ANNALES DE L'INSTITUT FOURIER

PAUL JOLISSAINT

Central sequences in the factor associated with Thompson's group F

Annales de l'institut Fourier, tome 48, nº 4 (1998), p. 1093-1106

<http://www.numdam.org/item?id=AIF_1998__48_4_1093_0>

Ann. Inst. Fourier, Grenoble
48, 4 (1998), 1093–1106

CENTRAL SEQUENCES IN THE FACTOR ASSOCIATED WITH THOMPSON'S GROUP F

by Paul JOLISSAINT

---

1. Introduction.

The group F is the following subgroup of the group of homeomorphisms of the interval $[0,1]$: it is the set of piecewise linear homeomorphisms of $[0,1]$ that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are integral powers of 2. It was discovered by R. Thompson in 1965 and rediscovered later by homotopy theorists. Its history is sketched in [6] where many results that we need here are proved. To begin with, it is known that F admits the following presentation

$$F = \langle x_0, x_1, \ldots \mid x_i^{-1} x_n x_i = x_{n+1}\ 0 \le i < n \rangle .$$

Since $x_n = x_0^{-(n-1)} x_1 x_0^{n-1}$ for $n \ge 2$, F is generated by x_0 and x_1, and in the geometric realization above, the corresponding homeomorphisms x_n are defined by

$$x_n(t) = \begin{cases} t & \text{if } 0 \le t \le 1 - 2^{-n} \\ \frac{t}{2} + \frac{1}{2}(1 - 2^{-n}) & \text{if } 1 - 2^{-n} \le t \le 1 - 2^{-n-1} \\ t - 2^{-n-2} & \text{if } 1 - 2^{-n-1} \le t \le 1 - 2^{-n-2} \\ 2t - 1 & \text{if } 1 - 2^{-n-2} \le t \le 1. \end{cases}$$

Geoghegan discovered the interest in knowing whether or not F is amenable, and he conjectured in 1979 [11] that

---

Key words: Type II_1 factors – Central sequences – Crossed products – Centrally free actions.
Math. classification: 46L10 – 46L55 – 20D25.

(1) F does not contain a non-Abelian free subgroup;

(2) F is not amenable.

The first conjecture was solved with an affirmative answer by Brin and Squier [5], but it is still unknown whether or not F is amenable. However, we remarked in [12] that F is *inner amenable*: this means that there exists an inner invariant mean on the algebra $l^\infty(F - \{e\})$, *i.e.* a mean m which is invariant under the action

$$(x \cdot f)(y) = f(x^{-1}yx).$$

Indeed, if ω is a free ultrafilter on $\mathbb{N}$, then the linear functional m on $l^\infty(F - \{e\})$ given by

$$m(f) = \lim_{n \to \omega} \frac{1}{n} \sum_{k=n+1}^{2n} f(x_k)$$

is an invariant mean. (Simply use the relations: $x_j^{-1} x_n x_j = x_{n+1}$ for all $n \geq 1$ and $j = 0$ and 1.)

Inner amenability was defined by E.G. Effros in [10] where he observed that if G is an *icc group* (*i.e.* all non trivial conjugacy classes of G are infinite) and if the factor $L(G)$ has *property gamma* of Murray and von Neumann (see below), then G is inner amenable.

Here we prove that the factors associated with F and with some of its subgroups have a stronger property: they are *McDuff factors*, which means that each such factor M is isomorphic to its tensor product $M \otimes R$ with the hyperfinite type II_1 factor R. It turns out that the latter property is equivalent to the existence of pairs of non commuting non trivial central sequences in M (see [13]), hence the title of our article. This is a weak form of amenability because it follows from Corollary 7.2 of [8] that for every countable amenable icc group G, the associated factor $L(G)$ is isomorphic to R. Before stating our main results, let us present some definitions:

In [3], D. Bisch extended that property to pairs $1 \in N \subset M$, where N is a type II_1 subfactor of M; so let us say that the pair $N \subset M$ has the *relative McDuff property* if there exists an isomorphism Φ of M onto $M \otimes R$ such that $\Phi(N) = N \otimes R$.

