ANNALES DE L'INSTITUT FOURIER

FERNANDO SANZ

Non oscillating solutions of analytic gradient vector fields

Annales de l'institut Fourier, tome 48, nº 4 (1998), p. 1045-1067

<http://www.numdam.org/item?id=AIF_1998__48_4_1045_0>

Ann. Inst. Fourier, Grenoble
48, 4 (1998), 1045–1067

NON OSCILLATING SOLUTIONS OF ANALYTIC GRADIENT VECTOR FIELDS

by Fernando SANZ(*)

1. Introduction and statement of the results.

Let ξ be an analytic vector field defined in an analytic real manifold M. This paper gives a contribution to the local study at a singular point P in the case of a gradient vector field $\xi = \nabla_g f$, where f is a real analytic function and g is an analytic riemannian metric on M. In view of an inequality of Łojasiewicz [10], any positive solution $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$ of $\nabla_g f$ contained in a neighbourhood of P has a finite length. Thus the ω-limit set $\omega(\gamma)$ is a single point. Suppose that $\omega(\gamma) = \{P\}$. René Thom [16] has asked if γ *has a well defined "tangent"* at this point. A positive answer to this question is known from the beginning for the two dimensional case and under certain hypotheses [6], [7], [11] for higher dimensions. Recently, a general proof of the existence of tangent for $M = \mathbf{R}^n$ and the euclidean canonical metric has been announced in [9].

Fix a positive solution $\gamma : t \mapsto \gamma(t)$, $t \geqslant 0$ of the vector field ξ such that $\omega(\gamma) = \{P\}$.

A condition stronger than the existence of a tangent is the *non oscillation property*: γ is non oscillating at P if for every locally closed analytic hypersurface H through P we have that either γ is contained in H or γ cuts H only finitely many times. If M is two dimensional then both properties are equivalent by using finiteness Khovanski's theory [12]. In view of Thom's question, R. Moussu has proposed the following.

(*) Partially supported by DGICYT; PB94-1124 and TMR; ERBFMRXCT96-0040.
Key words: Vector field – Gradient – Tangent – Oscillation – Blowing-up – Desingularization – Center manifold.
Math. classification: 34C05 – 34C45 – 32S45.

Strong gradient conjecture: Any solution γ of $\nabla_g f$ with a single ω-limit point is non oscillating.

For the rest of this paper we restrict ourselves to the case that M is three dimensional. The main result is the following:

THEOREM 1. — *Let M be a three dimensional analytic manifold and f an analytic function on M. Assume that the hessian $\operatorname{Hess} f(P)$ is not zero at any singular point P of f. Then for any analytic riemannian metric g on M the strong gradient conjecture is true for $\nabla_g f$.*

Non oscillation in terms of blowing-ups. We can make an interpretation of the non oscillation property for γ in terms of the existence of "iterated generalized tangents".

To say that γ has a tangent at P means that the lift γ_1 of γ by the blowing-up at P has a single ω-limit point P_1. Then γ_1 has also a tangent if its lift γ_2 by the blowing-up at P_1 has a single ω-limit point P_2 and so on. We say that γ has the property of *existence of all iterated tangents* at P if this process can be continued indefinitely. The sequence of points $\{P_i\}_{i\geqslant 0}$ so constructed is called the sequence of iterated tangents of γ and denoted by $IT(\gamma)$. Recall that for an irreducible formal curve Γ through P the sequence of tangents $IT(\Gamma)$ (or infinitely near points) is well defined.

The non oscillation property implies the existence of all iterated tangents. In fact, suppose that for a sequence of quadratic transformations (blowing-ups of points) $\tilde{\pi} : \tilde{M} \to M$, the lift of γ has two different ω-limit points P_1 and P_2 in the last exceptional divisor D. By the projective nature of the blowing-up, we can take a surface H through P such that the strict transform of H separates P_1 and P_2 in two different connected components. Thus, γ is oscillating with respect to H. Nevertheless, the non oscillation condition is generally stronger than the property of existence of iterated tangents. Take, for example the vector field ξ in $\mathbf{R}^3$ given by

$$\xi = (-x - yz)\frac{\partial}{\partial x} + (-y + xz)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}. \tag{1}$$

Any solution $\gamma : [0,\infty) \to \mathbf{R}^3$ not contained in the (x, y) plane or in the z-axis turns around this axis and is infinitely tangent to it. Then γ has all iterated tangents but it is oscillating with respect to any plane containing the z-axis.

In order to get the non oscillation condition we need to consider the *existence of the iterated generalized tangents*, given in terms of a large class of blowing-ups and not only quadratic ones. Let Y be either a point or a smooth analytic curve of M. We say that Y is admissible for (ξ, γ) if it is invariant by ξ and γ does not cut Y. Suppose that $\omega(\gamma)$ is a single point P. We say that γ has the property of existence of the iterated generalized tangents at P if for every sequence of admissible blowing-ups

$$M = M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} M_n \tag{2}$$

the last lift γ_n of γ has a single ω-limit point P_n. In §2 we give a proof of the following:

PROPOSITION 1. — *If γ has the property of existence of all the iterated generalized tangents at P then it is non oscillating at P.*

Non nilpotent analytic vector fields. In §3 we study the local dynamics of a vector field ξ in a neighbourhood of a singular point P which has a non nilpotent linear part $L_P(\xi)$ (semihyperbolic vector field). We first recall some known facts about local invariant manifolds through P, related to the stable, unstable and central parts of $L_P(\xi)$: the stable, unstable and center manifold. We look to the local topological reduction of the dynamics to the center manifold and we compare the velocities of the solutions outside and inside this invariant manifold. This allows us to reduce the problem of existence of a tangent to a lower dimensional case.

To look for the existence of iterated generalized tangents, we need to study also the asymptotic behaviour of the solutions of certain vector fields in a neighbourhood of a circle. These vector fields are naturally obtained by admissible blowing-ups.

Finally, in §4, we define a class $\mathcal{G}_0(M;P)$ of three dimensional semihyperbolic vector fields for which any solution that converges to P is non oscillating.

Gradient vector fields. In §5 we finish the proof of the main theorem showing that the gradient vector field $\xi = \nabla_g f$ belongs to the class $\mathcal{G}_0(M;P)$ for every singular point P of f. This justifies the letter $\mathcal{G}$ for this class. The subindex 0 indicates that we expect to find larger classes of non oscillating vector fields whose union contains all the gradient vector fields.

The results in this paper have been announced in [14].

2. Non oscillating curves and iterated generalized tangents.

In this section we give the proof of Proposition 1. Suppose, by contradiction, that γ has all iterated generalized tangents and it is oscillating with respect to an analytic surface H. Let $\pi : M' \to M$ be an admissible blowing-up. There is a unique analytic vector field ξ' on M' such that $\pi_*\xi' = \xi$, called the (total) transform of ξ by π. The lift γ' of γ by π is a solution of ξ'. If H' denotes the strict transform of H and $\omega(\gamma') = \{P'\}$, then $P' \in H'$ and γ' is oscillating with respect to H' at its unique ω-limit point $P' \in H'$.

First case: H non singular. Let $C \subset H$ be the set of points where ξ is tangent to H. Since H is not invariant by ξ we have $C \neq H$ and C is an analytic set of dimension $\leqslant 1$. We consider first the following situation:

(A) *The set C is contained in $D \cap H$, where $D \cup H$ is a normal crossings divisor and D is invariant by ξ.*

Fix local coordinates at P such that $H \cup D$ is union of coordinate planes. Since D is invariant, we have that $|\gamma| \cap D = \varnothing$ and the points where γ cuts H are all in a single connected component L of $H \setminus H \cap D$. Denote by z a local coordinate such that $H = (z = 0)$. The function $dz(\xi)$ has constant sign over L since $C \cap L = \varnothing$. Thus, the derivative of $t \mapsto z(\gamma(t))$ has a constant sign at any of its zeroes, contradicting the hypotheses that there exists at least two of them.

