

# ANNALES DE L'INSTITUT FOURIER

JARED WUNSCH

## **The trace of the generalized harmonic oscillator**

*Annales de l'institut Fourier*, tome 49, n° 1 (1999), p. 351-373

[http://www.numdam.org/item?id=AIF\\_1999\\_\\_49\\_1\\_351\\_0](http://www.numdam.org/item?id=AIF_1999__49_1_351_0)

© Annales de l'institut Fourier, 1999, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## THE TRACE OF THE GENERALIZED HARMONIC OSCILLATOR

by Jared WUNSCH

---

### 1. Introduction.

Let  $M$  be a compact manifold with boundary endowed with a *scattering metric*  $g$  as defined by Melrose [9]. Thus in a neighborhood of  $\partial M$ , we can write

$$(1.1) \quad g = \frac{dx^2}{x^4} + \frac{h}{x^2}$$

where  $x$  is a boundary-defining function for  $\partial M$ , *i.e.* is smooth, nonnegative, and vanishes exactly at  $\partial M$  with  $dx \neq 0$  at  $\partial M$ , and where  $h \in C^\infty(\text{Sym}^2(T^*M))$  restricts to be a metric on  $\partial M$ . Scattering metrics form a class of complete, asymptotically flat metrics that includes asymptotically Euclidian metrics on  $\mathbb{R}^n$ , radially compactified to the  $n$ -ball; this class also includes metrics on  $\mathbb{R}^n$  that are not asymptotically Euclidian but that look like arbitrary, non-round metrics on the sphere at infinity (see [9] for details).

We consider a generalization of the quantum-mechanical harmonic oscillator on the manifold  $M$ : let  $x$  be a boundary-defining function for  $\partial M$  with respect to which  $g$  has the form (1.1), *e.g.*  $|z|^{-1}$  on flat  $\mathbb{R}^n$  (modified to be a smooth function at  $z = 0$ ). For any  $\omega \in \mathbb{R}_+$ , we consider the associated time-dependent Schrödinger equation

$$(1.2) \quad \left( D_t + \frac{1}{2} \Delta + \frac{\omega^2}{2x^2} + v \right) \psi = 0$$

---

*Keywords:* Schrödinger equation – Harmonic oscillator – Propagation of singularities – Trace theorem – Scattering metric.

*Math. classification:* 58G17 – 58G25 – 58G15 – 35Q40.

where  $v$  is a formally self-adjoint perturbation term that can include both magnetic and electric potential terms. We will take  $v$  to be an error term in a sense to be made precise later on; potentials of the form  $v \in C^\infty(M)$  are certainly allowed. Note that for such a  $v$ ,

$$\frac{1}{2}\Delta + \frac{\omega^2}{2x^2} + v$$

is semi-bounded, hence the Friedrichs extension gives a self-adjoint operator on  $L^2(M)$  (with respect to the metric  $dg$ ). Our class of operators thus includes compactly supported metric and potential perturbations of the standard harmonic oscillator on  $\mathbb{R}^n$ .

Perturbations of the free-particle Schrödinger equation on manifolds with scattering metrics were studied in [13] using a calculus of pseudodifferential operators on manifolds with boundary called the *quadratic-scattering* (or *qsc*) calculus and denoted  $\Psi_{\text{qsc}}(M)$ . This calculus is a microlocalization of the Lie algebra of “quadratic-scattering vector fields” on  $M$ , given by

$$(1.3) \quad \mathcal{V}_{\text{qsc}}(M) = x^2 \mathcal{V}_b(M)$$

where

$$(1.4) \quad \mathcal{V}_b(M) = \{\text{vector fields on } M \text{ tangent to } \partial M\}.$$

Near  $\partial M$ ,  $\mathcal{V}_{\text{qsc}}(M)$  is locally spanned over  $C^\infty(M)$  by vector fields of the form  $x^3 \partial_x, x^2 \partial_{y_j}$  where  $x, y_j$  are product-type coordinates on  $M$  near  $\partial M$ , *i.e.* the  $y_j$ 's are coordinates on  $\partial M$ . The Lie algebra  $\mathcal{V}_{\text{qsc}}(M)$  can be written as the space of sections of a vector bundle:

$$\mathcal{V}_{\text{qsc}}(M) = C^\infty(M; {}^{\text{qsc}}TM);$$

we call  ${}^{\text{qsc}}TM$  the *quadratic scattering tangent bundle* of  $M$ . Let  ${}^{\text{qsc}}T^*M$  be the dual bundle (the *quadratic scattering cotangent bundle*). Let  ${}^{\text{qsc}}\bar{T}^*M$  be the unit-ball bundle over  $M$  obtained by radially compactifying the fibers of  ${}^{\text{qsc}}T^*M$  (see [9] or [13]). This is a manifold with corners. The principal symbols of operators in the *qsc*-calculus are conormal distributions on  ${}^{\text{qsc}}\bar{T}^*M$  with respect to the boundary (a precise definition of such distributions will be given in § 2). There is an associated wavefront set,  $WF_{\text{qsc}}$ , which is a closed subset of  $\partial({}^{\text{qsc}}\bar{T}^*M)$ .

In [13], propagation of  $WF_{\text{qsc}}$  was described for perturbations of the free particle Schrödinger equation on  $M$ . In this paper, we discuss

the analogous results for the harmonic oscillator, referring to [13] for all technical details. We can conclude from the propagation results that if there are no trapped geodesics on  $\overset{\circ}{M}$ , then except at a certain set of times

$$(1.5) \quad S_\omega = \left\{ \frac{L}{\omega} : \text{there exists a closed geodesic in } \partial M \text{ of length } \pm L \right\} \\ \cup \left\{ \pm \frac{n\pi}{\omega} : \text{there exists a geodesic } n\text{-gon in } M \right. \\ \left. \text{with vertices in } \partial M \right\} \cup \{0\},$$

there is no recurrence of  $WF_{\text{qsc}}$  for solutions to (1.2). In the above definition of  $S_\omega$  we adopt the convention that the sides of a geodesic  $n$ -gon in  $M$  with vertices in  $\partial M$  are maximally extended geodesics in  $\overset{\circ}{M}$  (which automatically have infinite length) and geodesics in  $\partial M$  of length  $\pi$ ; the latter geodesics appear naturally as limits of geodesics through  $\overset{\circ}{M} - cf.$  Prop. 1 of [10]. Using very general properties of the qsc calculus, in § 5 we use the non-recurrence result to conclude that if  $U(t)$  is the solution operator for the Cauchy problem for (1.2) then

$$(1.6) \quad \text{sing supp Tr } U(t) \subset S_\omega.$$

For example, if we have a compactly-supported potential perturbation of the standard harmonic oscillator on  $\mathbb{R}^n$ ,  $S_\omega = 2\pi\mathbb{Z}$ : If  $M$  is the radial compactification of  $\mathbb{R}^n$ ,  $\partial M$  is the unit  $(n - 1)$ -sphere. Geodesics on  $\overset{\circ}{M}$  connect antipodal points on  $\partial M$  and geodesics in  $\partial M$  are great circles, hence consecutive vertices of a geodesic  $n$ -gon are antipodal points and there exist geodesic  $n$ -gons iff  $n$  is even; closed geodesics in  $\partial M$  also only occur with lengths in  $2\pi\mathbb{Z}$ . Hence for a potential perturbation of the harmonic oscillator on  $\mathbb{R}^n$ , the trace of the solution operator can only be singular at multiples of  $2\pi$ . One can deduce this easily from Mehler’s formula in the unperturbed case.

The trace theorem (1.6) closely resembles a result of Chazarain [1] and Duistermaat-Guillemin [6] which says that on a compact Riemannian manifold without boundary,

$$\text{sing supp Tr } e^{it\sqrt{\Delta}} \subset \{\pm \text{lengths of closed geodesics}\} \cup \{0\};$$

related results of Colin de Verdière using heat kernels can be found in [3] and [4]. Chazarain [2] has also proved a semi-classical trace theorem for the time-dependent Schrödinger equation, in which the lengths of closed bicharacteristics of the total symbol appear. By contrast, the trace theorem

of this paper is a non-semi-classical result, and over  $S^*\mathring{M}$ , the relevant bicharacteristic flow is that of the symbol  $\frac{1}{2}|\xi|^2$  rather than the full symbol as in [2]. Results on singularities of perturbations of the harmonic oscillator have been obtained by Zelditch [15], Weinstein [12], Fujiwara [7], Yajima [14], Kapitanski-Rodnianski-Yajima [8], and Treves [11]. Periodic recurrence of singularities for perturbations of the harmonic oscillator on  $\mathbb{R}^n$  was demonstrated by Zelditch [15] and Weinstein [12], and the trace theorem (1.6) was proven by Zelditch for perturbations of the harmonic oscillator in  $\mathbb{R}^n$  by potentials in  $\mathcal{B}(\mathbb{R}^n)$ .

