ANNALES DE L'INSTITUT FOURIER

LÁSZLO LOVÁSZ
ALEXANDER SCHRIJVER

On the null space of a Colin de Verdière matrix

Annales de l'institut Fourier, tome 49, n° 3 (1999), p. 1017-1026
<http://www.numdam.org/item?id=AIF_1999__49_3_1017_0>

Ann. Inst. Fourier, Grenoble
49, 3 (1999), 1017–1026

ON THE NULL SPACE OF A COLIN DE VERDIÈRE MATRIX

by L. LOVÁSZ and A. SCHRIJVER

1. Introduction and results.

Let $G = (V, E)$ be an undirected graph, with vertex set $\{1, \ldots, n\}$. Let $\mathcal{M}(G)$ be the set of symmetric $n \times n$ matrices $M = (m_{i,j})$ satisfying

(1) (i) M has exactly one negative eigenvalue, of multiplicity 1;

(ii) for all i, j with $i \neq j$ one has :
if i and j are nonadjacent then $m_{i,j} = 0$,
if i and j are adjacent then $m_{i,j} < 0$.

Moreover, M is said to have the *Strong Arnold Property* (or to satisfy the *Strong Arnold Hypothesis*) if for each symmetric $n \times n$ matrix $X = (x_{i,j})$ satisfying $x_{i,j} = 0$ if $i = j$ or i and j are adjacent, and satisfying $MX = 0$, one has $X = 0$.

Yves Colin de Verdière [1] introduced the parameter $\mu(G)$, being the maximum corank of any matrix in $\mathcal{M}(G)$ having the Strong Arnold Property. (The *corank* of a matrix is the dimension of its null space.)

Stimulated by discussions with François Jaeger (who suggested an idea to show (ii) below), Colin de Verdière showed that $\mu(H) \leq \mu(G)$ if H

Keywords: Null space – Colin de Verdière matrix – Planar graph – Outerplanar graph.
Math. classification: 05C10 – 05C50.

is a minor of G, and that

(2) (i) $\mu(G) \leq 1$ if and only if G is a disjoint union of paths;

(ii) $\mu(G) \leq 2$ if and only if G is outerplanar;

(iii) $\mu(G) \leq 3$ if and only if G is planar.

In [5], we showed that

(3) (iv) $\mu(G) \leq 4$ if and only if G is linklessly embeddable.

(A graph G is *linklessly embeddable* if it can be embedded in $\mathbb{R}^3$ so that any two disjoint circuits in G form unlinked closed curves in $\mathbb{R}^3$.)

It was shown by Hein van der Holst [3] that if G is 3-connected and planar, then any matrix in $\mathcal{M}(G)$ has corank at most 3 — also those not having the Strong Arnold Property.

The main result in this paper is that if G is 3-connected and planar, then for any matrix $M \in \mathcal{M}(G)$ of corank 3, the null space $\ker M$ of M yields an embedding of G in the 2-sphere S^2. We also show the related but easier results that if G is a path, then for any matrix $M \in \mathcal{M}(G)$ of corank 1, the null space $\ker M$ of M yields an embedding of G in the line; furthermore, if G is 2-connected and outerplanar, then for any matrix $M \in \mathcal{M}(G)$ of corank 2, the null space $\ker M$ of M yields a representation of G as a convex polygon with non-crossing diagonals.

To make this more precise, assume that G is connected. In this case condition (1)(ii) implies that the eigenvector π belonging to the negative eigenvalue is (up to scaling) positive. We define the *null space representation* derived from the matrix M as follows. Let r be the corank of M, let $a_1, a_2, \ldots, a_r$ form a basis of $\ker M$, and for each vertex i of G, let $u_i := (a_{1,i}, a_{2,i}, \ldots a_{r,i})^T \in \mathbb{R}^r$. As π is orthogonal to each of $a_1, \ldots, a_r$ we have

$$\sum_i \pi_i u_i = 0.$$

This way we define a mapping $V \to \mathbb{R}^r$. Note that this mapping is determined only up to a linear transformation of $\mathbb{R}^r$. In the results below, we have to scale the vectors u_i. Note that arbitrary scaling of them can be achieved through appropriate scaling of the rows and columns of M. In particular, we consider the unit vectors

(4) $$v_i = \frac{1}{\|u_i\|} u_i ,$$

and the vectors

$$w_i = \frac{1}{\pi_i} u_i. \tag{5}$$

For (4) to make sense, we'll have to show that the u_i are non-null.

