ANNALES DE L'INSTITUT FOURIER

GERRY W. SCHWARZ
DAVID L. WEHLAU

Invariants of four subspaces

Annales de l'institut Fourier, tome 48, nº 3 (1998), p. 667-697
<http://www.numdam.org/item?id=AIF_1998__48_3_667_0>

Ann. Inst. Fourier, Grenoble
48, 3 (1998), 667–697

INVARIANTS OF FOUR SUBSPACES

by G.W. SCHWARZ and D.L. WEHLAU

Introduction.

1.1. — Our problem is to determine the invariant theory of four linear subspaces of a fixed vector space V (everything over $\mathbb{C}$). This is related to the problem of classifying r-tuples of subspaces of vector spaces. The latter problem (in the representation theory of quivers) is known to be "tame" if $r \leq 4$ and "wild" otherwise; for example, the determination of the normal pairs of $n \times m$ matrices (see [GP], [BGP], [Kac]).

1.2. — Let $k_1, \ldots, k_4$ be integers between 1 and $n-1$, inclusive, and let k denote one of the k_i. A k-dimensional subspace of $\mathbb{C}^n$ corresponds to an element of the projective space $P(D_k)$, where D_k denotes the set of decomposable elements of $\wedge^k(\mathbb{C}^n)$. We are interested in the SL_n-orbit structure of $P(D_{k_1}) \times \cdots \times P(D_{k_4})$, which one easily obtains from the SL_n-orbit structure of $X := D_{k_1} \times \cdots \times D_{k_4}$. The SL_n-variety X is the basic object of interest.

Note that the quotient of the $(\mathrm{SL}_n \times \mathrm{SL}_k)$-module $\mathbb{C}^n \otimes \mathbb{C}^k$ by the action of SL_k is just D_k (see 2.1). Thus (X, SL_n) is a quotient, as follows. Set $H = \mathrm{SL}_{k_1} \times \cdots \times \mathrm{SL}_{k_4}$ with its natural action on $W := \mathbb{C}^{k_1} \oplus \cdots \oplus \mathbb{C}^{k_4}$. Set $\vec{k} := (k_1, \ldots, k_4)$ and set $G := \mathrm{SL}_n \times H$ with its natural action on $V := V(n, \vec{k}) = \mathbb{C}^n \otimes W$. Then (X, SL_n) is the quotient $(V/\!/H,\, G/H)$. The

We thank Hanspeter Kraft, Michel Brion and Peter Littelmann for their hospitality in Basel and Luminy.
Research partially supported by grants from the NSF and NSERC.
Key words: Invariants – Equidimensional – Castling.
Math. classification: 20G15 – 20G20 – 22E15 – 22E46.

orbit structure of (V,G) determines that of (X,SL_n), so we concentrate on the former case. Set

$$\delta := 2n - k_1 - \cdots - k_4 \quad \text{(the defect)},$$
$$q := \dim \mathbb{C}[X]^{\mathrm{SL}_n} = \dim \mathbb{C}[V]^G.$$

Unless we are doing a numerical example or otherwise specified, we assume that

$$k_1 \le k_2 \le k_3 \le k_4.$$

1.3. — We first determine the closed orbits of (V,G); equivalently, we determine the algebra $\mathbb{C}[V]^G$ or the categorical quotient $V/\!/G$. It is more difficult to find the orbit structure of the fibers of the quotient mapping $\pi: V \to V/\!/G$. Classically, this has been done by using certain *covariants.* A covariant is just an element of an isotypic component of $\mathbb{C}[V]$. While we do not determine the orbit structure of the fibers, we do show that the covariants have a very nice structure: they are (almost always) a *free* $(G\text{-}\mathbb{C}[V]^G)$-module. We then say that (V,G) is cofree; equivalently, $V/\!/G$ is an affine space and π is equidimensional. We determine the principal isotropy group of (V,G) in the important cases (including the cases $\delta=0$, $q\ge 2$). This allows us to completely determine the structure of $\mathbb{C}[V]$ as a $(G\text{-}\mathbb{C}[V]^G)$-module [Sch2, 1.1] (but not as a *graded* $(G\text{-}\mathbb{C}[V]^G)$-module).

1.4. — The dimensions $(n,\vec{k})$ and representations $V(n,\vec{k})$ occur in series (obtained via castling 2.2). Eventually the elements of the series are either all cofree or all not cofree, but some series take longer than others to "settle down". Among these are the series containing $V(2,(1,1,1,2))$ and $V(3,(1,1,1,1))$. For other reasons, Gelfand and Ponomarev (see [GP]) found the indecomposables of the latter kind to be exceptional.

1.5. — The invariants of (X,SL_n) were recently determined by R. Howe and R. Huang [HoHu]. Their work complements and/or extends earlier work of Turnbull [Turn], Huang [Huan] and Ringel [Ring]. Howe and Huang used techniques from representation theory and combinatorics (the symbolic method) to obtain the following:

- an explicit description of $\mathbb{C}[X]^{\mathrm{SL}_n}$;
- a proof that $\mathbb{C}[X]^{\mathrm{SL}_n}$ is always regular (a polynomial ring).

There are slight inaccuracies in these results: One type of invariant is missing (see 3.17.4 and Example 2.7) and $\mathbb{C}[X]^{\mathrm{SL}_n} = \mathbb{C}[V]^G$ fails to be regular in one case (Theorem 2.9.3.d). We have approached the problem using ideas from invariant theory, and we are able to find shorter and more direct proofs of the results in [HoHu].

When $\delta \neq 0$, the technique of castling is decisive (see 2.2 and 3.1). We show that $q \leq 4$ and completely classify the cases $q = 3$ and $q = 4$. We show how the invariants arise from very simple ones twisted and made complicated by castling (see 2.2). In particular, we explain the "mysterious" type III invariants of [HoHu] and we also find a fourth type (Theorem 3.17 (4)). If $\delta = 0$, castling is not very effective. Instead we use invariant theory to compute the principal isotropy group of (V, G). This gives us q, and then we exhibit q explicit generators of $\mathbb{C}[V]^G$. The computation of the invariants is in Sections 3 and 4.

In Sections 5, 6 and 7 we determine necessary and sufficient conditions for $\mathbb{C}[V]$ to be a free (G-$\mathbb{C}[V]^G$)-module; equivalently, we determine when the quotient mapping $\pi_V : V \to V/\!/G$ is equidimensional. This is much harder than determining the invariants. The most important case is that of 4 medials $(V(2n,(n,n,n,n)))$. Here we use an induction to reduce to properties of the case $n = 2$. In general, the obvious method is to use the Luna-Richardson theorem to reduce to the 4 medials case. This does not work, but we find a variant which does.

We determine the principal isotropy groups of (V, G) in case $\delta = 0$ and $q \geq 2$. The cases where $\delta \neq 0$ and $q \geq 3$ are easily handled. As noted above, this information allows us to compute covariants and their multiplicities. For example, if the principal isotropy group is trivial, then every irreducible representation of G occurs as a free $\mathbb{C}[V]^G$-submodule of $\mathbb{C}[V]$ with multiplicity equal to its dimension. This occurs in the cases $V(5,(4,2,2,2))$ ($\delta = 0$, $q = 2$), $V(7,(3,3,3,3))$ ($\delta = 2$, $q = 4$) and many others.

Usually, $\pi_V : V \to V/\!/G$ is equidimensional if and only if $\pi_X : X \to X/\!/G$ is. Theorem 2.10 (proof in Section 8) lists the exceptions.

2. Main results.

2.1. — Let Y be an affine G-variety. Let $Y/\!/G$ denote the affine variety corresponding to the algebra of invariants $\mathbb{C}[Y]^G$ and $\pi_Y : Y \to Y/\!/G$ the morphism dual to the inclusion $\mathbb{C}[Y]^G \subset \mathbb{C}[Y]$. We say that (Y, G) is

• *coregular* if $\mathbb{C}[Y]^G$ is a polynomial algebra, equivalently; $Y//G$ is smooth (and isomorphic to affine space);

• *equidimensional* if π_Y is equidimensional;

• *cofree* if $\mathbb{C}[Y]$ is a free $\mathbb{C}[Y]^G$-module. (A G-module V is cofree if and only if it is coregular and equidimensional [Schl], 17.29);

• *stable* if there is a dense subset of closed orbits.

A subgroup $L \subset G$ is a

• *generic* isotropy group if there is an open dense subset Y' of Y such that G_y is conjugate to L for every $y \in Y'$;

• *principal* isotropy group if there is an open dense subset U of $Y//G$ such that G_y is conjugate to L for every *closed* orbit in $\pi_Y^{-1}(U)$.

2.2. — We make extensive use of *castling* (see Section 3.1): Let σ denote $k_1 + \cdots + k_4$ and set $\delta := 2n - \sigma$. (The problem is quite different, depending upon whether or not δ is zero.) We have castling transformations $\mathcal{C}_\nu$ and $\mathcal{C}_\kappa$ which transform $V(n, \vec{k})$ to $V(\sigma - n, \vec{k})$ and $V(n, \vec{k})$ to $V\big(n, (n - k_4, \ldots, n - k_1)\big)$. The transformation $\mathcal{C}_\kappa$ was already used in [HoHu] to reduce to the case that $\delta \geq 0$. The *class* $\mathrm{Cl}(V(n, \vec{k}))$ *of* $V(n, \vec{k})$ is the collection of representations which can be obtained from $V(n, \vec{k})$ by $\mathcal{C}_\nu$ and $\mathcal{C}_\kappa$. Castling transformations preserve algebras of invariants, (non) stability and generic isotropy groups (up to isomorphism). Thus principal isotropy groups are preserved in the stable case.

2.3. — Each class (with $\delta \neq 0$) has a linear ordering: Let h denote $\sigma + n$. If we apply $\mathcal{C}_\nu$, then h changes to $h - \delta$. Thus we say that $\mathcal{C}_\nu$ *is a castling up (resp. down)* if $\delta < 0$ (resp. $\delta > 0$). We define "up" and "down" for $\mathcal{C}_\kappa$ similarly, where $\mathcal{C}_\kappa$ sends h to $h + 2\delta$. We say that $V(n', \vec{k}')$ is *above* (resp. *strictly above*) $V(n, \vec{k})$ if $V(n', \vec{k}')$ is obtained from $V(n, \vec{k})$ by a (resp. nonempty) sequence of castlings up.

2.4. Definition. — *We say that* $V(n, \vec{k})$ is minimal *if it is minimal in its castling class. Equivalently,* $\mathcal{C}_\nu$ *cannot be applied or is a castling up, and the same for* $\mathcal{C}_\kappa$.

To determine the algebra of invariants, etc. of $V(n, \vec{k})$ we may reduce to the case that $V(n, \vec{k})$ is minimal. Here our questions are either easy to answer or we may reduce to an instance of three subspaces. We then use our complete analysis of the possibilities for two and three subspaces (Section 3.8).

2.5. — We say that an invariant f of $V(n, \vec{k})$ has *(reduced) degree* (a, b, c, d) if, as a function on $X = D_{k_1} \times \cdots \times D_{k_4}$, it is multihomogeneous of degree a in D_{k_1}, b in D_{k_2}, etc. Note that, as a function on $V(n, \vec{k})$, it has multidegree $(ak_1, \ldots, dk_4)$. By degree we will always mean the reduced degree. The total (reduced) degree of f is $a + b + c + d$. We will use similar notation and definitions for two and three subspaces of n-space. We always denote the dimension of $\mathbb{C}[V]^G$ by q.

