KRZYSZTOF KURDYKA

On gradients of functions definable in o-minimal structures


<http://www.numdam.org/item?id=AIF_1998__48_3_769_0>
ON GRADIENTS OF FUNCTIONS
DEFINABLE IN O-MINIMAL STRUCTURES

by Krzysztof KURDYKA

0. Introduction.

Many results in subanalytic or semialgebraic geometry of $\mathbb{R}^n$ hold true in a more general setting called “the theory of o-minimal structures on the real field” (see [DM]). This theory has presented a strong interest since 1991 when A. Wilkie [W1] proved that a natural extension of the family of semialgebraic sets containing the exponential function $((\mathbb{R}, \exp)$-definable sets) is an o-minimal structure. A similar extension of subanalytic sets $((\mathbb{R}_{an}, \exp)$-definable sets) was then treated by L. van den Dries, A. Macintyre, D. Marker in [DMM] (geometric proofs of these facts were found recently by J-M. Lion and J.-P. Rolin [LR1], [LR2]). Another type of o-minimal structure $((\mathbb{R}_{an}^K)$-definable sets) was obtained by C. Miller [Mi], by adding to subanalytic sets all functions $x \rightarrow x^r$, $r \in K$, where $K$ is a subfield of $\mathbb{R}$. We give a list and examples of o-minimal structures in section 1. An extension of semialgebraic and subanalytic geometry was also undertaken by M. Shiota [S1], [S2].

Theorem 1 (Section 2), the first main result of this paper, is an o-minimal generalization of the famous Łojasiewicz inequality $\|\nabla f\| \geq |f|^\alpha$ with $\alpha < 1$, where $f$ is an analytic function in a neighborhood of $a \in \mathbb{R}^n$, $f(a) = 0$. We prove that if $f$ is a differentiable function in a

This research was partially supported by KBN grant 0844/P3/94/07.

Key words: o-minimal structure – Subanalytic sets – Łojasiewicz inequalities – Trajectories of gradient.

bounded domain, definable in some o-minimal structure, then there exists a \( C^1 \) function \( \Psi \) in one variable such that \( \| \text{grad} \Psi \circ f \| \geq c > 0 \). It is rather surprising that the result holds also for infinitely flat functions. Theorem 1 implies that the set of asymptotic critical values of \( f \) is finite (Proposition 2). We recall in the beginning of the section the already known o-minimal version of another Lojasiewicz inequality for continuous definable functions on a compact set.

The main result of Section 3 is Theorem 2 which states: if \( U \) is an open, bounded subset of \( \mathbb{R}^n \), \( f : U \to \mathbb{R} \) is a \( C^1 \) function definable in some o-minimal structure, then all trajectories of \( -\text{grad} f \) (i.e. solutions of the equation \( \dot{x} = -\text{grad} f \)) have their length bounded by a constant independent of the trajectory. The function \( f \) may be unbounded and may not have a continuous extension on \( \bar{U} \). We prove also, that for a non negative definable \( g \), the flow of \( -\text{grad} g \) defines a deformation retraction onto \( g^{-1}(0) \). Some applications of this result in the real analytic case can be found in [Si], [Sj]. We finish the paper by a discussion of Thom's Gradient Conjecture for o-minimal structures.

In Section 1 we gather basic facts on o-minimal structures. To make the paper self-contained and accessible for a wider audience we add a proof of Lemma 2 (on definable functions in one variable). We give also an elementary proof (suggested by C. Miller and J-M. Lion) of the curve selection lemma, the crucial tool in the proof of Theorem 1.

General references of various facts, when not specified, will be as follows: for semialgebraic geometry – [BCR], for subanalytic geometry – [BM] or [L4], for o-minimal structures – [DM].

In this paper we take the gradient with respect to the canonical euclidian metric in \( \mathbb{R}^n \).

1. o-minimal structures on the real field.

Definition 1. — Let \( \mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n \), where each \( \mathcal{M}_n \) is a family of subsets of \( \mathbb{R}^n \). We say that the collection \( \mathcal{M} \) is an o-minimal structure on \( (\mathbb{R}, +, \cdot) \) if:

1. each \( \mathcal{M}_n \) is closed under finite set-theoretical operations;
2. if \( A \in \mathcal{M}_n \) and \( B \in \mathcal{M}_m \), then \( A \times B \in \mathcal{M}_{n+m} \).
(3) let $A \in \mathcal{M}_{n+m}$ and $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ be projection on the first $n$ coordinates, then $\pi(A) \in \mathcal{M}_n$;

(4) let $f, g_1, \ldots, g_k \in \mathbb{Q}[X_1, \ldots, X_n]$, then $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\} \in \mathcal{M}_n$;

(5) $\mathcal{M}$ consists of all finite unions of open intervals and points.

For a fixed $o$-minimal structure $\mathcal{M}$ on $(\mathbb{R}, +, \cdot)$ we say that $A$ is an $\mathcal{M}$-set if $A \in \mathcal{M}_n$ for some $n \in \mathbb{N}$. We say that a function $f : A \longrightarrow \mathbb{R}^m$, where $A \subset \mathbb{R}^n$, is an $\mathcal{M}$-function if its graph is an $\mathcal{M}$-set.