THEOREM 1.1. — *Let G be a path. Then the mapping $i \mapsto w_i$, together with the segments connecting w_i and w_j for $ij \in E(G)$, gives an embedding of G in the line.*

THEOREM 1.2. — *Let G be a 2-connected outerplanar graph. Then the mapping $i \mapsto v_i$, together with the segments connecting v_i and v_j for $ij \in E(G)$, gives an embedding of G in the plane as a convex polygon with non-crossing diagonals.*

THEOREM 1.3. — *Let G be a 3-connected planar graph. Then the mapping $i \mapsto v_i$, together with the geodesic curves on S^2 connecting v_i and v_j for $ij \in E(G)$, gives an embedding of G in S^2.*

Remarks. — 1. It seems curious that we had to scale differently in the case of the path as in the other two cases. It may be the case that for planar and outerplanar graphs the scaling (w_i) provides an embedding with interesting properties, say in the planar case an embedding in the skeleton of a convex polyhedron; but we cannot prove this.

2. Our motivation has been to derive a similar result for one dimension higher, that is, for linklessly embeddable graphs, where we would like to obtain a linkless embedding of G in S^3. Suppose that G is a 4-connected linklessly embeddable graph and M is a matrix in $\mathcal{M}(G)$ of corank 4. Perhaps $\ker M$ yields a proper embedding of G in S^3: vertices of G are embedded in S^3 in such a way that adding shortest arcs in S^3 connecting adjacent vertices, gives a linkless embedding of G in S^3. We do not know if this is true.

2. Proofs.

2.1. Van der Holst's lemma and its extensions.

For any vector $x \in \mathbb{R}^n$, the *support* of x, denoted by $\mathrm{supp}(x)$, is the set $\{i \in V | x_i \neq 0\}$. The *positive support* $\mathrm{supp}^+(x)$ of x is the set $\{i \in V | x_i > 0\}$, and the *negative support* $\mathrm{supp}^-(x)$ is the set $\{i \in V | x_i < 0\}$. For any $U \subseteq V$, $N(U)$ denotes the set of vertices $i \notin U$ adjacent to at least one vertex in U.

The following observation is useful:

(6) Let $M \in \mathcal{M}(G)$ and let $x \in \ker M$. Then each vertex in $N(\mathrm{supp}(x))$ belongs to both $N(\mathrm{supp}^+(x))$ and $N(\mathrm{supp}^-(x))$.

To see (6), consider any $i \in N(\mathrm{supp}(x))$ (so $x_i = 0$), and note that

(7) $\sum_j m_{i,j} x_j = 0,$

implying that, if $x_j > 0$ for some j adjacent to i, then $x_j < 0$ for some j adjacent to i, and conversely. (Here we use that $m_{i,j} < 0$ if j is adjacent to i, and $m_{i,j} x_j = 0$ otherwise.)

An important further tool we need is the following lemma proved by van der Holst [2]. A vector x in $\ker M$ is said to have *minimal support* if $x \neq 0$, and each nonzero vector $y \in \ker M$ with $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$ satisfies $\mathrm{supp}(y) = \mathrm{supp}(x)$.

LEMMA 2.1 [van der Holst]. — *Let G be connected, let $M \in \mathcal{M}(G)$ and let $x \in \ker M$ have minimal support. Then both $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ are nonempty and induce connected subgraphs of G.*

Geometrically, this lemma expresses the following nice property of the null space embedding: *every hyperplane in $\mathbb{R}^r$ spanned by some vectors u_i separates the graph into two non-empty connected subgraphs.*

We need the following variation on this lemma.