2.6. *Example.* — Consider $V\big(7,(3,3,3,2)\big)$. Then castling down we obtain

$$V\big(4,(3,3,3,2)\big), \quad V\big(4,(1,1,1,2)\big) \quad \text{and} \quad V\big(1,(1,1,1,2)\big).$$

Of course, $V\big(1,(1,1,1,2)\big)$ does not come from four subspaces of $\mathbb{C}^1$, but this is not a problem when computing invariants! The invariant of degree $(1,0,0,0)$ in $V\big(1,(1,1,1,2)\big)$ becomes one of degree $(2,1,1,1)$ in $V\big(7,(3,3,3,2)\big)$ (see (3.17)). The generators whose degree is bigger than $(1,1,1,1)$ are labeled "mysterious" in [HoHu]. As in this example, they all arise from simple invariants made complicated by castling.

2.7. *Example.* — Consider $V\big(4,(1,1,1,1)\big)$, which is generated by the obvious invariant of degree $(1,1,1,1)$. Castling we obtain

$$V\big(4,(3,3,3,3)\big) \quad \text{and} \quad V\big(8,(3,3,3,3)\big),$$

where the latter has invariants generated by an element in degree $(2,2,2,2)$.

Our main results are the following:

2.8. Theorem. — *Suppose that* $\delta \neq 0$. *Then*

(1) (V, G) *is coregular, and* $q \leq 4$.

(2) *If* $q = 3$, *then* V *is above*

(a) $V\big(n,(d,n,n,n)\big)$, $1 \leq d < n$, *or*

(b) $V\big(n,(a,b,n,n)\big)$ *where* $a + b = n$, $a < b$, *or*

(c) $V\big(n,(a,b,b,n)\big)$ *or* $V\big(n,(a,a,b,n)\big)$, $a + b = n$, $a < b$, *or*

(d) $V\big(2n,(n,n,n,d)\big)$, $d > 2n$ *or*

(e) $V\big(n,(n,n,n,d)\big)$, $d > n$.

(3) *If* $q = 4$, *then* V *is above*

(a) $V\big(n,(n,n,n,n)\big)$, *or*

(b) $V\big(2n,(n,n,n,2n)\big)$.

Concerning cofreeness (equivalently, equidimensionality) we have:

(4) *If* $q \leq 2$, *then* V *is cofree.*

(5) *The non-cofree representations with* $q = 3$ *are*

(a) *(from* 2.a, $d = 1$*) those above* $V\big(2n+1,(n+1,n+1,n+1,2n)\big)$, $n \geq 2$,

(b) *(from 2.d,* $n = 1$*)* $V\big(2,(1,1,1,d)\big)$ *and* $V\big(d+1,(1,1,1,d)\big)$, $d \geq 3$ *and*

(c) *(from 2.e,* $n = 1$*) those above* $V\big(d+2,(1,1,1,d)\big)$, $d \geq 2$.

(6) *The only non-cofree representations with* $q = 4$ *are*

(a) *(from* 3.a, $n = 1$*) those above* $V\big(3,(1,1,1,1)\big)$,

(b) *(from* 3.a, $n = 2$*) those above* $V\big(6,(2,2,2,2)\big)$ *and*

(c) *(from* 3.b, $n = 1$*)* $V\big(2,(1,1,1,2)\big)$, $V\big(3,(1,1,1,2)\big)$, $V\big(3,(2,2,2,1)\big)$, $V\big(4,(2,2,2,1)\big)$, $V\big(4,(2,2,2,3)\big)$ *and* $V\big(5,(2,2,2,3)\big)$.

2.9. THEOREM. — *Suppose that* $\delta = 0$.

(1) *If* $k_4 > n$, *then* $q = 0$ *and* V *is not stable.*

(2) *If* $k_4 = n$, *then* $q = 2$ *with generators of degrees* $(1,1,1,0)$ *and* $(0,0,0,1)$. *Moreover,* V *is cofree and stable with principal isotropy group* $\mathrm{SL}_{k_1} \times \mathrm{SL}_{k_2} \times \mathrm{SL}_{k_3}$.

(3) *If* $k_4 < n$, *let* a *denote* $\min\{k_1, n-k_4\}$. *Then*

(a) V *is stable with principal isotropy group (up to a finite cover)* $\mathrm{SL}_{n-k_1-k_2} \times \mathrm{SL}_{n-k_1-k_3} \times \mathrm{SL}_{|n-k_1-k_4|} \times T^{a-1}$, *where* T^r *denotes an* r*-torus* $(\mathbb{C}^*)^r$. *Furthermore,*

$$q = a + 1 + \delta_{k_1+k_4,n} + \delta_{k_1+k_3,n} + \delta_{k_1+k_2,n}.$$

(b) V *is cofree if* $V \neq V\big(2n,(n,n,n,n)\big)$, $n \leq 2$.

(c) $V\big(4,(2,2,2,2)\big)$ *is coregular, but not cofree.*

(d) $V\big(2,(1,1,1,1)\big)$ *is neither coregular nor cofree.*

Write $V = V(n,\vec{k}\,) = (\mathbb{C}^n \otimes W, \mathrm{SL}_n \times H)$ as in 1.2. Then $X = V/\!/H = D_{k_1} \times \cdots \times D_{k_4}$. We consider equidimensionality of (X, SL_n):

2.10. Theorem. — *Let (X,SL_n) and $V = V(n,\vec{k}\,)$ be as above.*

(1) *If V is equidimensional, then (X,SL_n) is equidimensional.*

(2) *If (X,SL_n) is equidimensional, then V is equidimensional, with the following exceptions :*

(a) $V = V\big(2n+1,(n+1,n+1,n+1,2n)\big)$, $n \geq 2$,

(b) $V = V\big(d+1,(1,1,1,d)\big)$, $d \geq 3$ *and*

(c) $V = V\big(5,(2,2,2,3)\big)$.

Theorem 2.8 follows from 3.11 and the results in Section 5. Theorem 2.9.3 follows from Proposition 4.5, Remark 4.6, Theorem 4.9 and Theorem 7.11. Parts 1 and 2 are left to the reader. Theorem 2.10 follows from Theorem 2.8, Theorem 2.9 and Theorem 5.3.

3. The case $\delta \neq 0$.

3.1. *Castling.* — Consider a representation $(\mathbb{C}^n \otimes U \oplus Z, \mathrm{SL}_n \times L)$ where $\dim U = n + n' > n$, SL_n acts trivially on Z and L is reductive. We assume that L acts trivially on $\wedge^{n+n'}(U)$. Then the invariants of $(\mathbb{C}^n \otimes U \oplus Z, \mathrm{SL}_n \times L)$ are isomorphic to those of $(\mathbb{C}^{n'} \otimes U^* \oplus Z, \mathrm{SL}_{n'} \times L)$. The only thing to observe is that the quotients by the special linear groups are the product of Z and the decomposable vectors in the isomorphic representations $\wedge^n(U)$ and $\wedge^{n'}(U^*)$. Furthermore, since the general SL_n and $\mathrm{SL}_{n'}$ orbits are closed with trivial stabilizer, the isotropy groups of non-zero points in the quotients are isomorphic. By projection to L, we obtain an isomorphism of generic isotropy groups. We will call a castling transformation of the form described above a *simple* castling transformation. Thus $\mathcal{C}_\nu$ is a simple transformation while $\mathcal{C}_\kappa$ is the composition of 4 simple transformations. However, unless otherwise specified, by "castling transformation" we are referring to $\mathcal{C}_\nu$ and $\mathcal{C}_\kappa$.

3.2. — As in §§ 1.2–2.3, let

$$V = V(n, \vec{k}), \quad \sigma := \sum_i k_i,$$
$$\delta := 2n - \sigma, \quad h = \sigma + n, \quad q = \dim \mathbb{C}[V]^G.$$

In this section we assume that $\delta \neq 0$. We only consider representations $V(n, \vec{k})$ where $n \geq 1$ and $k_i \geq 1$, $i = 1, \ldots, 4$. Castling sends

$$V(n, \vec{k}) = \mathbb{C}^n \otimes W \quad \text{to} \quad \mathbb{C}^{\sigma - n} \otimes W^*.$$

But we may replace W^* by W without changing anything of interest, and we obtain our transformation $\mathcal{C}_\nu$. Similarly, we obtain $\mathcal{C}_\kappa$ by castling each of the representations

$$(\mathbb{C}^{k_i} \otimes \mathbb{C}^n, \mathrm{SL}_{k_i} \times \mathrm{SL}_n) \quad \text{to} \quad (\mathbb{C}^{n-k_i} \otimes \mathbb{C}^n, \mathrm{SL}_{n-k_i} \times \mathrm{SL}_n).$$

We denote by $V' = V(n', \vec{k}')$ the representation obtained by applying $\mathcal{C}_\nu$ or $\mathcal{C}_\kappa$ to $V(n, \vec{k})$, and we set

$$\sigma' = \sum_i k_i', \quad \delta' = 2n' - \sigma', \quad h' = \sigma' + n'.$$

Then we have

3.3. Proposition.

(1) *$\mathcal{C}_\nu$ applies iff $\sigma > n$, in which case $h' = h - \delta$;*

(2) *$\mathcal{C}_\kappa$ applies iff $k_4 < n$, in which case $h' = h + 2\delta$;*

(3) *$\delta' = -\delta$;*

(4) *$\mathrm{Cl}(V)$ is totally ordered by the value of h.*

3.4. Corollary. — *Let*

$$M = \{m \in \mathbb{N}\,;\ m|\delta| < k_i < n - 2m|\delta|,\ i = 1, \ldots, 4\}.$$

If $m \in M$, then we can castle down from V to

$$V' = V\big(n - 2m|\delta|, (k_1 - m|\delta|, \ldots, k_4 - m|\delta|)\big).$$

If m is the maximal element of M, then V' is at most three castlings up from the minimal element.

Proof. — We may assume that $m \geq 1$. First suppose that $\delta > 0$. Applying $\mathcal{C}_\nu$ and then $\mathcal{C}_\kappa$ we obtain $V(n-\delta,(n-\delta-k_4,\dots,n-\delta-k_1))$, and another iteration gives $V(n-2\delta,(k_1-\delta,\dots,k_4-\delta))$. Induction then gives the result. If $\delta < 0$, then we may use the argument above after applying $\mathcal{C}_\kappa$. □

3.5. *Example.* — Consider $V\big(2r+1,(r,r,r,r)\big)$ where $\delta = 2$.

• If $r = 2k+1$ is odd, then applying 3.4 with $m = k$ we arrive at $V\big(1,(1,1,1,1)\big)$.

• If $r = 2k$ is even, we may take $m = k-1$ to obtain $V\big(3,(1,1,1,1)\big)$, and then apply $\mathcal{C}_\nu$ to get to $V\big(1,(1,1,1,1)\big)$. One easily sees that, in fact, $\{V\big(2r \pm 1,(r,r,r,r)\big)\}$ is the castling class of $V\big(1,(1,1,1,1)\big)$. Similarly,

$$\{V\big(2r,(r,r,r,r\pm 1)\big)\} \cup \{V\big(2r+1,(r,r,r,r+1)\big)\} \cup \{V\big(2r-1,(r-1,r,r,r)\big)\}$$

is the castling class of $V\big(2,(1,1,1,2)\big)$.