The author is grateful to Richard Melrose, who supervised the Ph.D. thesis of which this work formed a part. The comments of an anonymous referee were also helpful, as was Hubert Goldschmidt’s help in reducing the level of illiteracy of the French abstract. The work was supported by a fellowship from the Fanny and John Hertz Foundation.

## 2. The quadratic-scattering calculus.

In this section, we briefly review the properties of the algebra  $\Psi_{\text{qsc}}(M)$ , which was constructed in [13], and is closely related to the “scattering algebra” of Melrose [9].

Let  $\mathcal{V}_{\text{qsc}}(M)$  and  $\mathcal{V}_b(M)$  be defined by (1.3) and (1.4), and let  $\text{Diff}_{\text{qsc}}(M)$  and  $\text{Diff}_b(M)$  be the order-filtered algebras of smooth linear combinations of products of elements of  $\mathcal{V}_{\text{qsc}}(M)$  and  $\mathcal{V}_b(M)$  respectively. There exists a bi-filtered star-algebra  $\Psi_{\text{qsc}}(M)$ , the “quadratic-scattering calculus” of pseudodifferential operators on  $M$  such that

- $\text{Diff}_{\text{qsc}}^m(M) \subset \Psi_{\text{qsc}}^{m,0}(M)$ .
- $\Psi_{\text{qsc}}^{m,\ell}(M) = x^\ell \Psi_{\text{qsc}}^{m,0}(M) = \Psi_{\text{qsc}}^{m,0}(M)x^\ell$ .
- $\Psi_{\text{qsc}}^{m,\ell}(M) \subset \Psi_{\text{qsc}}^{m',\ell'}(M)$  if  $m \leq m'$  and  $\ell' - m' \leq \ell - m$ .
- $\bigcap_{m,\ell} \Psi_{\text{qsc}}^{m,\ell}(M) \equiv \Psi_{\text{qsc}}^{-\infty,\infty}(M)$  consists of operators whose Schwartz kernels are smooth functions on  $M \times M$ , vanishing to infinite order at  $\partial(M \times M)$ .
- Elements of  $\Psi_{\text{qsc}}^{0,0}(M)$  are bounded operators on  $L^2(M)$ .
- Given a sequence  $A_j \in \Psi_{\text{qsc}}^{m-j,\ell+j}(M)$  for  $j = 0, 1, 2, \dots$ , there exists an “asymptotic sum”  $A \in \Psi_{\text{qsc}}^{m,\ell}(M)$ , uniquely determined modulo  $\Psi_{\text{qsc}}^{-\infty,\infty}(M)$ , such that  $A - \sum_0^{N-1} A_j \in \Psi_{\text{qsc}}^{m-N,\ell+N}(M)$ .

Let  $C_{\text{qsc}}M = \partial(\text{qsc}\bar{T}^*M)$ . Let  $\sigma$  be a boundary defining function for the boundary face  $\text{qsc}S^*M$  of  $\text{qsc}\bar{T}^*M$  created by the fiber compactification. Let  $x$  be the lift of a boundary defining function on  $M$  to  $\text{qsc}\bar{T}^*M$  – thus  $x$  defines the boundary face  $\text{qsc}\bar{T}^*_{\partial M}M$ . Let  $\dot{C}^\infty(M)$  denote smooth functions on  $M$  vanishing to infinite order at  $\partial M$  and  $C^{-\infty}(M)$  the dual space to  $\dot{C}^\infty(M)$ -valued densities. Following Melrose [9], we define conormal distributions on  $\text{qsc}\bar{T}^*M$  with respect to  $C_{\text{qsc}}M$  as follows:

$\mathcal{A}^{p,q}(\text{qsc}\bar{T}^*M)$   
 $= \{u \in C^{-\infty}(\text{qsc}\bar{T}^*M) : \text{Diff}_b^k(\text{qsc}\bar{T}^*M)u \subset \sigma^p x^q L^\infty(\text{qsc}\bar{T}^*M) \text{ for all } k\};$   
 here  $\text{Diff}_b^k$  is defined on the manifold with corners  $\text{qsc}\bar{T}^*$  exactly as it was defined on manifolds with boundary: as the span of products of vector fields tangent to (all faces of) the boundary. Let

$$\mathcal{A}^{[m,\ell]}(C_{\text{qsc}}M) = \mathcal{A}^{m,\ell}(\text{qsc}\bar{T}^*M) / \mathcal{A}^{m-1,\ell+2}(\text{qsc}\bar{T}^*M).$$

There exists a symbol map

$$j_{\text{qsc},m,\ell} : \Psi_{\text{qsc}}^{m,\ell}(M) \rightarrow \mathcal{A}^{[-m,\ell-m]}(C_{\text{qsc}}M)$$

such that

- There is a short exact sequence

$$(2.1) \quad 0 \rightarrow \Psi_{\text{qsc}}^{m-1,\ell+1}(M) \longrightarrow \Psi_{\text{qsc}}^{m,\ell}(M) \xrightarrow{j_{\text{qsc},m,\ell}} \mathcal{A}^{[-m,\ell-m]}(C_{\text{qsc}}M) \rightarrow 0.$$

- The symbol map is multiplicative.
- The Poisson bracket extends continuously from the usual bracket defined on the interior of  $\text{qsc}\bar{T}^*M$  to  $\mathcal{A}^{[\cdot,\cdot]}$ , and

$$j_{\text{qsc},m_1+m_2-1,\ell_1+\ell_2}([P, Q]) = \frac{1}{i} \{j_{\text{qsc},m_1,\ell_1}(P), j_{\text{qsc},m_2,\ell_2}(Q)\}.$$

Furthermore, if  $a \in \mathcal{A}^{m,\ell}(\text{qsc}\bar{T}^*M)$ ,  $\{a, b\} = H_a(b)$  where  $H_a$  is the extension of the usual Hamilton vector field on the interior of  $\text{qsc}\bar{T}^*M$  to an element of  $\sigma^{-m+1}x^{\ell+2}\mathcal{V}_b(\text{qsc}\bar{T}^*M)$ . (We refer to the flow along  $H_a$  or  $\sigma^{m-1}x^{-\ell-2}H_a$  as bicharacteristic flow.)

- There exists a (non-unique) “quantization map”

$$\text{Op} : \mathcal{A}^{-m,\ell-m}(\text{qsc}\bar{T}^*M) \longrightarrow \Psi_{\text{qsc}}^{m,\ell}(M)$$

such that

$$j_{\text{qsc},m,\ell}(\text{Op}(a)) = [a] \in \mathcal{A}^{[-m,\ell-m]}(C_{\text{qsc}}M).$$

DEFINITION 2.1. — An operator  $P \in \Psi_{\text{qsc}}^{m,\ell}(M)$  is said to be elliptic at a point  $p \in C_{\text{qsc}}M$  if  $j_{\text{qsc},m,\ell}$  is locally invertible near  $p$ . The set of points at which  $P$  is elliptic is denoted  $\text{ell } P$ . If  $P$  is elliptic everywhere, it is simply said to be elliptic.

DEFINITION 2.2. — Let  $P \in \Psi_{\text{qsc}}^{m,\ell}(M)$ . A point  $p \in C_{\text{qsc}}M$  is in the complement of  $WF'_{\text{qsc}}P$  (the operator wavefront set or microsupport of  $P$ ) if there exists  $Q \in \Psi_{\text{qsc}}^{-m,-\ell}(M)$  such that  $Q$  is elliptic at  $p$  and  $PQ \in \Psi_{\text{qsc}}^{-\infty,\infty}(M)$ .

We can now define the *qsc wavefront set* of  $u \in C^{-\infty}(M)$  as the subset  $WF_{\text{qsc}}u$  of  $C_{\text{qsc}}M$  such that  $p \notin WF_{\text{qsc}}u$  if and only if there exists  $A \in \Psi_{\text{qsc}}^{0,0}(M)$  with  $p \in \text{ell } A$  such that  $Au \in \dot{C}^\infty(M)$ .