Axiom (5) will be called the $o$-minimality of $\mathcal{M}$.

Examples. — We give below a list of $o$-minimal structures on $(\mathbb{R}, +, \cdot)$ (see also [DM] for detailed definitions and comparisons between the above examples) with examples of functions definable in those $o$-minimal structures:

(1) Semialgebraic sets (by Tarski-Seidenberg); $f(x, y) = \sqrt{x^4 + y^4}$.

(2) Global subanalytic sets (by Gabrielov);

$f(x, y) = \frac{y}{\sin x}$, $x \in (0, \pi)$.

(3) $(\mathbb{R}, \exp)$-definable sets (by Wilkie);

$f(x, y) = x^2 \exp\left(-\frac{y^2}{x^4 + y^2}\right) \ln x$.

(4) $(\mathbb{R}_{an}, \exp)$-definable sets (by van den Dries, Macintyre, Marker);

$f(x, y) = x^{\sqrt{2}} \ln(\sin y)$, $x > 0, y \in (0, \pi)$.

(5) $(\mathbb{R}_{an}^R)$-definable sets (by Miller);

$f(x, y) = x^{\sqrt{2}} \exp\left(\frac{x}{y}\right)$, $0 < x < y < 1$.

Recently another example of an $o$-minimal structure was found by van den Dries and Speissegger [DS] which is larger than $\mathbb{R}_{an}^R$ but polynomially bounded (i.e any definable function in one variable is bounded by a polynomial at infinity). Finally we mention a result of Wilkie [W2] in which he gives a general method for construction of $o$-minimal structures; this method can be applied to Pfaffian functions.

In the rest of this paper $\mathcal{M}$ will denote some fixed, but arbitrary, $o$-minimal structure on $(\mathbb{R}, +, \cdot)$. We will give now several elementary properties of $\mathcal{M}$-sets and $\mathcal{M}$-functions.

Remark 1. — Let $E$ be an $\mathcal{M}$-set in $\mathbb{R}^{n+1}$. Axioms (1)--(4) imply
that the sets
\[ \{ x \in \mathbb{R}^n : \exists x_{n+1} \ (x, x_{n+1}) \in E \} \] and
\[ \{ x \in \mathbb{R}^n : \forall x_{n+1} \ (x, x_{n+1}) \in E \} \]
are \( \mathcal{M} \)-sets. Actually the first set is the image of \( E \) by projection, the second
is the complement of the image of the complement of \( E \) by projection.

Remark 2. — The sum, product, inverse, composition of \( \mathcal{M} \)-functions is again an \( \mathcal{M} \)-function. Also the image and inverse image of an
\( \mathcal{M} \)-set by an \( \mathcal{M} \)-function are again \( \mathcal{M} \)-sets. Proofs of these facts are quite
standard applications of Remark 1 and axioms (1)-(4) and actually the
same as in the semialgebraic case (see e.g. [BCR]).

**Lemma 1.** — Let \( f : A \rightarrow \mathbb{R} \) be an \( \mathcal{M} \)-function such that \( f(x) \geq 0 \)
for all \( x \in A \). Let \( G : A \rightarrow \mathbb{R}^m \) be an \( \mathcal{M} \)-mapping and define a function
\( \varphi : G(A) \rightarrow \mathbb{R} \) by
\[ \varphi(y) = \inf_{x \in G^{-1}(y)} f(x). \]
Then \( \varphi \) is an \( \mathcal{M} \)-function.

*Proof.* — Write a formula for the graph of the function \( \varphi \) and apply
Remark 1.

**Corollary 1.** — Let \( A \) be an \( \mathcal{M} \)-set in \( \mathbb{R}^n \). Then the distance
function \( d_A : \mathbb{R}^n \rightarrow \mathbb{R} \) is an \( \mathcal{M} \)-function, where \( d_A(x) = \inf_{y \in A} |x - y| \).

**Corollary 2.** — Let \( A \) be an \( \mathcal{M} \)-set in \( \mathbb{R}^n \). Then \( \overline{A} \) and \( \text{Int} A \) are
\( \mathcal{M} \)-sets.

*Proof.* — Actually by Corollary 1 we know that \( d_A \) is an \( \mathcal{M} \)-function,
hence \( \overline{A} = d_A(0)^{-1} \) is an \( \mathcal{M} \)-set. To prove that the interior of \( A \) is an \( \mathcal{M} \)-set
we use the fact that by axiom (1) the complement of an \( \mathcal{M} \)-set is an \( \mathcal{M} \)-set.

**Lemma 2** (Monotonicity Theorem). — Let \( f : (a, b) \rightarrow \mathbb{R} \) be an
\( \mathcal{M} \)-function. Then there exist real numbers \( a = a_0 < a_1 < \ldots < a_k = b \) such
that \( f \) is continuously differentiable on each interval \( (a_i, a_{i+1}) \). Moreover
\( f' \) is an \( \mathcal{M} \)-function and the function \( f \) is strictly monotone or constant on
every interval \( (a_i, a_{i+1}) \).