We will now reduce us to the situation (A) after some admissible blowing-ups. The behaviour of C under admissible blowing-up is described by the following:

LEMMA 1. — *Let $\pi : M' \to M$ be an admissible blowing-up with center $Y \subset H$. Let H' be the strict transform of H by π and define $C' \subset H'$ as the set of points where the transform ξ' is tangent to H'. Then C' is the union of the strict transform of C and, eventually, the curve $H' \cap \pi^{-1}(Y)$.*

The proof is immediate.

Let $IT(\gamma) = \{P_i\}_{i=0}^{\infty}$ be the sequence of tangents of γ. We have two possibilities:

a) The sequence $IT(\gamma)$ is not the sequence of infinitely near points of a branch of C. That is, for a certain index i_0, the point P_{i_0} is not in the corresponding strict transform of C. At this point, using Lemma 1 we have

$C \subset D \cap H$ where D is the invariant divisor created by the blowing-ups at the points P_i, $i < i_0$ and $D \cup H$ is a normal crossings divisor.

b) The sequence $IT(\gamma)$ coincides with the sequence of infinitely near points $IT(Y)$ of a branch Y of C. For a certain index i_0 we can suppose that Y is a smooth curve. The points P_i, $i \geqslant i_0$ are singular points not only for the corresponding (total) transform of ξ but for the strict transform at P_i (obtained after division as much as possible by an equation of the exceptional divisor at P_i). Then we have that Y is invariant and hence admissible. Blowing-up Y at the i_0-th stage we get the situation $C \subset D \cap H$ as in the previous case.

Second case: H singular. For any $Q \in M$ denote by $\nu_Q(H)$ the multiplicity of H at Q. We shall work by induction on $\nu = \nu_P(H)$.

Recall that an analytic curve $Y \subset H$ is called equimultiple if $\nu_Q(H)$ is constant for $Q \in Y$. We will reduce our problem to the same one with smaller multiplicity by means of a finite sequence of admissible and equimultiple blowing-ups. Our method is derived from the reduction of singularities of surfaces by using maximal contact theory and characteristic polygons [1], [4].

Maximal contact. We say that a germ of a non singular surface W through P has maximal contact with H iff locally at P we have that

i) If $\nu_Q(H) = \nu_P(H) = \nu$, then $Q \in W$.

ii) Let $\pi : M' \to M$ be the blowing-up with center either the point P or a non singular equimultiple curve $Y \subset H$. Denote by W', H' the strict transforms of W, H by π. For any $Q' \in H'$, we know that $\nu_{Q'}(H') \leqslant \nu$. If $\nu_{Q'}(H') = \nu$ then $Q' \in W'$. Moreover the same properties i) and ii) repeat at Q'.

We can get a maximal contact surface W to be the plane $(z = 0)$ if an equation for H is written as:

$$z^\nu + h^{\nu-2}(x,y)z^{\nu-2} + \cdots + h^0(x,y) = 0.$$

This can be obtained from the Weierstrass preparation theorem, followed by a Tschirnhäusen's transformation.

For any sequence of admissible and equimultiple blowing-ups (2), the successive strict transforms H_i of H satisfy

$$\nu_{P_i}(H_i) \leqslant \nu \tag{3}$$

where γ_i is the lift of γ_{i-1} by π_i and $\omega(\gamma_i) = \{P_i\}$. If the inequality in (3) is strict at one stage, then we are done. Hence we assume, by contradiction, that the equality holds in (3) for any sequence of admissible blowing-ups.

Let us fix a maximal contact surface W and let $C_\nu \subset H$ be the set of points where the multiplicity of H is equal to ν. It is an analytic set contained in W near P. If $\pi : M' \to M$ is a blowing-up with smooth equimultiple center $Y \ni P$ and $C_\nu' \subset H'$ is the set of points where the multiplicity of H' is equal to ν, then, by the properties of the Maximal Contact, C_ν' is the union of the strict transform of C_ν and, eventually, the curve $W' \cap \pi^{-1}(Y)$. From this, making only admissible and equimultiple blowing-ups, we reduce us, as in cases a) or b) above, to the situation

$$C_\nu \subset D \cap W \tag{4}$$

where $D \cup W$ is a normal crossings divisor and D is invariant by ξ. Furthermore, this property persists under admissible and equimultiple blowing-up near the unique ω-limit point of the lift of γ.

Let us construct now a particular sequence of admissible and equimultiple blowing-ups as follows: if there is an admissible curve $Y \subset C_\nu$ through $P_0 = P$ then blow-up it, otherwise blow-up the point P. Let P_1 be the unique ω-limit point of the lift of γ. We follow the same criteria to choose the next center through P_1 to blow-up and so on.

Take coordinates (x, y, z) at P such that $W = (z = 0)$ and D in (4) is contained in $(xy = 0)$. Let $(h = 0)$ be an equation for H near P and write

$$\begin{aligned} h(x,y,z) &= \sum_{i+j+k \geqslant \nu} h_{ijk} x^i y^j z^k \\ dz(\xi)(x,y,0) &= \sum_{i+j \geqslant 1} c_{ij} x^i y^j . \end{aligned}$$

Consider the following discrete set of points:

$$N = \left\{ \left(\frac{i}{\nu - k}, \frac{j}{\nu - k} \right) ; \, h_{ijk} \neq 0, \, \nu > k \right\} \cup \{(i,j) \, ; \, c_{ij} \neq 0\}$$

in the plane $\mathbf{R}^2_{(u,v)}$. The y-axis is admissible and contained in C_ν if and only if $N \subset \{u \geqslant 1\}$. Similar conditions for the x-axis and the set $\{v \geqslant 1\}$. Let Δ be the convex hull of $N + \mathbf{R}^2_0$ in $\mathbf{R}^2_0$. We call Δ to be the Characteristic Polygon at stage 0. Let $(\alpha, \beta) \in \Delta$ (resp. (α', β')) be the vertex of smallest abscissa of Δ (resp. smallest ordinate). The numbers β and α' will be the main invariants to control the singularity in our particular sequence of blowing-ups. Namely, we can choose coordinates (x_i, y_i, z_i) at P_i, inductively for $i = 1, 2, \ldots$, which are related to the ones at P_{i-1} by one of the following transformations:

$$\begin{aligned}
(\mathrm{T1},\zeta): &\quad x_{i-1} = x_i,\ y_{i-1} = x_i(y_i + \zeta),\ z_{i-1} = x_i z_i \\
(\mathrm{T2}): &\quad x_{i-1} = x_i y_i,\ y_{i-1} = y_i,\ z_{i-1} = y_i z_i \\
(\mathrm{T3}): &\quad x_{i-1} = x_i,\ y_{i-1} = y_i,\ z_{i-1} = x_i z_i \\
(\mathrm{T4}): &\quad x_{i-1} = x_i,\ y_{i-1} = y_i,\ z_{i-1} = y_i z_i.
\end{aligned}$$

Let Δ_i be the Characteristic Polygon at stage i, and the corresponding vertices (α_i, β_i), (α_i', β_i'), as above. Due to the fact that we choose a curve as a center when it is admissible and equimultiple, we get that $\beta_i \leqslant \beta_{i-1}$ and the inequality is strict in cases (T2) or (T4), (see [4]). So, we can suppose that all transformations are (T1,ζ) or (T3) and, thus, $\beta' < 1$. Furthermore, by a formal change of variables $y \mapsto y - \phi(x)$, all ζ can be assumed to be 0. Now we have

$$\alpha_i{}' < \alpha_{i-1}{}'$$

which is not possible infinitely many times. This is the desired contradiction.