The qsc wavefront set and microsupport enjoy the following properties:

- If  $A, B \in \Psi_{\text{qsc}}(M)$ , then  $WF'_{\text{qsc}}AB \subset WF'_{\text{qsc}}A \cap WF'_{\text{qsc}}B$  and  $WF'_{\text{qsc}}A^* = WF'_{\text{qsc}}A$ .
- Microlocal parametrices exist at elliptic points: if  $P \in \Psi_{\text{qsc}}^{m,\ell}(M)$  is elliptic at  $p \in C_{\text{qsc}}M$  then there exists  $Q \in \Psi_{\text{qsc}}^{-m,-\ell}(M)$  such that

$$p \notin WF'_{\text{qsc}}(PQ - I) \quad \text{and} \quad p \notin WF'_{\text{qsc}}(QP - I).$$

- Microlocality: let  $P \in \Psi_{\text{qsc}}(M)$  and  $u \in C^{-\infty}(M)$ . Then

$$WF_{\text{qsc}}Pu \subset WF'_{\text{qsc}}P \cap WF_{\text{qsc}}u.$$

- Microlocal elliptic regularity: Let  $P \in \Psi_{\text{qsc}}(M)$  and  $u \in C^{-\infty}(M)$ . Then

$$WF_{\text{qsc}}(u) \subset WF_{\text{qsc}}(Pu) \cup (\text{ell } P)^c.$$

- We can (and do) choose the map  $\text{Op}$  in such a way that

$$WF'_{\text{qsc}} \text{Op}(a) \subset \text{ess supp } a$$

( $\text{ess supp } a$  is the set of points in  $C_{\text{qsc}}M$  near which  $a$  does not vanish to infinite order).

We will also require a notion of qsc wavefront set that is uniform in a parameter.

DEFINITION 2.3. — Let  $u \in \mathcal{C}(\mathbb{R}; \mathcal{C}^{-\infty}(M))$ . For  $S \subset \mathbb{R}$  compact, we say that  $p \notin WF_{\text{qsc}}^S(u)$  if there exists a smooth family  $A(t) \in \Psi_{\text{qsc}}^{0,0}(M)$  such that  $A(t)$  is elliptic at  $p$  for all  $t \in S$  and  $Au \in \mathcal{C}(S; \dot{\mathcal{C}}^\infty(M))$ .

Associated to  $\Psi_{\text{qsc}}(M)$  is a family of Sobolev spaces

$$H_{\text{qsc}}^{m,\ell}(M) = \{u \in \mathcal{C}^{-\infty}(M) : \Psi_{\text{qsc}}^{m,-\ell}(M)u \subset L^2(M)\}$$

such that

- If  $A \in \Psi_{\text{qsc}}^{m',\ell'}(M)$  then

$$A : H_{\text{qsc}}^{m,\ell}(M) \longrightarrow H_{\text{qsc}}^{m-m',\ell+\ell'}(M)$$

is continuous for any  $m, \ell$ .

- For any  $\ell \in \mathbb{R}$ ,

$$\bigcap_m H_{\text{qsc}}^{m,\ell}(M) = \dot{\mathcal{C}}^\infty(M) \quad \text{and} \quad \bigcup_m H_{\text{qsc}}^{m,\ell}(M) = \mathcal{C}^{-\infty}(M).$$

- If  $a_n$  is a bounded sequence in  $\mathcal{A}^{-m,\ell-m}(M)$  and  $a_n \rightarrow a$  in some  $\mathcal{A}^{p,q}(M)$ , then  $\text{Op}(a_n) \rightarrow \text{Op}(a)$  in the strong operator topology on

$$\mathcal{B}(H_{\text{qsc}}^{M,L}(M), H_{\text{qsc}}^{M-m,M+\ell}(M))$$

for all  $M, L$ .

### 3. The propagation of $WF_{\text{qsc}}$ .

For details of all computations in this section, see [13], especially §11.

We consider the symbol and corresponding bicharacteristic flow for the operator

$$\mathcal{H} = \frac{1}{2} \Delta + \frac{\omega^2}{2x^2} + v$$

where

$$v \in \text{Diff}_{\text{qsc}}^{1,1}(M)$$

is formally self-adjoint and  $x$  is a boundary-defining function with respect to which  $g$  takes the form (1.1).

Let the canonical one-form on  ${}^{\text{qsc}}T^*M$  be

$$\lambda \frac{dx}{x^3} + \mu \frac{dy}{x^2}.$$

The joint symbol of  $\mathcal{H}$  is represented in  $\mathcal{A}^{[-2,-2]}(C_{\text{qsc}}M)$  by a conormal distribution of the form

$$(3.2) \quad j_{\text{qsc},2,0}(\mathcal{H}) = \frac{1}{2x^2} (\lambda^2 + |\mu|^2 + \omega^2 + xr(\lambda, \mu));$$

$$r(\lambda, \mu) \in \lambda^2 x C^\infty(x, y) + \lambda \mu C^\infty(x, y) + \mu^2 C^\infty(x, y)$$

where  $|\mu|$  denotes the norm of  $\mu$  with respect to the metric  $\bar{h} = h|_{\partial M}$ . Note that (3.1) shows that  $\mathcal{H}$  is an elliptic element of  $\Psi_{\text{qsc}}^{2,0}(M)$ ; the perturbation  $v$  does not enter into the expression (3.1) as it has lower order than  $\frac{1}{2}\Delta + \frac{\omega^2}{2x^2}$  in both indices. The Hamilton vector field of  $\mathcal{H}$  is

$$X = \tilde{X} + P$$

where

$$(3.2) \quad \tilde{X} = \lambda x \partial_x + (\lambda^2 - |\mu|^2 + \omega^2) \partial_\lambda + \langle \mu, \partial_y \rangle + 2\lambda \mu \cdot \partial_\mu - \frac{1}{2} \partial_y |\mu|^2 \cdot \partial_\mu$$

is the Hamilton vector field for the symbol  $\frac{1}{2x^2} (\lambda^2 + |\mu|^2 + \omega^2)$ , and

$$(3.3) \quad P = p_1 x^2 \partial_x + p_2 x \partial_y + q_1 x \partial_\lambda + q_2 x \partial_\mu$$

is the Hamilton vector field for the “error term”  $\frac{1}{2} x^{-1} r(\lambda, \mu)$ . Here we adopt the convention that

$$\langle a, b \rangle = \sum a_i b_j \bar{h}^{ij}(y) \quad \text{and} \quad a \cdot b = \sum a_i b_i.$$

The vector field  $P$  is identically zero if  $h$  is a function of  $y$  only, and always vanishes at  $x = 0$ .

Under the flow along  $\tilde{X}$ ,

$$\frac{d}{dt} (\lambda + i|\mu|) = (\lambda + i|\mu|)^2 + \omega^2,$$

hence

$$(3.4) \quad \lambda + i|\mu| = \omega \frac{\sin \omega(t - t_0) + iR \cos \omega(t - t_0)}{\cos \omega(t - t_0) - iR \sin \omega(t - t_0)}$$

for some  $R \in [0, 1]$ . For  $R > 0$ , this gives a periodic orbit with period  $\pi/\omega$ . On  $\{\mu \neq 0\}$  (i.e.  $R > 0$ ), we set  $\hat{\mu} = \mu/|\mu|$ , and introduce the rescaled time parameter  $s = \int |\mu| dt$  to rewrite the flow along  $\tilde{X}$  as

$$(3.5) \quad \frac{dy_i}{ds} = \bar{h}^{ij} \hat{\mu}_j, \quad \frac{d\hat{\mu}_i}{ds} = -\frac{1}{2} \hat{\mu}_j \hat{\mu}_k \partial_{y_i} \bar{h}^{jk},$$

$$(3.6) \quad \frac{d\lambda}{ds} = \frac{\lambda^2 - |\mu|^2}{|\mu|} + \omega^2, \quad \frac{d|\mu|}{ds} = 2\lambda,$$

$$(3.7) \quad \frac{dx}{ds} = \frac{\lambda x}{|\mu|}.$$

As the set  $\mu = 0$  plays an important role in the geometry of  $\tilde{X}$ , we give it a name:

DEFINITION 3.1. — Let  $\mathcal{N} \subset {}^{\text{qsc}}\bar{T}^*M$  be the set given in our coordinates by  $\{x = \mu = 0\}$ . Let  $\mathcal{N}_{\pm} \subset \mathcal{N}$  be the subsets on which  $\pm\lambda \geq 0$ . Let  $\mathcal{N}_{\pm}^c = \mathcal{N}_{\pm} \cap {}^{\text{qsc}}S^*M$  (i.e.  $\mathcal{N}^c$  is the intersection of  $\mathcal{N}$  with the corner). We refer to  $\mathcal{N}$  as the “normal set,” with  $\mathcal{N}_+$  being the “incoming normal set” and  $\mathcal{N}_-$  the “outgoing normal set.”

