ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 48, nº 5 (1998), p. 1345-1357
<http://www.numdam.org/item?id=AIF_1998__48_5_1345_0>

Ann. Inst. Fourier, Grenoble
48, 5 (1998), 1345–1357

INSTABILITY OF EQUILIBRIA IN DIMENSION THREE

by Marco BRUNELLA

Let us consider an analytic vector field v defined on a neighbourhood of 0 in $\mathbb{R}^n$ and having there an isolated singular point. We shall say that 0 is a *stable* singular point if it has a fundamental system of neighbourhoods which are invariant by v; this is equivalent to the Lyapunov stability in the past *and* in the future. In even dimension there are many examples of stable singular points: take, for instance, a hamiltonian vector field near a strict minimum point of its hamiltonian function. In odd dimension, however, examples are more difficult to construct (we don't know any), at least in the analytic category. For instance, if v is homogeneous then the nonvanishing of the Euler characteristic of an even dimensional sphere implies immediatly the existence of a trajectory of v which tends to 0 and consequently the instability of 0. Aim of this paper is to generalize this last remark when $n = 3$, or more generally when v is tangent to a three-dimensional analytic variety containing 0 and smooth outside 0.

THEOREM. — *Let v be an analytic vector field defined on a neighbourhood U of $0 \in \mathbb{R}^n$ and having at 0 an isolated singular point. Let $M \subset U$ be an analytic irreducible subvariety of dimension three, containing* 0 *and smooth outside* 0, *invariant by v. Suppose that the link of M at* 0 (*which is a compact connected surface*) *has nonvanishing Euler characteristic. Then there exists $p \in M \setminus \{0\}$ and $T \in \{-\infty, +\infty\}$ such that $\gamma_p(t) \to 0$ as $t \to T$, where γ_p denotes the trajectory of v througth p. In particular,* 0 *is an unstable singular point.*

Of course, if $n = 3$ we may choose $M = U$. If $n = 4$ we may apply the theorem to a hamiltonian vector field, provided the hamiltonian function H

Key words: Singularities of vector fields – Stability problems – Invariant manifolds – Blow-ups.
Math. classification: 14E15 – 34C05 – 34D20.

has an isolated critical point at 0 and $H^{-1}(H(0))$ has an irreducible component whose link at 0 has nonvanishing Euler characteristic. This happens, for example, if

$$H(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + F(q_1, q_2)$$

and F has an isolated critical point which is not a minimum nor a maximum. This gives the following refinement of a (not too) particular case of a theorem of Palamodov [Pa].

COROLLARY. — *Consider the second order differential equation*

$$\ddot{q} = -\operatorname{grad} F$$

where F is an analytic function on a neighbourhood V of $0 \in \mathbb{R}^2$ such that 0 is an isolated critical point which is not a minimum nor a maximum. Then there exists initial conditions $(q_0, \dot{q}_0) \in V \times \mathbb{R}^2$ (resp. $(q_1, \dot{q}_1) \in V \times \mathbb{R}^2$) such that the corresponding solution tends to $(0,0)$ as the time tends to $+\infty$ (resp. $-\infty$).

The fact that there are two trajectories tending to $(0,0)$, one in the future and the other in the past, follows from the reversibility of the differential equation. We notice that it is rather easy to prove the instability of $(0,0)$ when F has a maximum point, even in the strong form of our corollary (Maupertuis principle and Hopf-Rinow theorem). However in that case our theorem does not apply because the link of $\{\frac{1}{2}(p_1^2 + p_2^2) + F(q_1, q_2) = F(0,0)\}$ is a 2-torus.