*Proof* (Due essentially to van den Dries [vD]). — We may assume
that the set \( f((a, b)) \) is infinite. First we prove that \( D(f) \), the set of
discontinuity points of \( f \), is finite.
Writing the definition of continuity of a function at a point and using Remark 1 we deduce that $D(f)$ is an $M$-set in $\mathbb{R}$, hence by o-minimality, it is enough to prove that $f$ is continuous at some point of $(a, b)$. Since the set $f((a, b))$ is an infinite $M$-set it contains an open interval. Thus by induction we can construct a descending sequence of intervals $[\alpha_n, \beta_n] \subset (a, b)$ such that $\alpha_n < \alpha_{n+1}$, $\beta_{n+1} < \beta_n$, $\beta_n - \alpha_n < 1/n$ and $f([\alpha_n, \beta_n])$ is contained in an interval of length smaller than $1/n$. Clearly $f$ is continuous at the point $\bigcap_{n \in \mathbb{N}} [\alpha_n, \beta_n]$. So we have proved that the complement of $D(f)$ is dense in $(a, b)$, hence $D(f)$ is finite.

We can assume now that $f$ is continuous on $(a, b)$. To prove differentiability observe first that by o-minimality we have:

**Observation.** — For each $x \in (a, b)$ and each $c \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that $f(t) \geq f(x) + c(t - x)$ for all $t \in (x, x + \varepsilon)$ or $f(t) \leq f(x) + c(t - x)$ for all $t \in (x, x + \varepsilon)$.

Let us write $f'_-(x) = \lim_{t \to 0^-} \frac{1}{t}(f(x + t) - f(x))$ for $x \in (a, b]$ and $f'_+(x) = \lim_{t \to 0^+} \frac{1}{t}(f(x + t) - f(x))$ for $x \in [a, b)$. Note that $f'_+$ and $f'_-$ are $M$-functions, by Remark 1. From the above observation it is not difficult to obtain the following consequences:

i) for each $x \in (a, b)$ the values of $f'_-(x)$ and $f'_+(x)$ are well defined (possibly equal to $+\infty$ or $-\infty$),

ii) for each $x \in (a, b)$ there exists $y$ arbitrary close to $x$, $y > x$ such that $f'_+(y) \leq f'_+(x)$, $f'_-(y) \leq f'_-(x)$ or $f'_+(y) \geq f'_+(x)$, $f'_-(y) \geq f'_-(x)$.

Clearly the sets

$$\{x \in (a, b); f'_+(x) = +\infty\}, \{x \in (a, b); f'_+(x) = -\infty\}$$

are $M$-sets, hence are finite unions of open intervals and points. By ii) these sets are finite. So we can assume that $f'_+$ and $f'_-$ take values in $\mathbb{R}$. Since $f'_+$ and $f'_-$ are $M$-functions we may also assume that these functions are continuous on $(a, b)$. It follows easily now from ii) that $f'_+ = f'_-$ on $(a, b)$, but this means that $f$ is $C^1$ on $(a, b)$.

We proved also that $f'$ is an $M$-function, hence the claim on monotonicity follows from the fact that $\{f' = 0\}$ is an $M$-set and so is a finite union of points and open intervals.

Writing the definition of partial derivatives and using Remark 1 we obtain:
Lemma 3. — Let \( f : U \rightarrow \mathbb{R}^k \) be a differentiable \( \mathcal{M} \)-function, where \( U \) is open in \( \mathbb{R}^n \). Then \( \partial f / \partial x_j, j = 1, \ldots, n \) are \( \mathcal{M} \)-functions, and hence \( \text{grad} \ f \) is an \( \mathcal{M} \)-mapping.

Proposition 1 (Curve Selection Lemma). — Let \( A \) be an \( \mathcal{M} \)-set in \( \mathbb{R}^n \) and suppose that \( a \in A \setminus \{a\} \). Then there exists an \( \mathcal{M} \)-function \( \gamma : [0, \varepsilon) \rightarrow \mathbb{R}^n \) which is \( C^1 \) on \( [0, \varepsilon) \) and such that
\[
a = \gamma(0) \quad \text{and} \quad \gamma((0, \varepsilon)) \subset A \setminus \{a\}.
\]

Proof. — The key point is to construct a “definable” selection operator \( e \), which assigns to each nonempty set \( A \in \mathcal{M}_n \) an element \( e(A) \in A \). Let \( n = 1 \). Then \( e(A) \) is the smallest element of \( A \) if \( A \) has one. Otherwise, let \( a := \inf A \) and let \( b \in \mathbb{R} \cup \{+\infty\} \) be maximal such that \( (a, b) \subseteq A \). If \( a, b \in \mathbb{R} \), then \( e(A) := (a + b)/2 \). If \( a \in \mathbb{R} \) and \( b = +\infty \), then \( e(A) := a + 1 \). If \( a = -\infty \) and \( b \in \mathbb{R} \), then \( e(A) := b - 1 \). If \( a = -\infty \) and \( b = +\infty \) (i.e., \( A = \mathbb{R} \)), then \( e(A) := 0 \). Assume \( e(A) \) has been defined for all nonempty \( A \in \mathcal{M}_n \). Let \( B \in \mathcal{M}_{n+1} \) be nonempty, and let \( A \) be its image in \( \mathbb{R}^n \) under the projection map \( (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n) \). Put \( a := e(A) \). Then \( e(B) := (a, e(B_a)) \) where \( B_a := \{r \in \mathbb{R} : (a, r) \in B\} \).