Now let F' denote the commutator subgroup of F; it is known that F' is a simple group and that it consists in all elements of F that are trivial in neighbourhoods of 0 and 1 (notice that each element x in F fixes 0 and 1): see [6], Theorem 4.1. Let us also introduce the intermediate subgroup

D consisting in all elements of F which are trivial on a neighbourhood of 1:

$$D = \{x \in F; \exists \varepsilon > 0 \text{ such that } x(t) = t \ \ \forall t \in [1-\varepsilon; 1]\}.$$

Then F' and D are icc groups, so that the von Neumann algebras $L(F')$ and $L(D)$ are type II_1 factors. Notice that $F = D \rtimes_\alpha \mathbb{Z}$, where the action α is defined by: $\alpha^n(x) = x_0^n x x_0^{-n}$ $\forall x \in F$. This will be used in the proof of:

THEOREM A. — *The pairs of factors $L(F') \subset L(D)$ and $L(D) \subset L(F)$ have the relative McDuff property.*

Notice that even if F turned to be amenable, the above theorem is still of interest, because D. Bisch gave in [4] examples of pairs $N \subset M$ of hyperfinite II_1 factors, with finite index, which do not have the relative McDuff property.

In [15], S. Popa and M. Takesaki proved that the unitary group $U(M)$ of a type II_1 McDuff factor M is contractible with respect to the topology induced by the norm $\|\cdot\|_2$. Thus we obtain more precisely:

THEOREM B. — *Let $N \subset M$ be a pair which has the relative McDuff property. Then there exists a continuous map $\alpha : [0,\infty[\times U(M) \longrightarrow U(M)$ with the following properties:*

(1) $\alpha_t(U(N)) \subset U(N)$ $\forall t \geq 0$;

(2) $\alpha_0(u) = u$ *and* $\lim_{t\to\infty} \alpha_t(u) = 1$, $\forall u \in U(M)$;

(3) *each* α_t *is an injective endomorphism of* $U(M)$;

(4) $\alpha_s \circ \alpha_t = \alpha_{s+t}$, $\forall s,t \geq 0$;

(5) $\|\alpha_t(u) - \alpha_t(v)\|_2 = e^{-\frac{t}{2}}\|u-v\|_2$, $\forall t \geq 0$, $\forall u,v \in U(M)$.

COROLLARY C. — *The unitary groups of the pairs of factors $L(F') \subset L(D)$ and $L(D) \subset L(F)$ have the contractibility properties of Theorem B.*

Theorem B is a particular case of Theorem 1 of [15]: the latter states that if M is a type II_1 factor such that the tensor product $M \otimes B$ of M with the type I_∞ factor B admits a one parameter group $(\theta_s)_{s\in\mathbb{R}}$ scaling the trace of $M\otimes B$, i.e. $\mathrm{Tr} \circ \theta_s = e^{-s}\,\mathrm{Tr}$ for every $s \in \mathbb{R}$, then there exists a map α having properties (2)-(5) above. The proof uses the structure theorem for factors of type III and Connes-Takesaki duality theorem. However, if

M is a McDuff factor, the one parameter group $(\theta_s)_{s\in\mathbb{R}}$ comes from a one parameter group $(\sigma_s)_{s\in\mathbb{R}}$ on the hyperfinite factor of type II_∞, and it seems interesting to give a proof in this special case avoiding type III factors techniques. In order to do that, we use the realization of the hyperfinite factor of type II_∞ as the crossed product $L^\infty(\mathbb{R}^2) \rtimes_\alpha SL(2,\mathbb{Z})$ given by P.-L. Aubert in [1]. This allows us to get an explicit description of the one parameter group $(\sigma_s)_{s\in\mathbb{R}}$.

Notations. — Let M be a type II_1 factor; we denote by tr its normal, normalized, faithful trace and by $\|a\|_2 = \mathrm{tr}(a^*a)^{\frac{1}{2}}$ the associated Hilbertian norm. Let ω be a free ultrafilter on $\mathbb{N}$. Then

$$I_\omega = \left\{(a_n)_{n\geq 1} \in l^\infty(\mathbb{N}, M);\ \lim_{n\to\omega} \|a_n\|_2 = 0\right\}$$

is a closed two-sided ideal of the von Neumann algebra $l^\infty(\mathbb{N}, M)$ and the corresponding quotient algebra is denoted by M^ω (*cf.* [7], [8], [3], [13]). We will write $[(a_n)] = (a_n) + I_\omega$ for the equivalence class of (a_n) in M^ω, and we recall that M embeds naturally into M^ω, where the image of $a \in M$ is the class of the constant sequence $a_n = a$, $\forall n \geq 1$.