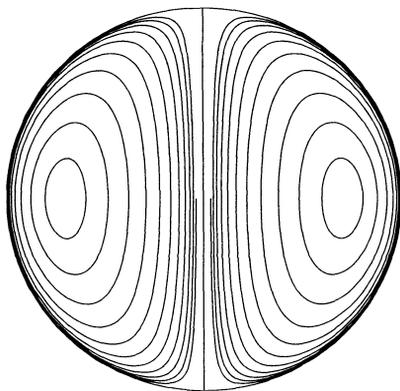


Figure 1. Integral curves of  $\tilde{X}$ , projected onto the  $(\lambda, \mu)$ -plane and radially compactified. The vertical line is the solution  $\mu = 0$ .

While  $(\lambda, |\mu|)$  are undergoing a flow described by (3.4) (see Fig. 1), then provided  $R \neq 0$ , (3.5) shows that  $(y, \hat{\mu})$  are undergoing *unit speed geodesic flow* in  $\partial M$  with rescaled time parameter  $s$ . For  $R = 0$ ,  $\mu$  is

identically zero,  $y$  is constant, and  $\lambda$  blows up at  $t - t_0 = \pm\pi/2\omega$ , *i.e.* the flow crosses  $\mathcal{N}$  from  $\mathcal{N}_-^c$  to  $\mathcal{N}_+^c$  in time  $\pi/\omega$ . More generally the integral curve starting at  $\mu = 0$ ,  $\lambda = \lambda_0$ , reaches the corner at time  $t = \omega^{-1} \arctan(\omega/\lambda_0)$ .

Note that all terms in  $X$  are homogeneous of degree 1 in  $(\lambda, \mu)$  except the term  $\omega^2\partial_\lambda$ , which is homogeneous of degree  $-1$ . If we let  $\sigma$  be the defining function for  ${}^{\text{qsc}}S^*M$  in  ${}^{\text{qsc}}\bar{T}^*M$  given by

$$\sigma = (\lambda^2 + |\mu|^2)^{-\frac{1}{2}}$$

and set

$$\bar{\lambda} = \sigma\lambda, \quad \bar{\mu} = \sigma\mu$$

then the vector field  $\sigma X$  is tangent to the boundary of  ${}^{\text{qsc}}\bar{T}^*M$ , and we have

$$(3.8) \quad \begin{aligned} \sigma X &= \bar{\lambda}x\partial_x - |\bar{\mu}|^2\partial_{\bar{\lambda}} + \langle \bar{\mu}, \partial_y \rangle \\ &\quad + \left(\bar{\lambda}\bar{\mu} - \frac{1}{2}\partial_y|\bar{\mu}|^2\right)\partial_{\bar{\mu}} - \bar{\lambda}\sigma\partial_\sigma + O(\sigma^2) + O(x) \end{aligned}$$

where  $O(\sigma^2)$  and  $O(x)$  denote error terms of the form  $\sigma^2 Y_1$  and  $xY_2$ , with  $Y_i$  tangent to  $\partial({}^{\text{qsc}}\bar{T}^*M)$ ; the  $O(\sigma^2)$  term is just  $\sigma\omega^2\partial_\lambda$ , while the  $O(x)$  term is what has above been denoted  $P$ .

The vector field  $X$  differs from the free-particle Hamilton vector-field  $X_{\text{fp}}$  described in [13]<sup>(1)</sup> only in the term  $\omega^2\partial_\lambda$ , hence since this term is  $O(\sigma)$ , we have

$$(3.9) \quad \sigma X|_{{}^{\text{qsc}}S^*M} = \sigma X_{\text{fp}}|_{{}^{\text{qsc}}S^*M}.$$

DEFINITION 3.2. — *A maximally extended integral curve of  $\sigma X$  on  ${}^{\text{qsc}}S^*M$  is said to be non-trapped forward/backward if*

$$\lim_{t \rightarrow \pm\infty} x(t) = 0.$$

*A point in  ${}^{\text{qsc}}S^*M \setminus \mathcal{N}^c$  is said to be non-trapped forward/backward if the integral curve through it is non-trapped. A point in  $\mathcal{N}^c$  is said to be non-trapped forward/backward if it is not in the closure of any*

---

<sup>(1)</sup> Unfortunately, this vector field is called  $X$  as well in [13].

forward-/backward-trapped integral curves. Let  $\mathcal{T}_\pm$  denote the set of forward-/backward-trapped points in  ${}^{\text{qsc}}S^*M$ .

The only zeros of  $\sigma X$  on  ${}^{\text{qsc}}S^*M$  are on the manifolds  $\mathcal{N}_-^c$  (attracting) and  $\mathcal{N}_+^c$  (repelling), so we can define

$$N_{\pm\infty} : {}^{\text{qsc}}S^*M \setminus (\mathcal{N}_\pm \cup \mathcal{T}_\pm) \longrightarrow \mathcal{N}_\mp^c$$

by

$$p \longmapsto \lim_{t \rightarrow \pm\infty} \exp(t\sigma X)[p].$$

We extend this definition of  $N_{\pm\infty}$  to  ${}^{\text{qsc}}\bar{T}^*M \setminus (\mathcal{N}_\pm \cup \mathcal{T}_\pm)$  by homogeneity. We further define

$$Y_{\pm\infty} : {}^{\text{qsc}}\bar{T}^*M \setminus (\mathcal{N}_\pm \cup \mathcal{T}_\pm) \longrightarrow \partial M$$

to be the projection of  $N_{\pm\infty}$  to  $\partial M$ .

**THEOREM 3.3.** —  $N_{\pm\infty}$  and  $Y_{\pm\infty}$  are smooth maps. If we let  $C_\pm^\epsilon$  be the submanifold of  ${}^{\text{qsc}}S^*M$  given by

$$C_\pm^\epsilon = \{x^2 + |\bar{\mu}|^2 = \epsilon, \bar{\lambda} \geq 0\}$$

then for  $\epsilon$  sufficiently small,  $C_\mp^\epsilon$  is a fibration over  $\partial M$  with projection map  $Y_{\pm\infty}$ , and every integral curve of  $\sigma X$  which is not trapped forward/backward passes through  $C_\mp^\epsilon$ . The sets  $\mathcal{T}_\pm \setminus \mathcal{N}_\pm^c$  are closed subsets of  ${}^{\text{qsc}}S^*M \setminus \mathcal{N}_\pm^c$ .

By (3.9), this theorem follows from Theorem 11.6 of [13].

We can thus define the scattering relation:

**DEFINITION 3.4.** — Let  $\mathcal{S} \subset \mathcal{N}_-^c \setminus \mathcal{T}_-$ . The scattering relation on  $\mathcal{S}$  is

$$\text{Scat}(\mathcal{S}) = N_{-\infty} (N_{+\infty}^{-1}(\mathcal{S})) \subset \mathcal{N}_+^c.$$

It is shown in [13] that  $\text{Scat}$  takes closed sets to closed sets and  $\text{Scat}^{-1}$  takes open sets to open sets.

*Example 3.5.* — If  $M$  is the radial compactification of  $\mathbb{R}^n$  with an asymptotically Euclidian metric, we can identify the manifolds  $\mathcal{N}_\pm^c$  with  $S^{n-1} = \partial M$ . Then for  $\theta \in S^{n-1}$ ,  $\text{Scat } \theta$  consists of all  $\theta' \in S^{n-1}$  such that there exists a geodesic  $\gamma$  in (uncompactified)  $\mathbb{R}^n$  with  $\lim_{t \rightarrow -\infty} \gamma'(t) = -\theta'$  and  $\lim_{t \rightarrow +\infty} \gamma'(t) = \theta$ . In other words,  $\text{Scat}$  consists of all directions in  $\mathbb{R}^n$  that can scatter to the direction  $\theta$ . In the Euclidian case,  $\text{Scat}$  is the antipodal map on  $S^{n-1}$ .

We now state theorems on propagation of  $WF_{\text{qsc}}$  that will suffice to obtain results on  $\text{sing supp Tr } U(t)$ . (Slightly more sophisticated theorems, corresponding to Theorems 12.1–12.5 of [13], in fact hold here as well.)