3.6. THEOREM. — *Let V be minimal with $\delta \neq 0$. Then we have the following possibilities:*

(1) *$\sigma < n$ and $q = 0$;*

(2) *$\sigma = n$ and $q = 1$ with a generator of degree $(1,1,1,1)$;*

(3) *$\sigma > 2n$ and $k_4 \geq n$. Then we have the generators of $V\big(n,(k_1,k_2,k_3)\big)$ and an additional generator of degree $(0,0,0,1)$ if $k_4 = n$.*

Proof. — Clearly (1) and (2) cover the possibilities for $\sigma \leq n$, so we may suppose that $\sigma > n$. Since $V(n,\vec{k})$ is minimal and $\sigma \neq 2n$, $\mathcal{C}_\nu$ must be a castling up, so that $\sigma > 2n$. If $k_4 < n$, then we may castle down by $\mathcal{C}_\kappa$, so we must be in case (3). □

3.7. COROLLARY. — *Let $V(n,\vec{k})$ be minimal where $\delta \neq 0$.*

(1) *If $n < \sigma \leq n+k_4$, then $\mathrm{Cl}(V(n,\vec{k})) = \{V(n,\vec{k}),\ \mathcal{C}_\nu(V(n,\vec{k}))\}$ is finite.*

(2) *If $n+k_4 < \sigma$ or $\sigma \leq n$, then $\mathrm{Cl}(V(n,\vec{k}))$ is infinite.*

Proof. — If $\sigma > n$, then $\sigma > 2n$ by minimality, hence $\mathcal{C}_\nu$ is a castling up. But we cannot apply $\mathcal{C}_\kappa$ to $V(n',\vec{k}') = \mathcal{C}_\nu(V(n,\vec{k}))$ if

$k_4' = k_4 \geq n' = \sigma - n$, so (1) holds. If $n + k_4 < \sigma$, then

$$V(n', \vec{k}') = \mathcal{C}_\kappa(\mathcal{C}_\nu(V(n, \vec{k})))$$

has $n' = \sigma - n$, $\sigma' = 3\sigma - 4n$ and $k_4' = \sigma - n - k_1$. Then

$$\sigma' - n' - (\max\{k_1', \ldots, k_4'\} = k_4') = \sigma - 2n + k_1 > 0,$$

and we may continue castling up indefinitely. Finally, if $\sigma \leq n$, then $V(n', \vec{k}') = \mathcal{C}_\kappa(V(n, \vec{k}))$ satisfies $\sigma' \geq 3n'$ and $\sigma' - n' - k_4' > 0$, so we may castle up indefinitely. □

3.8. *Invariants of two or three subspaces.* — If $V(n, \vec{k})$ is minimal, then it often happens that k_4 is larger than n, in which case we are reduced to computing the invariants of three subspaces. So, we consider the invariants of $V(n, \vec{\ell})$ where $n \geq 1$, $\ell_j \geq 1$ for $j = 1, 2, 3$. We adapt our terminology from four subspaces to this situation: We set

$$\tau = \ell_1 + \ell_2 + \ell_3.$$

The castlings $\mathcal{C}_\nu$ and $\mathcal{C}_\kappa$ send $V(n, \vec{\ell})$ to

$$V(\tau - n, \vec{\ell}) \quad \text{and} \quad V(n, (n - \ell_3, n - \ell_2, n - \ell_1)),$$

respectively (whenever they apply). Then $\mathcal{C}_\nu$ (or $\mathcal{C}_\kappa$) is a castling up if it increases $n + \tau$. In other words, $\mathcal{C}_\nu$ (resp. $\mathcal{C}_\kappa$) is a castling up if $\tau > 2n$ (resp. $3n > 2\tau$).

3.9. *Remark.* — It is no longer true that a castling class has a unique minimal element. For example, from $V(5,(2,3,4))$ we can castle down by $\mathcal{C}_\nu$ or by $\mathcal{C}_\nu \mathcal{C}_\kappa$ to obtain minimal elements $V(4,(2,3,4))$ and $V(1,(1,2,3))$, respectively.

3.10. Theorem. — *Let $\ell_j \geq 1$, $1 \leq j \leq 3$. Then minimal homogeneous generators of the invariants of $\mathbb{C}[V(n, \vec{\ell})]$ have, up to permutation of the ℓ_j, degrees $(1,0,0)$, $(1,1,0)$ or $(1,1,1)$, corresponding to equalities*

$$\ell_1 = n, \quad \ell_1 + \ell_2 = n \quad \text{and} \quad \ell_1 + \ell_2 + \ell_3 = n$$

(or $\ell_1 + \ell_2 + \ell_3 = 2n$ with ℓ_1, ℓ_2, $\ell_3 < n$). There are only the following possibilities (up to permutation of the ℓ_i) :

(1) $q = 0$: *no subset of* $\{\ell_1, \ell_2, \ell_3\}$ *consisting of numbers at most* n *adds up to* n *or* $2n$.

(2) $q = 1$ *and one of the following occurs:*

(a) $\ell_1 = n$;

(b) $\ell_1 + \ell_2 = n$;

(c) $\ell_1 + \ell_2 + \ell_3 = n$;

(d) $\ell_1 + \ell_2 + \ell_3 = 2n$ *and* $\ell_1, \ell_2, \ell_3 < n$.

(3) $q = 2$ *and the generators are as in* 2.a *and/or* 2.b.

(4) $q = 3$ *and* $V = V\big(n,(n,n,n)\big)$, $n \geq 1$, *or* $V\big(2n,(n,n,n)\big)$, $n \geq 1$.

Moreover,

(5) $V(n,\vec{\ell})$ *is coregular.*

(6) *Only* $V\big(2,(1,1,1)\big)$ *is not cofree.*

Proof. — Assume that $\ell_1 \leq \ell_2 \leq \ell_3$. If $\ell_3 \geq n$, then we obtain (at most) an invariant of type 2 (a) and the invariants of $V = V(n,(\ell_1,\ell_2))$. The only possibility to get something new is if $\ell_1, \ell_2 < n < \ell_1 + \ell_2$. But then $\mathcal{C}_\kappa$ gives $V(n,(n-\ell_2, n-\ell_1))$ where $n - \ell_2 + n - \ell_1 < n$, so there are no new types of invariants. Note that we get one possibility here for $q = 3$, namely $V(n,(n,n,n))$.

It is easy to see that $\mathcal{C}_\nu$, when it applies, can only interchange invariants of types 2 (a) and 2 (b). Similarly, if $\mathcal{C}_\kappa$ applies, then it only interchanges invariants of types 2 (c) and 2 (d). Hence we may reduce to finding the invariants of the minimal V.

We may assume that $\ell_3 < n$ and that V is minimal. Since $\mathcal{C}_\kappa$ is not a castling down, we have $\tau \leq \frac{3}{2}n$. If $\mathcal{C}_\nu$ applies it is a castling down, so we must have $\tau \leq n$. We are in case 2 (c) or case 1, hence 1, 2 and 3 hold. Part 4 is the observation that the castling class of $V(n,\vec{n})$ is $\{V(n,\vec{n}),\ V(2n,\vec{n})\}$.

If $q \leq 2$, then $V(n,\vec{\ell})$ is cofree: Equidimensionality is easy, and coregularity follows from [Kempf]. Obviously $V\big(n,(n,n,n)\big)$ is coregular (and cofree), hence its castling transform $V\big(2n,(n,n,n)\big)$ is coregular, and we have (5). It is well-known that $V\big(2,(1,1,1)\big)$ is not cofree, since its null cone has codimension 2 while $q = 3$. To establish (6), we need to show that $V\big(2n,(n,n,n)\big)$ is cofree, $n \geq 2$. But in Section 6 we show that $V\big(2n,(n,n,n,n)\big)$ is cofree for $n \geq 3$, hence so is $V\big(2n,(n,n,n)\big)$. If $n = 2$, one uses the idea of Example 4.10 to reduce to the fact that $(3\mathbb{C}^6, \mathrm{SO}(6))$ is cofree [Sch2]. □

3.11. *Proof of Theorem 2.8 (1)–(3).* — Let $V(n,\vec{k})$ be minimal. Applying Theorem 3.6 we may assume that $k_4 \geq n$. Parts 1, 2 and 3 are then immediate from Theorem 3.10. □

3.12. *Effects of castling.* — Let f be an invariant of degree $(\vec{\imath} := (i_1, i_2, i_3, i_4))$. Since f is SL_n-invariant, $\vec{\imath} \cdot \vec{k} = \sum_j i_j k_j$ is a multiple s of n. We call s the *n-degree of f*. Let $|\vec{\imath}|$ denote $\sum_j i_j$. Let $V(n', \vec{k}')$, f', s', etc. denote the result of applying $\mathcal{C}_\nu$ or $\mathcal{C}_\kappa$ to $V(n,\vec{k})$, f, s, etc.

3.13. Lemma. — *Suppose that $\mathcal{C}_\nu$ applies to $V(n,\vec{k})$. Then $\mathcal{C}_\nu(f)$ has degree $(s - i_1, \ldots, s - i_4)$.*

Proof. — Since $\wedge^n(W)$ is a sum of terms

$$\wedge^{\vec{r}} := \wedge^{r_1}(\mathbb{C}^{k_1}) \otimes \cdots \otimes \wedge^{r_4}(\mathbb{C}^{k_4}) \quad \text{where } |\vec{r}| = n,$$

its dual $\wedge^{\sigma - n}(W)$ is a sum of corresponding terms $\wedge^{\vec{k} - \vec{r}}$. The invariant f lies in sums of tensor products $\wedge^{\vec{r}^{(1)}} \otimes \cdots \otimes \wedge^{\vec{r}^{(s)}}$ with $\sum_{j=1}^{s} r_1^{(j)} = i_1 k_1$, etc. It follows that $\mathcal{C}_\nu(f)$ lies in sums of terms $\wedge^{\vec{k} - \vec{r}^{(1)}} \otimes \cdots \otimes \wedge^{\vec{k} - \vec{r}^{(s)}}$ where $\sum_{j=1}^{s} (k_1 - r_1^{(j)}) = (s - i_1)k_1$, etc. Thus $\mathcal{C}_\nu(f)$ has degree $(s - i_1, \ldots, s - i_4)$. □

3.14. Corollary. — *Let f, $\vec{\imath}$, etc. be as above.*

(1) *If $\mathcal{C}_\nu$ applies, then $\vec{\imath}' = (s - i_1, \ldots, s - i_4)$ and $s' = s$.*

(2) *If $\mathcal{C}_\kappa$ applies, then $\vec{\imath}' = \vec{\imath}$ and $s' = |\vec{\imath}| - s$.*

Proof. — Part (1) is immediate from Lemma 3.13, and (2) is a simple calculation. □

3.15. *Types of generators.* — Let f be a minimal multihomogeneous generator of $\mathbb{C}[V]^G$, where $\delta \neq 0$.

- We say that f *is of type* (1) if its total degree is 2, e.g., its degree is $(1, 1, 0, 0)$.

- Otherwise we say that f *is of type* (2).

Our usage of "type" is different than that in [HoHu]. We call $\{d, s\}$ the *degree pair* of f, where d (resp. s) is the degree (resp. n-degree) of f.