A sequence $(a_n) \in l^\infty(\mathbb{N}, M)$ is a *central sequence* if

$$\lim_{n\to\infty} \|[a, a_n]\|_2 = 0$$

for every $a \in M$, where $[a,b] = ab - ba$; two central sequences (a_n) and (b_n) are *equivalent* if

$$\lim_{n\to\infty} \|a_n - b_n\|_2 = 0;$$

finally, a central sequence is *trivial* if it is equivalent to a scalar sequence, and a factor M has *property gamma* if it admits a non trivial central sequence.

Let now G be an icc (countable) group; let λ denote the left regular representation of G on $l^2(G)$

$$(\lambda(x)\xi)(y) = \xi(x^{-1}y)$$

for all $x, y \in G$ and $\xi \in l^2(G)$. The bicommutant $\lambda(G)''$ in the algebra of all linear, bounded operators on $l^2(G)$ is then a type II_1 factor denoted by $L(G)$, whose trace is

$$\mathrm{tr}(a) = \langle a\delta_e, \delta_e\rangle,$$

where δ_e is the characteristic function of $\{e\}$. Recall finally that if H is a subgroup of G, then $L(H)$ embeds into $L(G)$ in a natural way.

2. Relative McDuff property.

Let M be a type II_1 factor with separable predual and let $1 \in N$ be a subfactor of M. Theorem 3.1 of [3] states that the following properties are equivalent:

(1) there exists an isomorphism Φ of M onto $M \otimes R$ such that $\Phi(N) = N \otimes R$;

(2) the algebra $M' \cap N^\omega$ is noncommutative;

(3) for all $a_1, \ldots, a_n \in M$, for every $\varepsilon > 0$, there exist a type I_2 subfactor B of N, with the same unit, and a system of matrix units $(e_{ij})_{1 \leq i,j \leq 2}$ of B such that $\|[a_k, e_{ij}]\|_2 < \varepsilon$ for all $k = 1, \ldots, n$ and all $i, j = 1, 2$.

We point out that in condition (2) above, $M' \cap N^\omega$ is the subalgebra of all $[(a_n)_{n \geq 1}] \in M^\omega$ such that $a_n \in N\ \forall n$ and $\lim\limits_{n \in \omega} \|[a_n, a]\|_2 = 0\ \forall a \in M$. Let us say that the pair $N \subset M$ has the *relative McDuff property* if it satisfies these conditions. (See also [13].)

The aim of this section is to prove:

THEOREM 2.1. — *The subgroups F' and D of F are icc groups, and the pairs of* II_1 *factors $L(F') \subset L(D)$ and $L(D) \subset L(F)$ have both the relative McDuff property.*

We are going to prove Theorem 2.1 into two steps, according to its statement.

The next lemma is Lemma 4.2 of [6], and we restate it for the convenience of the reader:

LEMMA 2.2. — *If $0 = a_0 < a_1 < \ldots < a_n = 1$ and $0 = b_0 < b_1 < \ldots < b_n = 1$ are partitions of $[0,1]$ consisting of dyadic rational numbers, then there exists $x \in F$ such that $x(a_i) = b_i$ for $i = 0, \ldots, n$. Moreover, if $a_{i-1} = b_{i-1}$ and $a_i = b_i$ for some $1 \leq i \leq n$, then x can be taken to be trivial on the interval $[a_{i-1}, a_i]$.*

From this we deduce:

LEMMA 2.3. — *The subgroups F' and D are icc groups and hence $L(F')$ and $L(D)$ are type* II_1 *subfactors of $L(F)$.*

Proof. — We give the proof for F'. Let $x \in F'$, $x \neq e$. There exist dyadic rational numbers $a, b, c, d \in]0, 1[$ such that

(1) $0 < c < a, b < d < 1$ and $a \neq b$;
(2) $x(a) = b$;
(3) $x(t) = t$ for every $t \in [0, c] \cup [d, 1]$.

Assume for instance that $b > a$, and let $N > 1$ be an integer such that $b + 2^{-N} < d$. Using Lemma 2.2, for every integer $n \geq N$, there exists $y_n \in F$ such that:

(4) $y_n(t) = t$ for every $t \in [0, c] \cup [d, 1]$;
(5) $y_n(a) = a$ and $y_n(b) = b + 2^{-n}$.