This selection operator \( e \) has several applications, and Curve Selection is only one of them: let \( A \in \mathcal{M}_n \) and \( a \in A \setminus \{a\} \). By \( \mathcal{O} \)-minimality the set \( \{|a - x| : x \in A\} \in \mathcal{M}_1 \) contains an interval \( (0, \varepsilon), \varepsilon > 0 \). For \( 0 < t < \varepsilon \), let \( \gamma(t) := e(\{x \in A : |a - x| = t\}) \). It is routine to check that \( \gamma : (0, \varepsilon) \rightarrow A \) belongs to \( \mathcal{M} \). By the monotonicity theorem \( \gamma \) is \( C^1 \) after suitable shrinking of \( \varepsilon \). After composition on the right with a sufficiently flat (at 0) function in \( \mathcal{M} \) (e.g. the inverse of the biggest component of \( \gamma \)) we can further arrange that \( \gamma \) extends to a \( C^1 \)-function on \( [0, \varepsilon) \).

2. Lojasiewicz inequalities for \( \mathcal{O} \)-minimal structures.

We begin this section recalling an already well-known generalization of the Lojasiewicz inequality for continuous \( \mathcal{M} \)-functions on a compact set. This result was observed by T. Loi [Lo] for \( (\mathbb{R}, \exp) \)-definable sets (actually his version is more precise than the theorem stated below); M. Shiota [S1], [S2] and L. van den Dries and C. Miller [DM] also noticed this fact.

Theorem 0. — Let \( K \) be a compact subset of \( \mathbb{R}^n \) and let \( f,g : K \rightarrow \mathbb{R} \) be two continuous \( \mathcal{M} \)-functions. If \( f^{-1}(0) \subset g^{-1}(0) \), then there
exists a strictly increasing positive $\mathcal{M}$-function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ of class $C^1$, such that for any $x \in K$ we have
\[ |f(x)| \geq \sigma(g(x)). \]

The idea of the proof goes back to the original argument of Łojasiewicz (see [L2], [KLZ]). Let $\Sigma \subset \mathbb{R}^2$ be the image of $K$ by the mapping $K \ni u \to (g(u), f(u)) = (x, y)$. Clearly $\Sigma$ is an $\mathcal{M}$-set; moreover it is compact and $\Sigma \cap \{y = 0\} = \{(0,0)\}$. It is not difficult to find (by Lemma 2) a strictly increasing positive $\mathcal{M}$-function $\sigma : \mathbb{R}_+ \to \mathbb{R}$ of class $C^1$, such that $\Sigma \subset \{y \geq \sigma(x), x \geq 0\}$. It is proved in [DM] that for each $k \in \mathbb{N}$ one can find $\sigma$ of class $C^k$.

We state now the main result of this section. Recall that $\mathcal{M}$ is any fixed o-minimal structure on $(\mathbb{R}, +, \cdot)$.

**Theorem 1.** Let $f : U \to \mathbb{R}$ be a differentiable $\mathcal{M}$-function, where $U$ is an open and bounded subset of $\mathbb{R}^n$. Suppose that $f(x) > 0$ for all $x \in U$. Then there exists $c > 0, \rho > 0$ and a strictly increasing positive $\mathcal{M}$-function $\Psi : \mathbb{R}_+ \to \mathbb{R}$ of class $C^1$, such that
\[ \|\text{grad}(\Psi \circ f)(x)\| \geq c, \]
for each $x \in U$, $f(x) \in (0, \rho)$.

The proof is given in the end of the section. We shall see now that in the subanalytic case our Theorem 1 is equivalent to the classical Łojasiewicz inequality for gradients of analytic functions (see [L1], [L2], [BM]). We state this result in the form generalized in [KP]:

**Theorem (LI).** Let $f : \Omega \to \mathbb{R}$ be a subanalytic function which is differentiable in $\Omega \setminus f^{-1}(0)$, where $\Omega$ is an open bounded subset of $\mathbb{R}^n$. Then there exist $C > 0, \rho > 0$ and $0 \leq \alpha < 1$ such that:
\[ \|\text{grad} f(x)\| \geq C|f(x)|^\alpha, \]
for each $x \in \Omega$ such that $|f(x)| \in (0, \rho)$. If in addition $\lim_{x \to a} f(x) = 0$ for some $a \in \overline{\Omega}$ (which holds in the classical case, where $f$ is analytic and $a \in \Omega$, $f(a) = 0$), then the above inequality holds for each $x \in \Omega \setminus f^{-1}(0)$ close to $a$.

