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About $G$-bundles over elliptic curves


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ABOUT G-BUNDLES OVER ELLIPTIC CURVES

by Yves LASZLO(*)

1. Introduction.

In this note, we study principal bundles over a complex elliptic curve $X$ with reductive structure group $G$. As in the vector bundle case, we first show that a non semistable bundle has a canonical semistable $L$-structure with $L$ some Levi subgroup of $G$ reducing the study of $G$-bundles to the study of semistable bundles (Proposition 3.2). We then look at the coarse moduli space $M_G$ of topologically trivial semistable bundles on $X$ (there is not any stable topologically trivial $G$-bundle) and prove that it is isomorphic to the quotient $[\Gamma(T) \otimes \mathbb{Z} X]/W$ where $\Gamma(T)$ is the group of one parameter subgroups of a maximal torus $T$ and $W = N(G,T)/T$ is the Weyl group (Theorem 4.16). Suppose that $G$ is simple and simply connected and let $\theta$ be the longest root (relative to some basis $(\alpha_1,\ldots,\alpha_l)$ of the root system $\Phi(G,T)$). The coroot $\theta^\vee$ of $\theta$ has a decomposition $\theta^\vee = \sum g_i \alpha_i^\vee$ with $g_i$ a positive integer. Using Theorem 4.16 and Looijenga’s isomorphism

$$[\Gamma(T) \otimes \mathbb{Z} X]/W \cong \mathbb{P}(1,g_1,\ldots,g_l),$$

one gets that $M_G$ is isomorphic to the weighted projective space $\mathbb{P}(1,g_1,\ldots,g_l)$ (see 4.17), generalizing the well-known isomorphism $M_{\text{SL}_{l+1}} \cong \mathbb{P}^l$ (see [T] for instance). One recovers for instance the Verlinde formula in this case.

We know that these results are certainly well-known from experts, but we were unable to find any reference in the literature, except of course when $G$ is either $\text{SL}$ or $\text{GL}$. For another point of view, see [BG].

During the referee process of this paper, an independent paper of Friedman, Morgan and Witten has appeared in Duke’s eprints (see [FMW]), where the link between Looijenga’s result and bundles on elliptic curves is studied from another point of view.

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Notations. — By scheme, we implicitly mean a complex scheme. Let $G$ be a reductive group with a Borel subgroup (resp. a maximal torus) $B$ (resp. $T \subset B$). The corresponding Lie algebras will be denoted by $t, b$ and $g$.

We denote by $W = N(G, T)/T$ the Weyl group and by $\Gamma(T)$ the $W$-module $\text{Hom}(G_m, T)$.

2. Review on the Harder-Narasimhan reduction.

Let $X$ be an algebraic curve which is smooth, projective and connected and $E$ be a $G$-bundle on $X$. Recall that $E$ is semistable if and only if the adjoint bundle $\mathcal{E} = \text{Ad}(E)$ is a semistable vector bundle. Following [AB], let me recall how to define the Harder-Narasimhan reduction of $E$.

Pick a non degenerate invariant quadratic form $q$ on the Lie algebra of $G$. Then $q$ defines a non degenerate quadratic form on $\mathcal{E}$.

**Lemma 2.1.** — The length $r - 1$ of the Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

of $\mathcal{E}$ is even.

**Proof.** — Let

$$0 = \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

be the Harder-Narasimhan filtration of $\mathcal{E}$. The Harder-Narasimhan filtration of $\mathcal{E}^*$ is

$$0 \subset \ldots \subset (\mathcal{E}/\mathcal{E}_{r-1})^* \subset \ldots \subset \mathcal{E}^*.$$ 