LEMMA 2.2. — *Let $M \in \mathcal{M}(G)$ and let $x \in \mathbb{R}^V$ satisfy $Mx < 0$. Then $\mathrm{supp}^+(x)$ is nonempty and induces a connected subgraph of G.*

One way to get a vector x with $Mx < 0$ is to choose a vector in the nullspace of M and add a positive multiple of π to it. In this special case, Lemma 2.2 has a nice geometric formulation: *the nodes i for which*

w_i belongs to an open halfspace in $\mathbb{R}^r$ containing the origin induce a nonempty connected subgraph.

We give a proof of a more general lemma, which also includes van der Holst's lemma and its extension given in [4]. To formulate this, we need to describe a class of exceptions. Let $G = (V, E)$ be a graph, $M \in \mathcal{M}(G)$ and $S \subset V$. Let $G_1, \ldots, G_r$ $(r \geq 2)$ be the connected components of $G - S$, and let $S_i = V(G_i)$. We say that S is a *regular cutset* (in G, with respect to M), if there exist non-zero, non-negative vectors $x_1, \ldots, x_r$ and y in $\mathbb{R}^V$ such that

(i) $\operatorname{supp}(x_i) = S_i$ and $\operatorname{supp}(y) \subseteq S$;

(ii) $Mx_1 = Mx_2 = \ldots = Mx_r = -y$.

Note that if M_i and x_i' are the restrictions of M and x_i to S_i, then $x_i' > 0$ and $M_i x_i = 0$. Thus by the Perron-Frobenius Theorem, the x_i are uniquely determined up to positive scaling.

LEMMA 2.3. — *Let $M \in \mathcal{M}(G)$ and let $x \in \mathbb{R}^V$ satisfy $Mx \leq 0$. Then $\operatorname{supp}^+(x)$ is nonempty. Furthermore, if $\operatorname{supp}^+(x)$ induces a disconnected graph then G has a regular cutset S such that x is a linear combination*

$$x = \sum_i \alpha_i x_i$$

of the corresponding vectors $x_1, \ldots, x_r$, where $\sum_i \alpha_i \geq 0$, and at least two α_i are positive.

Proof. — We may assume that the negative eigenvalue of M is -1. Then

$$\pi^T x = -\pi^T M x \geq 0,$$

and since $x \neq 0$, it follows that $\operatorname{supp}^+(x)$ is nonempty.

Suppose next that $\operatorname{supp}^+ x$ can be decomposed into two disjoint nonempty sets A and B such that no edge connects A and B. Let $C := \operatorname{supp}^-(x)$ (this may be empty), and $S = V \setminus \operatorname{supp}(X)$. Let a, b and c arise from x by setting to 0 the entries with index out of A, B and C, respectively. Define

$$w := (\pi^T b)a - (\pi^T a)b.$$

Then $\pi^T w = 0$, and hence by our assumption that M has exactly one negative eigenvalue with eigenvector π,

$$w^T M w \geq 0.$$

On the other hand, we have $a^TMb = 0$ (as there are no edges between A and B) and $a^TMc \geq 0$ (as $a \geq 0$, $c \leq 0$ and M is non-positive outside the diagonal). Furthermore, $Mx \leq 0$ implies that

$$a^TMa \leq -a^TMb - a^TMc \leq 0.$$

Thus

$$w^TMw = (\pi^Tb)^2a^TMa - 2(\pi^Tb)(\pi^Ta)a^TMb + (\pi^Ta)^2b^TMb \leq 0.$$

This implies that $w^TMw = 0$. Moreover, we must have equality in $a^TMc \geq 0$, which implies that there are no edges between A and C, and similarly, between B and C.

Thus $(Ma)_i = (Mx)_i \leq 0$ for $i \in A$, and so $a^TMa = 0$ implies that $(Ma)_i = (Mx)_i = 0$ for $i \in A$. Similarly, $(Mb)_i = (Mx)_i = 0$ for $i \in B$. Thus $\mathrm{supp}(Ma), \mathrm{supp}(Mb). \mathrm{supp}(Mc) \subseteq S$.