**THEOREM 3.6** (propagation over the boundary). — *Let  $p$  in  $({}^{\text{qsc}}\overline{T}_{\partial M}^* M)^\circ$  and assume*

$$\exp(TX)[p] \in ({}^{\text{qsc}}\overline{T}_{\partial M}^* M)^\circ.$$

*Then  $p \notin WF_{\text{qsc}}\psi(0)$  if and only if there exists  $\delta > 0$  such that  $\exp(TX)[p] \notin WF_{\text{qsc}}^{[T-\delta, T+\delta]}\psi$ .*

**THEOREM 3.7** (propagation into the interior). — *Let  $p \in {}^{\text{qsc}}S^* M \setminus \mathcal{N}_-^c$  be non-backward-trapped and let  $T \in (0, \pi/\omega)$ . If*

$$\exp(-TX)[N_{-\infty}(p)] \notin WF_{\text{qsc}}\psi(0)$$

*then there exists  $\delta > 0$  such that  $p \notin WF_{\text{qsc}}^{[T-\delta, T+\delta]}\psi$ .*

**THEOREM 3.8** (scattering across the interior). — *Let  $q \in \mathcal{N}_-^c$  be non-backward-trapped. If*

$$\exp(-T_0 X)[\text{Scat}(q)] \cap WF_{\text{qsc}}\psi(0) = \emptyset$$

*for some  $T_0 \in (0, \pi/\omega)$ , then for every  $T \in (T_0, T_0 + \pi/\omega)$ , there exists  $\delta > 0$  such that  $\exp((T - T_0)X)[q] \notin WF_{\text{qsc}}^{[T-\delta, T+\delta]}\psi$ .*

**THEOREM 3.9** (global propagation into the boundary). — *Let  $q \in \mathcal{N}_-^c$  be non-backward-trapped. If*

$$\overline{N_{+\infty}^{-1}(q)} \cap WF_{\text{qsc}}\psi(0) = \emptyset$$

(closure taken in  ${}^{\text{qsc}}S^*M$ ), then for  $T \in (0, \pi/\omega)$ , there exists  $\delta > 0$  such that

$$\exp(TX)[q] \notin WF_{\text{qsc}}^{[T-\delta, T+\delta]} \psi.$$

The proofs are by the same positive-commutator arguments used in [13] (which were in turn adapted from Craig-Kappeler-Strauss [5]), although the symbol constructions need to be slightly modified from those in [13] because the maps  $Y_{\pm\infty}$  are not exactly constant along the flow of  $X$ ; we discuss these issues in an appendix.

### 4. Non-recurrence of singularities.

We assume throughout this section that there are no trapped geodesics in  $\overset{\circ}{M}$ .

This section is devoted to proving

**THEOREM 4.1.** — *Let  $S_\omega$  be defined by (1.5). For  $T \notin S_\omega$  and for any  $p \in C_{\text{qsc}}M$ , there exists an open neighborhood  $\mathcal{O}$  of  $p$  and  $\epsilon > 0$  such that if  $WF_{\text{qsc}}\psi(0) \subset \mathcal{O}$  then  $WF_{\text{qsc}}^{[T-\epsilon, T+\epsilon]}\psi \cap \mathcal{O} = \emptyset$ .*

In order to deduce this theorem from Theorems 3.6–3.9, we first define a relation on  $C_{\text{qsc}}M$  which describes from what points singularities may reach a point  $p \in C_{\text{qsc}}M$ :

**DEFINITION 4.2.** — *Let  $p, q \in C_{\text{qsc}}M$ . We write  $p \overset{t}{\sim} q$  if there exists a continuous path  $\gamma$  from  $p$  to  $q$  in  $C_{\text{qsc}}M$  that is a concatenation of maximally extended integral curves of  $\sigma X$  such that*

$$(4.1) \quad \sum (\text{lengths of integral curves in } {}^{\text{qsc}}\bar{T}_{\partial M}^*M) = t,$$

where we define the length of an integral curve in  ${}^{\text{qsc}}\bar{T}_{\partial M}^*M$  to be its length as an integral curve of  $X$  (and hence a finite number).

Then for  $S \subset C_{\text{qsc}}M$ , let

$$\mathcal{G}_t(S) = \{p \in C_{\text{qsc}}M : p \overset{t}{\sim} q \text{ for some } q \in S\}.$$

If  $p \overset{s}{\sim} q$  and  $q \overset{t}{\sim} r$ , then  $p \overset{s+t}{\sim} r$ , hence

$$(4.2) \quad \mathcal{G}_{s+t}(S) = \mathcal{G}_s \circ \mathcal{G}_t(S).$$

We also have

$$(4.3) \quad \mathcal{G}_t(S \cup T) = \mathcal{G}_t(S) \cup \mathcal{G}_t(T).$$

The relation  $p \overset{t}{\sim} q$  is closed in the following sense:

LEMMA 4.3. — *Let  $R \subset C_{\text{qsc}}M \times C_{\text{qsc}}M \times \mathbb{R}$  be defined by*

$$(p, q, t) \in R \quad \text{iff} \quad p \overset{t}{\sim} q.$$

*Then  $R$  is a closed subset of  $C_{\text{qsc}}M \times C_{\text{qsc}}M \times \mathbb{R}$ .*

*Proof.* — Suppose  $p_i \rightarrow p$ ,  $q_i \rightarrow q$ , and  $t_i \rightarrow t$  as  $i \rightarrow \infty$ , and that  $(p_i, q_i, t_i) \in R$ . We will show that  $(p, q, t) \in R$ .

For simplicity, we reformulate (4.1) as follows: let  $k$  be a Riemannian metric on the manifold  $({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ$  such that the norm of  $X$  with respect to  $k$  is one. (As  $X = O(\sigma^{-1})$ ,  $k$  vanishes at  ${}^{\text{qsc}}S_{\partial M}^*M$ .) Let  $\theta = k(\cdot, X) \in \Omega^1({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ$ ; extend  $\theta$  to be zero on the interior of the boundary face  ${}^{\text{qsc}}S^*M$ . Then the condition (4.1) is equivalent to

$$(4.4) \quad \int_{\gamma} \theta = t.$$

Now by hypothesis there exists a sequence  $\gamma_i$  of paths as in Definition 4.2 such that  $\gamma_i(0) = p_i$ ,  $\gamma_i(1) = q_i$ , and  $\int_{\gamma_i} \theta = t_i$  for all  $i$ . As the  $\gamma_i$  are all integral curves of  $\sigma X$ , we apply Ascoli-Arzelà to obtain a path  $\gamma$  between  $p$  and  $q$ , made up of integral curves of  $\sigma X$  with  $\int_{\gamma} \theta = t$ .  $\square$

DEFINITION 4.4. — *Let*

$$\mathcal{G}_t^{-1}S = \{p : \mathcal{G}_t(p) \subset S\}.$$

We now prove that  $\mathcal{G}_t$  is, in an appropriate sense, a continuous set map.

LEMMA 4.5. — *If  $K \subset \mathbb{R}$  is compact then*

$$\bigcup_{t \in K} \mathcal{G}_t$$

*takes closed sets to closed sets, and*

$$\bigcap_{t \in K} \mathcal{G}_t^{-1}$$

*takes open sets to open sets.*

*Proof.* — Let  $\pi_L$  and  $\pi_R$  denote the projections of  $C_{\text{qsc}}M \times C_{\text{qsc}}M \times \mathbb{R}$  onto the “left” and “right” factors of  $C_{\text{qsc}}$  and let  $\pi_t$  denote projection to  $\mathbb{R}$ . Then we can write

$$\bigcup_{t \in K} \mathcal{G}_t(S) = \pi_L(\pi_R^{-1}S \cap \pi_t^{-1}K \cap R)$$

and

$$\bigcap_{t \in K} \mathcal{G}_t^{-1}(S) = [\pi_R(\pi_L^{-1}(S^c) \cap \pi_t^{-1}K \cap R)]^c$$

hence the result follows from Lemma 4.3. □

Theorems 3.6–3.9 can now be conveniently recast as

MAIN PROPAGATION THEOREM. — *If  $S \subset C_{\text{qsc}}M$  and*

$$\mathcal{G}_t(S) \cap WF_{\text{qsc}}\psi(0) = \emptyset$$

*then there exists  $\epsilon > 0$  such that*

$$S \cap WF_{\text{qsc}}^{[T-\epsilon, T+\epsilon]}\psi = \emptyset.$$

*Proof.* — By (4.3), it suffices to prove the result for  $S = \{p\}$ , a single point in  $C_{\text{qsc}}M$ . By (4.2), it suffices to prove the result for small  $t$ ; we take  $t < \pi/\omega$  for simplicity. If

$$p \in ({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ \setminus \mathcal{N},$$

then for any  $t$ , as discussed in §3,  $\mathcal{G}_t(p)$  is a single point in  $({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ$ , and the result follows from Theorem 3.6.