Then in fact y_n belongs to F', and if $n \neq m$ are integers larger than N one has

$$y_n x y_n^{-1}(a) = b + 2^{-n} \neq b + 2^{-m} = y_m x y_m^{-1}(a).$$

This proves that the conjugacy class of x in F' is infinite. □

To prove the first half of Theorem 2.1, we need the following general result which is analoguous to Lemma 7 of [9]:

PROPOSITION 2.4. — *Let G be a countable icc group and let H be an icc subgroup of G with the following property: for every finite subset E of G, there exist elements g and h in $H - \{e\}$ such that*

(1) *$xg = gx$ and $xh = hx$ for every $x \in E$;*

(2) *$gh \neq hg$.*

Then the pair $L(H) \subset L(G)$ has the relative McDuff property.

Proof. — Let $(E_n)_{n\geq 1}$ be an increasing sequence of finite subsets of G such that $G = \bigcup_{n\geq 1} E_n$. For every $n \geq 1$, choose g_n and h_n in $H - \{e\}$ satisfying properties (1) and (2) with respect to E_n. Set $a = [(\lambda(g_n))_{n\geq 1}]$ and $b = [(\lambda(h_n))_{n\geq 1}]$, which are both elements of $L(H)^\omega$, for any ultrafilter ω on $\mathbb{N}$.

Then $ab \neq ba$ because $\|\lambda(g_n h_n g_n^{-1} h_n^{-1}) - 1\|_2 = \sqrt{2}$ for every $n \geq 1$. Moreover, if $c \in L(G)$ and if $\varepsilon > 0$ are given, there exists $N \geq 1$ such that

$$\sum_{x \notin E_n} |c(x)|^2 < \frac{\varepsilon^2}{4}$$

for every $n \geq N$. We get for these n:

$$\begin{aligned}
\|[a_n, c]\|_2 &= \|\lambda(g_n)c\lambda(g_n^{-1}) - c\|_2 \\
&= \left(\sum_{x \notin E_n} |c(g_n^{-1}xg_n) - c(x)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{x \notin E_n, x \neq g_n^{-1}xg_n} |c(g_n^{-1}xg_n) - c(x)|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{g_n^{-1}xg_n \notin E_n} |c(g_n^{-1}xg_n)|^2\right)^{\frac{1}{2}} + \left(\sum_{x \notin E_n} |c(x)|^2\right)^{\frac{1}{2}} \\
&< \varepsilon,
\end{aligned}$$

since, if $x \notin E_n$ and if $g_n^{-1}xg_n \neq x$, then $g_n^{-1}xg_n \notin E_n$.

This proves that $\lim_{n\to\infty} \|[a_n, c]\|_2 = 0$, and similarly, $\lim_{n\to\infty} \|[b_n, c]\|_2 = 0$. This implies that a and b belong to $L(G)' \cap L(H)^\omega$, which means that it is a noncommutative algebra. □

The next corollary proves the first half of Theorem 2.1:

COROLLARY 2.5. — *The pair $L(F') \subset L(D)$ has the relative McDuff property.*

Proof. — Given a finite subset E of D, there exists a positive integer N such that $x(t) = t$ for $t \in [1 - 2^{-N}, 1]$. Let Γ be the subgroup of all elements $g \in D$ such that $g(t) = t$ for $t \in [0, 1 - 2^{-N}]$. Clearly, Γ is contained in F' and in the centralizer of E. Moreover, Γ is non abelian because it is actually isomorphic to D. This proves the existence of the required pair $g, h \in \Gamma$ to apply Proposition 2.4. □

It is noticed in the introduction that F is isomorphic to the semidirect product group $D \rtimes_\alpha \mathbb{Z}$; this implies that $L(F)$ is spatially isomorphic to the crossed product $L(D) \rtimes_\alpha \mathbb{Z}$, where $\alpha^m = Ad(\lambda(x_0^m))$, $\forall m \in \mathbb{Z}$. This fact plays a crucial role in the proof of the second half of Theorem 2.1, for which we need to recall the following definitions: Let M be a type II_1 factor and let θ be an automorphism of M; then θ is *centrally trivial* if one has for every central sequence $(a_n)_{n\geq 1}$

$$\lim_{n\to\infty} \|\theta(a_n) - a_n\|_2 = 0.$$

The group of centrally trivial automorphisms of M is denoted by $Ct(M)$.

If G is a countable group and if α is an action of G on M, then α is called *centrally free* if $\alpha_g \notin Ct(M)$ for every $g \in G - \{e\}$.

