catlin [4] proved that every graph with chromatic number ≥ 4 contains a subdivision of the complete graph K_4 such that each triangle of K_4 is subdivided to form an odd cycle (see [10] and [11]). If the graph G was not 3-colorable, then a partial graph G' would be a subdivision of K_4 of the type described above. However $\tau_e(G') = 2$ is different from $\nu_e(G') = 1$. A contradiction.

Similar results can also be obtained by using some basic results of Hypergraph Theory. Let H_G be the hypergraph whose *vertices* are the edges of G and whose *edges* are the elementary odd cycles of G. So, using usual notations τ and ν for the two main coefficients of a hypergraph (see for instance [1]), we can write $\tau_e(G) = \tau(H_G)$ and $\nu_e(G) = \nu(H_G)$.

Every hypergraph H satisfies $\tau(H) \geq \nu(H)$. If $\tau(H) = \nu(H)$, the hypergraph H is said to have the *König property.* A hypergraph H has the *Helly property* if every family of pairwise intersecting edges has a non-empty intersection. For a hypergraph H with the Helly property, the most usual way to prove that H satisfies the König property is to show that the line-graph (or "intersection graph") $L(H)$ is a perfect graph; but unfortunately, as we have seen above, the hypergraph H_G does not necessarily satisfy the Helly property.

The property of G described in the statement of Theorem 2 implies that H_G is a *normal* hypergraph (Lovász [6]). Fournier and Las Vergnas [5] have shown that the vertex-set of a normal hypergraph can be split into two classes, the "red" vertices and the "blue" vertices, so that no edge of the hypergraph is monochromatic. This shows that *this graph G can be split into two partial graphs which are bipartite.* This result is implied by Theorem 2. However, perhaps this alternative approach, via Hypergraph Theory, can shed some light on the problem.

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