Let  $\arctan_+$  denote the branch of  $\arctan$  taking values in  $[0, \pi)$ . If  $p \in \mathcal{N}^\circ$ , then for  $t \in (0, \omega^{-1}\arctan_+(\lambda(p)/\omega))$ ,  $\mathcal{G}_t(p)$  is again a point in  $\mathcal{N}^\circ$ , and again the theorem follows from Theorem 3.6. At  $t = \omega^{-1}\arctan_+(\lambda(p)/\omega)$ ,  $\exp(-tX)[p] \in \mathcal{N}_+^c$ , and

$$\mathcal{G}_t(p) = \overline{N_{+\infty}^{-1}(\exp(-tX)[p])} \subset {}^{\text{qsc}}S^*M,$$

hence Theorem 3.9 takes care of this case. For

$$\omega^{-1}\arctan_+(\lambda(p)/\omega) < t < \pi/\omega,$$

we once again have  $\mathcal{G}_t(p) \subset ({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ$ , and Theorem 3.8 finishes the proof.

If, on the other hand,  $p \in {}^{\text{qsc}}S^*M$ ,  $\mathcal{G}_t(p) \subset ({}^{\text{qsc}}\bar{T}^*M)^\circ$  for  $t \in (0, \pi/\omega)$ :  $\mathcal{G}_t(p)$  is a single point if  $p \notin \mathcal{N}_+^c$ , or a whole set, given by the scattering relation, if  $p \in \mathcal{N}_+^c$ . The theorem then follows from Theorem 3.7 in the former case, and Theorem 3.8 in the latter. □

The relation  $\mathcal{G}_t$  is non-recurrent except at certain times:

LEMMA 4.6. — *For any  $T \notin S_\omega$  and  $p \in C_{\text{qsc}}M$ , there exists an open neighborhood  $\mathcal{O}$  of  $p$  and  $\epsilon > 0$  such that*

$$\mathcal{G}_t(\mathcal{O}) \cap \mathcal{O} = \emptyset \quad \text{for all } t \in [T - \epsilon, T + \epsilon].$$

*Proof.* — By compactness of  $\partial M$ ,  $S_\omega$  is closed. Hence if  $T \notin S_\omega$ , there exists  $\epsilon > 0$  such that

$$K = [T - \epsilon, T + \epsilon] \subset \mathbb{R} \setminus S_\omega.$$

By Lemma 4.5,  $\bigcup_{t \in K} \mathcal{G}_t(p)$  is closed. If this set does not contain  $p$  then we can choose an open set  $\mathcal{U}$  containing  $\bigcup_{t \in K} \mathcal{G}_t(p)$  but such that  $p \notin \bar{\mathcal{U}}$ .

By Lemma 4.5, we can then set

$$\mathcal{O} = \bigcap_{t \in K} \mathcal{G}_t^{-1}(\mathcal{U}) \setminus \bar{\mathcal{U}}.$$

Thus it will suffice to prove that for  $t \notin S_\omega$ ,  $p \notin \mathcal{G}_t(p)$ .

First we take the case  $p \in {}^{\text{qsc}}S^*M \setminus \mathcal{N}_\pm^c$ . Then for  $t \in (0, \pi/\omega)$ ,

$$\mathcal{G}_t(p) = \exp(-tX)[N_{-\infty}(p)] \subset ({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ,$$

and this set certainly doesn't contain  $p$ . Let  $\mathcal{I}$  be the involution of  $\mathcal{N}^c$  swapping  $\mathcal{N}_+^c$  and  $\mathcal{N}_-^c$ . Then

$$\mathcal{G}_{\pi/\omega}(p) = \overline{N_{+\infty}^{-1} \circ \mathcal{I} \circ N_{-\infty}(p)},$$

and this set doesn't contain  $p$  unless  $Y_{+\infty}(p) = Y_{-\infty}(p)$ , *i.e.* unless  $p$  lies on a geodesic 1-gon with vertex in  $\partial M$ . For  $t \in (\pi/\omega, 2\pi/\omega)$ ,

$$\mathcal{G}_t(p) = \exp(-(t - \pi/\omega)X) [\text{Scat} \circ \mathcal{I} \circ N_{-\infty}(p)],$$

again a subset of  $({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ$ . The set

$$\mathcal{G}_{2\pi/\omega}(p) = \overline{N_{+\infty}^{-1} \circ \mathcal{I} \circ \text{Scat} \circ \mathcal{I} \circ N_{-\infty}(p)},$$

and this set certainly does contain  $p$ . Continuing in this manner, we find that if  $t = n\pi/\omega + r$  with  $r \in (0, \pi/\omega)$  then

$$\mathcal{G}_t(p) = \exp(-rX) (\text{Scat} \circ \mathcal{I})^n N_{-\infty}(p) \subset ({}^{\text{qsc}}\bar{T}_{\partial M}^*M)^\circ,$$

while

$$\mathcal{G}_{n\pi/\omega}(p) = \overline{N_{+\infty}^{-1} \circ \mathcal{I} \circ (\text{Scat} \circ \mathcal{I})^n \circ N_{-\infty}(p)},$$

hence  $p \in \mathcal{G}_t(\omega)$  iff there exists a geodesic  $n$ -gon passing through  $p$  with vertices in  $\partial M$  (this is always the case for  $n$  even, as we are allowed to repeat edges).

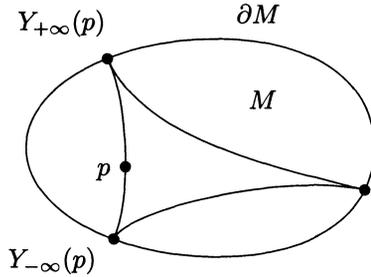


Figure 2. A point  $p$  on a geodesic triangle with vertices in  $\partial M$ .

Now we take the case  $p \in (\text{qsc}\bar{T}_{\partial M}^* M)^\circ \setminus \mathcal{N}$ . The flow of  $X$  in  $(\text{qsc}\bar{T}_{\partial M}^* M)^\circ \setminus \mathcal{N}$  is, as discussed in §3, given by unit speed geodesic flow in  $\partial M$  with time parameter  $s = \int |\mu| dt$ , while  $(\lambda, |\mu|)$  undergo the motion (3.4). The only fixed-point of the  $(\lambda, |\mu|)$  flow is given by  $\lambda = 0, |\mu| = \omega$ ; all other orbits are periodic with period  $\pi/\omega$ . Hence if  $(\lambda(p), |\mu(p)|) \neq (0, \omega)$  and  $t \notin (\pi/\omega)\mathbb{Z}$  then  $p \notin \mathcal{G}_t(p)$ , since the  $(\lambda, |\mu|)$  coordinates distinguish between these two points. If, on the one hand,  $t = n\pi/\omega$ , we have by (3.4)

$$\begin{aligned} (4.5) \quad s &= \int_0^{n\pi/\omega} |\mu| dt \\ &= \Im \int_0^{n\pi/\omega} \omega \frac{\sin \omega(t - t_0) + iR \cos \omega(t - t_0)}{\cos \omega(t - t_0) - iR \sin \omega(t - t_0)} dt \\ &= n\omega \Im \int_{-\pi/2\omega}^{\pi/2\omega} \frac{\tan \omega t - iR}{1 + iR \tan \omega t} dt \\ &= n\pi \end{aligned}$$

(recall that  $R = 0$  only on  $\mathcal{N}$ ). Thus by (3.5), for  $(\lambda, |\mu|) \neq (0, \omega)$ ,  $p = \mathcal{G}_{n\pi/\omega}(p)$  only if there is a closed geodesic of length  $n\pi$  in  $\partial M$ . On the other hand, if  $(\lambda(p), |\mu(p)|) = (0, \omega)$ ,  $(\lambda, |\mu|)$  remains constant along the flow, so  $p = \mathcal{G}_t(p)$  only if there is a closed geodesic in  $\partial M$  of length  $\omega t$ . This proves the result for  $p \in (\text{qsc}\bar{T}_{\partial M}^* M)^\circ \setminus \mathcal{N}$ .