Since M is positive semidefinite on vectors orthogonal to π, $w^TMw = 0$ implies $Mw = 0$. This says that

$$\frac{1}{(\pi^Ta)}Ma = \frac{1}{(\pi^Tb)}Mb. \tag{8}$$

To conclude, let $S_1, S_2, \ldots S_r$ be the connected components of $G - S$. The fact that $\mathrm{supp}(Mc) \subseteq S$ implies that if $z \in \mathbb{R}^V$ is defined by $z_i = |x_i|$, then $Mz \leq 0$. Thus we can apply the above argument choosing any two components S_i as A and B. If z_i denotes the restriction of z to S_i, and $x_i = (1/\pi^Tz_i)z_i$, then (8) above implies that $Mx_1 = \ldots = Mx_r$. We have also seen that this vector has support in S, and since its support is disjoint from the support of the x_i, it is non-positive. Clearly x is a linear combination of the x_i, and the conditions on the coefficients are trivially verified. □

2.2. Proof of Theorem 1.1.

Look at the null space representation (u_i) of $M \in \mathcal{M}(P)$ with corank 1, where P is a path with n nodes labelled $1, 2, \ldots, n$ (in this order on the path), and the scaled version (w_i). Note that the w_i are numbers now. We claim that either $w_1 < w_2 < \ldots < w_n$ or $w_1 > w_2 > \ldots > w_n$. It suffices to argue that $w_{i-1} < w_i \geq w_{i+1}$ cannot occur. There are 3 cases to consider:

(a) $w_i > 0$. Then the vector $x \in \mathbb{R}^n$ defined by $x_j = (w_i - w_j)\pi_j$ satisfies $Mx < 0$, and has $x_i = 0$, $x_{i-1} > 0$, and $x_{i+1} \geq 0$. Let

$$y_j = \begin{cases} x_j - \varepsilon, & \text{if } j = i+1, \\ x_j, & \text{otherwise,} \end{cases}$$

for a small positive ε. Then $My < 0$, and $y_i = 0$, $y_{i-1}, y_{i+1} > 0$, which means that $\mathrm{supp}^+(y)$ is disconnected, contradicting Lemma 2.3.

(b) $w_i = 0$. It is trivial from $Mu = 0$ that if a node i with $u_i = 0$ has a neighbor j with $u_j < 0$, then it must also have a neighbor k with $u_k > 0$ (by (6)), which is not satisfied here.

(c) $w_i < 0$. We know by Lemma 2.1 that there is a node k with $w_k > 0$. If $i < k$, then let j be the index with $i < j < k$ and w_j minimum; this j then violates (a) (with signs flipped). If $k > i$, the argument is similar. □

2.3. Proof of Theorem 1.2.

As G is outerplanar, we may fix an embedding of G in $\mathbb{R}^2$. When speaking of faces below, we mean faces with respect to this embedding. Similarly for the outer face. We can assume that the vertices $1, ..., n$ occur in this order along the outer face.

We first show:

(9) if a and b are consecutive vertices along the outer face, then u_a and u_b are linearly independent.

Suppose not. Then there exists a nonzero vector $x \in \ker M$ such that $x_a = x_b = 0$. Take such an x with $\mathrm{supp}(x)$ minimal. Lemma 2.1 implies that both $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ are nonempty and induce connected subgraphs of G.

As G is 2-connected, G has vertex-disjoint paths P_1 and P_2, each starting in $N(\mathrm{supp}(x))$, and ending in a and b respectively. By (6), P_i starts in a vertex that is both in $N(\mathrm{supp}^+(x))$ and $N(\mathrm{supp}^-(x))$. However, since $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ induce connected subgraphs of G, this is topologically not possible, showing (9).