We now sketch the main elements of the proof of the theorem. To give the idea, suppose $n = 3$ (and so $M = U$) and consider the case where after a blow-up of 0 the transformed vector field $\tilde{v}$ is tangent to the exceptional divisor $D \simeq \mathbb{S}^2$ and has only *isolated* singularities with *nonnilpotent* linear part. By Poincaré-Hopf formula, there exists a singular point $s \in D$ of $\tilde{v}$ whose Poincaré-Hopf index w.r.t. to $\tilde{v}_{|D}$ is not zero. The linear part of $\tilde{v}$ at s is either a rotation or (partially) hyperbolic. Using the result of [BD] in the first case or centre manifold splitting plus the simple structure of two-dimensional singularities [AI] in the second case, we obtain a trajectory of $\tilde{v}$ which tends to s and which is outside D. The blow-down of this trajectory gives the desired γ_p. Remark, moreover, that this trajectory has a well defined tangent line at the origin.

The general case requires: 1) the (local) desingularization theorem of Cano [Ca1], [Ca2], which roughly speaking says that we can always reduce the singularities of a three-dimensional vector field to nonnilpotent ones; 2) a definition of Poincaré-Hopf index w.r. to the exceptional divisor D of a nonisolated singularity of $\tilde{v}$ on D. Thanks to an equidesingularization theorem of Sancho de Salas [Sa], we show that if $\gamma \subset D$ is a singular curve of $\tilde{v}$ then there is a finite set of "exceptional" points on γ which have a "consistent" index w.r. to D (the other points will have index zero, by definition). These indices will satisfy Poincaré-Hopf formula relatively to D. Of course, the location and the indices of these exceptional points will depend on the full structure of $\tilde{v}$ (not only $\tilde{v}_{|D}$) near γ: intuitively, the dynamics of $\tilde{v}$ outside D induce a "canonical" perturbation of $\tilde{v}_{|D}$ having only isolated singularities in correspondence of the exceptional points of γ, and then we can take the usual Poincaré-Hopf indices of these isolated singularities.

We remark that, as the proof will show, the trajectory γ_p that we find approaches the origin in a rather regular way, for instance with a well defined tangent line. In order to exclude many trivial cases, the proof will be done by contradiction. A "positive" proof (left to the reader) will show that there are in fact at least $|\chi(\text{link of } M)|$ trajectories which tend to 0.

It is a pleasure to thank R. Moussu for his constant help during the elaboration of this work. Thanks also to F. Cano for giving me some useful references about desingularization in dimension three.

1. Reduction of singularities in dimension three.

In this section we survey some results concerning the reduction of singularities of three-dimensional vector fields; for more details and results we refer to the works of Cano and Sancho de Salas.

Let w be an analytic vector field on an analytic 3-manifold N, whose singular set $\mathrm{Sing}(w)$ has at most dimension 1 and so it is a locally finite union of points and analytic curves (possibly singular). There are two types of blow-up:

1) If $p \in \mathrm{Sing}(w)$ we may do a *spherical* blow-up centered at p, $\pi_p : \widetilde{N} \to N$, which consists in replacing the point p by the set (diffeomorphic to $\mathbb{S}^2$) of half-lines of T_pN; $\widetilde{N}$ is an analytic manifold with boundary $\partial\widetilde{N} \simeq \mathbb{S}^2$, π_p maps that boundary to the point p, and $\pi_p : \widetilde{N} \setminus \partial\widetilde{N} \to N \setminus \{p\}$ is an analytic diffeomorphism; $\pi_p^*(w)$ is an analytic vector field (up to the boundary) which is frequently identically zero on $\partial\widetilde{N}$, but after division

by an analytic function vanishing on $\partial\widetilde{N}$ and positive on $\widetilde{N} \setminus \partial\widetilde{N}$ we can obtain an analytic vector field $\widetilde{w}$ on $\widetilde{N}$ whose singular set $\mathrm{Sing}(\widetilde{w})$ has again dimension at most one.