3.16. Lemma. — *Let f be a multihomogeneous minimal generator of $\mathbb{C}[V]^G$.*

(1) *Suppose that f is of type* (1), *i.e.,* $|\vec{\imath}| = 2$. *Then $s = 1$ and the same holds true for $\mathcal{C}_\nu(f)$ and $\mathcal{C}_\kappa(f)$.*

(2) *Suppose that $\delta < 0$ and that f has degree pair*

$$\{(2r-1), (r-1, r-1, r-1, r)\}, \quad \{(2r), (r, r, r, r-1)\},$$

or $\{(2r-1), (r, r, r, r)\}$. *Then $\mathcal{C}_\nu(f)$ has the same n-degree and degree* $(r, r, r, r-1)$, $(r, r, r, r+1)$, *or* $(r+1, r+1, r+1, r+1)$, *respectively.*

(3) *Suppose that $\delta > 0$ and that f has degree pair*

$$\{(2r-2), (r-1, r-1, r-1, r)\}, \quad \{(2r-1), (r, r, r, r-1)\},$$

or $\{(2r-2), (r, r, r, r)\}$. *Then $\mathcal{C}_\kappa(f)$ has degree pair*

$$\{(2r-1), (r, r-1, r-1, r-1)\}, \quad \{(2r), (r-1, r, r, r)\},$$

or $\{(2r), (r, r, r, r)\}$, *respectively.*

Proof. — Part (1) is easy and (2) follows from 3.14. For (3), consider the case where

$$\vec{\imath} \cdot \vec{k} = (r-1)|\vec{k}| + k_4 = (2r-2)n.$$

Then

$$\begin{aligned} \vec{\imath} \cdot (\vec{n} - \vec{k}) &= (r-1)\big(4n - |\vec{k}|\big) + (n - k_4) \\ &= \big(4(r-1)+1\big)n - (2r-2)n = (2r-1)n. \end{aligned}$$

The other cases are similar. The change in the degrees of the invariants is due to the fact that $n - k_4 \leq n - k_3 \ldots$ $\square$

3.17. THEOREM. — *Let $V(n,\vec{k})$ be minimal. Let f be a minimal non-constant multihomogeneous generator of $\mathbb{C}[V]^G$, and let $r \geq 0$.*

(1) *If f is of type* (1), *then so are any castling transforms of f.*

(2) *Suppose that f has degree $(0,0,0,1)$. Then $(\mathcal{C}_\kappa\mathcal{C}_\nu)^{2r}(f)$ has degree $(r,r,r,r+1)$, $\mathcal{C}_\nu(\mathcal{C}_\kappa\mathcal{C}_\nu)^{2r-1}f$ has degree $(r+1,r,r,r)$, $(\mathcal{C}_\kappa\mathcal{C}_\nu)^{2r+1}(f)$ has degree $(r,r+1,r+1,r+1)$ and $\mathcal{C}_\nu(\mathcal{C}_\kappa\mathcal{C}_\nu)^{2r}f$ has degree $(r+1,r+1,r+1,r)$.*

(3) *If f has degree $(1,1,1,0)$, then we are in case* 3.7.1. *One can only apply $\mathcal{C}_\nu$, and $\mathcal{C}_\nu(f)$ has degree $(0,0,0,1)$.*

(4) *If f has degree $(1,1,1,1)$, then $(\mathcal{C}_\nu\mathcal{C}_\kappa)^r f$ and $\mathcal{C}_\kappa(\mathcal{C}_\nu\mathcal{C}_\kappa)^r f$ have degree $(r+1,r+1,r+1,r+1)$.*

4. The case $\delta = 0$.

When $\delta = 0$, we cannot get very far with our usual castling transformations, since $\mathcal{C}_\nu$ is the identity and $\mathcal{C}_\kappa$ has order 2. However, there are some alternative means to simplify things. We eventually land in the case of four medials, i.e., in the case $V\big(2n,(n,n,n,n)\big)$.

4.1. *Four medials.* — If V is a G-module, we denote the principal isotropy group by $\mathrm{PIG}(V)$ (or $\mathrm{PIG}(V,G)$) and its identity component by $\mathrm{PIG}(V)^0$. Let $L = \mathrm{PIG}(V)$. We denote by N (or N_L) the quotient $N_G(L)/L$, which has a natural action on V^L. See [Slod] for the notion of *slice representation* used in the proof below.

4.2. PROPOSITION. — *Let $n \geq 1$. Then $(V,G) = V\big(2n,(n,n,n,n)\big)$ is stable with principal isotropy group $T \simeq (\mathbb{C}^*)^{n-1}$, where T lies in G as*

$$\big\{(t,t,t,t,\mathrm{diag}(t^{-1},t^{-1})) \in (\mathrm{SL}_n)^4 \times \mathrm{SL}_{2n}\ ;\ t \in (\mathbb{C}^*)^{n-1}\big\}.$$

Proof. — We use the symbols n, n', n'', n''' to distinguish our four copies of $\mathbb{C}^n$ and SL_n. The subrepresentation $(\mathbb{C}^n \oplus \mathbb{C}^{n'}) \otimes \mathbb{C}^{2n}$ is stable with one dimensional quotient (the determinant is the generator of the invariants), and the principal isotropy group is

$$\big\{(g,h,k) \in \mathrm{SL}_n \times \mathrm{SL}_{n'} \times \mathrm{SL}_{2n}\ ;\ k = \mathrm{diag}((g^{-1})^t,(h^{-1})^t)\big\}.$$

The slice representation is, ignoring trivial factors,

$$\big((\mathbb{C}^n \oplus \mathbb{C}^{n'})^* \otimes (\mathbb{C}^{n''} \oplus \mathbb{C}^{n'''}), \mathrm{SL}_n \times \mathrm{SL}_{n'} \times \mathrm{SL}_{n''} \times \mathrm{SL}_{n'''}\big).$$

We can now take slice representations of the various pairings of the SL_n's to obtain, at the penultimate stage, the adjoint representation of the group

$$\mathrm{SL}_n \simeq \{(g,g,g,g) \in (\mathrm{SL}_n)^4\,;\ g \in \mathrm{SL}_n\}.$$

Thus the principal isotropy group is a maximal torus T^{n-1}. Since the final slice representation is stable, the original representation was stable. □

4.3. *Principal isotropy group.* — Suppose that we are given k_1, etc. as usual. Define

$$a := \min\{k_1, n-k_4\}, \quad r = n - k_1 - k_2,$$
$$s = n - k_1 - k_3, \qquad t = |n - k_1 - k_4|.$$

Note that, since $V_1 + V_2$ has codimension at least r in $\mathbb{C}^n$, the intersection $V_3 \cap V_4$ has dimension at least r. Similarly, $V_2 \cap V_4$ has dimension at least s and $V_2 \cap V_3$ (resp. $V_1 \cap V_4$) has dimension at least t if $k_1 + k_4 \leq n$ (resp. $k_1 + k_4 > n$).

We now find the generic stabilizer of $V(n, \vec{k})$: We may assume that all intersections we consider are as generic as possible. That is, $\dim V_1 + V_2 = k_1 + k_2$, etc. If $k_1 + k_4 \leq n$ define

(1) $W_r := V_3 \cap V_4$,

(2) $W_s := V_2 \cap V_4$,

(3) $W_t := V_2 \cap V_3$,

(4) $W_{2a} := (V_1 + V_2) \cap (V_1 + V_3) \cap (V_1 + V_4)$.

4.4. LEMMA. — *Generically, we have*

(1) $\dim W_{2a} = 2a$, $\dim W_r = r$, *etc.*,

(2) $\mathbb{C}^n = W_{2a} \oplus W_r \oplus W_s \oplus W_t$,

(3) $V_1 \subset W_{2a}$,

(4) $V_2 \subset W_{2a} \oplus W_s \oplus W_t$,

(5) $V_3 \subset W_{2a} \oplus W_r \oplus W_t$ *and*

(6) $V_4 \subset W_{2a} \oplus W_r \oplus W_s$.

Proof. — First of all, the configuration above is possible: Given a decomposition of $\mathbb{C}^n$ as in 2, choose a-dimensional subspaces U_i of W_{2a} such that $W_{2a} = U_i \oplus U_j$, $1 \le i < j \le 4$. Then $V_1 := U_1$, $V_2 := U_2 \oplus W_s \oplus W_t$, etc. do the job. Generically, $V_1 + V_2$ has codimension $n - k_1 - k_2 = k_3 + k_4 - n = r$ in $\mathbb{C}^n$, and similarly we obtain generic codimensions s and t for $V_1 + V_3$ and $V_1 + V_4$. It follows that W_{2a} generically has codimension $r + s + t$. Similar arguments show that $V_2 \cap W_{2a}$ generically has codimension $s + t$ in V_2, so we have (4). The other cases are similar. □

Now we consider the isotropy group of such a collection of subspaces. Suppose that $n = 2a + r + s + t$ where $a > 0$, r, s, $t \ge 0$. Replacing each $\mathbb{C}^{k_i}$ by its dual, we may consider that we have homomorphisms $\varphi_i : \mathbb{C}^{k_i} \to \mathbb{C}^n$ with images V_i, $i = 1, \ldots 4$. Our point in $V(n, \vec{k})$ is denoted $\varphi := (\varphi_1, \ldots, \varphi_4)$.

4.5. Proposition. — *Let n, W_{2a}, etc. be as above. Let $U_1, \ldots, U_4$ be a-dimensional subspaces of W_{2a} such that $U_i + U_j = W_{2a}$, $1 \le i < j \le 4$. Assume that*

(1) $\varphi_1 : \mathbb{C}^a \to U_1$,

(2) $\varphi_2 : \mathbb{C}^{a+s+t} = (\mathbb{C}^a)' \oplus \mathbb{C}^s \oplus \mathbb{C}^t \to U_2 \oplus W_s \oplus W_t$,

(3) $\varphi_3 : \mathbb{C}^{a+r+t} = (\mathbb{C}^a)'' \oplus \mathbb{C}^r \oplus (\mathbb{C}^t)' \to U_3 \oplus W_r \oplus W_t$ *and*

(4) $\varphi_4 : \mathbb{C}^{a+r+s} = (\mathbb{C}^a)''' \oplus (\mathbb{C}^r)' \oplus (\mathbb{C}^s)' \to U_4 \oplus W_r \oplus W_s$

are injective homomorphisms which respect the direct sum decompositions (the generic case). Then the isotropy group G_φ is isomorphic to

$$L := (\mathbb{C}^*)^{a-1} \times \mathrm{SL}_r \times \mathrm{SL}_s \times \mathrm{SL}_t.$$

Moreover, L is a principal isotropy group, the representation is stable and the quotient has dimension

$$q = a + 1 + \delta_{r0} + \delta_{s0} + \delta_{t0} = a + 1 + \delta_{k_1+k_4,n} + \delta_{k_1+k_3,n} + \delta_{k_1+k_2,n}.$$

Proof. — Since the images V_i of the φ_i determine the direct sum decomposition $\mathbb{C}^n = W_{2a} \oplus W_r \oplus W_s \oplus W_t$, the projection $\pi(G_\varphi)$ of G_φ to SL_n lies in $(\mathrm{GL}_{2a} \times \mathrm{GL}_r \times \mathrm{GL}_s \times \mathrm{GL}_t) \cap \mathrm{SL}_n$. By choosing appropriate bases of W_t, $\mathbb{C}^t$ and $\mathbb{C}^{t'}$ we may assume that $\varphi_{2|\mathbb{C}^t}$ and $\varphi_{3|\mathbb{C}^{t'}}$ are multiples of the identity map, and similarly for r and s.