3. Solutions near center manifolds and circles.

Let M be an analytic real manifold and let ξ be an analytic vector field on M. Assume that $\xi(P) = 0$ for a point $P \in M$ and that the linear part of ξ at P, denoted by $L_P(\xi)$, is non nilpotent. Let N^u, N^s, N^c be the eigenspaces in $T_P M$ of $L_P(\xi)$ corresponding to the eigenvalues with positive, negative and zero real part, respectively. For any $k \in \mathbf{N}$ there exists a neighbourhood V of P and invariant manifolds for ξ through P in V of class $\mathcal{C}^k$

$$W^u, W^s, W^c, W^{cu}, W^{cs}$$

whose tangent spaces at P are respectively N^u, N^s, N^c, $N^u \oplus N^c$, $N^s \oplus N^c$. They are called the unstable, stable, center, center-unstable and center-stable manifold. The stable and unstable ones are unique and analytic. Solutions of ξ starting at a point of W^s (resp. W^u) tend to P exponentially as t goes to ∞ (resp. $-\infty$). Also, if V is sufficiently small then any positive solution γ of ξ such that $\mid \gamma \mid \subset V$ is contained in all the center-stable manifolds in V. Similarly, any center manifold W^c contains the solutions defined for all $t \in \mathbf{R}$ that remain bounded in a neighbourhood of P. For instance, any singular point near P belongs to W^c. (See e.g. [5], [15]).

As we can see in [2], the k-jet of any center manifold of class $\mathcal{C}^l$ with $l \geqslant k$ is uniquely determined. Hence there is a unique formal manifold $\widehat{W^c}$ formally invariant by ξ and tangent to N^c. It is called the formal center manifold of ξ at P.

LEMMA 2. — *Let ξ be an analytic vector field in a neighbourhood of a singularity $P \in M$. Assume that*

$$\dim N^c(P) > \dim_P \mathrm{Sing}\,(\xi) \geqslant 0.$$

Then there is an integer $k \geqslant 0$ and a center manifold W^c of class $\mathcal{C}^k$ such that $\xi^c = \xi\mid_{W^c}$ is a vector field of class $\mathcal{C}^k$ for which the k-jet $j^k\xi^c(P)$ is not zero.

Proof. — Let us reason by contradiction, assuming that $j^k\xi\mid_{W^c}(P) = 0$ for all center manifold W^c of class $\mathcal{C}^k$, $k \geqslant 0$. This means that if $\widehat{W^c}$ is the formal center manifold of ξ at 0 then the formal restriction of ξ to $\widehat{W^c}$ satisfies

$$\xi\mid_{\widehat{W^c}} \equiv 0.$$

So we have $\widehat{W^c} \subset \mathrm{Sing}\,(\xi)$ formally. This contradicts our hypotheses about the dimension of the singular set of ξ.

There is a global version of the topological reduction to a center manifold ([13]) along an invariant compact $\mathcal{C}^1$-submanifold S of M. We state here a special version.

PROPOSITION 2. — *Let ξ be an analytic vector field defined in a neighbourhood U of $S = \mathbf{S}^1 \times \{0\} \subset \mathbf{S}^1 \times \mathbf{R}^n \times \mathbf{R}^m$, written in coordinates (θ, x, y) as*

$$\xi = \Theta(\theta,x,y)\frac{\partial}{\partial\theta} + (Ax + f(\theta,x,y))\frac{\partial}{\partial x} + (By + g(\theta,x,y))\frac{\partial}{\partial y},$$

where A, B are matrices whose eigenvalues have zero and negative real parts, respectively; the functions Θ, f, g are 2π-periodic in the variable θ and zero over S and f, g are of order two in the variables x, y. Then:

1) There is a center manifold $W^c(S)$ of class $\mathcal{C}^2$, in a neighbourhood of S, invariant by ξ and tangent to $\mathbf{S}^1 \times \mathbf{R}^n \times 0$ at any point of S.

2) Let $W^c(S)$ be such a center manifold and $\gamma : [0,\infty) \to U$ a solution of ξ such that $\omega(\gamma) \subset S$, there is $t_0 \geqslant 0$ and a unique solution $\sigma : [t_0,\infty) \to W^c(S)$ of $\xi \mid_{W^c(S)}$ that approaches γ in the sense that

$$|\gamma(t) - \sigma(t)| \leqslant Ke^{-\varepsilon t}, \quad t \geqslant t_0 \tag{5}$$

for some $K, \varepsilon > 0$.

Remark 1. — If ξ does not depend on θ then S reduces to a singular point P and 2) holds for a local center manifold through P. We also deduce from the proof of Proposition 2 that σ in 5 reduces to the singular point P if and only if γ is contained in the stable manifold of ξ at P.

The solutions contained in a center manifold can not accumulate exponentially to the singular point, as we see in the following lemma, deduced also from the proof of Proposition 2.

LEMMA 3. — *Let ξ be a C^2-vector field singular at the origin of $\mathbf{R}^n$ whose linear part has all eigenvalues with zero real part. If $\sigma : [0,\infty) \to \mathbf{R}^n$ is a non constant solution such that $\omega(\sigma) = \{0\}$, then for any $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} e^{\varepsilon t} |\sigma(t)| = \infty. \tag{6}$$

Using the estimations (5) and (6) and Remark 1 we can show the following:

COROLLARY 1. — *Suppose that $\omega(\gamma) = \{P\}$ and γ is not contained in the stable manifold. Then γ is tangent to any center manifold W^c. That is, blowing-up the singular point P, the lift γ' of γ accumulates to the projectivized of the tangent space of W^c at P. Moreover, if the (non constant) solution σ in the center manifold that approaches γ has a well defined tangent then so does γ.*

Limit sets on a circle. Consider a circle $S = \mathbf{S}^1 \times 0 \subset \mathbf{S}^1 \times \mathbf{R}^n$ and let ξ be a vector field of class C^k, $k \geqslant 1$, defined in a neighbourhood V of S. Assume that S is an invariant circle for ξ and that ξ has only finitely many singular points over S. Fix a solution $\gamma : [0,\infty) \to V$ such that $\varnothing \neq \omega(\gamma) \subset S$. We consider the covering

$$\begin{array}{ccc} \mathbf{R} \times \mathbf{R}^n & \to & \mathbf{S}^1 \times \mathbf{R}^n \\ (\alpha, x) & \mapsto & (\exp i\alpha, x). \end{array}$$

Then there is a lift $\tilde{\gamma} : [0, \infty) \to \mathbf{R} \times \mathbf{R}^n$ of γ given by $\tilde{\gamma}(t) = (\theta(t), x(t))$. In this situation the ω-limit set $\omega(\gamma)$ is either a single point or the whole circle S. Moreover, if $\omega(\gamma) = S$ then γ "turns" around S. More precisely, we have the following result:

LEMMA 4. — *The limit* $\lim_{t\to\infty} \theta(t)$ *exists and it is either finite, or* $+\infty$, *or* $-\infty$.

Proof. — Let $\tilde{\xi}$ be the lift of ξ to $\mathbf{R} \times \mathbf{R}^n$. Then $\tilde{\gamma}$ is a solution of $\tilde{\xi}$. Moreover $\tilde{\xi}$ has a discrete set of singular points over the invariant line $\mathbf{R} \times \{0\}$. We know that if $\tilde{\gamma} = (\theta(t), x(t))$ then

$$\lim_{t\to\infty} x(t) = 0$$

due to the fact that $\varnothing \neq \omega(\gamma) \subset S$. Assume, by contradiction that $\theta(t)$ accumulates in two (finite or infinite) points as $t \to \infty$, call them α_1, α_2, $\alpha_1 < \alpha_2$. There is a non singular point $P = (\alpha, 0)$ of $\tilde{\xi}$ such that $\alpha_1 < \alpha < \alpha_2$. Taking a flow-box around P for $\tilde{\xi}$, we get $\delta > 0$ such that if $|x(t)| \leqslant \delta$ and $\theta = \alpha$ then $\dfrac{d\theta}{dt}(t)$ has a constant sign, for instance positive. We take $t \geqslant t_0$ so that $|x(t)| \leqslant \delta$. There is a point $t_1 > t_0$ such that $\theta(t_1) > \alpha$ since $\theta(t)$ accumulates in α_2. Now, for any $t \geqslant t_1$ we have that $\theta(t) \geqslant \alpha$ since $\theta(t)$ is an increasing function when $\theta(t) = \alpha$. So α_1 can not be an accumulation value of $\theta(t)$.