2) If $N_0 \subset N$ is open and $\gamma \subset \mathrm{Sing}(w) \cap N_0$ is a smooth open interval which is closed in N_0, then we may also do a *cylindrical* blow-up centered at γ, $\pi_\gamma : \widetilde{N}_0 \to N_0$, which consists in replacing each point p of γ by the set (diffeomorphic to a circle) of half-planes of T_pN_0 containing $T_p\gamma$; the boundary of $\widetilde{N}_0$ is now diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and projects by π_γ to the curve γ, and π_γ realizes a diffeomorphism between $\widetilde{N}_0 \setminus \partial\widetilde{N}_0$ and $N_0 \setminus \gamma$; as before, we obtain after division an analytic vector field $\widetilde{w}$ on $\widetilde{N}_0$, whose singular set has dimension at most one.

We remark that, in the cylindrical case, the localization on an open subset $N_0 \subset N$ may be unavoidable, because the component of $\mathrm{Sing}(w)$ which contains γ may be singular. Remark also that it is *not* required that $\mathrm{Sing}(w)$ is smooth at every point of γ, but only that γ is a smooth subset of $\mathrm{Sing}(w)$.

In both cases, the *exceptional divisor* $D = \pi_p^{-1}(p)$ or $D = \pi_\gamma^{-1}(\gamma)$ can be non invariant by $\widetilde{w}$: this is the so-called *dicritical case.* In fact, we shall not be concerned with such a possibility, because if we find a dicritical situation after some sequence of blow-ups then we find also plenty of trajectories of the initial vector field which tend to the initial singular point. Let us also observe that $\pi_p^{-1}(p)$ has dimension two, whereas $\pi_\gamma^{-1}(p)$ has dimension one.

In order to consider sequences of blow-ups, we have to work in the category of manifolds with boundary and corners, *i.e.* locally modelled on open sets of $[0,1)^3$. If N is such a manifold, we denote by $\partial_0 N$ the smooth part of ∂N, $\partial_1 N$ the codimension one singular subset of ∂N (a union of smooth curves) and $\partial_2 N$ the codimension two singular subset of ∂N (isolated points). In the case of a spherical blow-up, the center p may still be any singular point of w; of course, $\pi_p^{-1}(p)$ will be only a closed portion of a two-sphere if $p \in \partial N$. In the case of a cylindrical blow-up, we will require that the curve γ is an open, closed or half-open smooth interval which has *normal crossing* with ∂N_0, in order to ensure that $\widetilde{N}_0$ shall still be a manifold with boundary and corners. See [Ca1] for the precise meaning of normal crossing.

We can now state the local desingularization theorem of Cano and the equidesingularization theorem of Sancho de Salas.

Cano's theorem is formulated in terms of a game between two players A and B. The starting point is an analytic vector field w on an analytic 3-manifold with boundary and corners N, with $\dim \mathrm{Sing}(w) \leq 1$. Player A chooses a singular point p of w with nilpotent linear part and then player B performs a blow-up centered at p or along a (permitted) curve γ containing p. After this blow-up, A chooses again a singular point $\tilde{p}$ of $\widetilde{w}$ with nilpotent linear part on $\pi^{-1}(p)$, where $\pi = \pi_p$ or π_γ depending on the choice of B, then B chooses a blow-up throught $\tilde{p}$, and so on. The game stops, and B wins, when A cannot do his choice, *i.e.* all the singularities of the transformed vector field on the "last" $\pi^{-1}(p)$ have nonnilpotent linear part.

THEOREM (see [Ca1], [Ca2]). — *There exists a winning strategy for the player B.*

This means that if player B does the "good" choice (between spherical and cylindrical blow-up) at every step, depending on the previous choice of A, then the game stops after a finite number of steps. It is a *local* desingularization theorem because after every blow-up the player A relocalizes the problem at a point of $\pi^{-1}(p)$. A *global* desingularization theorem is still lacking, however the theorem of Sancho de Salas that we now describe says that a *generic* point on a singular curve of w has a good desingularization.