Because $\mathcal{E}$ is self-dual, the Harder-Narasimhan filtration is self-dual and one has an isomorphism

$$\mathcal{E}_i \cong (\mathcal{E}/\mathcal{E}_{r-i})^*$$

inducing isomorphisms

$$gr_i \cong (gr_{r+1-i})^*, \; i = 1, \ldots, r$$

where $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. Assume that the length $r - 1$ is odd. Let us consider the morphisms

$$m_k : \mathcal{E}_{r/2} \otimes \mathcal{E}_k \rightarrow \mathcal{E}/\mathcal{E}_{k-1}, \; 0 \leq k \leq r - 1$$
deduced from the Lie bracket of $\mathcal{E}$. The equality (2.2) gives the inequalities

\[ \mu_1 > \ldots > \mu_{r/2} > -\mu_{r/2} > \ldots > -\mu_1 \]

where $\mu_i$ is the slope of the semistable vector bundle $gr_i = \mathcal{E}_i/\mathcal{E}_{i-1}$. In particular, the slopes $\mu_i + \mu_j$ of the subquotients $gr_i \otimes gr_j, i \leq r/2$ and $j \leq k$ which appear in $\mathcal{E}_{r/2} \otimes \mathcal{E}_k$ are not less than $\mu_{r/2} + \mu_k > \mu_k$ though the slopes of the subquotients $gr_j, i \geq k$ which appear $\mathcal{E}_0/\mathcal{E}_{k-1}$ are $\leq \mu_k$. This shows that $m_k$ is zero for all $k$ and that all elements of $\mathcal{E}_{r/2}$ are nilpotent. By (2.1), this algebra is also lagrangian. Suppose that the center of $G$ is of positive dimension. Then $\mathcal{E}$ contains a trivial sub-bundle (of positive rank) as a direct summand which implies that some of the $\mu_i$'s is zero, contradicting (2.3). The Lie algebra bundle $\mathcal{E}$ is therefore semisimple and therefore has no non trivial lagrangian sub-Lie algebra (with respect of the Killing form form) consisting of nilpotent elements, just by a dimension argument.

It follows that one can index the Harder-Narasimhan filtration of $\mathcal{E}$ such that

\[ \mathcal{E}_- = \mathcal{E}_{-r} \subset \mathcal{E}_{-r+1} \subset \ldots \subset \mathcal{E}_{-1} \subset \mathcal{E}_0 \subset \ldots \subset \mathcal{E}_{r-1} = \mathcal{E} \]

where $\mathcal{E}_{-j}$ is the orthogonal of $\mathcal{E}_{j-1}$. One checks that $\mathcal{E}_0$ is a subalgebra of $\mathcal{E}$. Notice that $\mathcal{E}_0/\mathcal{E}_{-1}$ is self-dual and therefore has slope zero. In particular, the slope of $gr_{-j}, j > 0$ is $> 0$. As in the proof of the preceding lemma, this immediately implies the sequence of inclusions

\[ [\mathcal{E}_{-j}, \mathcal{E}_{-1}] \subset \mathcal{E}_{-j-1} \text{ for all } j. \]

In particular, all elements of $\mathcal{E}_{-1}$ are nilpotent. For sake of completeness, let me prove this easy lemma (cf. Th VIII.10.1 of [Bo]).

**Lemma 2.2.** — Let $\mathfrak{g}$ be a reductive algebra endowed with an invariant non degenerate bilinear form. Let $\mathfrak{n}'$ be a subalgebra of $\mathfrak{g}$ whose elements are nilpotent. Then, if the orthogonal of $\mathfrak{n}'$ is a sub-Lie algebra of $\mathfrak{g}$, it is parabolic.

**Proof.** — The Lie algebra $\mathfrak{n}'$ is nilpotent. Let $\mathfrak{b}$ be a maximal solvable Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{n}'$ and $\mathfrak{n}$ its nilpotent ideal. By [Bo], Definition VIII.3.3.1, $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. Because all elements of $\mathfrak{n}'$ are nilpotent, $\mathfrak{n}'$ is contained in $\mathfrak{n}$. By [Bo], proposition VII.1.3.10 (iii), the orthogonal of $\mathfrak{n}$ is $\mathfrak{b}$ and the lemma follows.