To see that in the subanalytic case (LI) $\Rightarrow$ Theorem 1 it is enough to put $\Psi(t) = t^{1-\alpha}$. To prove the converse in the subanalytic case, recall first that every subanalytic function in one variable is actually
semianalytic (see [L2], [KLZ]). Hence $\Psi$ has the Puiseux expansion of the form $\Psi(t) = \sum_{\nu=0}^{\infty} a_{\nu} t^{\nu}$. Thus, for $t$ small enough we have $|\Psi'(t)| \leq D t^k$ for some $D > 0$. The last inequality and Theorem 1 yield

$$\|\nabla f(x)\| = \frac{\|\nabla (\Psi \circ f)(x)\|}{|\Psi'(f(x))|} \geq \frac{c}{D} |f(x)|^{1-k}.$$  

Remark. — The above argument and Theorem 1 imply that (LI) holds in any polynomially bounded o-minimal structure on $(\mathbb{R}, +, \cdot)$.

We discuss now a consequence of Theorem 1. Let $f : U \to \mathbb{R}$ be a differentiable function, where $U$ is an open subset of $\mathbb{R}^n$. We shall say that $\lambda \in \mathbb{R} \cup \{-\infty, +\infty\}$ is an asymptotic critical value of $f$ if there exists a sequence $x_n \in U$ such that

$$f(x_n) \to \lambda \quad \text{and} \quad \nabla f(x_n) \to 0.$$  

Clearly any “true” critical value of $f$ (i.e. $\lambda = f(x)$ and $\nabla f(x) = 0$, for some $x \in U$) is also an asymptotic critical value. Notice that this notion depends heavily on the domain $U$, in particular on whether $U$ is bounded or not.

Suppose now that $U$ is bounded and that our $f$ is an $\mathcal{M}$-function, where $\mathcal{M}$ is an o-minimal structure on $(\mathbb{R}, +, \cdot)$. Let $\lambda$ be an asymptotic critical value of $f$. It follows immediately from Theorem 1 that $f$ has no asymptotic critical values in $(\lambda - \rho, \lambda) \cup (\lambda, \lambda + \rho)$ for some $\rho > 0$. But on the other hand the set of all asymptotic critical values of $f$ is an $\mathcal{M}$-subset of $\mathbb{R}$, so it must be finite. Thus we have proved:

**Proposition 2.** — If $U$ is bounded and $f$ is an $\mathcal{M}$-function, then the set of all asymptotic critical values of $f$ is finite.

It is easily seen that $-\infty$ and $+\infty$ cannot be an asymptotic critical value of an $\mathcal{M}$-function defined in a bounded set. As the following example shows the assumption of boundness on $U$ is necessary.

**Example.** — The function $f(x, y) = \frac{x}{y}$ on $U = \{y > 0\} \subset \mathbb{R}^2$, being semialgebraic, belongs to any o-minimal structure on $(\mathbb{R}, +, \cdot)$. But clearly any $\lambda \in \mathbb{R}$ is an asymptotic critical value of $f$.

**Proof of Theorem 1.** — It follows from Lemma 3 that $\nabla f(x) \to 0$ for any $x \in U$ is a $\mathcal{M}$-function. We may suppose that $f^{-1}(t) \neq \emptyset$ for any $t \in \mathbb{R}$. Theorem 1. — It follows from Lemma 3 that $\nabla f(x) \to 0$ for any $x \in U$ is a $\mathcal{M}$-function. We may suppose that $f^{-1}(t) \neq \emptyset$ for any $t \in \mathbb{R}$.
small enough \( t > 0 \), since otherwise, by \( \alpha \)-minimality, the theorem is trivial. Hence the function

\[
\varphi(t) = \inf \{ \| \text{grad} f(x) \| : x \in f^{-1}(t) \}
\]

is well-defined in some interval \((0, \varepsilon)\). By Lemma 1, \( \varphi \) is an \( \mathcal{M} \)-function.

**Claim.** There exists \( \varepsilon' > 0 \) such that \( \varphi(t) > 0 \) for any \( t \in (0, \varepsilon') \).

Assume that this is not the case and put

\[
\Sigma = \{ x \in U : \| \text{grad} f(x) \| < (f(x))^2 \}.
\]

Clearly \( \Sigma \) is an \( \mathcal{M} \)-set. Let \( f|_\Sigma \) denote the graph of \( f \) restricted to \( \Sigma \). If the claim doesn’t hold, then there exists a sequence of positive numbers \( t_n \to 0 \) such that \( \varphi(t_n) = 0 \) for all \( n \in \mathbb{N} \). Let \( x_n \in \Sigma \) be a sequence such that \( f(x_n) = t_n \), in other words \( (x_n, t_n) \in f|_\Sigma \). Let \( b \) be an accumulation point of \( \{x_n\} \), then \((b, 0)\) belongs to the closure of the set \((f|_\Sigma \setminus \{(b, 0)\})\).