Suppose that $\pi(G_\varphi)$ is not in the product $\mathrm{SL}_{2a} \times \cdots \times \mathrm{SL}_t$. Then there is a nontrivial cyclic subgroup C of SL_n acting via scalars $\xi_{2a}, \dots, \xi_t$ on $W_{2a}, \dots, W_t$, respectively, where $\xi_{2a}^{2a}\, \xi_r^r\, \xi_s^s\, \xi_t^t = 1$. Since

$$\wedge^a(\varphi_1)\colon \wedge^a(\mathbb{C}^a) \longrightarrow \wedge^a(U_1) \subset \wedge^a(W_{2a})$$

is $\pi(G_\varphi)$-fixed, we must have that $\xi_{2a}^a = 1$. Similarly, invariance of $\wedge^{a+s+t}(\varphi_2)$ gives that $\xi_{2a}^a\, \xi_s^s\, \xi_t^t = 1$. Hence $\xi_r^r = 1$. Similarly, $\xi_s^s = \xi_t^t = 1$, so our cyclic subgroup lies in $\mathrm{SL}_{2a} \times \cdots \times \mathrm{SL}_t$.

Since we have isomorphisms

$$\rho := \varphi_{2|\mathbb{C}^t}\colon \mathbb{C}^t \longrightarrow W_t \quad \text{and} \quad \rho' := \varphi_{3|\mathbb{C}^{t'}}\colon \mathbb{C}^{t'} \longrightarrow W_t,$$

we obtain a contribution to the isotropy group of

$$\big\{(g, g', g'') \in \mathrm{SL}_t \times \mathrm{SL}_{t'} \times \mathrm{SL}(W)\,;\; g''\rho\, g^{-1} = \rho,\; g''\rho'(g')^{-1} = \rho'\big\}.$$

Similarly we obtain diagonal copies of SL_r and SL_t in L. Finally, we obtain the principal isotropy group of the four generic injections of $\mathbb{C}^a \times \mathbb{C}^{a'} \times \mathbb{C}^{a''} \times \mathbb{C}^{a'''} \to W_{2a} \simeq \mathbb{C}^{2a}$, which we already know is $(\mathbb{C}^*)^{a-1}$. Thus L is a generic isotropy group.

By a theorem of Popov [Po] (see also [LuVu]), since G is semisimple and there is an open dense subset of orbits with reductive stabilizer (L), the representation is stable, and then clearly our L is a principal isotropy group. Making note that the dimension of SL_m is $m^2 - 1 + \delta_{m0}$ we obtain that the dimension of $\mathbb{C}[V]^G$ is

$$\begin{aligned}
\dim V - \dim G &+ \dim \mathrm{PIG}(V, G)\\
&= 2n^2 - \Big(n^2 - 1 + \sum_{i=1}^{4}(k_i^2 - 1)\Big)\\
&\qquad + \Big(k_1 - 1 + \sum_{i=2}^{4}(n - k_1 - k_i)^2 - 1 + \delta_{k_1+k_i,n}\Big)\\
&= k_1 + 1 + \delta_{k_1+k_2,n} + \delta_{k_1+k_3,n} + \delta_{k_1+k_4,n}.
\end{aligned}$$

□

4.6. *Remark.* — Suppose that $k_1 + k_4 > n$. We set

$$r := k_3 + k_4 - n \quad \text{and} \quad s := k_2 + k_4 - n$$

as before, and

$$t := k_1 + k_4 - n.$$

Then

(1) $k_1 = a + t$,

(2) $k_2 = a + s$,

(3) $k_3 = a + r$, and

(4) $k_4 = a + r + s + t$.

In the generic case, we obtain

$$\mathbb{C}^n = W_{2a} \oplus W_r \oplus W_s \oplus W_t$$

where $W_r = V_3 \cap V_4$, $W_s = V_2 \cap V_4$, $W_t = V_1 \cap V_4$ and

$$W_{2a} = (V_1 + V_2) \cap (V_1 + V_3) \cap (V_2 + V_3).$$

The results and proofs are now as above. Alternately, one can apply $\mathcal{C}_\kappa$ to change to the case where $k_1 + k_4 \leq n$.

4.7. *Invariants.* — We compute generators of $\mathbb{C}[V]^G$. They are all of degree $(1,1,1,1)$, $(1,1,0,0)$, or $(1,0,1,0)$, etc. We assume at first that $k_1 + k_4 \leq n$.

Clearly, whenever $k_i + k_j = n$, $i < j$, we have a "determinant" invariant f_{ij} of total degree 2. Now the symmetric algebra $\operatorname{Sym}(D_a) := \mathbb{C}[D_a]^*$ is just $\oplus_j \varphi_a^j$, and the U-invariants (covariants) of the tensor product $\operatorname{Sym}(D_a) \otimes \operatorname{Sym}(D_b)$, $a \leq b \leq n - a$, are a polynomial algebra generated by

$$\varphi_a,\ \varphi_b,\ \varphi_{a-1}\varphi_{b+1}, \ldots,\ \varphi_{a+b}.$$

Since $k_1 + k_3 \leq n$, we have invariants $\{g_p\}_{p=0}^{k_1}$ which contract the copy of $\varphi_{k_1-p}\varphi_{k_3+p}$ with its dual $\varphi_{k_2-q}\varphi_{k_4+q}$ where $k_4 + q + k_1 - p = n$, i.e., $q = p + n - k_1 - k_4 \geq p$.

To find formulas for the g_p, one proceeds as follows: Let $v_j^{(i)}$ denote $\varphi_i(e_j)$, $1 \leq j \leq k_i$. We assume that $v_j^{(1)} = v_j^{(3)}$, $1 \leq j \leq k_1 - p$ and that $v_j^{(2)} = v_j^{(4)}$, $1 \leq j \leq k_2 - q$. Then the dimension of $V_1 + V_3$ is at most $k_3 + p$, which implies that the invariants $g_{p+1}, \ldots, g_{k_1}$ vanish. Moreover, it is clear that the value of g_p in this case has to be (up to constants) the following product of determinants:

$$\left[v_1^{(1)}, \ldots, v_{k_1-p}^{(1)}, v_1^{(4)}, \ldots, v_{k_4}^{(4)}, v_{k_2-q+1}^{(2)}, \ldots, v_{k_2}^{(2)}\right] \cdot \left[v_1^{(3)}, \ldots, v_{k_3}^{(3)}, v_{k_1-p+1}^{(1)}, \ldots, v_{k_1}^{(1)}, v_1^{(2)}, \ldots, v_{k_2-q}^{(2)}\right].$$

The formula for g_p in general is obtained by polarizing the formula above in the variables which we assumed equal.

Note that if $t := k_1 + k_4 - n > 0$, then we only have invariants g_p for $p \geq t$.

4.8. PROPOSITION.

(1) *If* $k_1 + k_3 < k_1 + k_4 = n$, *then* $g_0 = f_{14}f_{32}$.

(2) *If* $k_1 + k_2 < k_1 + k_3 = n$, *then* $g_0 = f_{14}f_{32}$ *and* $g_{k_1} = f_{13}f_{24}$.

(3) *If* $k_1 + k_2 = n$ *(four medials), then* $g_0 = f_{14}f_{32} + f_{12}f_{34}$, $g_{k_1} = f_{13}f_{24}$ *and* $g_1 \equiv -f_{14}f_{32} + f_{12}f_{34}$ *modulo* $g_3, g_5, \ldots$

Proof. — We prove (3) and leave the rest to the reader. Set $m := \frac{1}{2}n$. Since g_0 is the contraction of two copies of φ_m^2, it is the (double) polarization of the invariant f_{12}^2 with respect to the pairs of indices $\{1,3\}$ and $\{2,4\}$, which gives, up to constants, $f_{14}f_{32} + f_{12}f_{34}$. The expression for g_m is obvious.

When $m > 1$, consider the case where

$$v_j^{(1)} = v_j^{(3)} = e_{j+2} \quad \text{and} \quad v_j^{(2)} = v_j^{(4)} = e_{m+j+1}, \quad 1 \leq j \leq m-1.$$

Write $v_m^{(i)} = \sum_{j=1}^{n} a_j^{(i)} e_j$. Then

$$\begin{aligned} g_1 &= \left[e_3, \ldots, e_{m+1}, e_{m+2}, \ldots, e_{2m}, v_m^{(2)}, v_m^{(4)}\right] \\ &\qquad \cdot \left[e_3, \ldots, e_{m+1}, v_m^{(1)}, v_m^{(3)}, e_{m+2}, \ldots, e_{2m}\right] \\ &= \left[w^{(2)}, w^{(4)}\right] \cdot \left[w^{(1)}, w^{(3)}\right], \end{aligned}$$

where $w^{(i)} = (a_1^{(i)}, a_2^{(i)}) \in \mathbb{C}^2$. Moreover,

$$\begin{aligned} &-f_{14}f_{32} + f_{12}f_{34} \\ &\quad = -\left[e_3, \ldots, e_{m+1}, v_m^{(1)}, e_{m+2}, \ldots, e_{2m}, v_m^{(4)}\right] \\ &\qquad \cdot \left[e_3, \ldots, e_{m+1}, v_m^{(3)}, e_{m+2}, \ldots, e_{2m}, v_m^{(2)}\right] \\ &\qquad + \left[e_3, \ldots, e_{m+1}, v_m^{(1)}, e_{m+2}, \ldots, e_{2m}, v_m^{(2)}\right] \\ &\qquad \cdot \left[e_3, \ldots, e_{m+1}, v_m^{(3)}, e_{m+2}, \ldots, e_{2m}, v_m^{(4)}\right] \\ &\quad = -\left[w^{(1)}, w^{(4)}\right] \cdot \left[w^{(3)}, w^{(2)}\right] + \left[w^{(1)}, w^{(2)}\right] \cdot \left[w^{(3)}, w^{(4)}\right] \\ &\quad = \left[w^{(1)}, w^{(3)}\right] \cdot \left[w^{(2)}, w^{(4)}\right]. \end{aligned}$$

Thus g_1 and $-f_{14}f_{32} + f_{12}f_{34}$ agree in our special case, which, since the functions are G-invariant, implies that they are equal modulo $g_2, \dots, g_m$. However, since both are skew in V_1 and V_3, only g_p for p odd are involved in the relation. □

As an immediate corollary we get the following:

4.9. THEOREM. — *If $\delta = 0$, then $V(n, \vec{k})$ is coregular, except in the case $V\big(2,(1,1,1,1)\big)$. If $k_1 + k_4 \leq n$, then $\mathbb{C}[V]^G$ is generated by $\{g_2, \dots, g_{k_1-1}\}$ (the empty set if $k_1 \leq 2$, also, omit g_1 below if $a = 1$) and*

(1) *g_0, g_1 and g_a if $k_1 + k_4 \neq n$;*

(2) *$\{f_{14}, f_{23}\}$, g_1 and g_a if $k_1 + k_3 < k_1 + k_4 = n$;*

(3) *$\{f_{14}, f_{23}, f_{13}, f_{24}\}$ and g_1 if $k_1 + k_2 < k_1 + k_3 = n$;*

(4) *$\{f_{14}, f_{23}, f_{13}, f_{24}, f_{12}, f_{34}\}$ in the case $k_1 + k_2 = n$ (four medials).*

If $t := k_1 + k_4 - n > 0$, then $\mathbb{C}[V]^G$ is generated by g_t, $g_{t+1}, \dots, g_{k_1}$.