The following proposition makes precise Lemma 5 of [2], and it relies on the deep Theorem 8.5 of [14], where the hypotheses are the same as ours:

PROPOSITION 2.6. — *Let N be a McDuff factor of type* II_1 *with separable predual, let G be an amenable countable group and let α be a centrally free action of G on N. Then the pair $N \subset N \rtimes_\alpha G$ has the relative McDuff property.*

Proof. — Set $M = N \rtimes_\alpha G$. Every element $a \in M$ is expressed as a series

$$a = \sum_{g \in G} a(g)u(g)$$

that converges in the following sense:

$$\|a\|_2^2 = \sum_{g \in G} \|a(g)\|_2^2$$

where $a(g) \in N \ \forall g$, and $u : G \longrightarrow U(M)$ is a homomorphism that implements the action α:

$$u(g)bu(g^{-1}) = \alpha_g(b) \ \ \forall b \in N, \ \ \forall g \in G.$$

Moreover, there is a unique normal, faithful conditional expectation E_N from M onto N such that

$$E_N\left(\sum_g a(g)u(g)\right) = a(e) \ \ \forall a \in M.$$

In particular, the trace tr on M is $\mathrm{tr}_N \circ E_N$ and the coefficients $a(g)$ of a are given by

$$a(g) = E_N(au(g^{-1})).$$

This implies that $\|a(g)\| \le \|a\| \ \ \forall g \in G$.

Let us prove first that for every finite subset E of G, for all $a_1, \dots, a_n \in M$ such that $a_l(g) = 0 \ \ \forall l$ and $\forall g \notin E$, and for every $\varepsilon > 0$, there exist a type I_2 subfactor B of N and a system of matrix units $(e_{ij})_{1 \le i,j \le 2}$ in B satisfying

$$\|[a_l, e_{ij}]\|_2 \le \varepsilon \ \ \forall l, i, j.$$

We assume that $\|a_l\| \leq 1 \ \ \forall l = 1, \ldots, n$. By Theorem 8.5 of [14], there exist a unitary α-cocycle $v : G \longrightarrow U(N)$ and a hyperfinite subfactor R of N such that

(1) $N = R \vee (R' \cap N)$;

(2) $\operatorname{Ad} v_g \circ \alpha_g | R = \mathrm{id}_R \ \ \forall g \in G$;

(3) $\|v_g - 1\|_2 \leq \varepsilon / 5|E| \ \ \forall g \in E$.

Using (1), for every $l = 1, \ldots, n$ and every $g \in G$, there exists $b_l(g)$ in the *-algebra generated by R and $R' \cap N$, with $b_l(g) = 0$ for $g \notin E$, such that

$$\|b_l(g)\| \leq \|a_l(g)\| \leq 1$$

and

$$\|b_l(g) - a_l(g)\|_2 \leq \frac{\varepsilon}{5|E|}.$$

Moreover, since R is hyperfinite, there exist a type I_2 subfactor B of R and a system of matrix units (e_{ij}) in B such that

$$\|[b_l(g), e_{ij}]\|_2 \leq \frac{\varepsilon}{5|E|} \quad \forall l, i, j, \text{ and } g \in G.$$

Set $b_l = \sum_{g \in G} b_l(g) u(g)$.

Then fix $l \in \{1, \ldots, n\}$ and $1 \leq i, j \leq 2$; we have

$$\begin{aligned} \|[a_l, e_{ij}]\|_2 &\leq 2 \sum_{g \in E} \|a_l(g) - b_l(g)\|_2 + \|[b_l, e_{ij}]\|_2 \\ &\leq \frac{2\varepsilon}{5} + \|[b_l, e_{ij}]\|_2. \end{aligned}$$

In order to estimate $\|[b_l, e_{ij}]\|_2$, observe that by (2) and (3) above, we have

$$u(g) e_{ij} u(g^{-1}) = \alpha_g(e_{ij}) = v_g^{-1} e_{ij} v_g$$

and

$$\|v_g^{-1} e_{ij} v_g - e_{ij}\|_2 \leq 2\|v_g - 1\|_2 \leq \frac{2\varepsilon}{5|E|}.$$