By the curve selection lemma (Proposition 1) we have an \( \mathcal{M} \)-function (arc) \( \tilde{\gamma} : (-\delta, \delta) \to \mathbb{R}^n \times \mathbb{R} \) of class \( C^1 \), such that \( \tilde{\gamma}(0) = (b, 0) \), and \( \tilde{\gamma}(0, \delta) \subseteq f|_\Sigma \). Write \( \tilde{\gamma}(s) = (\gamma(s), f \circ \gamma(s)) \), where \( \gamma(s) \in \Sigma \subseteq \mathbb{R}^n \). Let \( h(s) = f \circ \gamma(s) \) for \( s \in (0, \delta) \), then clearly \( \lim_{s \to 0^+} h(s) = 0 = \lim_{s \to 0^+} h'(s) \), since \( \gamma(s) \in \Sigma \). Of course \( h \) and \( h' \) are \( \mathcal{M} \)-functions, so by Lemma 2 we may suppose that \( h \) and \( h' \) are monotone; actually they must be strictly increasing. Thus we have

\[
0 < h'(s) \leq A(h(s))^2, \quad \text{for} \quad s \in (0, \delta),
\]

where \( A \) is a bound for \( \| \gamma'(s) \| \). But by the Mean Value Theorem we have \( h(s) \leq sh'(s) \), because \( h' \) is increasing. Finally, we get \( 0 < h'(s) \leq As^2(h'(s))^2 \) for any \( s \in (0, \delta) \), which is impossible since \( \lim_{s \to 0^+} h'(s) = 0 \).

So we have proved that \( \varphi(t) > 0 \) for all \( t \in (0, \varepsilon) \), provided that \( \varepsilon > 0 \) is small enough. We define now:

\[
\Delta = \{ x \in U \setminus f^{-1}(0) : f(x) < \varepsilon, \| \text{grad} f(x) \| \leq 2\varphi(f(x)) \}.
\]

Observe that \( \Delta \) is also an \( \mathcal{M} \)-set and moreover \( \Delta \cap f^{-1}(t) \neq \emptyset \) for every \( t \in (0, \varepsilon) \). Hence as before there exists \( d \in \overline{U} \) such that \( (d, 0) \in f|_\Delta \setminus \{(d, 0)\} \).

Applying again the curve selection lemma to \( f|_\Delta \) at the point \((d, 0)\) we obtain an \( \mathcal{M} \)-function (arc) \( \tilde{\eta} : (-\delta, \delta) \to \mathbb{R}^n \) of class \( C^1 \), such that \( \tilde{\eta}(0) = (d, 0) \), and \( \tilde{\eta}(0, \delta) \subseteq f|_\Delta \). Write as before \( \tilde{\eta}(s) = (\eta(s), f \circ \eta(s)) \), where \( \eta(s) \in \Delta \subseteq \mathbb{R}^n \). Let \( g(s) = f \circ \eta(s) \) for \( s \in (0, \delta) \), then clearly \( \lim_{s \to 0^+} g(s) = 0 \) and \( g(s) > 0 \) for each \( s \in (0, \delta) \). It follows from Lemma 2 that for \( \delta' > 0 \) small enough the function \( g : (0, \delta') \to \mathbb{R} \) is a diffeomorphism onto \((0, \rho)\), for some \( \rho > 0 \). We put

\[
\Psi(t) = g^{-1}(t) \quad \text{for} \quad t \in (0, \rho).
\]
We shall check now the inequality claimed in Theorem 1. Let $B$ be some bound for $|\eta'(s)|$ in $(0, \delta')$. Take any $x \in U$ such that $t = f(x) \in (0, \rho)$, and write $s = \Psi(t) = g^{-1}(t)$. Then we have
\[
\|\text{grad} \Psi \circ f(x)\| = \Psi'(f(x))\|\text{grad} f(x)\| \\
\geq \Psi'(t) \frac{1}{2} \|\text{grad} f(\eta(s))\| \geq \frac{\Psi'(t)}{2B} (f \circ \eta)'(s) = \frac{1}{2B} = c,
\]
since $\|\text{grad} f(\eta(s))\| \|\eta'(s)\| \geq \langle \text{grad} f(\eta(s)), \eta'(s) \rangle = (f \circ \eta)'(s)$ and $B \geq \|\eta'(s)\|$. Theorem 1 follows.

3. Trajectories of gradients of $\mathcal{M}$-functions.

Let $f : U \rightarrow \mathbb{R}$ be a $C^1$ function, where $U$ is an open subset of $\mathbb{R}^n$. We shall consider a vector field,
\[
U \ni x \mapsto -\text{grad} f(x) \in \mathbb{R}^n.
\]
Let $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$. We shall say that $\gamma : (\alpha, \beta) \rightarrow U$ is a trajectory of the vector field $-\text{grad} f$ if it is a maximal differentiable curve verifying $\gamma'(t) = -\text{grad} f(\gamma(s))$. Actually we shall consider $\gamma$ as an equivalence class of all curves obtained from $\gamma$ by a strictly increasing $C^1$ reparametrization. Observe that if $\psi$ is an increasing $C^1$ diffeomorphism between two intervals in $\mathbb{R}$, then the trajectories of $-\text{grad} \psi \circ f$ and those of $-\text{grad} f$ are the same.

Let $a, b \in \gamma$. We denote by $|\gamma(a, b)|$ the length of $\gamma$ between $a$ and $b$.