More precisely, by a *tower of cylindrical blow-ups* we mean the following data:

1) an open set $N_0 \subset N$ and a smooth open interval $\gamma_0 \subset \mathrm{Sing}(w) \cap N_0$ closed in N_0 and having normal crossing with ∂N_0 (that is, by openness, $\gamma_0 \subset \mathrm{int}\, N_0$ or $\partial_0 N_0$ or $\partial_1 N_0$),

2) a sequence of cylindrical blow-ups $\pi_i : N_i \to N_{i-1}$, $i = 1, \dots, k$, with π_i centered along a smooth open interval $\gamma_{i-1} \subset \mathrm{Sing}(w_{i-1}) \cap N_{i-1}$ which projects diffeomorphically onto γ_0 by $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_{i-1}$ (here w_{i-1} is the transformed vector field on N_{i-1}, and $w_0 = w$).

We will say that the singularities of w on a smooth open interval $\gamma_0 \subset \mathrm{Sing}(w)$ are *equidesingularizable* if $\mathrm{Sing}(w)$ is smooth at every point of γ_0 and there is a tower of cylindrical blow-ups $\{N_0, \gamma_0, \pi_i\}_{i=1}^k$ such that

1) $\mathrm{Sing}(w_k) \cap (\pi_1 \circ \cdots \circ \pi_k)^{-1}(\gamma_0)$ is a collection of smooth open intervals, each one of which projects diffeomorphically onto γ_0 by $\pi_1 \circ \cdots \circ \pi_k$;

2) every singular point of w_k on $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(\gamma_0)$ has nonnilpotent linear part.

THEOREM (see [Sa]). — *There exists a discrete set $S \subset \mathrm{Sing}(w)$ such that $\mathrm{Sing}(w) \setminus S$ is a collection of equidesingularizable smooth open intervals.*

This is a sort of "parametric" version of the classical (Seidenberg's) desingularization theorem in dimension two [AI], and the proof of Sancho de Salas is in fact virtually bidimensional.

2. Definition of the index.

Let us again consider an analytic vector field w on a 3-manifold with boundary and corners N, with $\dim \mathrm{Sing}(w) \leq 1$. Let us suppose moreover that $\mathrm{Sing}(w) \subset \partial N$ and that w is tangent to ∂N. Let γ_0 be an equidesingularizable open interval of $\mathrm{Sing}(w)$, contained in $\partial_0 N$ or $\partial_1 N$, and let $\pi : \widetilde{N}_0 \to N_0$ be the composition of the sequence of cylindrical blow-ups which desingularize w along γ_0. Let us assume that the lifted-divided vector field $\widetilde{w}$ is still tangent to $\partial \widetilde{N}_0$. We now want to analyze the structure of $\widetilde{w}$ near a singular interval $\tilde{\gamma} \subset \pi^{-1}(\gamma_0)$ under the following additional hypothesis: *no orbit of $\widetilde{w}$ outside $\partial \widetilde{N}_0$ tends, in the past or in the future, to a point of $\tilde{\gamma}$.*

We firstly consider the case where $\tilde{\gamma}$ is in $\partial_0 \widetilde{N}_0$. If $p \in \tilde{\gamma}$ then the linear part of $\widetilde{w}$ at p is nonnilpotent, and cannot be a rotation around $\tilde{\gamma}$ because $\partial \widetilde{N}_0$ is $\widetilde{w}$-invariant. Hence it has at least one nonzero real eigenvalue. The corresponding eigenspace must be contained in $T_p(\partial_0 \widetilde{N}_0)$, otherwise there would be orbits of $\widetilde{w}$ outside $\partial \widetilde{N}_0$ tending to p, by the (un)stable manifold theorem [AI]. It follows that the linear part of $\widetilde{w}$ at p has *exactly one* nonzero eigenvalue, with multiplicity one and whose eigenspace is in $T_p(\partial_0 \widetilde{N}_0)$ and transverse to $T_p \tilde{\gamma}$. In particular, $\widetilde{w}$ has near p an invariant center manifold W which is two-dimensional, transverse to $\partial_0 \widetilde{N}_0$, and intersects $\partial_0 \widetilde{N}_0$ along $\tilde{\gamma}$ (see [AI]). Because no orbit of $\widetilde{w}$ on $W \setminus \tilde{\gamma}$ tends to a point of $\tilde{\gamma}$, we are (up to reversing time) in one of the two situations of figure 1 (and remember that the topological structure of the flow is a product of the flow on the center manifold and a hyperbolic singularity [AI]).