Then

$$\begin{aligned}\|[b_l, e_{ij}]\|_2 &\le \sum_{g\in E} \|b_l(g)u(g)e_{ij} - e_{ij}b_l(g)u(g)\|_2 \\ &= \sum_{g\in E} \|b_l(g)u(g)e_{ij}u(g^{-1}) - e_{ij}b_l(g)\|_2 \\ &\le \sum_{g\in E} \left\{\|b_l(g)(v_g^{-1}e_{ij}v_g - e_{ij})\|_2 + \|[b_l(g), e_{ij}]\|_2\right\} \\ &\le \sum_{g\in E} \left\{\|b_l(g)\|\, \|v_g^{-1}e_{ij}v_g - e_{ij}\|_2 + \frac{\varepsilon}{5|E|}\right\} \\ &\le \frac{3\varepsilon}{5}.\end{aligned}$$

This implies that $\|[a_l, e_{ij}]\|_2 \le \varepsilon \ \forall l = 1, \ldots, n$ and $1 \le i, j \le 2$.

The *–subalgebra $M_0 = \left\{ \sum_{\text{finite}} a(g)u(g);\ a(g) \in N \right\}$ of M is $\|\cdot\|_2$–dense, and using Kaplansky's density theorem, the same conclusion holds for all $a_1, \ldots, a_n \in M$. □

The next corollary gives the second half of Theorem 2.1:

COROLLARY 2.7. — *The pair $L(D) \subset L(F)$ has the relative McDuff property.*

Proof. — We are going to use existence and uniqueness of a normal form for every non trivial element $x \in F$ (see [5], p.369): x can be written in a unique way as

$$x = x_{i_1} \ldots x_{i_k} x_{j_m}^{-1} \ldots x_{j_1}^{-1}$$

where $0 \le i_1 \le \ldots \le i_k$, $0 \le j_1 \le \ldots \le j_m$, $i_k \ne j_m$, and if x_i and x_i^{-1} appear in the decomposition of x, then so does x_{i+1} or x_{i+1}^{-1}.

Since $L(D)$ is a McDuff factor, it suffices to check that the action α of $\mathbb{Z}$ on $L(D)$ given by

$$\alpha(a) = \lambda(x_0)a\lambda(x_0^{-1})$$

is centrally free.

Fix a positive integer m; we have to exhibit a central sequence $(a_n)_{n\ge 1}$ in $L(D)$ such that

$$\liminf_n \|\alpha^m(a_n) - a_n\|_2 > 0.$$

For $n \geq 1$, set $a_n = \lambda(x_{n+m}x_{n+m+1}^{-1}) \in L(D)$. Then

$$\begin{aligned} \alpha^m(a_n) &= \lambda(x_0^m x_{n+m}x_{n+m+1}^{-1}x_0^{-m}) \\ &= \lambda(x_n x_{n+1}^{-1}) \\ &\neq \lambda(x_{n+m}x_{n+m+1}^{-1}) \end{aligned}$$

by uniqueness of normal forms.

This implies that

$$\|\alpha^m(a_n) - a_n\|_2 = \sqrt{2} \ \forall n \geq 1.$$

Finally, $(a_n)_{n\geq 1}$ is a non trivial central sequence because for every finite subset E of D, $x_n x_{n+1}^{-1}$ commutes with every $x \in E$ for n large enough: see the proof of Proposition 2.4. □

Remark. — We are indebted to the referee for the following observation: F is a semidirect product of F' with $\mathbb{Z}^2$, therefore the pair $L(F') \subset L(F)$ has also the relative Mc Duff property, using again Proposition 2.6. However, one has to check that the action of $\mathbb{Z}^2$ on $L(F')$ is centrally free. Indeed, define $\varphi : F \longrightarrow \mathbb{Z}^2$ and $\sigma : \mathbb{Z}^2 \longrightarrow F$ by $\varphi(x_0) = (1,0)$, $\varphi(x_n) = (0,1)$ for $n \geq 1$, and $\sigma((1,-1)) = x_0x_1^{-1}$, $\sigma((0,1)) = x_2$. Since $x_0x_1^{-1}$ commutes with x_k for every $k \geq 2$, σ is a homomorphism and a section for φ. The action α of $\mathbb{Z}^2$ on $L(F')$ is given by

$$\alpha_{(m,n)}(\lambda(x)) = \lambda((x_0x_1^{-1})^m x_2^n x x_2^{-n}(x_0x_1^{-1})^{-m}),$$