Lojasiewicz derived (see [L1], [L3]) from (LI) that all trajectories of $-\text{grad} f$ are of finite length, when $f$ is analytic in a neighborhood of a compact $\overline{U}$. We have:

\textbf{Theorem 2.} — Let $f : U \rightarrow \mathbb{R}$ be a function of class $C^1$, where $U$ is an open and bounded subset of $\mathbb{R}^n$. Suppose that $f$ is an $\mathcal{M}$-function, for some o-minimal structure $\mathcal{M}$.

a) Then there exists $A > 0$ such that all trajectories of $-\text{grad} f$ have length bounded by $A$.

b) More precisely, there exists $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a continuous strictly increasing $\mathcal{M}$-function, with $\lim_{t \rightarrow 0} \sigma(t) = 0$, such that if $\gamma$ is a trajectory of $-\text{grad} f$ and $a, b \in \gamma$, then
\[
|\gamma(a, b)| \leq \sigma(|f(b) - f(a)|).\]
Proof of theorem 2. — Taking, if necessary the composition $\psi \circ f$, where $\psi(t) = \frac{t}{\sqrt{1+t^2}}$, we may suppose that $f$ is bounded; more exactly that the image of $f$ lies in $(-1,1)$. We consider again the $\mathcal{M}$-function $\varphi : (-1,1) \to \mathbb{R}$ defined by

$$\varphi(t) = \inf\{\|\text{grad } f(x)\| : x \in f^{-1}(t)\},$$

when $f^{-1}(t) \neq \emptyset$, and $\varphi(t) = 1$ when $f^{-1}(t) = \emptyset$. Let $\Sigma$ be the set of all asymptotic critical values of $f$. Observe that $\lambda \in \Sigma$ if $\varphi(\lambda) = 0$, or $\lim_{t \searrow \lambda} \varphi(t) = 0$, or $\lim_{t \nearrow \lambda} \varphi(t) = 0$.

Let $I \subset (-1,1)$ be an open interval. Assume that $\varphi$ is bounded from below in $I$ by some $c > 0$. Let $\gamma$ be a trajectory of $-\text{grad } f$ and $a, b \in \gamma$. Suppose that the part of $\gamma$ lying between $a$ and $b$ is contained in $f^{-1}(I)$. We parametrise $\gamma$ by arc-length (i.e $\|\gamma'(s)\| = 1$), so by the Mean Value Theorem we have that $|f \circ \gamma(\beta) - f \circ \gamma(\alpha)| \geq c |\beta - \alpha|$, in other words

$$|\gamma(a, b)| \leq \frac{1}{c} |f(b) - f(b)|.$$

This observation explains the idea of the proof. By a partition $-1 = t_0 < t_1 < \ldots < t_k = 1$ we shall decompose $(-1,1)$ in such a way that $\varphi$ is strictly monotone on $(t_i, t_{i+1})$. Moreover we shall distinguish between two disjoint types of intervals, namely

1. there exists $c_i > 0$ such that $\varphi(t) \geq c_i$ on $(t_i, t_{i+1})$ (we write $i \in I_1$ in this case), or
2. one of $t_i, t_{i+1}$ is an asymptotic critical value of $f$, hence by Theorem 1, there exist $c_i > 0$ and $\Psi_i : (t_i, t_{i+1}) \to \mathbb{R}$ a strictly increasing, bounded $C^1$ function such that,

$$\|\text{grad } (\Psi_i \circ f)(x)\| \geq c_i$$

for all $x \in f^{-1}(t_i, t_{i+1})$ (we write $i \in I_2$ in this case).

Take now any trajectory $\gamma$ of $-\text{grad } f$, and let $\gamma_i = \gamma \cap f^{-1}(t_i, t_{i+1})$. We denote by $|\gamma|$ (resp. $|\gamma_i|$) the length of $\gamma$ (resp. $\gamma_i$). Clearly $|\gamma_i| \leq \frac{1}{c_i} |t_i - t_{i+1}|$ if $i \in I_1$. Extending by continuity, we may suppose that each $\Psi_i$ is defined also at $t_i$ and $t_{i+1}$. Hence for $i \in I_2$ we have $|\gamma_i| \leq \frac{1}{c_i} |\Psi_i(t_i) - \Psi_i(t_{i+1})|$, since the trajectories of $-\text{grad } (\Psi_i \circ f)$ and $-\text{grad } f$ are the same in $f^{-1}(t_i, t_{i+1})$. Finally, we can write

$$|\gamma| = \sum_{i=0}^{k-1} |\gamma_i| \leq \sum_{i \in I_1} \frac{1}{c_i} |t_i - t_{i+1}| + \sum_{i \in I_2} \frac{1}{c_i} |\Psi_i(t_i) - \Psi_i(t_{i+1})| = A,$$
which proves part a) of Theorem 2.

We are now going to construct the function \( \sigma \) of part b). For \( i \in I_2 \) we put

\[
\sigma_i(r) = \frac{1}{c_i} \sup \{ |\Psi_i(p) - \Psi_i(q)| : p, q \in (t_i, t_{i+1}), r = p - q \},
\]

and \( \sigma_i(r) = \frac{r}{c_i} \) for \( i \in I_1 \). Extend each \( \sigma_i \) to a continuous strictly increasing \( \mathcal{M} \)-function on \( \mathbb{R} \). It is easily seen that \( \sigma = \sup \sigma_i \) satisfies b) of Theorem 2.

We finish this section by a short discussion of some consequences of Theorem 2, which extend and generalize those known in the real analytic (compact) setting.