Consider now the case where the interval $\tilde{\gamma}$ is contained in $\partial_1 \widetilde{N}_0$. If $p \in \tilde{\gamma}$, it is now possible that the linear part of $\widetilde{w}$ at p has *two* nonzero real

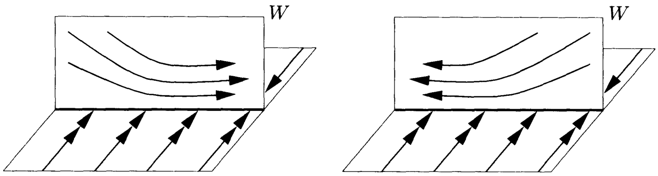

Figure 1

eigenvalues, one positive and the other negative, and whose corresponding invariant manifolds are contained in $\partial\widetilde{N}_0$. Then the structure of $\widetilde{w}$ is the following one:

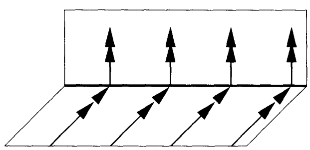

Figure 2

Otherwise, the linear part of $\widetilde{w}$ at p has only one nonzero real eigenvalue, and there will be again a two-dimensional center manifold W, contained in $\partial\widetilde{N}_0$. Then one can easily show that, outside a discrete set $T(\tilde{\gamma})$ of points of $\tilde{\gamma}$, we are (up to reversing time) in one of the following three situations:

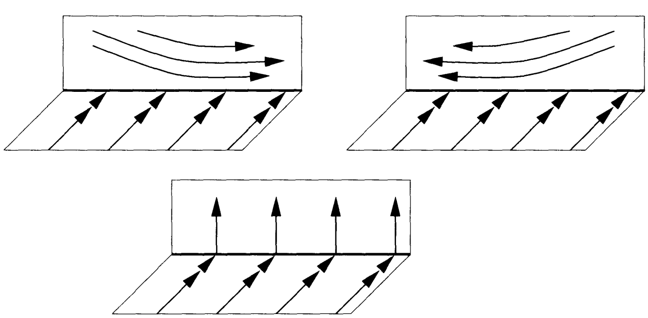

Figure 3

The exceptional set $T(\tilde{\gamma})$ projects by π to a discrete subset of γ_0. Doing the same for every interval of $\mathrm{Sing}(\widetilde{w}) \cap \pi^{-1}(\gamma_0)$, we obtain a discrete subset $T \subset \gamma_0$. In fact, T is discrete not only in γ_0 but also in $\mathrm{Sing}(w)$, and so we may include it in the discrete set S which appears in Sancho's theorem.

We can resume this discussion in the following proposition.

PROPOSITION 1. — *Let w be an analytic vector field on a 3-manifold with boundary and corners N, with* $\mathrm{Sing}(w)$ *at most one-dimensional and contained in ∂N. Suppose that no orbit of w outside ∂N tends, in the past or in the future, to a point of ∂N. Then there is a discrete set $S \subset \mathrm{Sing}(w)$ such that $\mathrm{Sing}(w) \setminus S$ is composed by equidesingularizable smooth open intervals, and moreover the local model of the desingularized vector field $\widetilde{w}$ near a singular point p over $\mathrm{Sing}(w) \setminus S$ is one of figures* 1, 2, 3.

We can now define, under the assumptions and notations of the previous proposition, an *index of $s \in S$ w.r. to ∂N*, $I(s, w, \partial N)$. The easiest case is when s is an isolated point of $\mathrm{Sing}(w)$ (observe that *all* isolated singular points of w are contained in S). Even if ∂N is not necessarily smooth at p (because s can be, of course, in $\partial_1 N$ or $\partial_2 N$), we may consider the Poincaré-Hopf index of the restriction $w_{|\partial N}$ at the isolated singular point $s \in \partial N$. By definition, we set $I(s, w, \partial N)$ equal to that index.