for $(m,n) \in \mathbb{Z}^2$ and $x \in F'$. Fix $(m,n) \neq (0,0)$. If $n \neq 0$, set $a_k = \lambda(y_k)$, for $k \geq 1$, where $y_k = [x_k, x_{k+1}] := x_k x_{k+1} x_k^{-1} x_{k+1}^{-1} = x_k x_{k+1} x_{k+2}^{-1} x_k^{-1}$ in normal form. Since $y_k(t) = t$ for $t \leq 1 - 2^{-k}$ (as a function on $[0,1]$), $(a_k)_{k\geq 1}$ is a central sequence in $L(F')$. Moreover, $\alpha_{(m,n)}(a_k) = a_{k-n}$ for every $k \geq |m| + |n| + 3$. Finally, if $n = 0$, set $z_k = x_0^{k+1}x_2^2x_3^{-1}x_1^{-1}x_0^{-k-1}$ and $b_k = \lambda(z_k)$ for $k \geq 1$. In the geometrical realization of F, $z_k = \theta(y_k)$, where θ is the automorphism of F given by $\theta(x)(t) = 1 - x(1-t)$. Then it can be proved that $z_k \in F'$, that $z_k(t) = t$ for $t \geq 2^{-k}$ and that $\alpha_{(m,0)}(b_k) = b_{k+m}$ for large k. Using the same arguments as in the proof of Corollary 2.7, this shows that α is centrally free.

3. Contractibility of unitary groups.

In the introduction, we noticed that the following result follows from Theorem 1 of [15], but we think it interesting to give a proof avoiding type

III factors techniques, and based on a clever realization of the hyperfinite factor of type II_∞ due to P.-L. Aubert [1].

THEOREM 3.1. — *Let M be a type II_1 factor with separable predual and let N be a subfactor of M such that the pair $N \subset M$ has the relative McDuff property. Then there exists a continuous map $\alpha : [0,\infty[\times U(M) \longrightarrow U(M)$ with the following properties:*

(1) $\alpha_t(U(N)) \subset U(N)\ \ \forall t \geq 0$;

(2) $\alpha_0(u) = u$ *and* $\lim_{t\to\infty} \alpha_t(u) = 1$, $\forall u \in U(M)$;

(3) *each* α_t *is an injective endomorphism of* $U(M)$;

(4) $\alpha_s \circ \alpha_t = \alpha_{s+t}$, $\forall s,t \geq 0$;

(5) $\|\alpha_t(u) - \alpha_t(v)\|_2 = e^{-\frac{t}{2}}\|u-v\|_2$, $\forall t \geq 0$, $\forall u,v \in U(M)$.

COROLLARY 3.2. — *The unitary groups of the pairs of factors $L(F') \subset L(D)$ and $L(D) \subset L(F)$ have contractibility properties of Theorem 3.2. In particular, they are contractible.*

Proof. — Let A denote the Abelian von Neumann algebra $L^\infty(\mathbb{R}^2)$ relative to the Lebesgue measure μ on $\mathbb{R}^2$. The group $\Gamma = SL_2(\mathbb{Z})$ acts canonically on $\mathbb{R}^2$ and preserves μ.

We denote by α the associated action of Γ on A:

$$\alpha_\gamma(a) = a \circ \gamma^{-1}\ \ \forall a \in A,\ \gamma \in \Gamma.$$

Then set $R_\infty = A \rtimes_\alpha \Gamma$; it is proved in [1] that R_∞ is the hyperfinite factor of type II_∞ with separable predual. Denote by E_A the natural conditional expectation from R_∞ onto A. Then the semifinite trace Tr on $(R_\infty)_+$ is

$$\mathrm{Tr}(x) = \int_{\mathbb{R}^2} E_A(x)d\mu\ \ \forall x \in R_\infty.$$

Set $e_0 = \chi_{I\times I}$, where $I = [0,1]$; e_0 is a projection belonging to A, and the reduced factor $e_0 R_\infty e_0$ is the hyperfinite II_1-factor R because $\mathrm{Tr}(e_0) = 1$, and the normalized trace on R is thus $tr(x) = \mathrm{Tr}(e_0 x e_0)$.