We are going to use this lemma in the following situation:

COROLLARY 2. — *Assume that* $\dim M = 3$. *Let* ξ *be an analytic vector field defined in a neighbourhood* V *of* $P \in M$. *Consider a solution* γ *of* ξ *such that* $\omega(\gamma) = \{P\}$ *and let* $\pi : M' \to V$ *be the blowing-up with center an admissible curve* $Y \subset V$ *through* P. *Assume that*

1. *The transform* ξ' *of* ξ *has only finitely many singular points over* $F = \pi^{-1}(P)$.

2. *There is a germ of* C^1*-surface* H' *at a point of* F *which is transversal to* F *and does not cut the lifted solution* γ'.

Then $\omega(\gamma')$ *is a single point.*

Proof. — Take the cylindric blowing-up $p : \mathbf{S}^1 \times U \subset \mathbf{S}^1 \times \mathbf{R}^2 \to V$ of center Y. The transform $\bar{\xi}$ of ξ by p and any of the two lifts $\bar{\gamma}$ of γ satisfy the hypotheses of Lemma 4 where the circle S is the fiber $p^{-1}(P) = \mathbf{S}^1 \times \{0\}$. The strict transform of $\pi(H')$ by p gives two C^1 local surfaces transversal to S at antipodal points which do not cut $\bar{\gamma}$. They separate a neighbourhood of

S in two connected components, each containing points of S. Thus $\omega(\bar{\gamma}) \neq S$ and it is a single point.

Also, in the two dimensional case, Lemma 4 gives us the following corollary:

COROLLARY 3. — *Let ξ be a $\mathcal{C}^k$ vector field in a neighbourhood of $0 \in \mathbf{R}^2$. Suppose that $\xi(0) = 0$ and that $j^\ell \xi(0) \neq 0$ for some $\ell < k$. Assume that there is a $\mathcal{C}^2$ invariant regular curve through 0. Let γ be a non singular solution of ξ such that $\omega(\gamma) = \{0\}$. Then γ has a well defined tangent.*

4. A class of non oscillating vector fields.

Assume that $\dim M = 3$ and fix a point $P \in M$. We denote by $\mathcal{S}R(M;P)$ the set of analytic vector fields ξ such that $\xi(P) = 0$, the linear part $L_P(\xi)$ is non nilpotent and its eigenvalues are real ones (the symbols $\mathcal{S}R$ stand for semihyperbolic real). Given an element $\xi \in \mathcal{S}R(M;P)$, denote by $c(\xi;P)$ the dimension of the center manifolds of ξ at P. Obviously we have that $c(\xi;P) \in \{0,1,2\}$.

PROPOSITION 3. — *Let $\xi \in \mathcal{S}R(M;P)$ and $\pi : M' \to M$ a blowing-up with center Y, non singular and invariant. Let ξ' be the transform of ξ by π and consider a point $P' \in \pi^{-1}(P)$ such that $\xi'(P') = 0$. Then $\xi' \in \mathcal{S}R(M';P')$.*

The proof is an easy computation.

DEFINITION 1. — *Given $\xi \in \mathcal{S}R(M;P)$ we say that ξ belongs to $\mathcal{NO}(M;P)$ if any solution γ of ξ such that $\omega(\gamma) = \{P\}$ is non oscillating at P.* In this paragraph we prove the following theorem:

THEOREM 2. — *Take ξ a vector field in $\mathcal{S}R(M;P)$. Then $\xi \in \mathcal{NO}(M;P)$ if and only if one of the following conditions holds:*

A. *The dimension of the center manifolds is $c(\xi;P) = 0$ (hyperbolic case).*

B. *The dimension of the center manifolds is $c(\xi;P) = 1$ and one of the following is true:*

B-1. *The vector field ξ does not have an analytic center manifold at P.*

B-2. *The two non zero eigenvalues of $L_P(\xi)$ are different or they are equal and positive.*

B-3. *There is an analytic center manifold W of ξ, the linear part $L_P(\xi)$ has a double negative eigenvalue and, moreover, there is a non singular germ of analytic surface $N \supset W$ such that any solution γ with $\omega(\gamma) = \{P\}$ is non oscillating with respect to N.*

C. *The dimension of the center manifolds is $c(\xi; P) = 2$ and there is a center manifold W such that any solution γ contained in W with $\omega(\gamma) = \{P\}$ has a well defined tangent.*

Let $\mathcal{G}_0(M;P)$ be the class of vector fields $\xi \in \mathcal{S}R(M;P)$ such that ξ satisfies one of the conditions A, B or C. It is easy to see that

$$\mathcal{N}O(M;P) \subset \mathcal{G}_0(M;P).$$

Now we show the other inclusion $\mathcal{G}_0(M;P) \subset \mathcal{N}O(M;P)$. In fact we prove the following:

PROPOSITION 4. — *Consider an analytic vector field $\xi \in \mathcal{G}_0(M;P)$ and a solution γ such that $\omega(\gamma) = \{P\}$. Then γ has the property of existence of iterated generalized tangents. In particular, γ is non oscillating. More precisely, if $\pi : M' \to M$ is an admissible blowing-up for (ξ, γ), there is a unique point $P' \in \pi^{-1}(P)$ such that $\omega(\gamma') = \{P'\}$ and $\xi' \in \mathcal{G}_0(M';P')$ where γ' denotes the lift of γ and ξ' is the transform of ξ by π.*

Proof of Proposition 4. — Before the proof we make the following useful remark:

Remark 2. — Consider $\xi \in \mathcal{S}R(M;P)$. Then $\xi \in \mathcal{G}_0(M;P)$ if we check the following conditions. These are naturally obtained the most part of the cases after admissible blowing-up.

a) Assume that $c(\xi; P) = 1$ and conditions B-1, B-2 do not hold. Then we get condition B-3 if there is an analytic smooth invariant surface N transversal to the stable manifold W^s of ξ at P. In fact, in this case N must contain the formal center manifold $\hat{W}^c$ which coincides with the analytic one W since B-1 is not satisfied. Thus B-3 holds.

b) Assume that $c(\xi; P) = 2$. If $\dim_P \mathrm{Sing}(\xi) = 2$ then there is a center manifold W of ξ at P made of singular points. Thus C trivially holds and $\xi \in \mathcal{G}_0(M;P)$. Suppose that $\dim_P \mathrm{Sing}(\xi) < 2$ and there is an analytic smooth invariant surface D transversal to the formal center manifold $\hat{W}^c$. Then for any center manifold W^c of class $\mathcal{C}^2$ we get a regular $\mathcal{C}^2$-curve $D \cap W^c$ inside W^c. Take W^c such that $\xi \mid_{W^c}$ has a non zero finite jet (by using Lemma 2) and apply Corollary 3. Then condition C holds and $\xi \in \mathcal{G}_0(M;P)$.

Define the following partition of $\mathcal{S}R(M;P)$:

$\eta \in \mathcal{S}R^{(1)}(M;P)$ iff $c(\eta;P) = 0$ and not all the eigenvalues have the same sign,

$\eta \in \mathcal{S}R^{(2)}(M;P)$ iff $c(\eta;P) = 1$ and all the eigenvalues are different,

$\eta \in \mathcal{S}R^{(3)}(M;P)$ iff $c(\eta;P) = 2$,

$\eta \in \mathcal{S}R^{(4)}(M;P)$ iff $c(\eta;P) = 1$ and there is a double non zero eigenvalue,

$\eta \in \mathcal{S}R^{(5)}(M;P)$ iff $c(\eta;P) = 0$ and all the eigenvalues have the same sign.