(9) implies that each u_i is nonzero, and hence v_i is defined. To show that $v_1, \ldots, v_k$ occur in this order along S^1, it suffices to show that (using the symmetry of $1, \ldots, k$):

(10) let l be the line through v_1 and 0. Then there is a k such that $v_2, \ldots, v_{k-1}$ all are at one side of l and $v_{k+1}, \ldots, v_n$ are at the other side of l.

Suppose this is not true. Let x be a nonzero vector $x \in \ker M$ with $x_1 = 0$. Then there are h, i, j with $2 \leq h < i < j \leq n$ such that $x_h > 0$, $x_i \leq 0$, and $x_j > 0$. Since $\mathrm{supp}^+(x)$ induces a connected subgraph of G, there is a path P connecting h and j and traversing vertices in $\mathrm{supp}^+(x)$ only. Then 1 and i are in different components of $G - P$. Both components contain vertices m with $x_m \leq 0$ (namely 1 and i), and hence vertices $m \in N(\mathrm{supp}^+(x))$ (using the 2-connectivity of G). Any such vertex belongs to $\mathrm{supp}^-(x)$ or is adjacent to a vertex in $\mathrm{supp}^-(x)$. So both regions intersect $\mathrm{supp}^-(x)$. This contradicts the fact that $\mathrm{supp}^-(x)$ induces a connected subgraph of G, which proves (10). □

2.4. Proof of Theorem 1.3.

As G is planar, we may fix an embedding of G in S^2. When speaking of faces below, we mean faces with respect to this embedding.

We first show:

(11) if a, b, c are distinct vertices on a face of G, then u_a, u_b and u_c are linearly independent.

Suppose not. Then there exists a nonzero vector $x \in \ker M$ such that $x_a = x_b = x_c = 0$. Take such an x with $\mathrm{supp}(x)$ minimal. Lemma 2.1 implies that both $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ are nonempty and induce connected subgraphs of G.

As G is 3-connected, G has pairwise vertex-disjoint paths P_1, P_2, P_3, each starting in $N(\mathrm{supp}(x))$ and ending in a, b and c respectively. By (6), the P_i start in a vertex that are both in $N(\mathrm{supp}^+(x))$ and $N(\mathrm{supp}^-(x))$. As $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ induce connected subgraphs of G, we can contract each of $\mathrm{supp}^+(x)$, $\mathrm{supp}^-(x)$, P_1, P_2, and P_3 to one vertex, so as to obtain a $K_{2,3}$ with the three vertices of degree 2 on one common face, which is not possible. This shows (11).

This implies that each u_i is nonzero, and hence v_i is defined. It also implies that v_i and v_j are linearly independent if i and j are adjacent. So for adjacent vertices i and j there exists a unique shortest geodesic on S^2 connecting v_i and v_j. This gives an extension of $i \to u_i$ to a mapping $\psi : G \to S^2$. We will show that this is in fact an embedding.

To this end we show:

(12) Let (say) $1, \ldots, k$ be the vertices around a face F of G, in this cyclic order. Then $u_1, \ldots, u_k$ are the extremal rays of the convex cone C they generate in $\mathbb{R}^3$, in this cyclic order.

Consider two consecutive vertices along F, say 1 and 2. Let H be the plane spanned by u_1, u_2 and 0. It suffices to prove that $u_3, \ldots, u_k$ are all at the same side of H. Let x be a nonzero vector in $\ker M$ satisfying $x_1 = x_2 = 0$ (x is unique up to scalar multiplication, as u_1 and u_2 are linearly independent.) By (11), $3, \ldots, k$ belong to $\mathrm{supp}(x)$, and hence none of $u_3, \ldots, u_k$ are on H. Suppose that they are not all at the same side of H. That is, both $\mathrm{supp}^+(x)$ and $\mathrm{supp}^-(x)$ intersect $\{3, \ldots, k\}$. As G is 3-connected, there exist vertex-disjoint paths P_1 and P_2 starting in $N(\mathrm{supp}(x))$ and ending in 1 and 2 respectively. Contracting each of $\mathrm{supp}^+(x)$, $\mathrm{supp}^-(x)$, P_1, and P_2 to one vertex, we obtain a K_4 embedded in S^2 with all four vertices on one face, which is not possible. This shows (12).