If $s \in S$ is not isolated in $\mathrm{Sing}(w)$, let us consider a small disc D in ∂N centered at s. We may assume that $\mathrm{Sing}(w) \cap D$ is the union of s and a collection of smooth intervals $\gamma_1, \ldots, \gamma_k \subset \mathrm{Sing}(w) \setminus S$ joining s and ∂D. We may simultaneously equidesingularize w over $D_0 = D \setminus \{s\}$; let $\pi : \widetilde{N}_0 \to N_0$ (where $\partial N_0 = N_0 \cap \partial N = D_0$) be the corresponding map (a composition of cylindrical blow-ups) and $\widetilde{w}$ the lifted-divided vector field, whose singular set has the local structure of figures 1, 2, or 3. Remark that $\widetilde{D}_0 = \pi^{-1}(D_0) = \partial \widetilde{N}_0$ is still a disc $\widetilde{D}$ minus a point $\tilde{s}$, and $\mathrm{Sing}(\widetilde{w})$ is a union of intervals $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ converging to that removed point.

We can perturb $\widetilde{w}_{|\widetilde{D}_0}$ to a nonsingular vector field u on $\widetilde{D}_0$ by removing each singular interval $\tilde{\gamma}_i$ in the following "canonical" way, mainly dictated by the dynamics of $\widetilde{w}$ on the center manifold (and we also smooth corners):

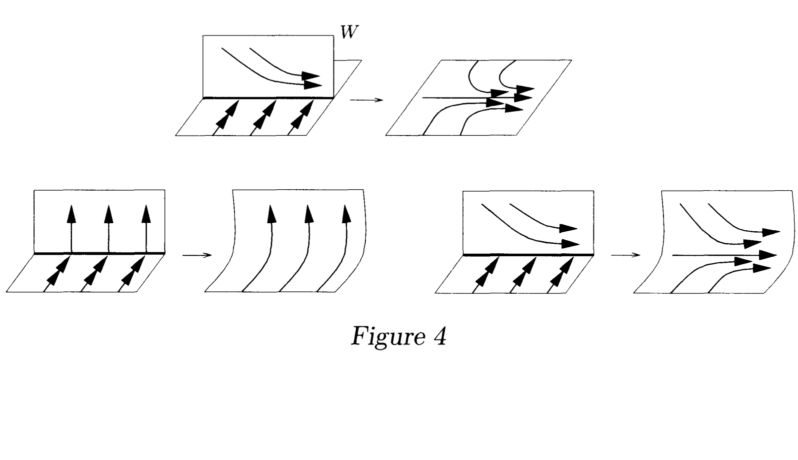

Figure 4

Now we can consider the Poincaré-Hopf index of u on $\widetilde{D}_0 \subset \widetilde{D}$ with respect to the removed point $\tilde{s}$. By definition, this index is set equal to $I(s, w, \partial N)$.

Because the structure of w is uniform over every interval in $\mathrm{Sing}(w) \setminus S$, we deduce the following Poincaré-Hopf formula.

PROPOSITION 2. — *Under the same hypotheses and notations of Proposition* 1, *assume moreover that ∂N is compact* (*and so, in particular, S is finite*). *Then*

$$\sum_{s \in S} I(s, w, \partial N) = \chi(\partial N)$$

where $\chi(\,\cdot\,)$ denotes the Euler characteristic.

Of course, there is also a relative version of this formula, for a compact subsurface $\Sigma \subset \partial N$ whose boundary is disjoint from S.

In order to uniformize the notations we will set $I(p, w, \partial N) = 0$ for every $p \in \mathrm{Sing}(w) \setminus S$; this is clearly coherent with the definition for $s \in S$.

3. Analysis of nonnilpotent singularities.

The final step of the proof of the theorem will require the following proposition.