Put now

$$\mathcal{G}_0^{(i)}(M;P) = \mathcal{G}_0(M;P) \cap \mathcal{S}R^{(i)}(M;P), \quad i = 1,2,3,4,5$$

and denote by $\mathcal{S}R^{(2)}(M;P)^{\pm}$ the subclass of $\xi \in \mathcal{S}R^{(2)}(M;P)$ such that the non zero eigenvalues of $L_P(\xi)$ have different sign. Note that $\mathcal{G}_0^{(i)}(M;P) = \mathcal{S}R^{(i)}(M;P)$ for $i = 1,2,5$.

First case. — The blowing-up π is quadratic, that is, the center of π is the point P.

I) We consider first the following situation: the stable manifold W^s of ξ at P has dimension $\leqslant 2$ and $|\gamma| \subset W^s$. Then $\xi|_{W^s}$ is an hyperbolic vector field (in smaller dimension) with real eigenvalues and γ is a solution of $\xi|_{W^s}$. This implies that γ has a well defined tangent as a curve in W^s and hence as a curve in M. The fact that P' is in the strict transform of W^s and that there are at most two negative eigenvalues ($\dim W^s \leqslant 2$) gives after elementary computation

$$\xi' \in \mathcal{S}R^{(1)}(M';P') \cup \mathcal{S}R^{(2)}(M';P')$$

and hence $\xi' \in \mathcal{G}_0(M';P')$. Moreover, if $\xi \in \mathcal{S}R^{(1)}(M;P) \cup \mathcal{S}R^{(2)}(M;P)^{\pm}$ and the transform ξ' is in $\mathcal{S}R^{(2)}(M';P')$ then also $\xi' \in \mathcal{S}R^{(2)}(M';P')^{\pm}$.

II) Assume that $|\gamma| \not\subset W^s$. Then $\xi \in \mathcal{G}_0^{(i)}(M;P)$ for $i \in \{2,3,4\}$. In view of Corollary 1 we can assume that γ is contained in a fixed center manifold W^c in order to prove the result. The fact that $\omega(\gamma') = \{P'\}$ is a consequence of the definition of $\mathcal{G}_0(M;P)$ in case $i = 3$ and of the fact that the center manifold is of dimension 1 if $i = 2,4$. Moreover, P' is in the projectivized tangent space of the formal center manifold of ξ at P. This implies that the linear parts $L_P(\xi)$ and $L_{P'}(\xi')$ have the same eigenvalues (same characteristic polynomials) and thus

$$\xi \in \mathcal{S}R^{(i)}(M;P) \Leftrightarrow \xi' \in \mathcal{S}R^{(i)}(M';P')$$

for $i = 2, 3, 4$. If $i = 2$ then $\xi' \in \mathcal{SR}^{(2)}(M'; P') = \mathcal{G}_0^{(2)}(M'; P')$ and we are done. (Note also that $\xi \in \mathcal{SR}^{(2)}(M; P)^{\pm} \Leftrightarrow \xi' \in \mathcal{SR}^{(2)}(M'; P')^{\pm}$). If $i = 3$, the (two dimensional) formal center manifold $\widehat{W^c}'$ of ξ' at P' cuts transversally the exceptional divisor $D = \pi^{-1}(P)$. Then $\xi' \in \mathcal{G}_0^{(3)}(M'; P')$ by Remark 2, b). Finally, if $i = 4$, conditions B-1, B-2 or B-3 for ξ' at P' are given directly by the same conditions for ξ at P by taking transforms by the blowing-up π. Thus $\xi' \in \mathcal{G}_0^{(4)}(M'; P')$.

III) Assume that $\dim W^s = 3$. Thus $\xi \in \mathcal{SR}^{(5)}(M; P)$. In the case that ξ is a linear vector field, we get directly that $\omega(\gamma') = \{P'\}$. Since all eigenvalues of $L_P(\xi)$ are real non zero and negative, we have a $\mathcal{C}^1$-conjugacy between ξ and its linear part (this is a strong version of the Hartman-Grobman theorem for the case that all the eigenvalues have real parts of the same sign, see [3]). Thus $\omega(\gamma') = \{P'\}$. The point P' is the projectivized of a proper line of $L_P(\xi)$. We know that $\xi' \in \mathcal{SR}(M'; P')$.

If $\xi' \in \mathcal{SR}^{(3)}(M'; P')$ then $D = \pi^{-1}(P)$ is a center manifold of ξ' at P'. Moreover, there is an invariant projective line $l \subset D$ through P', obtained as the projectivized of an invariant plane of the linear part $L_P(\xi)$. Then condition C holds and $\xi' \in \mathcal{G}_0^{(3)}(M'; P')$ either because $\xi' \mid_D \equiv 0$ or applying Corollary 3.

If $\xi' \in \mathcal{SR}^{(4)}(M'; P')$ then D is invariant and transversal to the stable manifold of ξ' at P'. Thus $\xi' \in \mathcal{G}_0^{(4)}(M'; P')$ by Remark 2, a).

Second case. — The blowing-up π is monoidal, that is, the center Y of π is an admissible curve for (ξ, γ) such that $P \in Y$. Then the curve Y is tangent to a proper line of the linear part $L_P(\xi)$ and it has normal crossings with the stable and the unstable manifolds of ξ at P. Denote by $D = \pi^{-1}(Y)$, $F = \pi^{-1}(P)$ the exceptional divisor and the fiber over P, respectively. Both are invariant by the transform ξ' and we have that $\omega(\gamma') \subset F$.

I) Assume that $\dim W^s \leqslant 2$ and $|\gamma| \subset W^s$. We consider two cases:

I-1. The center Y is not tangent to W^s. Then the tangent direction of γ is not tangent to Y. This implies that $\omega(\gamma') = \{P'\}$ where $P' \in F$ is in the strict transform of W^s by π. The following description is obtained by elementary computation:

If $\xi \in \mathcal{G}_0^{(1)}(M; P)$ then Y is contained in W^u and hence $\xi' \in \mathcal{SR}^{(1)}(M'; P') \cup \mathcal{SR}^{(2)}(M'; P')^{\pm}$.

If $\xi \in \mathcal{G}_0^{(2)}(M;P)$ then either $Y = W^u$ or Y is a center manifold of ξ at P. In the first case $\xi' \in \mathcal{SR}^{(1)}(M';P')$ and, in the second case, $\xi' \in \mathcal{SR}^{(2)}(M';P') \cup \mathcal{SR}^{(4)}(M';P')$. Moreover, if $\xi' \in \mathcal{SR}^{(4)}(M';P')$ then D is transversal to the stable manifold of ξ' at P'. Thus in any case $\xi' \in \mathcal{G}_0(M';P')$ by Remark 2, a). Also, if $\xi \in \mathcal{SR}^{(2)}(M;P)^{\pm}$ then $\xi' \in \mathcal{SR}^{(2)}(M';P')^{\pm}$.

If $\xi \in \mathcal{G}_0^{(3)}(M;P)$ then $\dim W^s = 1$ and Y is tangent to the formal center manifold. We have that $\xi' \in \mathcal{SR}^{(2)}(M';P')^{\pm}$.

If $\xi \in \mathcal{G}_0^{(4)}(M;P)$ then Y is a center manifold and we have that $\xi' \in \mathcal{SR}^{(3)}(M';P')$. The divisor D is a center manifold of ξ' at P' and $F \subset D$ is an invariant curve through P'. Thus $\xi' \in \mathcal{G}_0^{(3)}(M';P')$ either because $\xi' \mid_D \equiv 0$ or by Corollary 3.

I-2. The center Y is contained in W^s. If $\dim W^s = 1$ then Y is not admissible for (ξ, γ) since $|\gamma| \subset W^s$. Assume that $\dim W^s = 2$. The strict transform of W^s by π cuts F in a unique point P'. Thus $\omega(\gamma') = \{P'\}$. We have that $\xi' \in \mathcal{SR}^{(1)}(M';P')$ in all possible cases.