4.10. *Example.* — Let $n = 2$. Then the D_i are just the rank 2 elements of $\wedge^2(\mathbb{C}^4)$, which we can interpret as the null cone of the representation $(\mathbb{C}^6, \mathrm{SO}(6))$. The generating invariants of the coregular representation $(4\mathbb{C}^6, \mathrm{SO}(6))$ are the inner products h_{ij}, $1 \leq i \leq j \leq 4$, and $\prod_i D_i$ is the zero set of the h_{ii}. Since $f_{ij} = h_{ij}$ for $i < j$, the f_{ij} are algebraically independent.

5. Cofreeness when $\delta \neq 0$.

We first determine all the castling classes. Equivalently, we find all minimal representations $V(n, \vec{k})$. We then determine which castlings up of the minimal representations are cofree. Recall that cofreeness is automatic when $q \leq 2$.

5.1. PROPOSITION. — *Suppose that $\delta \neq 0$ and that $V = V(n, \vec{k})$ is minimal in its castling class and that $q = \dim \mathbb{C}[V]^G \geq 3$. Then, up to permutation of the k_i, we have the following possibilities:*

(1) *$q = 3$ and*

(a) *$V = V\big(n,(d,n,n,n)\big)$, $1 \leq d < n$, or*

(b) $V = V(n,(a,b,n,n))$ *where* $a+b=n$, $a<b$, *or*

(c) $V = V(n,(a,b,b,n))$ *or* $V = V(n,(a,a,b,n))$ *where* $a+b=n$, $a<b$, *or*

(d) $V = V(2n,(n,n,n,d))$, $d>2n$ *or*

(e) $V = V(n,(n,n,n,d))$, $d>n$;

(2) $q=4$ *and*

(a) $V = V(n,(n,n,n,n))$, *or*

(b) $V = V(2n,(n,n,n,2n))$.

Proof. — Since V is minimal and $q \geq 3$, we have that $\sigma > 2n$ and $k_4 \geq n$.

If $q=4$, then Proposition 3.10 shows that we have $k_4 = n$ and that we are in either in case 2 (a) or case 2 (b). If $q=3$ and $k_4=n$, then by Proposition 3.10.3–4, we are in case 1 (a), 1 (b) or 1 (c). Finally, if $k_4 > n$, one similarly obtains that 1 (d) or 1 (e) holds. □

5.2. *Castling up.* — We use the following result (compare [Wehl, Prop. 4.8.1]).

5.3. Theorem. — *Let* $(V,G) = (\mathbb{C}^n \otimes U \oplus Z, \mathrm{SL}_n \times L)$ *where* U *and* Z *are representations of* L. *Castling we obtain* $(V',G') = (\mathbb{C}^{n'} \otimes U^* \oplus Z, \mathrm{SL}_{n'} \times L)$ *where* $n' = \dim U - n$. *We assume that* L *acts trivially on* $\wedge^{n+n'}(U)$. *Denote the variety* $V/\!/\mathrm{SL}_n = D_n \oplus Z$ *by* Y *and consider the conditions:*

(a) $q-1 \leq n + \dim \mathbb{C}[Z]^L$,

(b) $q-1 \leq n' + \dim \mathbb{C}[Z]^L$.

Then

(1) *If* (a) *fails, then* (V,G) *is not equidimensional.*

(2) *If* (b) *fails, then* (V',G') *is not equidimensional.*

(3) *If* (a) *holds, then* (V,G) *is equidimensional if and only if* (Y,L) *is equidimensional.*

(4) *If* (a) *and* (b) *hold, then* (V,G) *is equidimensional* (*resp. cofree*) *if and only if* (V',G') *is equidimensional* (*resp. cofree*).

Proof. — For $k \geq n$, the fibres of the morphism $k\mathbb{C}^n \to k\mathbb{C}^n/\!/\mathrm{SL}_n$ have dimension $n^2 - 1 = \dim \mathrm{SL}_n$, except that the zero fiber has codimension $k - n + 1$. Parts 1, 2 and 3 are then immediate. Since castling preserves coregularity, we also have (4). □

5.4. Corollary. — *Suppose that $\mathcal{C}_\nu(V = V(n, \vec{k}))$ is a castling up. If $q - 1 > n$, then $\mathcal{C}_\nu(V)$ is not cofree, otherwise $\mathcal{C}_\nu(V)$ is cofree if and only if V is cofree.*

5.5. Corollary. — *Let $q_1 = q - \dim W_1/\!/H_1$ where $(W_1, H_1) = V(n,(k_2, k_3, k_4))$, and define q_2, etc. similarly. If $q_i - 1 > k_i$ for some i, then $\mathcal{C}_\kappa(V = V(n, \vec{k}))$ is not cofree, otherwise $\mathcal{C}_\kappa(V)$ is cofree if and only if V is.*

5.6. Corollary. — *Let $V = V(n, \vec{k})$. Suppose that $q - 1 \leq \min\{k_1, n - k_4\}$. Then all sequences of castlings up from V preserve (non) equidimensionality.*

Proofs. — Corollaries 5.4 and 5.5 are immediate from 5.3. Corollary 5.6 combines 5.4 and 5.5 with the observation that $\min\{k_1, n - k_4\}$ increases under castling up and that $n - k_4 < n$. □

5.7. Lemma. — *Let V_0 be one of the representations in Proposition 5.1. Then*

(1) *V_0 is cofree, except for case 1.d with $n = 1$. Here $V_0 = V(2,(1,1,1,2))$ is not cofree.*

(2) *If $q = 4$ and $n \geq 3$, then the whole castling class of V_0 is cofree.*

(3) *Suppose that $q = 3$, that $n \geq 2$ and that $d \geq 2$ in 5.1.1 (a). Then V_0 and its castling class are cofree.*

Proof. — It is obvious that V_0 is cofree except for 5.1.2 (b) and 5.1.1 (d). But $V(2n,(n,n,n))$ is cofree for $n \geq 2$, and this implies that $V(2n,(n,n,n,2n))$ and $V(2n,(n,n,n,d))$, $d > 2n$, are cofree. This gives (1). □

Consider $V_0 = V(n,(1, n-1, n-1, n))$ (5.1.1.c with $a = 1$). Note that $n \geq 3$ since $1 \neq n - 1$. By 5.4, $V_1 = V(2n,(1, n-1, n, n))$ is cofree. Since $q_1 = 2$, we may apply 5.5 to get that $V_2 = V(2n,(n, n, n+1, 2n-1))$ is cofree. After this corollary 5.6 applies. All the other cases are similar.

We handle the remaining cases in

5.8. PROPOSITION. — *The following are the noncofree (equivalently, nonequidimensional) representations with $\delta \neq 0$:*

(1) *V is above $V\big(3,(1,1,1,1)\big)$;*

(2) *V is above $V\big(6,(2,2,2,2)\big)$;*

(3) *V is above $V\big(2n+1,(n+1,n+1,n+1,2n)\big)$ where $n \geq 2$;*

(4) *V is above $V\big(d+2,(1,1,1,d)\big)$ where $d \geq 2$;*

(5) *$V = V\big(2,(1,1,1,d)\big)$ or $V\big(d+1,(1,1,1,d)\big)$, $d \geq 3$;*

(6) *$V = V\big(2,(1,1,1,2)\big)$, $V\big(3,(1,1,1,2)\big)$, $V\big(3,(2,2,2,1)\big)$, $V\big(4,(2,2,2,1)\big)$, $V\big(4,(2,2,2,3)\big)$ or $V\big(5,(2,2,2,3)\big)$.*

Proof. — We begin with the cases where $q = 4$. Consider $V_0 = V\big(1,(1,1,1,1)\big)$, which is cofree. Then $V_1 = V\big(3,(1,1,1,1)\big)$ is not cofree, and $V_2 = V\big(3,(2,2,2,2)\big)$ is also not cofree, since quotienting by the SL_2's essentially gives V_1 back again. Now one can apply 5.4 and 5.5 followed by any number of applications of 5.6. When $V_0 = V\big(2,(2,2,2,2)\big)$, 5.4 shows that $V\big(6,(2,2,2,2)\big)$ is not cofree. All castlings up after this preserve non-cofreeness.

From the non-cofree case $V_0 = V\big(2,(1,1,1,2)\big)$ we obtain

$$V_1 = V\big(3,(1,1,1,2)\big),\quad V_2 = V\big(3,(2,2,2,1)\big),\quad V_3 = V\big(4,(2,2,2,1)\big),$$
$$V_4 = V\big(4,(2,2,2,3)\big),\quad V_5 = V\big(5,(2,2,2,3)\big),\quad V_6 = V\big(5,(3,3,3,2)\big).$$

Amazingly enough, V_0 through V_5 are not cofree, while V_6, V_7, etc. are cofree. One can see the non-cofreeness of V_1 through V_2 by quotienting by the copies of SL_2 to obtain the coregular and non-cofree representations $(3\mathbb{C}^3 + (\mathbb{C}^3)^*, \mathrm{SL}_3)$ and its dual. The non-cofreeness of V_3 (with $q_4 = 1$), V_4 and V_5 follow from 5.5 and 5.4. We show below that V_6 is cofree, and then the cofreeness of V_7, etc. follow from 5.6. The case $V_0 = V\big(4,(2,2,2,4)\big)$ leads to no non-cofree representations, as one easily sees.

We now consider the cases with $q = 3$. There are only three left.

- Case 1: $V_0 = V\big(n,(1,n,n,n)\big)$, $n \geq 2$.

Then $V_1 = V\big(2n+1,(1,n,n,n)\big)$ is cofree by 5.4. Since $q_1 = 3$, 5.5 shows that $V_2 = V\big(2n+1,(n+1,n+1,n+1,2n)\big)$ is not cofree. Thereafter, 5.6 applies.

• Case 2: $V_0 = V\big(2,(1,1,1,d)\big)$, $d \geq 3$.

Then V_0 is not cofree. Since $q_4 = 3$, $V_1 = V\big(d+1,(1,1,1,d)\big)$ is not cofree. However, $V_2 = V\big(d+1,(1,d,d,d)\big)$ is cofree: Since $q_i = 1$ for $i \geq 2$, V_2 is cofree if and only if quotient by the copies of SL_d is cofree. But this quotient is $(3(\mathbb{C}^{d+1})^* \oplus \mathbb{C}^{d+1}, \mathrm{SL}_{d+1})$, which is cofree. Further castlings up are cofree by 5.4–5.6.

• Case 3: $V_0 = V\big(1,(1,1,1,d)\big)$, $d \geq 2$.

