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BICUBIC PLANAR MAPS (*)

by **William T. TUTTE**

Called to lecture at a conference that honours the memory of François Jaeger I reflect that his interests in graph theory overlapped considerably with my own. Thus I have been interested in nowhere-zero flows and François proved the 8-flow conjecture for them. We have both worked on edge-3-colourings of cubic graphs and he dealt with more general edge-colourings too. I have considered a 2-variable polynomial for graphs and he imported it into knot theory. We both had something to say about planar bicubic maps. Perhaps it would be appropriate today to put our two sayings together.

A cubic graph is a graph in which each vertex is trivalent. Such a graph is "bicubic" if its vertex-set can be partitioned into two complementary subsets U and V such that each edge has one end in each subset. It follows that every circuit in a bicubic graph is of even length. A bicubic graph is *planar* if it can be drawn in the plane (or on the sphere) with no crossing. And when it is so drawn there results a "bicubic planar map".

Let M be a plane bicubic map and G its graph. If G is connected each residual domain in the sphere is simply connected. But if G is disconnected this rule no longer holds.

In the theory of cubic graphs it is often convenient to admit "loose edges" that is edges incident with no vertex. A loose edge is conveniently represented in a diagram by a simple closed curve. A loose edge all by itself is said to constitute a cubic and even a bicubic graph. It is referred to as a 0-circuit and as such said to have length zero. As part of a bicubic graph it is exempted from the requirement of having one end in U and one in V.

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We can get a useful theorem about bicubic planar maps by applying the Euler formula to each component of G not constituted by a single loose edge. As with any cubic map in the sphere there has to be a face of valency ≤ 5 in that component. But all circuits in a bicubic graph are even. Hence:

THEOREM 1. — *A planar bicubic map M has either a loose edge or a face that is a 2-gon or a 4-gon.*

Referring to a paper by François we find a list of operations whereby simpler bicubic maps are derived from a given one [3]. They are the following operations (i), (ii) and (iii):

(i) A loose edge is deleted from G, forming G' and M'.

(ii) An edge is deleted from a digon and the resultant 2-valent vertices are suppressed, forming G' and M'.

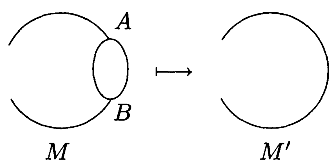

Figure 1

(iii) Two opposite edges of a 4-gon are deleted and the resulting 2-valent vertices are suppressed. This can be done in two ways, forming M' and M'' (and G' and G'').

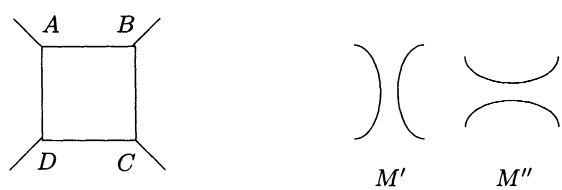

Figure 2

François now looks for functions of planar bicubic maps having recursion formulæ simply related to his three operations. The first one he presents is

$$T(G) = T(M),$$

the number of edge-3-colourings of G in three colours a, b and c. This function is of course defined for all cubic graphs. And for planar cubic maps it is well known that T is non-zero if and only if the map is face-4-colourable [5]. I once used the theory of these "Tait-colourings" in a new proof of Smith's Theorem, that the number of Hamiltonian circuits through any edge of a cubic graph, planar or non-planar, is even [6]. In their improved form of the Four Colour Theorem [4] Robertson, Sanders, Seymour and Thomas reformulate the problem in terms of Tait colourings.

For bicubic planar maps François points out that

- $T(M) = 3T(M')$ for operation (i);
- $T(M) = 2T(M')$ for operation (ii);
- $T(M) = T(M') + T(M'')$ for operation (iii).