II) Assume that $\dim W^s \leqslant 2$, $|\gamma| \not\subset W^s$ and $\xi \notin \mathcal{G}_0^{(4)}(M;P)$. We already know that γ has a well defined tangent in the projectivized of the tangent space of the formal center manifold $\widehat{W^c}$. We study first the easier cases $\xi \in \mathcal{G}_0^{(i)}(M;P)$ for $i = 2, 3$ and leave the case $\xi \in \mathcal{G}_0^{(4)}(M;P)$ for the end of the paragraph. We distinguish the two situations:

II-1. The curve Y is not tangent to $\widehat{W^c}$. Then $\omega(\gamma') = \{P'\}$, where $P' \in F$ is in the strict transform by π of the tangent space of $\widehat{W^c}$. The eigenvalues of the linear part $L_{P'}(\xi')$ coincide with those of $L_P(\xi)$ and hence ξ' belongs to $\mathcal{SR}^{(2)}(M';P')$ or $\mathcal{SR}^{(3)}(M';P')$ just as ξ does. In this last case we also have that $\xi' \in \mathcal{G}_0^{(3)}(M';P')$ by Remark 2, b), since D is transversal to the formal center manifold $\widehat{W^c}'$ of ξ' at P'.

II-2. The curve Y is tangent to $\widehat{W^c}$.

Assume that $\xi \in \mathcal{G}_0^{(2)}(M;P)$. Then Y is a center manifold of ξ at P. Since the two non zero eigenvalues λ_1, λ_2 of $L_P(\xi)$ are different, there are only two singular points P_1, P_2 of ξ' over F. The linear part $L_{P_i}(\xi')$ has eigenvalues $0, \lambda_i, \lambda_j - \lambda_i$ for $\{i, j\} = \{1, 2\}$. The last written one (non zero) is associated to the tangent direction of F at P_i. Thus F is not tangent to the center manifolds of ξ' at P_1 or P_2. We have that

$$\xi' \in \mathcal{SR}^{(i_1)}(M';P_1) \cap \mathcal{SR}^{(i_2)}(M';P_2) \tag{7}$$

where $i_1, i_2 \in \{2,4\}$. Moreover, at one of these points, say P_1, the two non zero eigenvalues of the linear part have different sign ($\Rightarrow i_1 = 2$). In particular there are center-stable and center-unstable manifolds of ξ' at P_1 of dimension two and of class $\mathcal{C}^1$. They cut in a center manifold so one of them is transversal to F at P_1. This shows, by applying Corollary 2 that $\omega(\gamma') = \{P_1\}$ or $\omega(\gamma') = \{P_2\}$. To see that ξ' belongs to the class $\mathcal{G}_0$ at the ω-limit point of γ' we only have to consider the case $i_2 = 4$ and $\omega(\gamma') = \{P_2\}$. The result follows as usual by using the divisor D and Remark 2, a).

Assume that $\xi \in \mathcal{G}_0^{(3)}(M;P)$. There are only two singular points P_1, P_2 of ξ' over F where we have

$$\xi' \in \mathcal{S}R^{(2)}(M';P_1)^{\pm} \cap \mathcal{S}R^{(3)}(M';P_2).$$

At P_1 we find an invariant regular $\mathcal{C}^1$-surface transversal to F (a center-stable or a center-unstable manifold as above). Thus $\omega(\gamma')$ is a single point by Corollary 3. If $\omega(\gamma') = \{P_1\}$ then $\xi' \in \mathcal{G}_0^{(2)}(M';P_1)^{\pm}$ and we are done. If $\omega(\gamma') = \{P_2\}$ then $\xi' \in \mathcal{G}_0^{(3)}(M';P_2)$ by applying Remark 2, b) since D is transversal to the formal center manifold of ξ' at P_2.

III) Assume that $\dim W^s = 3$, that is, $\xi \in \mathcal{G}_0^{(5)}(M;P)$. We have several possibilities depending on the eigenvalues of $L_P(\xi)$ not associated to the tangent direction of Y.

III-1. If they are different, there are two singular points P_1, P_2 of ξ' in F such that

$$\xi' \in \mathcal{S}R^{(1)}(M';P_1) \cap \mathcal{S}R^{(5)}(M';P_2).$$

In this case we only have to show that $\omega(\gamma')$ is a single point. We see that the stable manifold of ξ' at P_1 is a regular invariant surface transversal to F. Thus we finish by Corollary 2.

III-2. If they coincide then either F has a single singular point of ξ' or F is made of singularities. In any case, for any $P' \in F$ such that $\xi'(P') = 0$ we have

$$\xi' \in \mathcal{S}R^{(i)}(M';P'), \quad i = 2 \text{ or } 4. \tag{8}$$

Moreover, at any such singular point, F is a center manifold of ξ' and the stable manifold is two dimensional and transversal to F. By Corollary 2 again, $\omega(\gamma')$ is a unique point P'. It remains to show that $\xi' \in \mathcal{G}_0(M';P')$. This is obvious if $i = 2$ in (8) and holds if $i = 4$ by Remark 2, a) applied to the invariant divisor D, transversal to the stable manifold of ξ' at P'.

Remark 3. — The subclass $\bar{\mathcal{G}}_0(M;P) \subset \mathcal{G}_0(M;P)$ given by

$$\bar{\mathcal{G}}_0(M;P) = \mathcal{G}_0^{(1)}(M;P) \cup \mathcal{G}_0^{(2)}(M;P)^{\pm} \cup \mathcal{G}_0^{(3)}(M;P)$$

is stable by admissible blowing-up. Namely, in the hypothesis of Proposition 4, if $\xi \in \bar{\mathcal{G}}_0(M;P)$ then $\omega(\gamma') = \{P'\}$ and $\xi' \in \bar{\mathcal{G}}_0(M';P')$. Hence, at this moment of the proof of Proposition 4, if $\xi \in \mathcal{G}_0^{(3)}(M;P)$, then any solution γ of ξ with $\omega(\gamma) = \{P\}$ is non oscillating.

IV) Assume that $|\gamma| \not\subset W^s$ and $\xi \in \mathcal{G}_0^{(4)}(M;P)$. We can suppose that ξ does not have a center manifold consisting of singular points. Otherwise, only the solutions contained in W^s can accumulate to the point P.

IV-1. Assume first that the center of the blowing-up Y is tangent to $\widehat{W^c}$. Then Y is itself an analytic center manifold of ξ at P and condition B-1 does not hold. Thus condition B-2 or B-3 is true for ξ at P.

Condition B-2 means that the double non zero eigenvalue of $L_P(\xi)$ is positive. In this case the center-stable manifolds are the center ones and, since $\omega(\gamma) = \{P\}$, $|\gamma|$ is contained in any center manifold. In particular, Y can not be admissible for (ξ, γ), so B-2 does not hold.

Assume then that B-3 holds. We show first that the lift γ' accumulates to a single point of F.

Take $p : \mathbf{S}^1 \times U \subset \mathbf{S}^1 \times \mathbf{R}^2 \to M$ the cylindric blowing-up of center Y and denote by S the circle $p^{-1}(P) = \mathbf{S}^1 \times \{0\}$. Let $\bar{\xi}$ be the transform of ξ by p. Then S is invariant and, either it is made of singularities of $\bar{\xi}$ or it has finitely many of them. In order to prove that $\omega(\gamma') = \{P'\}$ it is enough to see that the ω-limit set $\omega(\bar{\gamma})$ of any of the two lifts $\bar{\gamma}$ of γ by p is a single point. We will show that either $\bar{\gamma}$ accumulates to a single point of S or it "turns" around S as $t \to \infty$. This last possibility can not occur because, by condition B-3, $\bar{\gamma}$ must cut the strict transform $\bar{N}$ of N by p finitely many times if $|\bar{\gamma}| \not\subset \bar{N}$.