Now, for $t \in \mathbb{R}$, let $\sigma_t \in \mathrm{Aut}(A)$ be defined by

$$\sigma_t(a)(x,y) = a(e^{\frac{t}{2}}x, e^{\frac{t}{2}}y)\ \ \forall (x,y) \in \mathbb{R}^2,\ \forall a \in A.$$

Then $(\sigma_t)_{t\in \mathrm{R}}$ is a one parameter group of automorphisms of A and

$$\sigma_t \circ \alpha_\gamma = \alpha_\gamma \circ \sigma_t\ \ \forall t \in \mathbb{R},\ \ \forall \gamma \in \Gamma.$$

Hence σ_t extends to an automorphism of R_∞, still denoted by σ_t, such that

$$\sigma_t(x) = \sum_{\gamma\in\Gamma} \sigma_t(x(\gamma))u(\gamma) \ \ \forall x \in R_\infty.$$

Now let $e_t = \sigma_t(e_0)$, which is the projection of A corresponding to the characteristic function of $[0, e^{-\frac{t}{2}}] \times [0, e^{-\frac{t}{2}}]$, so that $e_t \leq e_0 \quad \forall t \geq 0$. Moreover, if $a \in (R_\infty)_+$, we have

$$\begin{aligned}\mathrm{Tr}\circ\sigma_t(a) &= \int E_A(\sigma_t(a))(x,y)d\mu(x,y)\\ &= \int E_A(a)(e^{\frac{t}{2}}x, e^{\frac{t}{2}}y)d\mu(x,y)\\ &= e^{-t}\,\mathrm{Tr}(a) \ \ \forall t \in \mathbb{R}.\end{aligned}$$

Finally, let Φ be an isomorphism from M onto $M \otimes e_0R_\infty e_0$ such that $\Phi(N) = N \otimes e_0R_\infty e_0$; set $\theta_t = id_M \otimes \sigma_t$, $p_0 = 1 \otimes e_0 \in M \otimes e_0R_\infty e_0$ and $p_t = \theta_t(p_0) = 1 \otimes e_t \leq p_0$ for $t \geq 0$.

Following [15], define $\alpha_t : U(M) \longrightarrow U(M)$ by

$$\alpha_t(u) = \Phi^{-1}(p_0 - p_t + \theta_t(\Phi(u))).$$

One checks easily that α has the required properties (1)–(5). □

BIBLIOGRAPHY

[1] P.-L. AUBERT, Deux actions de $SL(2,\mathbb{Z})$, In Théorie ergodique, Monographie de l'E.N.S. Math., 1981.

[2] E. BÉDOS, On actions of amenable groups on II_1-factors, J. Funct. Anal., 91 (1990), 404–414.

[3] D. BISCH, On the existence of central sequences in subfactors, Trans. Amer. Math. Soc., 321 (1990), 117–128.

[4] D. BISCH, Central sequences in subfactors II, Proc. Amer. Math. Soc., 121 (1994), 725–731.

[5] M.G. BRIN, and C.C. SQUIER, Groups of piecewise linear homeomorphisms of the real line, Invent. Math., 79 (1985), 485–498.

[6] J.W. CANNON, and W.J. FLOYD, and W.R. PARRY, Introductory notes on Richard Thompson's groups, E.N.S. Math., 42 (1996), 215–256.

[7] A. CONNES, Outer conjugacy classes of automorphisms of factors, Ann. Scient. Ec. Norm. Sup., 8 (1975), 383–420.

[8] A. CONNES, Classification of injective factors, Ann. of Math., 104 (1976), 73–115.

[9] J. DIXMIER, and E.C. LANCE, Deux nouveaux facteurs de type II_1, Invent. Math., 7 (1969), 226–234.

[10] E.G. EFFROS, Property Γ and inner amenability, Proc. Amer. Math. Soc., 47 (1975), 483–486.

[11] S.M. GERSTEN, and J.R. STALLINGS (eds), Combinatorial Group Theory and Topology, in Annals of Math. Studies 111, Princeton University Press, 1987.

[12] P. JOLISSAINT, Moyennabilité intérieure du groupe F de Thompson, C.R. Acad. Sci. Paris, Série I, 325 (1997), 61–64.

[13] D. MCDUFF, Central sequences and the hyperfinite factor, Proc. London Math. Soc., 21 (1970), 443–461.

[14] A. OCNEANU, Actions of discrete amenable groups on von Neumann algebras, Lect. Notes in Math. 1138, Springer Verlag, 1985.

[15] S. POPA and M. TAKESAKI, The topological structure of the unitary and automorphism groups of a factor, Comm. Math. Phys., 155 (1993), 93–101.

Manuscrit reçu le 15 octobre 1997,
accepté le 9 février 1998.

Paul JOLISSAINT,
Université de Neuchâtel
Institut de Mathématiques
Émile-Argand 11
CH-2000 Neuchâtel.
paul.jolissaint@maths.unine.ch