Now $V_1 = V\big(d+2,(1,1,1,d)\big)$ is not cofree by 5.4, and further castlings up are not cofree by 5.4–5.6. □

5.9. Lemma. — *$V = V\big(5,(2,3,3,3)\big)$ is cofree.*

Proof. — Since the quotient $(\mathbb{C}^2 \otimes \mathbb{C}^5)/\!/\mathrm{SL}_2$ has dimension 7, we may assume that we have a non-zero element there, which we can put in the normal form $e_1 \wedge e_2$ where e_1, e_2 span a copy of $\mathbb{C}^2$ in $\mathbb{C}^5$. Let $\mathbb{C}^3$ denote a complementary subspace in $\mathbb{C}^5$. Then the elements of $\wedge^3(\mathbb{C}^2 \oplus \mathbb{C}^3)$ which have trivial wedge product with $e_1 \wedge e_2$ have the form $e_1 \wedge e_2 \wedge f_1 + e_1 \wedge f_2 \wedge f_3$ for some $f_i \in \mathbb{C}^3$. For this element to be decomposable we must have that f_1 is a linear combination of f_2 and f_3. Now the projection of these forms to $e_1 \wedge f_2 \wedge f_3$ is surjective with fibres of the form $e_1 \wedge e_2 \wedge f$ where $f \in \mathrm{span}\{f_1, f_2\}$ (except over 0 we get $e_1 \wedge e_2 \wedge \mathbb{C}^3$, of dimension 3). It follows that the zero set of the invariants of degrees $(1,1,0,0)$, $(1,0,1,0)$ and $(1,0,0,1)$ is irreducible. But then by Lemma 2.3 of [Sch2], the fourth invariant cannot vanish identically on the zero set of the first three, hence the null cone has codimension 4 and (V,G) is cofree. □

6. Cofreeness for four medials.

6.1. Lemma. — *Let $(V,G) = V\big(4,(2,2,2,2)\big)$. Then in $\mathbb{C}[V]$, f_{12}, f_{14}, f_{23}, f_{34} are a regular sequence, and we may obtain a regular sequence of length 5 by adding f_{13}, f_{24} or $f_{13}f_{24}$. However (as we already know), all 6 of the f_{ij} do not form a regular sequence.*

Proof. — We consider $V/\!/H$, where $H = (\mathrm{SL}_2)^4$. If we are in the null cone of the action of, say, the first copy of SL_2 acting on $4\mathbb{C}^2$, we are in codimension three, so that setting the invariants f_{23} and $f_{34} = 0$ certainly gives us codimension 5, so we may go to the quotient by H and assume that we have four non-zero decomposable 2-forms $\alpha_1, \ldots, \alpha_4$. Since $f_{12} = 0$, we must have that $\alpha_1, \alpha_2 \in v_{12} \wedge \mathbb{C}^4$ for some v_{12}. Considering the other pairs we obtain vectors v_{ij}, $i = 1, 3$ and $j = 2, 4$. Now, generically, all these four vectors are linearly independent, and we have that (up to ± 1):

(1) $\alpha_1 = v_{12} \wedge v_{14}$,

(2) $\alpha_2 = v_{12} \wedge v_{23}$,

(3) $\alpha_3 = v_{23} \wedge v_{34}$, and

(4) $\alpha_4 = v_{14} \wedge v_{34}$.

If the vectors are not linearly independent (or two of the α_i are proportional), we are in a situation of codimension 5 or more. Now $f_{13} = 0$ is equivalent to the span of our v_{ij} being three dimensional, as is $f_{34} = 0$ or $f_{12}f_{34} = 0$. Hence we have the lemma. □

6.2. Theorem. — *$V\big(2n,(n,n,n,n)\big)$ is cofree if $n \geq 3$.*

Proof. — We consider homomorphisms $\varphi_i: \mathbb{C}^n \to \mathbb{C}^{2n}$, $i = 1, \ldots, 4$ as in 4.3, where $V_i = \varphi_i(\mathbb{C}^n)$ and $v_j^{(i)} = \varphi_i(e_j)$, $i = 1, \ldots, 4$, $1 \leq j \leq n$. If φ_1 is not injective, we are in a situation of codimension $n+1$, and the invariants f_{23}, f_{24} and f_{34} give another codimension 3, totalling $n + 4 = \dim V/\!/G$. Thus we may assume that all the φ_i are injective. We may also assume that $V_1 \cap \ldots \cap V_4 = 0$. Otherwise, the V_i contain a common line and we are in codimension $2n+1$, where $2n + 1 \geq n + 4$ for $n \geq 3$.

Consider the common zeroes of f_{13} and f_{24}, which surely is of codimension 2. On this set, $V_1 \cap V_3$ has dimension at least 1, and similarly for $V_2 \cap V_4$. Using the group actions we can reduce to the case that $v_1^{(1)} = v_1^{(3)} = e_1$ and $v_1^{(2)} = v_1^{(4)} = e_2$. Since it does not change the invariants, we may assume that $v_j^{(i)} \in \operatorname{span}\{e_3 \ldots, e_{2n}\}$ for $j > 1$. Let φ_i' denote the restriction of φ_i to $\operatorname{span}\{e_2, \ldots, e_n\} \simeq \mathbb{C}^{n-1}$ which has image in $\operatorname{span}\{e_3, \ldots, e_{2n}\} \simeq \mathbb{C}^{2n-2}$.

The invariants f_{12}, f_{34}, f_{23} and f_{14} restrict to functions having the same zeroes as the corresponding f'_{ij} of

$$V' = V\big(2n-2,(n-1, n-1, n-1, n-1)\big).$$

The restrictions of $g_2, \ldots, g_{n-2}$ have the same zeroes as the corresponding $g'_2, \ldots, g'_{n-2}$, and the restriction of g_{n-1} has the same zeroes as $f'_{13}f'_{24}$ by 4.8. Thus V is cofree if the invariants

$$\{f'_{12}, f'_{14}, f'_{23}, f'_{24}, f'_{13}f'_{24}, g'_2, \ldots, g'_{n-2}\}$$

of $V\big(2n-2,(n-1,n-1,n-1,n-1)\big)$ form a regular sequence. This is certainly true if V' is itself cofree, so our whole induction rests on the case $V' = V\big(4,(2,2,2,2)\big)$. Now apply the lemma above. □

7. Cofreeness when $\delta = 0$.

7.1. *Luna-Richardson.* — A generalization of the Chevalley Restriction Theorem, due to Luna and Richardson [LR], says the following:

7.2. Theorem. — *Let V be a G-module with $L = \mathrm{PIG}(V)$. Then the inclusion $V^L \to V$ induces an isomorphism $V^L/\!/N \simeq V/\!/G$. Moreover, $\mathrm{PIG}(V^L, N)$ is trivial.*

7.3. *Remark.* — If (V^L, N) is cofree, then (V, G) is also cofree, so that one can apply Theorem 7.2 to establish cofreeness in certain cases.

The obvious thing to do at this point is to apply Theorem 7.2 to our representations to arrive at the case of four medials. Recall that $L \simeq \mathrm{SL}_r \times \mathrm{SL}_s \times \mathrm{SL}_t \times T^{a-1}$. There are two problems with this. Firstly, N contains many central tori, which makes cofreeness doubtful, and secondly, it can turn out that $L = \{e\}$, in which case Luna-Richardson is no help at all.

7.4. *Example.* — Consider $V = V\big(3,(1,1,2,2)\big)$, which has trivial principal isotropy groups. We now "force" nontriviality. Let ν denote a copy of $\mathbb{C}^*$, and denote its one-dimensional module of weight p by ν_p. We let $G \times \nu = \mathrm{SL}_2 \times \mathrm{SL}_{2'} \times \mathrm{SL}_3 \times \nu$ act on V as

$$\big((\mathbb{C}^1 \otimes \nu_{-2} \oplus \mathbb{C}^{1'} \otimes \nu_{-2} \oplus \mathbb{C}^2 \otimes \nu_1 \oplus \mathbb{C}^{2'} \otimes \nu_1) \otimes \mathbb{C}^3\big).$$

Let σ (resp. σ') be a torus acting on $\mathbb{C}^2$ (resp. $\mathbb{C}^{2'}$) with weights 3 and -3, and let η act on $\mathbb{C}^3$ as $\mathbb{C}^2 \otimes \nu_2 \oplus \nu_{-4}$. Then

$$\begin{aligned}(V^L, N) \simeq \big(&2\mathbb{C}^2 \otimes \eta_2 \oplus \mathbb{C}^2 \otimes \sigma_{-3} \otimes \eta_2 \oplus \\ &\mathbb{C}^2 \otimes \sigma'_{-3} \otimes \eta_2 \oplus \sigma_3 \otimes \eta_{-4} \oplus \sigma'_3 \otimes \eta_{-4}, \mathrm{SL}_2 \times \sigma \times \sigma' \times \eta\big).\end{aligned}$$

(Actually, one should divide N by the ineffective part of the action.) Note that quotienting by the actions of σ and σ' one obtains the representation

$$V' := \big(2\mathbb{C}^2 \oplus 2(\mathbb{C}^2)^*, \mathrm{GL}_2\big)$$

which is cofree. However, (V^L, N) is not cofree since all nontrivial invariants vanish when $\sigma_3 \otimes \eta_{-4} \oplus \sigma_3' \otimes \eta_{-4}$ is zero. Our techniques below combine castling, Luna-Richardson and the idea of increasing the group to establish cofreeness. In our current example one obtains V' without passing through any non-cofree representations.

7.5. LEMMA. — *Let $(V, G) := (k(\mathbb{C}^n \oplus (\mathbb{C}^n)^*), \mathrm{SL}_n)$ where $k \leq n-1$, and set $L = \mathrm{PIG}(V, G)$.*

(1) *If $k \leq n-2$, then $L = \mathrm{SL}_{n-k}$ and $(V^L, N) = (k(\mathbb{C}^k \oplus (\mathbb{C}^k)^*), \mathrm{GL}_k)$.*

(2) *The invariants of (V, G) are the same as those of $(V', G') = (k(\mathbb{C}^n \oplus (\mathbb{C}^n)^*, \mathrm{GL}_n)$.*

(3) *If $k = n-1$, then $L' := \mathrm{PIG}(V', G') \simeq \mathbb{C}^*$, and $((V')^{L'}, N_{G'}(L')/L')$ is isomorphic to $((n-1)(\mathbb{C}^{n-1} \oplus (\mathbb{C}^{n-1})^*), \mathrm{GL}_{n-1})$.*

We now consider what happens when we apply the above to $V(n, \vec{k})$ via castling:

7.6. COROLLARY. — *Suppose that $r := k_3 + k_4 - n > 0$ (otherwise we are in the case of four medials). Set*

$$(V', G') := V\big(n - r, (k_1, k_2, k_3 - r, k_4 - r)\big).$$

Let η denote a copy of $\mathbb{C}^*$ and let $(V', G' \times \eta)$ denote a copy of V' with the usual G'-action and the η-action on $\mathbb{C}^{k_1}$, $\mathbb{C}^{k_2}$, $\mathbb{C}^{k_3-r}$ and $\mathbb{C}^{k_4-r}$ with weights $(k_3 - r)(k_4 - r)$, $(k_3 - r)(k_4 - r)$, $-(k_4 - r)^2$ and $-(k_3 - r)^2$, respectively. Then

(1) $\mathbb{C}[V]^G \simeq \mathbb{C}[V']^{G' \times \eta}$.