If S has finitely many singular points then either $\bar{\gamma}$ accumulates to a single point of S or it "turns" around S, by Lemma 4. Suppose that S consists of singular points of $\bar{\xi}$. The restriction of $\bar{\xi}$ to a neighbourhood of S satisfies the hypothesis of Proposition 2. Moreover, the divisor $E = p^{-1}(Y)$ is a center manifold of $\bar{\xi}$ along S. We have that the restriction $\bar{\xi}\mid_E$ is not identically zero since we have supposed that Y does not consists of singularities of ξ. After dividing $\bar{\xi}\mid_E$ by a power of an equation of S we can suppose that it has finitely many singular points over S and thus, it is in the hypotheses of Lemma 4. Then the solutions σ of $\bar{\xi}\mid_E$ outside S with

$\omega(\sigma) \subset S$ either accumulate to a single point or they "turn" around S. By using Proposition 2, the same happens with $\bar{\gamma}$.

Hence $\omega(\gamma') = \{P'\}$ where $P' \in F$ is a singular point of ξ'. We have that

$$\xi' \in \mathcal{SR}^{(3)}(M'; P')$$

and the divisor D is a center manifold of ξ' at P'. Moreover F is an invariant smooth curve through P', so $\xi' \in \mathcal{G}_0^{(3)}(M'; P')$ by Corollary 3.

IV-2. Assume finally that Y is transversal to the formal center manifold $\widehat{W^c}$ of ξ at P. Then Y is not tangent to the tangent direction of γ and $\omega(\gamma') = \{P'\}$ where $P' \in F$ is the unique point in the strict transform of $\widehat{W^c}$ by π. Also, the eigenvalues of the linear parts $L_P(\xi)$ and $L_{P'}(\xi')$ are the same. Thus we have $\xi' \in \mathcal{SR}^{(4)}(M'; P')$. To get $\xi' \in \mathcal{G}_0(M'; P')$ it remains to show that condition B-1, B-2 or B-3 hold.

Condition B-2 holds at P' directly if it does at P. Condition B-1 holds at P' if it does at P by means of the map π. Finally, condition B-3 at P' can be obtained from the same condition at P by taking the strict transform N' of N by π if N is transversal to the center Y. Otherwise N' can be singular. The fact that N is transversal to Y can be assumed if condition B-3 holds for any smooth surface containing the analytic center manifold W and not only for a fixed one N. More precisely, we have the following:

LEMMA 5. — *Consider $\xi \in \mathcal{SR}^{(4)}(M; P)$ and assume that B-1 and B-2 do not hold for ξ at P. Let W be the analytic center manifold and consider a solution γ of ξ such that $\omega(\gamma) = \{P\}$ and $|\gamma| \not\subset W$. Let $\pi : M' \to M$ be the blowing-up with center W and suppose that the lift γ' of γ by π accumulates to a single point $P' \in \pi^{-1}(P)$. Then γ is non oscillating with respect to any analytic smooth surface containing W. In particular $\xi \in \mathcal{G}_0^{(4)}(M; P)$.*

Proof. — We know that $\xi' \in \mathcal{G}_0^{(3)}(M'; P')$ so, by Remark 3, γ' is non oscillating at P' as a solution of the vector field ξ'. Suppose, by contradiction, that there is an analytic smooth surface $N_1 \supset W$ such that γ is oscillating with respect to N_1. If ${N_1}'$ denotes the strict transform then $P' \in {N_1}'$ and γ' is oscillating with respect to ${N_1}'$ at P'.

This lemma ends the proof of Proposition 4 and Theorem 2.

5. The case of a gradient vector field.

In this section we proof Theorem 1 about the non oscillation property of the gradient vector field $\xi = \nabla_g f$. Let P be a singular point of f. We will show that ξ belongs to the class $\mathcal{G}_0(M;P)$ and apply Theorem 2.

The eigenvalues of the linear part $L_P(\xi)$ are precisely the ones of the hessian quadratic form $\text{Hess}\, f(P)$ with respect to coordinates for which the matrix of $g(P)$ is the identity. Then we have that ξ belongs to $\mathcal{S}R(M;P)$.

If the rank of the hessian is 3 then $\xi \in \mathcal{S}R^{(1)}(M;P) \cup \mathcal{S}R^{(5)}(M;P)$ and we are done.

Assume that the rank of $\text{Hess}\, f(P)$ is equal to 1. Then $\xi \in \mathcal{S}R^{(3)}(M;P)$. We can suppose that there is no center manifold consisting only of singularities, otherwise condition C holds automatically. Let W^c be a center manifold for ξ at P of class $\mathcal{C}^k$ such that the k-jet of $\xi^c = \xi\mid_{W^c}$ is not zero. Then ξ^c is the gradient vector field of a function of class $\mathcal{C}^k$ with non zero k-jet at P. In the same way as Thom's proof [11] for the analytic two dimensional case, the strict transform of ξ^c by means of the polar blowing-up $p : \bar{W} \to W^c$ at $P \in W^c$ restricts to a gradient vector field over the exceptional divisor $p^{-1}(P) \simeq \mathbf{S}^1$ with only finitely many singularities. This allows us to prove that any solution of ξ^c that accumulates to P has a well defined tangent. Then, ξ satisfies condition C at P and $\xi \in \mathcal{G}_0^{(3)}(M;P)$.

Finally, assume that the rank of the hessian is 2. Then $\xi \in \mathcal{S}R^{(4)}(M;P)$. Suppose that conditions B-1 and B-2 do not hold for ξ at P and let W be the analytic (one dimensional) center manifold, which we can suppose not made of singularities. Let $\pi : M' \to M$ be the blowing-up with center W. By Lemma 5, it suffices to proof that for any solution γ of ξ with $\omega(\gamma) = \{P\}$, its lift by π converges to a single point. We proof now this property.

The fiber $F = \pi^{-1}(P)$ is a projective line made of singular points of the transform ξ'. At any of these points, the exceptional divisor $D = \pi^{-1}(W)$ is a center manifold and, by means of a cylindric blowing-up instead of π, the global situation of ξ' around F is like in Proposition 2. We only have to show that the solutions of ξ' inside D that accumulates to F have a single ω-limit point. The rest of this paragraph is devoted to prove this result.

Take local coordinates (x, y, z) at P such that $W \equiv (x = y = 0)$. For any series $q \in \mathbf{R}\{x, y, z\}$ denote by $\bar{q}(z) = q(0, 0, z)$. The number $\nu(\bar{q})$ denotes its usual order. Write the vector field ξ in these coordinates as:

$$(xa_1 + ya_2)\frac{\partial}{\partial x} + (xb_1 + yb_2)\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}. \tag{9}$$

Consider the usual chart (x', y', z') of M such that the blowing-up π is written as $x = x'$, $y = x'y'$, $z = z'$. The exceptional divisor D is given by $(x' = 0)$ and the fiber F is the y'-axis. The restriction of ξ' to D is in this chart

$$\xi \mid_D = (m_0(z') + y'm_1(z') - y'^2 m_2(z'))\frac{\partial}{\partial y'} + \bar{c}(z')\frac{\partial}{\partial z'}, \tag{10}$$

where we put $m_0 = \bar{b}_1; m_1 = \bar{b}_2 - \bar{a}_1; m_2 = \bar{a}_2$. Note that $\bar{c}$ has finite order since we have supposed that the center manifold is not made of singular points.

There are several favorable cases for which we are done:

i) Suppose that $l_1 = \nu(\bar{c}) \leqslant \min\{\nu(m_0), \nu(m_1), \nu(m_2)\} = l_2$. Then, dividing (10) by z'^{l_1} we obtain a new vector field in D transversal to F whose integral curves coincide with those of $\xi \mid_D$ outside F.

ii) Suppose that $l_2 < l_1$ and that $\nu(m_1) < \min\{\nu(m_0), \nu(m_2)\}$. Then, we can divide this time by z'^{l_2} so that the resulting vector field has a semihyperbolic singularity at the origin $P' = (y' = 0, z' = 0)$. The curve F is a real separatrix through P' and there is another one (or a center manifold) which is transversal to it. Thus, solutions of (10) can not accumulate to the whole fiber F.

iii) Suppose that $l_2 < l_1$, $\nu(m_0) = \nu(m_2) \leqslant \nu(m_1)$ and that the initial parts of m_0 and m_2 coincide. Then, dividing again (10) by z'^{l_2} we obtain two semihyperbolic singularities in F and we reason as above.