If we conventionally put

$$T(M) = 3$$

when G consists of a single loose edge we can compute T for any M by repeated use of these three recursions, without leaving the realm of planar bicubic maps. François now goes on to discuss the flow polynomial $F(G, \lambda)$ of G. I now quote from his paper:

> "For positive integer values of λ this polynomial counts the flows (in an arbitrary orientation) of G with non-zero values in an arbitrary abelian group of λ elements. For instance, by considering edge-3-colourings as flows with non-zero values in $\mathbb{Z}_2 \times \mathbb{Z}_2$ we see that $F(G, 4) = T(G)$. For our purposes we can be satisfied with the following definition: $F(G, \lambda)$ is the chromatic polynomial of the dual of G divided by λ."

Apparently François was unable to fit $F(G, \lambda)$ to his scheme for every λ but succeeded in doing so for the special case $\lambda = \tau + 1$, where τ is the golden ratio $\frac{1}{2}(1 + \sqrt{5})$. There are some curious theorems about chromatic polynomials that apply only to that value of λ (see [8]). I have worked on them myself and I was much gratified to find François using one of them.

To cut the story short he showed

- for his operation (i),

$$F(G, \tau + 1) = \tau F(G', \tau + 1);$$

- furthermore for operation (ii),

$$F(G, \tau + 1) = (\tau - 1)F(G', \tau + 1);$$

- and for operation (iii), with the help of one of these special theorems,

$$F(G,\tau+1) = (-3\tau+5)\{F(G',\tau+1) + F(G'',\tau+1)\}.$$

Again we have recursion formulæ enabling us to calculate $F(M,\tau+1)$ form the conventional $F(M,\tau+1) = \tau$ of the single-loose-edge case.

The paper now invites us to consider these two map-functions as a special case of a more general function $U(M)$ that satisfies:

(o) $U(O) = 1$ for the loose-edge graph O;

(i) $U(G) = xU(G')$ with operation (i);

(ii) $U(G) = yU(G')$ with operation (ii);

(iii) $U(G) = z(U(G') + U(G''))$ with operation (iii).

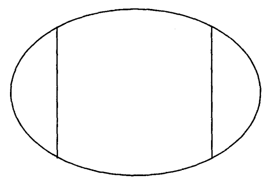

Figure 3

However by evaluating U for the map shown in Figure 3 in two ways, first by two applications of Rule (ii) and then by Rule (iii) it is found that a U is possible only when

$$y^2 = z(x+1).$$

But the paper goes on to give a proof that whenever this restriction holds a U exists. There we have a new invariant.

Having got so far into the paper I began to wonder where further information about planar cubic maps could be found. I bethought me of a paper by Brooks, Smith, Stone and Tutte [2] published in 1975 (see also [9]). In this connection let me remind you of a theorem of P.J. Heawood, that a connected bicubic planar map can be face-3-coloured in essentially only one way. So let us take an M, this time with no loose edges, and 3-colour the faces in red, blue and yellow (R, B and Y). Take the bipartition (U, V) calling U and V respectively positive and negative. We can do this so that the cyclic sequence (RBY) goes anticlockwise around the positive vertices (on the sphere) and clockwise around the negative ones.

Note that there is a way of constructing a plane directed graph D_R from M. We put one vertex of D_R in each red face. Each edge of M that goes from one red face to another is extended into each red face to end at the corresponding vertices of D_R. Thus we get the edges of D_R. We direct each from the positive vertex of M on it to the negative.

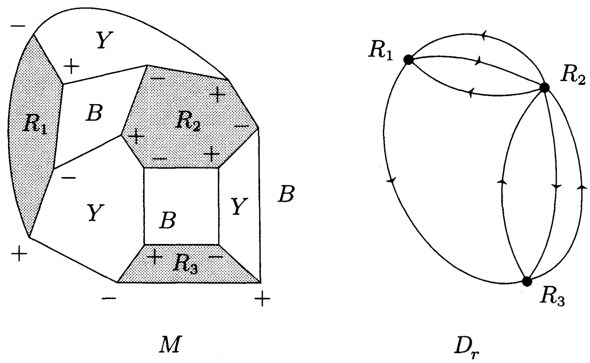

Figure 4

Note that D_R in this plane drawing is "alternating"; incoming and outgoing edges alternate at each vertex.