Because the orthogonal of $\mathcal{E}_{-1}$ is $\mathcal{E}_0$, the Lie subalgebra $\mathcal{E}_0$ is therefore parabolic with radical $\mathcal{E}_{-1}$ (see [AB], p. 589). Let $P$ be the unique standard
parabolic subgroup defined by $\mathcal{E}_0$. If $F$ is the bundle of local trivialization $G_S \to E_S (S \to X \text{ étale})$ whose differential sends $\text{Lie}(P)_S$ to $\mathcal{E}_0$, then $F$ is a $P$-structure of $E$. Let us denote by $U$ the unipotent radical of $P$ and by $\tilde{P}$ (resp. $\tilde{F}$) the quotient $P/U$ (resp. $F/U$). By construction, $\tilde{F}$ is semistable (because $\text{Ad}(\tilde{F}) = \mathcal{E}_0/\mathcal{E}_{-1}$).

**Definition 2.3.** — With the notation above, the $P$-bundle $F$ is the Harder-Narasimhan reduction of $E$.

**Remark 2.4.** — It is easy to check that the filtration and therefore the corresponding reduction does not depend on the particular choice of the invariant non degenerate quadratic form on $\text{Lie}(G)$.

**Example 2.5.** — Suppose that $E$ is the $\text{GL}_n$-bundle of local frames of a vector bundle $\mathcal{E}$ on $X$ with Harder-Narasimhan filtration $0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_k = \mathcal{E}$. Let $P$ be the quasi-triangular subgroup of $\text{GL}_n$ defined by the partition $[r_i = \text{rk}(\mathcal{E}_{i+1}) - \text{rk}(\mathcal{E}_i)]_{0 \leq i < k}$ of $\text{rk}(\mathcal{E})$. Then $F$ is the $P$-bundle of local frames compatible with the filtration and $\tilde{F}$ is the $\times_i \text{GL}_{r_i}$-bundle of local frames of $\bigoplus \mathcal{E}_{i+1}/\mathcal{E}_i$.

We suppose once for all that $X$ is an elliptic curve.

### 3. Non semistable $G$-bundles.

Let $E$ be a $G$-bundle on $X$ and let $F$ be the Harder-Narasimhan reduction of $E$. Let us consider a Levi factor of $P$ thought as a section $\sigma : \tilde{P} \to P$ of the canonical projection $P \to \tilde{P} = P/U$.

**Remark 3.1.** — Following Humphreys (see [Hu], 30.2) a Levi factor is a factor of the unipotent radical and not of the radical itself (as in Bourbaki for instance).

**Proposition 3.2.** — With the notations above, the $P$-bundle $\sigma_*(\tilde{F})$ is isomorphic to $F$.

**Remark 3.3.** — This is the generalization of the well-known (and easy) fact that any vector bundle on $X$ is a direct sum of semistable vector bundles.

**Proof.** — Let us denote by

\[ 1 \to U \to P \to \tilde{P} \to 1 \]
be the twist of
\[ 1 \to U \to P \to \tilde{P} \to 1 \]
by \( F \) (see [S], chap. I & 5). Geometrically, \( P \) (resp. \( \tilde{P} \)) is the group scheme \( \text{Aut}_P(F) \) (resp. \( \text{Aut}_{\tilde{P}}(\tilde{F}) \)). The twisted group \( U \) is the unipotent radical of \( P \) and is isomorphic \( F \times_P U \) (\( P \) acts on the normal subgroup \( U \) by conjugation). As usual, the map
\[
\begin{align*}
\{ H^1(X, P) & \to H^1(X, P) \} \\
F'' & \mapsto \text{Isom}_P(F, F'')
\end{align*}
\]
are bijective. The image of \( \text{Isom}_P(F, \sigma_* \tilde{F}) \) in \( H^1(X, \tilde{P}) \) is the trivial torsor \( \text{Isom}_P(F, \tilde{F}) \) and it is enough to show the equality \( H^1(X, U) = \{ [U] \} \) to prove the isomorphism \( F \sim \sigma_* \tilde{F} \). With the notations of (2.4), the Lie algebra of \( U \) is \( \mathcal{E}_{-1} \). By (2.5), the Lie algebra \( \mathcal{E}_{-j}/\mathcal{E}_{-j-1} \) is abelian for any \( j \geq 1 \). This induces a filtration
\[ 1 = U_0 \subset U_1 \subset \cdots \subset U_2 \subset U_{-1} = U \]
by unipotent group schemes where the exponential defines isomorphisms
\[ U_{-j}/U_{-j-1} \sim \mathcal{E}_{-j}/\mathcal{E}_{-j-1} j \geq 1 \]
of abelian group schemes. By construction, \( \mathcal{E}_{-j}/\mathcal{E}_{-j-1}, j \geq 1 \) is semistable of positive slope and therefore
\[ H^1(X, \mathcal{E}_{-j}/\mathcal{E}_{-j-1}) = 0, j \geq 1 \]
because \( g(X) = 1 \). This implies the equality
\[ H^1(X, U) = \{ [U] \}. \]