PROPOSITION 3. — *Let w be an analytic vector field on a 3-manifold with boundary and corners N, whose singular set is at most one-dimensional and contained in ∂N. Suppose that no orbit of w outside ∂N tends to a point of ∂N. Let $p \in \mathrm{Sing}(w)$ be a point where the linear part of w is nonnilpotent. Then*

$$I(p, w, \partial N) = 0.$$

The proof is a case-by-case analysis.

Consider firstly the case where the linear part of w at p is a rotation R. Then, because w is tangent to ∂N, the point p belongs to $\partial_0 N$ and the axis of R is transverse to $\partial_0 N$; in particular, p is an isolated singularity of w. By a theorem of Bonckaert and Dumortier [BD] the axis of R is tangent to a trajectory of w which tends to p, contradicting our hypothesis. Hence this case do not occur. More generally, the spectrum of the linear part cannot contain a pair of conjugate nonreal eigenvalues.

Consequently, the nonnilpotent linear part of w at p must have a real nonzero eigenvalue, say a negative one $\lambda < 0$. Henceforth we shall assume that this is the only nonzero eigenvalue (with multiplicity one), and that p is nonisolated in $\mathrm{Sing}(w)$. This is the more degenerate situation, the other (easier) cases are left to the reader.

If p belongs to $\partial_0 N$, then the (one-dimensional) λ-eigenspace must be contained in $T_p(\partial_0 N)$ (compare the discussion in Section 2), in particular $\mathrm{Sing}(w)$ is smooth at p, transversely hyperbolic with respect to the restriction $w_{|\partial_0 N}$, and it coincides (locally) with the transverse intersection between $\partial_0 N$ and a center manifold W. As in Section 2, the fact that no orbit of w outside ∂N tends to a point of ∂N implies a local structure like that drawed in figure 1. We see that the index $I(p, w, \partial N)$ is equal to 0.

Suppose now that $p \in \partial_1 N$. If the λ-eigenspace is not contained in $T_p(\partial_1 N)$, the situation is again close to that of the previous section: $\mathrm{Sing}(w)$ locally coincides with $\partial_1 N$, the center manifold W is inside ∂N and the pictures are either those of figure 3 or the following ones:

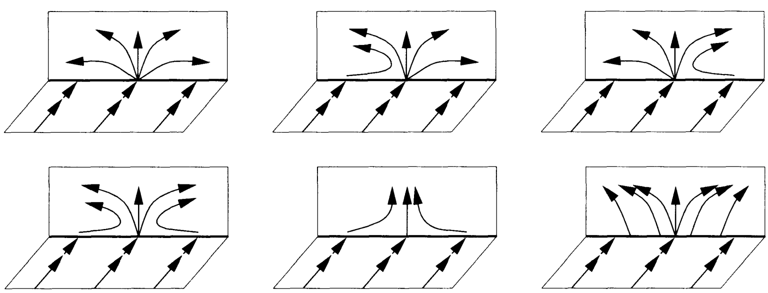

Figure 5

This happens because W cannot contain an orbit positively asymptotic to $\mathrm{Sing}(w)$, otherwise, by the local product structure of the flow near p, there would also be an orbit outside ∂N positively asymptotic to $\mathrm{Sing}(w)$. Hence the flow on W has only nodal sectors of repelling type or hyperbolic sectors whose attracting side is in $\partial_1 N$ (see [AI] for the notion of sectors of plane singularities).

Incidentally, let us observe that situations like those in figure 5 occur also at the points of the set T which was introduced after figure 3.