Observe that if \( \gamma : (\alpha, \beta) \to U \) is a trajectory then \( x_0 = \lim_{s \to \beta} \gamma(s) \) exists, and in general \( x_0 \) belongs to \( \overline{U} \). Notice that if \( x_0 \in U \), then \( x_0 \) is a critical point of \( f \). Let us take \( E \) a closed \( \mathcal{M} \)-subset in an open set \( U \); by 4.22 of [DM], \( E \) is the zero set of an \( \mathcal{M} \)-function \( f : U \to \mathbb{R} \) of class \( C^2 \). Let \( g = f^2 \). We want to show that the flow of \( -\text{grad} g \) defines a strong deformation retraction of a neighborhood of \( E \) onto \( E \). This is actually a new result even in the subanalytic case since the retraction is global and \( E \) is not necessarily compact. By Proposition 2, taking a neighborhood of \( E \), we may suppose that 0 is the only asymptotic critical value of \( g \) in \( U \).

Clearly the set

\[
V = \{ x \in U : \text{dist}(x, \partial U) < \sigma(g(x)) \}
\]

is an \( \mathcal{M} \)-set, it is an open neighborhood of \( E \). For each \( x \in V \) we denote by \( \gamma_x : (\alpha_x, \beta_x) \to U \) the trajectory passing through \( x \). It is clearly unique if \( g(x) \neq 0 \) and constant (hence unique) if \( g(x) = 0 \). Put \( R(x) = \lim_{s \to \beta_x} \gamma_x(s) \), and observe that \( R(x) \in E \). We have:

**Proposition 3.** — There exists an open neighborhood \( V_1 \) of \( E \) such that \( R : V_1 \to E \) is a strong deformation retraction.

**Proof.** — First we shall prove that \( R \) is continuous. Take \( x_0 \in V \) and \( \Omega_0 \) a neighborhood of \( R(x_0) \). Let \( x_1 \notin E \) be close to \( R(x_0) \) so that there is (by Theorem 2 b)) a neighborhood \( \Omega_1 \) of \( x_1 \) with the following property: any trajectory passing through \( \Omega_1 \) has its limit in \( \Omega_0 \). By continuity of the flow of \( -\text{grad} g \) there exists a neighborhood \( G \) of \( x_0 \) such that any trajectory passing by \( G \) must cross \( \Omega_1 \). So we have \( R(G) \subset \Omega_0 \), which proves the continuity of \( R \).
Let $\gamma$ be the trajectory passing through $x$. Let $\gamma_x$ be the part of $\gamma$ between $x$ and the limit $R(x)$. Assume that $\gamma_x : [0, \beta_x] \to U$ is parametrized by arc-length; moreover that $\gamma_x(0) = x$, and $\gamma_x(\beta_x) = R(x)$. Clearly $\beta_x$ is the length of $\gamma_x$. Notice that the argument in the proof of continuity of $R$ yields that the function $V \ni x \to \beta_x$ is continuous. Let $V_1$ be the set of all $x \in V$ such that $\gamma_x$ lies in $V$. We define a homotopy $F : [0,1] \times V_1 \to V_1$ as follows: $F_t(x) = \gamma_x(t\beta_x)$.

In general the retraction $R$ is not an $\mathcal{M}$-mapping. Take $g(x,y) = (x^2 - y^3)^2$; it was observed by Hu [Hu] that the retraction $R$ is not hoelderian (at $(0,0)$) in this case, hence it cannot be subanalytic. Observe also that, in general, the set $V_1$ is not an $\mathcal{M}$-set. It would be interesting to prove that actually $R$ belongs to some larger $\mathcal{O}$-minimal structure. Even a weaker problem is open (also in the subanalytic case):

**Conjecture (F).** — Let $\gamma$ be a trajectory of $-\text{grad} \, f$, where $f$ is an $\mathcal{M}$-function of class $C^1$, and let $H$ be any $\mathcal{M}$-subset. Then $\gamma \cap H$ has a finite number of connected components.

This is connected with the Gradient Conjecture of R. Thom, proved recently in [KM]. R. Thom asked whether for an analytic function $f$ every trajectory $\gamma$ of $-\text{grad} \, f$ has a tangent at the limit point (i.e. whether $\lim_{s \to \beta_x} \frac{\gamma(s) - R(x)}{|\gamma(s) - R(x)|}$ exists). We can of course ask the same question for a trajectory of the gradient of any $\mathcal{M}$-function of class $C^1$.

It is easily seen that (F) implies that $\lim_{s \to \beta_x} \frac{\gamma'(s)}{|\gamma'(s)|}$ exists, thus that the tangent to $\gamma$ at the limit point exists.
BIBLIOGRAPHY


Manuscrit reçu le 15 septembre 1997,
accepté le 13 janvier 1998.

Krzystof KURDYKA,
Université de Savoie
Laboratoire de Mathématiques
Campus Scientifique
73376 Le Bourget-du-Lac Cedex (France).
kurdyka@univ-savoie.fr

and

Uniwersytet Jagielloński
Instytut Matematyki
ul. Reymonta 4
30-059 Kraków (Poland).
kurdyka@im.uj.edu.pl