The proof for  $p \in \mathcal{N}$  (including  $\mathcal{N}^c$ ) proceeds like the proof for  $p \in {}^{\text{qsc}}S^*M \setminus \mathcal{N}_-$ ; certainly if  $t \notin (\pi/\omega)\mathbb{Z}$ ,  $p \notin \mathcal{G}_t(p)$ , as  $\lambda$  is constant on  $\mathcal{G}_t(p)$  at fixed  $t$ , and equals  $\lambda(p)$  only for  $t \in (\pi/\omega)\mathbb{Z}$ . The same geometrical discussion used in the proof for points in  $({}^{\text{qsc}}S^*M)^\circ$  also shows that  $p \notin \mathcal{G}_{n\pi/\omega}(p)$  unless there is a geodesic  $n$ -gon with vertices in  $\partial M$ , with one vertex at  $y(p)$ .  $\square$

*Proof of Theorem 4.1.* — The theorem follows directly from the Main Propagation Theorem and Lemma 4.6.  $\square$

From Theorem 4.1, we deduce the following, which is the key result for our trace theorem.

**COROLLARY 4.7.** — *Given  $T \notin S_\omega$ , there exists  $\epsilon > 0$ ,  $k \in \mathbb{Z}_+$ , and  $A_i \in \Psi_{\text{qsc}}^{0,0}(M)$ ,  $i = 1, \dots, k$  such that*

$$A_i U_\omega(t) A_i \in C^\infty([T - \epsilon, T + \epsilon]; \Psi_{\text{qsc}}^{-\infty, \infty}(M))$$

and

$$I = \sum_{i=1}^k A_i^2 + R$$

( $I$  denotes the identity operator) with  $R \in \Psi_{\text{qsc}}^{-\infty, \infty}(M)$ .

*Proof.* — By Theorem 4.1, we can find a partition of unity  $(b_{1,i})^2$ , subordinate to a cover  $\mathcal{O}_i$  of  $C_{\text{qsc}}M$ , such that  $WF_{\text{qsc}}\psi(0) \subset \mathcal{O}_i$  implies that  $WF_{\text{qsc}}^{[T-\epsilon, T+\epsilon]}\psi \cap \mathcal{O}_i = \emptyset$ . Extend the  $b_{1,i}$  to be smooth functions on  ${}^{\text{qsc}}\bar{T}^*M$  with  $\text{ess sup } b_{1,i} \subset \mathcal{O}_i$ . Set  $B_{1,i} = \text{Op}(b_{1,i})$ . Then

$$\sum_i B_{1,i}^2 - I = C_1 \in \Psi_{\text{qsc}}^{-1,1}(M).$$

Let  $c_1$  denote a representative of the symbol of  $C_1$  in  $\mathcal{A}^{1,2}({}^{\text{qsc}}\bar{T}^*M)$ . Setting  $b_{2,i} = -\frac{1}{2}c_1 b_{1,i}$  and  $B_{2,i} = \text{Op}(b_{2,i})$ , we have

$$\sum_i (B_{1,i} + B_{2,i})^2 - I = C_2 \in \Psi_{\text{qsc}}^{-2,2}(M).$$

Now let  $c_2$  represent the symbol of  $C_2$ , set  $b_{3,i} = -\frac{1}{2}c_2 b_{1,i}$  and  $B_{3,i} = \text{Op}(b_{3,i})$ , and continue in this manner, defining  $B_{j,i}$  inductively. Then use asymptotic summation to obtain

$$A_i \sim \sum_j B_{j,i},$$

with  $WF'_{\text{qsc}}A_i \subset \mathcal{O}_i$  and  $I = \sum A_i^2 + R$  with  $R \in \Psi_{\text{qsc}}^{-\infty, \infty}(M)$ .

By our construction of  $\mathcal{O}_j$ , for all  $i = 1, \dots, k$  we have

$$WF'_{\text{qsc}} A_i \cap WF_{\text{qsc}}^{[T-\epsilon, T+\epsilon]} U(t) A_i \psi(0) = \emptyset$$

for  $t \in [T - \epsilon, T + \epsilon]$ , hence by microlocality,

$$WF_{\text{qsc}}^{[T-\epsilon, T+\epsilon]} A_i U(t) A_i \psi(0) = \emptyset$$

for any  $\psi(0) \in C^{-\infty}(M)$ , *i.e.*  $A_i U(t) A_i \in \mathcal{C}([T - \epsilon, T + \epsilon]; \Psi_{\text{qsc}}^{-\infty, \infty}(M))$ . Smoothness in  $t$  follows similarly, as

$$D_t^k A_i U(t) A_i = A_i (-\mathcal{H})^k U(t) A_i,$$

and since  $\mathcal{H} \in \Psi_{\text{qsc}}(M)$ ,

$$WF_{\text{qsc}}^{[T-\epsilon, T+\epsilon]} (-\mathcal{H})^k U(t) A_i \psi(0) \subset WF_{\text{qsc}}^{[T-\epsilon, T+\epsilon]} U(t) A_i \psi(0). \quad \square$$

### 5. The trace.

We begin the study of  $\text{Tr} U(t)$  by showing that it exists as a distribution:

PROPOSITION 5.1. — For  $\phi \in \mathcal{S}(\mathbb{R})$ ,

$$\int \phi(t) U(t) dt \in \Psi_{\text{qsc}}^{-\infty, \infty}(M)$$

and

$$\phi \longmapsto \text{Tr} \int \phi(t) U(t) dt$$

is a tempered distribution on  $\mathbb{R}$ .

*Proof.* — The structure of the argument is standard — see, for example, part II of [2]. We reproduce it only owing to the slight novelty of the Sobolev spaces involved.

Choose  $\kappa \in \mathbb{R}$  below the spectrum of  $\mathcal{H}$ . Then by ellipticity of  $\mathcal{H}$ ,

$$(\kappa + \mathcal{H})^{-k} : H_{\text{qsc}}^{0,0}(M) \longrightarrow H_{\text{qsc}}^{2k,0}(M).$$

Since

$$U(t) = (\kappa + \mathcal{H})^k (\kappa + \mathcal{H})^{-k} U(t) = (\kappa - D_t)^k (\kappa + \mathcal{H})^{-k} U(t),$$

we can write

$$(5.1) \quad \int \phi(t)U(t) dt = \int (\kappa - D_t)^k \phi(t) (\kappa + \mathcal{H})^{-k} U(t) dt.$$

$U(t)$  is unitary on  $H_{\text{qsc}}^{0,0}(M)$ , so

$$(\kappa + \mathcal{H})^{-k} U(t) : H_{\text{qsc}}^{0,0}(M) \longrightarrow H_{\text{qsc}}^{2k,0}(M)$$

is bounded uniformly in  $t$ . Since  $\bigcap_k H_{\text{qsc}}^{2k,0}(M) = \dot{C}^\infty(M)$ , (5.1) shows that

$$\int \phi(t)U(t) dt : \mathcal{C}^{-\infty}(M) \longrightarrow \dot{C}^\infty(M),$$

*i.e.*

$$\int \phi(t)U(t) dt \in \Psi_{\text{qsc}}^{-\infty,\infty}(M).$$

Furthermore, if we take  $k$  large enough so that  $(\kappa + \mathcal{H})^{-k} U(t)$  is trace-class, we see that  $\phi \mapsto \text{Tr} \int \phi(t)U(t) dt$  is a tempered distribution of order at most  $k$ . □

We are now in a position to prove our main theorem:

**THEOREM 5.2.** — *If there are no trapped geodesics in  $\mathring{M}$  then*

$$\text{sing supp Tr } U(t) \subset S_\omega.$$

*Proof.* — Let  $\phi \in \mathcal{C}^\infty(\mathbb{R})$  be 0 for  $x > 2$  and 1 for  $x < 1$ . Set

$$W_n = \text{Op}[(1 - \phi(nx))(1 - \phi(n\sigma))] \in \Psi_{\text{qsc}}^{0,0}(M);$$

then  $W_n \rightarrow I$  strongly on  $L^2(M)$ . We regularize  $\text{Tr } U(t)$  by examining instead  $\text{Tr } U(t)W_n$ ; this is a smooth function on  $\mathbb{R}$  since  $D_t^p \text{Tr } U(t)W_n = \text{Tr}(-\mathcal{H})^p U(t)W_n$ .