Now we prove that no further possibilities can arrive for a gradient vector field. Let $G = (g^{ij})_{1 \leqslant i,j \leqslant 3}$ be the inverse of the matrix of the metric g with respect to the given coordinates (x, y, z). The components of (9) are obtained as the product of G with the column vector of all the partial derivatives of f. We can suppose the coordinates chosen so that $G(0)$ is the identity matrix and $f = \frac{\lambda}{2}(x^2 + y^2) + h(x, y, z)$, where λ is negative and h has order $\geqslant 3$. Write

$$h = \sum_{i+j \geqslant 0} h_{ij}(z) x^i y^j.$$

The invariance of the z-axis and the fact that g is nowhere degenerated implies $l_1 = \nu(h_{00}) - 1$, which is a finite number. A calculation shows that the series

$$(11) \qquad \begin{aligned} m'_1 &= \lambda(\bar{g}^{22} - \bar{g}^{11}) + 2(\bar{g}^{22}h_{02} - \bar{g}^{11}h_{20}) \\ m'_0 &= \lambda\bar{g}^{12} + 2\bar{g}^{12}h_{20} + \bar{g}^{22}h_{11} \\ m'_2 &= \lambda\bar{g}^{12} + 2\bar{g}^{12}h_{02} + \bar{g}^{11}h_{11} \end{aligned}$$

project to the same element in the ring $\mathbf{R}\{z\}/(z^{l_1})$ as the series m_1, m_0, m_2, respectively.

Assume that the case i) above is not true. Then we have that

$$l_2 = \min_{0\leqslant i\leqslant 2}\{\nu(m'_i)\} < l_1.$$

Suppose that $\nu(m'_0) \neq \nu(m'_2)$ or that $\nu(m'_0) = \nu(m'_2)$ but the initial parts of m'_0 and m'_2 are different. Then we will show that $\nu(m'_1) < \min\{\nu(m'_0), \nu(m'_2)\}$. This completes the proof because either the case ii) or iii) is true.

If, for example, $\nu(m'_0) \leqslant \nu(m'_2)$, the assumption implies that it is equal to the order of $m'_2 - m'_0$. In view of the formulae (11) put $p = 2h_{02} - 2h_{20}$, $q = \bar{g}^{11} - \bar{g}^{22}$, $s = \bar{g}^{12}$, $t = h_{11}$, $u = \bar{g}^{22}$, $v = \lambda + 2h_{20}$ and apply the following lemma.

Lemma 6. — *Let p, q, s, t, u, v be series in one variable such that p, q, s, t are of order $\geqslant 1$ and u, v are unities. If $\nu(sp + tq) = \nu(ut + vs) = \nu < \infty$ then $\nu(up - vq) < \nu$.*

Proof. — Assume by contradiction that $\nu(up - vq) \geqslant \nu$. We write $\epsilon = \sum_{i=0}^{\infty} \epsilon_i z^i$ for any series $\epsilon \in \mathbf{R}\{z\}$. Write for short $A = sp+tq$, $B = ut+vs$, $C = up - vq$. We want to show that $A = 0$, contradicting the hypotheses.

Suppose that for $k \geqslant 1$ we have $A_\ell = 0$, $\ell \leqslant k$. Since $\nu(A) = \nu(B) \leqslant \nu(C)$, the same is true for B and C. We can construct nested matrices $\Delta_1, \dots, \Delta_k$ such that for $\ell \leqslant k$:

$$\underline{t}_\ell = \Delta_\ell\, \underline{s}_\ell$$
$$\underline{p}_\ell = -\Delta_\ell\, \underline{q}_\ell$$

where, for $\epsilon = p, q, s, t$, we denote by $\underline{\epsilon}_\ell$ the transpose of the vector $(\epsilon_1, \dots, \epsilon_\ell)$. Denote also by $\underline{\epsilon}_\ell{}'$ the column vector obtained by reversing the order of the indices. Write $\Delta_k = (\delta^k_{\alpha,\beta})$. It is easy to see that

$$\delta^k_{\alpha,\beta} = \delta^k_{\alpha+l,\beta+l}, \quad \forall l$$

so that, if $\underline{t}_k{}' = \Delta_k{}'\underline{s}_k{}'$ then $\Delta_k{}'$ is the transpose of Δ_k. Finally we have

$$\begin{aligned} A_{k+1} &= \quad < \underline{s}_k, \underline{p}_k{}' > + < \underline{t}_k, \underline{q}_k{}' > \quad = \\ &= < \underline{s}_k, -\Delta_k{}'\underline{q}_k{}' > + < \Delta_k\underline{s}_k, \underline{q}_k{}' > \quad = \\ &= < -\Delta_k\underline{s}_k, \underline{q}_k{}' > + < \Delta_k\underline{s}_k, \underline{q}_k{}' > \quad = 0 \end{aligned}$$

so $A = 0$ as wanted.

BIBLIOGRAPHY

[1] J.M. AROCA, H. HIRONAKA, J.L. VICENTE, The theory of the maximal contact, Mem. Mat. Inst. Jorge Juan, Madrid, 29 (1975).

[2] J. CARR, Applications of Center Manifolds Theory Applied Mathematical Sciences, vol. 35, Springer-Verlag, New York (1981).

[3] P. HARTMAN, On Local Homeomorphisms of Euclidean Spaces, Bol. Soc. Mat. Mexicana, 5 (1960), 220-241.

[4] H. HIRONAKA, Desingularization of Excellent Surfaces, Adv. Sci. Seminar (1967), Bodwoin College. Lect. Notes in Math., 1101, Springer-Verlag (1984).

[5] M.W. HIRSCH, C.C. PUGH, M. SHUB, Invariant Manifolds, Lect. Notes in Math., 583, Springer-Verlag (1977).

[6] H. XING LIN, Sur la structure des champs de gradients de fonctions analytiques réelles, Thèse, Paris VII (1992).

[7] F. ICHIKAWA, Thom's conjecture on singularities of gradient vector fields, Kodai Math. Journal, 15 (1992), 134-140.

[8] Al. KELLEY, Stability of the Center-Stable Manifold, Jour. of Math. Anal. and Appl., 18 (1967), 336-344.

[9] K. KURDYKA, T. MOSTOWSKI, The gradient conjecture of R, Thom Preprint (1996).

[10] S. ŁOJASIEWICZ, Sur les trajectoires du gradient d'une fonction analytique Seminari di Geometria, Bologna (1983), 115-117.

[11] R. MOUSSU, Sur la dynamique des gradients, Existence de variétés invariantes Math. Ann., 307 (1997), 445-460.

[12] R. MOUSSU, C. ROCHE, Théorèmes de finitude pour les variétés pfaffiennes, Ann. Inst. Fourier, Grenoble, 42-1 & 2 (1992), 393-420.

[13] J. PALIS, F. TAKENS, Topological equivalence of normally hyperbolic dynamical systems, Topology, 16 (1977), 335-345.

[14] F. SANZ, Trajectoires non-oscillantes des champs de vecteurs gradients. C. R. Acad. Sci. Paris, t. 325, Série I (1997), 429-432.

[15] M. SHUB, Stabilité globale des systèmes dynamiques, Astérisque, vol. 56. Soc. Math. de France (1978).

[16] R. THOM, Problèmes rencontrés dans mon parcours mathématique : un bilan, Publ. IHES, 70 (1989), 200-214.

Manuscrit reçu le 15 octobre 1997,
accepté le 24 février 1998.

Fernando SANZ,
Universidad de Valladolid
Facultad de Ciencias
Departamento de Álgebra
Geometría y Topología
47005 Valladolid. (España).
fsanz@cpd.uva.es