Likewise we have D_B and D_Y.

Now the four authors of the 1975 paper had much to do with alternating directed maps. Each such map, to them, represented a dissection of an equilateral triangle into equilateral triangles. With such a dissection there are three choices for which side of the dissected triangle is to be called horizontal. The three choices give three alternating directed maps [7]. It seemed to us that this association of alternating directed maps in triads was analogous of that of undirected maps in dual pairs.

It would be too long a digression to give a full theory relating an alternating map to a dissected triangle. Very briefly, the head of one directed edge is detached from its vertex. Then each directed edge is expanded as an equilateral triangle, the arrow having its tail in the horizontal side and its head at the opposite vertex. If the sizes of the triangles are made to satisfy some evident equations a dissection of an equilateral triangle is obtained.

For a long time we lacked a description of just how three "trine" alternating maps were related. And then Smith realized that they were the D_R, D_B and D_Y of some bicubic map.

For each alternating directed map D we had been interested in the number of spanning trees converging to a vertex v, that is having every edge directed to v in the tree. There were also spanning trees diverging from v, that is with every edge directed away from v. We had theorems saying that the number of directed trees converging to v was equal to the number diverging from it, that this number was independent of v, and that it was the same for each member of a triad. We called it the "diplexity" of the alternating maps and of the triad. The triad being determined by a bicubic map M, the diplexity could be seen as a function of bicubic planar maps.

It was Smith who found a simple characterization of the diplexity in terms of the structure of a bicubic planar map M. This is explained in the BSST paper [2] as follows.

For each cubic map (whereof M is one) there is an integer n such that the map has $3n$ edges, $2n$ vertices and $n + 2$ faces. Let us distinguish one positive vertex v of M as the root-vertex and the three faces incident with v as the root-faces, one of each colour. There are now $n - 1$ non-root positive vertices and the same number of non-root faces.

A pairing of each non-root face with an incident non-root positive vertex (if such is possible) is a "vertex-face matching" (with respect to the root-vertex v). It is pointed out that if M is disconnected no such mapping exists. For a component without v has too many faces. Theorem 9.1 of the paper asserts that the number of vertex-face mappings with respect to v is independent of the choice of v. Moreover it is unaltered by an interchange of black and white vertices. So it is an invariant of M.

The paper then proceeds to the proof of the main theorem, the equating of this invariant to the diplexity of each of the three associated alternating directed maps.

The last section of the paper presents some recursion formulæ for the invariant — call it the *match-number* $m(M) = m(G)$ of M. They can be associated with François' operations (ii) and (iii). We can extend the list by the following trivial rule for operation (i):

$$m(G) = 0 \cdot m(G').$$

This is conventional but it fits the rule that $m(G) = 0$ when G is disconnected. There is no harm now in writing

$$m(G) = 1$$

for the loose-edge graph. (It does embarrass us when we try to calculate $m(G)$ for the null-graph.)

The rules from the BSST paper are:

- $m(G) = m(G')$ for operation (ii);
- $m(G) = m(G') + m(G'')$ for operation (iii).

So the match number is a special case of the Jaeger function U. It has

$$x = 0, \quad y = 1, \quad z = 1.$$

I am happy to pay contribute to François' memory by drawing attention to this slight extension of the theory of his paper.

A theory of dissected triangles was published in 1948. But the existence of such a theory is briefly noted in an earlier paper of Brooks, Smith, Stone and Tutte [1] published in 1940.

Nineteen forty! Some years before François was born. How that emphasizes for me the shortness of his life and the tragic loss to Mathematics imposed by his premature death.

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