Given  $T \notin S_\omega$ , we choose  $A_i, i = 1, \dots, k$  as in Corollary 4.7, and write

$$\text{Tr } U(t)W_n = \text{Tr } IU(t)W_n = \sum_{i=1}^k \text{Tr } A_i^2 U(t)W_n + \text{Tr } RU(t)W_n.$$

$A_i U(t)W_n$  is trace-class, so we may now rewrite

$$\text{Tr } U(t)W_n = \sum_{i=1}^k \text{Tr } A_i U(t)W_n A_i + \text{Tr } RU(t)W_n.$$

As  $n \rightarrow \infty$ ,  $D_t^p RU(t)W_n$  converges to  $D_t^p RU(t)$  in the norm topology on operators  $H_{\text{qsc}}^{m,\ell}(M) \rightarrow H_{\text{qsc}}^{m',\ell'}(M)$  for any  $m, \ell, m', \ell'$ , and any  $p \in \mathbb{Z}_+$ ; thus  $\text{Tr } RU(t)W_n$  approaches a smooth function as  $n \rightarrow \infty$ . Thus, if we can also show that

- 1)  $\lim_{n \rightarrow \infty} \text{Tr } U(t)W_n = \text{Tr } U(t)$ , and
- 2)  $\lim_{n \rightarrow \infty} \text{Tr } A_i U(t)W_n A_i = \text{Tr } A_i U(t)A_i$  for all  $i = 1, \dots, k$ ,

in the sense of distributions, we will have  $\text{Tr } U(t) \in \mathcal{C}^\infty([T - \epsilon, T + \epsilon])$  for some  $\epsilon > 0$ , and we will be done.

Both 1) and 2) follow from the following identity, which holds, in the distributional sense, for any  $A \in \Psi_{\text{qsc}}^{p,q}(M)$  (and any  $p, q$ ):

$$\lim_{n \rightarrow \infty} \text{Tr } AU(t)W_n A = \text{Tr } AU(t)A.$$

To prove this, let  $\phi \in \mathcal{S}(\mathbb{R})$  be a test function, let  $\kappa$  lie below the spectrum of  $\mathcal{H}$ , and write

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int \phi(t) \text{Tr } AU(t)W_n A \, dt \\ &= \lim_{n \rightarrow \infty} \text{Tr} \int \phi(t)U(t)W_n A^2 \, dt \\ &= \lim_{n \rightarrow \infty} \text{Tr} \int \phi(t)(\kappa - D_t)^m (\kappa + \mathcal{H})^{-m} U(t)W_n A^2 \, dt \\ &= \lim_{n \rightarrow \infty} \int [(\kappa - D_t)^m \phi(t)] \text{Tr} [(\kappa + \mathcal{H})^{-m} U(t)W_n A^2] \, dt \\ &= \lim_{n \rightarrow \infty} \int [(\kappa - D_t)^m \phi(t)] \text{Tr} [A(\kappa + \mathcal{H})^{-m} U(t)W_n A] \, dt \\ &= \int [(\kappa - D_t)^m \phi(t)] \text{Tr} [A(\kappa + \mathcal{H})^{-m} U(t)A] \, dt \\ &= \int \phi(t) \text{Tr } AU(t)A \, dt; \end{aligned}$$

here we take  $m$  large enough that  $(\kappa + \mathcal{H})^{-m}U(t)$  is trace-class; the penultimate equality follows from the norm convergence

$$A(\kappa + \mathcal{H})^{-m}U(t)W_n A \rightarrow A(\kappa + \mathcal{H})^{-m}U(t)A$$

as operators  $H_{\text{qsc}}^{0,0}(M) \rightarrow H_{\text{qsc}}^{2m-2p-\epsilon, 2q-\epsilon}(M)$  for all  $p, q$  and all  $\epsilon > 0$ .  $\square$

### Appendix: the propagation theorems.

As noted above, the only obstacle to proving Theorems 3.6–3.9 in exactly the same manner as Theorems 12.1–12.5 of [13] is the fact that  $(Y_{\pm\infty})_*X \neq 0$  in the harmonic oscillator case; we merely have

$$(Y_{\pm\infty})_*X = O(\sigma).$$

This makes no difference in proving Theorems 3.6 or 3.8, but we must modify the constructions of the symbols  $a_{\pm}$  and  $\tilde{a}_{\pm}$  used to prove the other three theorems.

We modify the symbols  $a_+^{m,\ell}$  and  $\tilde{a}_+^{m,\ell}$  defined in §13 of [13] by replacing the factor  $\psi_{-\infty} = \phi(d(Y_{-\infty}(p), y_0))$  ( $\phi$  is a cutoff function) by

$$\tilde{\psi}_{-\infty} = \phi(d(Y_{-\infty}(p), y_0)^2 - \epsilon\sigma).$$

Since  $X\sigma = -\bar{\lambda} + O(\sigma^2) + O(x) = -1 + O(\sigma^2) + O(x) + O(|\bar{\mu}|^2)$  and since  $(Y_{-\infty})_*X = O(\sigma)$ ,

$$\begin{aligned} -X(\tilde{\psi}_{-\infty}) \\ = -\phi'(d(Y_{-\infty}(p), y_0)^2 - \epsilon\sigma) [O(\sigma) + \epsilon + O(\sigma^2) + O(x) + O(|\bar{\mu}|^2)]. \end{aligned}$$

The quantity in square brackets is strictly positive for  $x, \sigma, \bar{\mu}$  sufficiently small, and the constructions of  $a_+$  and  $\tilde{a}_+$  in [13] go through as before, with  $\tilde{\psi}_{-\infty}$  replacing  $\psi_{-\infty}$ , and  $b_+$  constructed so as to ensure that  $\sigma$  is small on  $\text{supp } a_+$ .

Similarly, in the construction of  $a_-$  and  $\tilde{a}_-$ , we replace  $\psi_{+\infty}(q) = \phi(d(Y_{+\infty}(q), y_0))$  with

$$\tilde{\psi}_{+\infty}(q) = \phi(d(Y_{+\infty}(q), y_0)^2 + \epsilon\sigma).$$

### BIBLIOGRAPHY

- [1] J. CHAZARAIN, Formule de Poisson pour les variétés riemanniennes, *Inv. Math.*, 24 (1974), 65–82.
- [2] J. CHAZARAIN, Spectre d'un Hamiltonien quantique et mécanique classique, *Comm. PDE*, 5 (1980), 599–644.
- [3] Y. COLIN DE VERDIÈRE, Spectre de laplacien et longueurs des géodésiques périodiques I, *Comp. Math.*, 27 (1973), 83–106.
- [4] Y. COLIN DE VERDIÈRE, Spectre du laplacien et longueurs des géodésiques périodiques II, *Comp. Math.*, 27 (1973), 159–184.

- [5] W. CRAIG, T. KAPPELER, W. STRAUSS, Microlocal dispersive smoothing for the Schrödinger equation, *Comm. Pure Appl. Math.*, 48 (1995), 769–860.
- [6] J.J. DUISTERMAAT, V.W. GUILLEMIN, The spectrum of positive elliptic operators and periodic bicharacteristics, *Inv. Math.*, 29 (1975), 39–79.
- [7] D. FUJIWARA, Remarks on the convergence of the Feynman path integrals, *Duke Math. J.*, 47 (1980), 559–600.
- [8] L. KAPITANSKI, I. RODNIANSKI, K. YAJIM, On the fundamental solution of a perturbed harmonic oscillator, *Topol. Methods Nonlinear Anal.*, 9 (1997), 77–106.
- [9] R.B. MELROSE, Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces, *Spectral and scattering theory*, M. Ikawa ed., Marcel Dekker, 1994.
- [10] R.B. MELROSE, M. ZWORSKI, Scattering metrics and geodesic flow at infinity, *Inv. Math.*, 124 (1996), 389–436.
- [11] F. TREVES, Parametrices for a class of Schrödinger equations, *Comm. Pure Appl. Math.*, 48 (1995), 13–78.
- [12] A. WEINSTEIN, A symbol class for some Schrödinger equations on  $\mathbb{R}^n$ , *Amer. J. Math.*, 107 (1985), 1–21.
- [13] J. WUNSCH, Propagation of singularities and growth for Schrödinger operators, *Duke Math. J.*, to appear.
- [14] K. YAJIMA, Smoothness and non-smoothness of the fundamental solution of time-dependent Schrödinger equations, *Comm. Math. Phys.*, 181 (1996), 605–629.
- [15] S. ZELDITCH, Reconstruction of singularities for solutions of Schrödinger equations, *Comm. Math. Phys.*, 90 (1983), 1–26.

Manuscrit reçu le 3 juin 1998,  
accepté le 11 septembre 1998.

Jared WUNSCH,  
Columbia University  
Department of Mathematics  
2990 Broadway, Mailcode 4406  
New York NY 10027 (USA).  
jwunsch@math.columbia.edu