In all these cases one clearly has the vanishing of $I(p, w, \partial N)$, for instance:

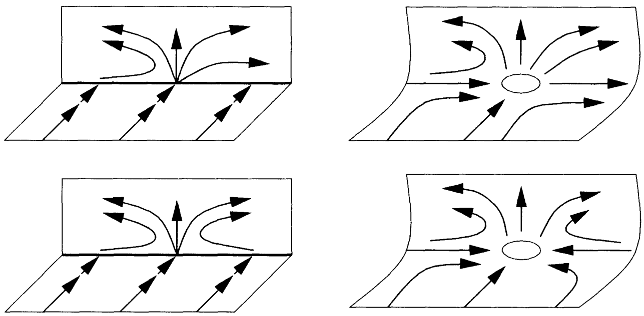

Figure 6

If $p \in \partial_1 N$ but the λ-eigenspace is in $T_p(\partial_1 N)$, then the center manifold W is transverse to $\partial_1 N$. By the same arguments as before, w on W has the dynamics of a hyperbolic sector and there are only two possibilities, depending on the number of branches of $\mathrm{Sing}(w)$ converging to p:

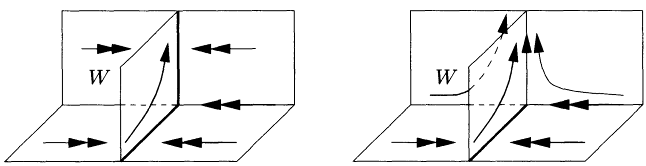

Figure 7

and the index still vanishes:

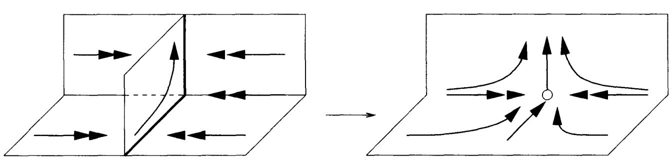

Figure 8

Finally, the case $p \in \partial_2 N$. Using again the center manifold theorem and the structure of two-dimensional singular points, one finds the following possibilities, and in all cases it is easy to verify that $I(p, w, \partial N) = 0$ (we only draw what happens when there are two branches of the singular set converging to p, the other case, with a single branch, being totally similar):

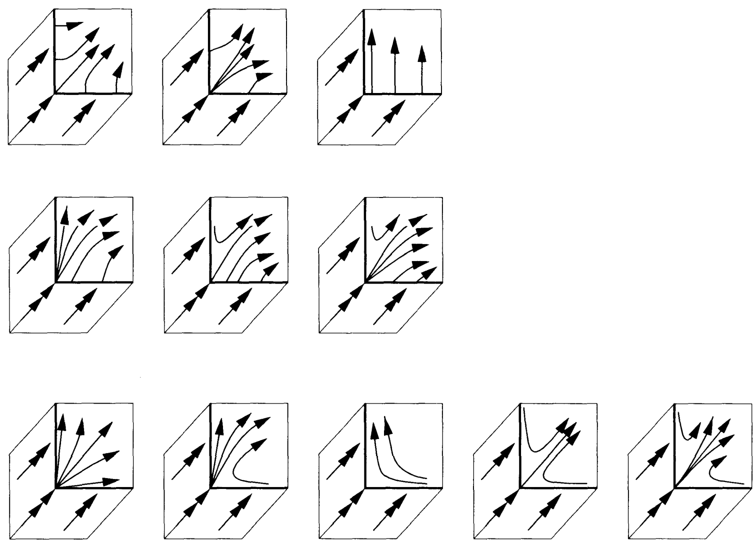

Figure 9

This ends the proof of Proposition 3.

Proof of the theorem.

We can now give the proof of our theorem.

By Hironaka's resolution of singularities theorem, we can take a sequence of blow-ups $\widetilde{M} \to M$ such that $\widetilde{M}$ is a 3-manifold with boundary and corners whose boundary $\Sigma = \partial\widetilde{M}$ is homeomorphic to a compact connected surface with nonvanishing Euler characteristic (the link of M at 0). Even if M is smooth at 0 we take at least one blow-up, in order to have a nonempty boundary.

Let w be the lifted-divided vector field on $\widetilde{M}$. We want to prove that there is an orbit of w outside Σ which converges to a point of Σ, in the past or in the future. To do this, we assume by contradiction that no such orbit exists. By Proposition 2 we find a point $p \in \Sigma \cap \mathrm{Sing}(w)$ whose index w.r. to Σ is not zero.