(2) *(V, G) is cofree if $(V', G' \times \eta)$ is cofree.*

(3) *There are minimal generating sets for $\mathbb{C}[V']^{G' \times \eta}$ of the form $f_{12} f_{34}, h_1, \ldots, h_q$ such that $f_{12}, f_{34}, h_1, \ldots, h_q$ minimally generate $\mathbb{C}[V']^{G'}$.*

(4) *If (V', G') is cofree, then so is (V, G).*

Proof. — First we castle $\mathbb{C}^{k_3} \otimes \mathbb{C}^n$ and $\mathbb{C}^{k_4} \otimes \mathbb{C}^n$ to $\mathbb{C}^{n-k_3} \otimes (\mathbb{C}^n)^*$ and $\mathbb{C}^{n-k_4} \otimes (\mathbb{C}^n)^*$, respectively (leaving the first two terms alone), obtaining (V_1, G_1). We know the generators of $\mathbb{C}[V]^G \simeq \mathbb{C}[V_1]^{G_1}$, and applying Theorem 5.3 we see that cofreeness is preserved. Now apply Lemma 7.5 to obtain

$$V_2 := (\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}) \otimes \mathbb{C}^{n-r} \otimes \nu_1 \oplus (\mathbb{C}^{n-k_3} \oplus \mathbb{C}^{n-k_4}) \otimes (\mathbb{C}^{n-r})^* \otimes \nu_{-1}$$

with the obvious action of $G_2 := \mathrm{SL}_{n-r} \times \mathrm{SL}_{k_1} \times \mathrm{SL}_{k_2} \times \mathrm{SL}_{n-k_3} \times \mathrm{SL}_{n-k_4} \times \nu$. If (V_2, G_2) is cofree, then so is (V_1, G_1). Note that $n - k_4 = k_3 - r$ and that $n - k_3 = k_4 - r$. The weights of ν on the quotient by $\mathrm{SL}_{k_1} \times \cdots \times \mathrm{SL}_{n-k_4}$ are k_1, k_2, $-(k_4 - r)$ and $-(k_3 - r)$. Now we castle back (ignoring ν for the moment) to obtain (V', G'). On $(\mathbb{C}^{k_1} \otimes \mathbb{C}^{n-r})/\!/\mathrm{SL}_{k_1}$ the weight of η is wk_1, where $w = (k_3 - r)(k_4 - r)$. Similarly, η acts on the other quotients with weights wk_2, $-w(k_4 - r)$ and $-w(k_3 - r)$. But these are just w times the weights we obtained for ν. This gives (1), and (3) is a weight calculation. Now Theorem 5.3 shows that we can determine whether or not (V_2, G_2) is cofree by first quotienting by the $\mathrm{SL}_{k_1} \times \cdots \times \mathrm{SL}_{n-k_4}$ action, and similarly for $(V', G' \times \eta)$. The two quotient spaces are isomorphic $\mathrm{SL}_{n-r} \times \mathbb{C}^*$-varieties (after one divides by the kernel of the action), hence (V_2, G_2) is cofree if and only if $(V', G' \times \eta)$ is cofree, and (2) follows. Finally, (4) follows from (1), (2) and (3). □

7.7. *Remark.* — If $k_1 + k_4 \leq n$, then $k_1 \leq k_3 - r \leq k_4 - r \leq k_2$, and if $k_1 + k_4 \geq n$, then $k_3 - r \leq k_1 \leq k_2 \leq k_4 - r$.

Applying the lemma three times we obtain:

7.8. THEOREM. — *Suppose that $k_1 + k_4 \leq n$. Write $k_1 = a$, $k_2 = a + s + t$, $k_3 = a + r + t$ and $k_4 = a + r + s$ where $r, s, t \geq 0$. Then $V(n, \vec{k})$ is cofree if (V', G') is, where*

$$\begin{aligned} V' = \mathbb{C}^a \otimes \mathbb{C}^{2a} \otimes \big(&\rho_{(a+s)(a+t)}\sigma_{a(a+t)}\tau_{a^2} \oplus \rho_{(a+s)(a+t)}\sigma_{-a^2}\tau_{-a^2} \\ &\oplus \rho_{-(a+s)^2}\sigma_{a(a+t)}\tau_{-a^2} \oplus \rho_{-(a+t)^2}\sigma_{-(a+t)^2}\tau_{a^2}\big), \\ G' = (\mathrm{SL}_a)^4 &\times \mathrm{SL}_{2a} \times \rho \times \sigma \times \tau. \end{aligned}$$

Here we omit the action of $\rho \times \sigma \times \tau$ if $r = 0$ (four medials!), the action of $\sigma \times \tau$ if $s = 0$, and the action of τ if $t = 0$.

7.9. *Remark.* — The actions of the tori guarantee the following. Consider the invariants of $V\big(2a,(a, a, a, a)\big)$. When $r \neq 0$, the action of ρ only allows invariants whose degrees in the first two copies of $\mathbb{C}^a \otimes \mathbb{C}^{2a}$ are the same as those in the second two copies. In other words, only f_{12} and f_{34} are not allowed, but $f_{12}f_{34}$ is allowed. Similarly, via σ, $s \neq 0$ rules out f_{13} and f_{24}, and via τ, $t \neq 0$ rules out f_{14} and f_{23} . Thus if all of r, s and t are not zero, the generators we see are $g_2, \dots, g_{a-1}$, $f_{12}f_{34}$, $f_{13}f_{24}$ and $f_{14}f_{23}$ or, equivalently, $g_0, \dots, g_a$.

7.10. *Example.* — Consider $V = V\big(20,(4,9,13,14)\big)$. Then (up to a finite cover),

$$\mathrm{PIG}(V) = \mathrm{SL}_7 \times \mathrm{SL}_3 \times \mathrm{SL}_2 \times T^3,$$

and $\mathbb{C}[V]^G$ is generated by $g_0, \dots, g_4$. Theorem 7.8 reduces the computation to the $\rho\times\sigma\times\tau$-invariants of $V\big(8,(4,4,4,4)\big)$, giving $f_{12}f_{34}$, $f_{13}f_{24}$, $f_{14}f_{23}$, g_2 and g_3.

7.11. Theorem. — *If $\delta = 0$, $V(n, \vec{k})$ is cofree, except in the two cases $V\big(2,(1,1,1,1)\big)$ and $V\big(4,(2,2,2,2)\big)$.*

Proof. — We may always apply $\mathcal{C}_\kappa$ to arrive at the case where $k_1 + k_4 \leq n$. Now apply Theorem 7.8 to $V(n, \vec{k})$. If $a \geq 3$, then we know that $V\big(2a,(a,a,a,a)\big)$ is cofree, hence the generators f_{ij} and the g_l, $\ell = 2, \dots, a-1$ are a regular sequence. When one (or several) of r, s and t are not zero, we eliminate a pair of f_{ij}, say f_{12} and f_{34}, but keep their product $f_{12}f_{34}$. But, obviously, this process still results in a (shorter) regular sequence.

Suppose that $a = 2$. Then by Theorem 7.8, the only problematical case is that of four medials! So, finally, suppose that $a = 1$. If $r = s = t = 0$, then we have the non-cofree and non-coregular case $V\big(2,(1,1,1,1)\big)$ where $q = 5$. If $q = 2$, then $t \neq 0$, and our generators are $f_{12}f_{34}$ and $f_{13}f_{24}$. If $q = 3$, we have f_{14}, f_{23} and $f_{13}f_{24}$. It is easy to check by hand that these are regular sequences. Finally, if $q = 4$, then $s = t = 0$ and $r \neq 0$, so we came from the case $V\big(r+2,(1,1,r+1,r+1)\big)$. It is then easy to see that the invariants f_{13}, f_{14}, f_{23} and f_{24} form a regular sequence. Alternately, $V\big(r+2,(1,1,r+1,r+1)\big)$ is cofree if its quotient by $\mathrm{SL}_{r+1} \times \mathrm{SL}_{r+1}$ is cofree, and this is the representation $(2\mathbb{C}^{r+2} \oplus 2(\mathbb{C}^{r+2})^*, \mathrm{SL}_{r+2})$, and this representation is cofree for $r \geq 1$ by [Sch2]. □

8. Equidimensionality of $D_{k_1} \times D_{k_2} \times D_{k_3} \times D_{k_4}$.

8.1. *Proof of Theorem 2.10.* — Let $q_1, \dots, q_4$ be as in Corollary 5.5, and consider the condition

(A) $$n - k_i + q_i \geq q - 1, \qquad i = 1,2,3,4.$$

From Theorem 5.3 we know that

(1) If V is equidimensional, then (A) holds and (Y, SL_n) is equidimensional.

(2) If (A) holds, then (Y, SL_n) equidimensional implies that V is. Therefore the representations we seek are among those where (A) fails. By Theorem 7.11, we need only consider the case $\delta \neq 0$, and then V is automatically coregular.

We consider the list of noncofree representations in Proposition 5.8, and look at the small number of cases where (A) does not hold. We find that V not equidimensional and (Y, SL_n) equidimensional occurs exactly in the cases:

(1) $V\big(2n+1,(n+1,n+1,n+1,2n)\big)$, $n \geq 2$;

(2) $V\big(d+1,(1,1,1,d)\big)$, $d \geq 3$;

(3) $V\big(5,(2,2,2,3)\big)$.

In each case (A) fails, of course, but $\mathcal{C}_\kappa(V)$ is cofree, so that (Y, SL_n) is equidimensional. □

BIBLIOGRAPHY

[BGP] I.N. BERNSTEIN, I.M. GEL'FAND and V.A. PONOMAREV, Coxeter functions and Gabriel's theorem, Russian Math. Surveys, 28 (1973), 17–32.

[GP] I.M. GEL'FAND and V.A. PONOMAREV, Quadruples of subspaces of a finite–dimensional vector space, Dokl. Akad. Nauk SSSR, 197 (1971), 762–765 = Soviet Math. Doklady, 12 (1971), 535–539.

[HoHu] R. HOWE and R. HUANG, Projective invariants of four subspaces, Adv. in Math., 118 #2 (1996), 295–336.

[Huan] R. HUANG, Invariants of sets of linear varieties, Proc. Natl. Acad. Sci. USA, 87 #12 (1990), 4557–4560.

[Kac] V.G. KAC, Infinite root systems, representations of graphs and invariant theory, Inv. Math., 56 (1980), 57–92.

[Kempf] G. KEMPF, Some quotient surfaces are smooth, Mich. Math. J., 27 (1980), 295–299.

[LuRi] D. LUNA and R.W. RICHARDSON, A generalization of the Chevalley restriction theorem, Duke Math. J., 46 (1979), 487–496.

[LuVu] D. LUNA and Th. VUST, Un théorème sur les orbites affines des groupes algébriques semi–simples, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 27 (1973), 527–535.

[Na] L.A. NAZAROVA, Representation of a tetrad, Izv. Akad. Nauk SSSR, Ser., 31 (1967), 1361–1378 = Math. USSR Izv., 1 (1967), 1305–1322.

[Po] V.L. POPOV, Stability criteria for actions of a semisimple group on a factorial manifold., Math. USSR Izv., 4 (1970), 527–535.

[Ring] C.M. RINGEL, The rational invariants of the tame quivers, Invent. Math., 58 (1980), 217–239.

[Sch1] G.W. SCHWARZ, Lifting smooth homotopies of orbit spaces, Publ. Math. IHES, 51 (1980), 37–135.

[Sch2] G.W. SCHWARZ, Representations of simple Lie groups with a free module of covariants, Invent. Math., 50 (1978), 1–12.

[Slod] P. SLODOWY, Der Scheibensatz für algebraische Transformationsgruppen, in Algebraic Transformation Groups and Invariant Theory, DMV Seminar, 13 (1989), Birkhäuser Verlag, Basel–Boston, 89–113.

[Turn] H.W. TURNBULL, The projective invariants of four medials, Proc. Edinb. Math. Soc. II, Ser. 7 (1942), 55–72.

[Wehl] D. WEHLAU, Equidimensional Representations of 2–Simple groups, J. Algebra, 154 (2) (1993), 437–489.

Manuscrit reçu le 23 janvier 1998,
accepté le 10 février 1998.

G.W. SCHWARZ,
Brandeis University
Department of Mathematics
Waltham MA 02254–9110 (USA).
schwarz@math.brandeis.edu
&
D.L. WEHLAU,
Royal Military College
Department of Mathematics and Computer Science
Kingston K7K 7B4 (Canada) and
Queen's University
Department of Mathematics and Statistics
Kingston K7L 3N6 (Canada).
wehlau@mast.queensu.ca