Next:

(13) Let $a \in V$, and let (say) $1, \ldots, k$ be the vertices adjacent to a, in this cyclic order. Then the geodesics on S^2 connecting v_a to $v_1, \ldots, v_k$ issue from v_a in this cyclic order.

Consider a nonzero vector $x \in \ker M$ with $x_a = x_1 = 0$. We claim

(14) there are no h, i, j with $2 \leq h < i < j \leq k$ such that $x_h > 0$, $x_i \leq 0$, and $x_j > 0$.

Suppose that such i, j and h do exist. Since $\mathrm{supp}^+(x)$ induces a connected subgraph of G, there is a path P connecting h and j and traversing vertices in $\mathrm{supp}^+(x)$ only. Together with a this gives a circuit C, dividing S^2 into two (open) regions. Both regions contain vertices m with $x_m \leq 0$ (namely 1 and i), and hence vertices $m \in N(\mathrm{supp}^+(x))$ (using the 3-connectivity of G). Any such vertex belongs to $\mathrm{supp}^-(x)$ or is adjacent to a vertex in $\mathrm{supp}^-(x)$. So both regions intersect $\mathrm{supp}^-(x)$. This contradicts the fact that $\mathrm{supp}^-(x)$ induces a connected subgraph of G. This proves (14).

This implies that v_2 and v_k are on different sides of the plane H through v_a, v_1, and 0. Otherwise we can assume that $x_2 > 0$ and $x_k > 0$. As $a \in N(\mathrm{supp}(x))$, we know that $a \in N(\mathrm{supp}^-(x))$. So $x_i < 0$ for some i with $2 < i < k$. This contradicts (14). Moreover, (14) implies that the nodes i with $x_i > 0$ occur contiguously, and also the i with $x_i < 0$ occur contiguously. This implies (13) (using the symmetry of the $1, \ldots, k$).

Now it is easy to complete the proof. (12) and (13) imply that we can extend the mapping $\psi : G \to S^2$ to a mapping $\psi : S^2 \to S^2$ such that ψ is locally one-to-one. We show that ψ is one-to-one. To see this, note that there is a number k such that $|\psi^{-1}(p)| = k$ for each $p \in S^2$. Now let H be any 3-connected planar graph embedded in S^2, with v vertices, e edges, and f faces. Then $\psi^{-1}(H)$ is a graph embedded in S^2, with kv vertices, ke edges, and kf faces. Hence by Euler's formula, $2k = kv - ke + kf = 2$, and so $k = 1$. Therefore, ψ is one-to-one, and ψ embeds $|G|$ in S^2. □

BIBLIOGRAPHY

[1] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, Journal of Combinatorial Theory, Series B, 50 (1990), 11–21 [English translation: On a new graph invariant and a criterion for planarity, in: Graph Structure Theory (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, 137–147].

[2] H. van der Holst, A short proof of the planarity characterization of Colin de Verdière, Journal of Combinatorial Theory, Series B, 65 (1995), 269–272.

[3] H. van der Holst, Topological and Spectral Graph Characterizations, Ph.D. Thesis, University of Amsterdam, Amsterdam, 1996.

[4] H. van der Holst, L. Lovász, A. Schrijver, On the invariance of Colin de Verdière's graph parameter under clique sums, Linear Algebra and its Applications, 226 (1995), 509–517.

[5] L. Lovász, A. Schrijver, A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs, Proceedings of the American Mathematical Society, 126 (1998), 1275–1285.

L. LOVÁSZ,
Department of Computer Science
Yale University
New Haven CT O6520 (USA).

A. SCHRIJVER,
CWI
Kruislaan 413
1098 SJ Amsterdam (The Netherlands).
&
University of Amsterdam
Department of Mathematics
Plantage Muidergracht 24
1018 TV Amsterdam (The Netherlands).