4. The coarse moduli space \( M_G \).

Let \( M_G \) be the coarse moduli space of semistable \( G \)-bundle of trivial topological type (what is the same, the component containing the trivial torsor \( G_X \)). Recall that the (closed) points of \( M_G \) are \( S \)-equivalence classes of semistable \( G \)-bundles. The only thing which will be needed about this equivalence relation is the following (cf. [Ra1], Corollary 3.12.1):

4.1. Every class \( \xi \) defines a Levi subgroup \( L \) such that there exists a stable \( L \)-bundle \( F \) with \( F(G) \in \xi \). Moreover, \( F(G) \) is well defined up to isomorphism.
Remark 4.2. — Ramanathan’s construction of $M_G$ is written for a curve of genus $\geq 2$, but the construction can be made in general (see for instance [LeP] in the case of $G = GL_n$ from which the general case follows).

4.3. We denote by $a \otimes b$ the product of two $T$-bundles $a$ and $b$ (for the natural structure of abelian group of $H^1(X,T)$). Let $\psi = (\psi_i)_{i \in I}$ be a finite family of one parameter subgroups and $L = (L_i \in I)$ a family of line bundles of degree 0 on $X$ (thought as $G_m$-torsors). Then, $\otimes_{i \in I} L_i(\psi_i)$ is a $T$-structure of a $G$-bundle $L_{\psi}$ on $X$ which is semistable. This defines a morphism of abelian groups

$$p : \Gamma(T) \otimes_{\mathbb{Z}} X \to H^1(X,T).$$

Chose a (closed) point $x$ of $X$ which defines an isomorphism $\text{Pic}^0(X) \cong X$ and a Poincaré line bundle $\mathcal{P}$ on $X \times \text{Pic}^0(X)$. This allows to construct a universal semistable $T$-bundle $L$ on $X \times \Gamma(T) \otimes_{\mathbb{Z}} X$.

Remark 4.4. — The theta line bundle $\Theta$ on $X = \text{Pic}^1(X)$ becomes through the isomorphism $X \cong \text{Pic}^0(X)$ the determinant bundle $\det(R\Gamma\mathcal{P})^*$. The family of semistable bundles $L(G)$ defines a morphism of (reduced) schemes

$$\Gamma(T) \otimes_{\mathbb{Z}} X \to M_G.$$

The action of the Weyl group $W$ on $\Gamma(T)$ defines an action $\Gamma(T) \otimes_{\mathbb{Z}} X$ such that $w.L_{\psi} \sim L_{\psi}$ for all $w \in W$. Let

$$\pi : [\Gamma(T) \otimes_{\mathbb{Z}} X]/W \to M_G$$

be the induced morphism. We want to prove that $\pi$ is an isomorphism.

4.5. Let us prove that $\pi$ is finite. Let $G \to \text{SL}_N$ be a faithful representation of $G$ inducing a morphism $M_G \to M_{\text{SL}_N}$. Let $L$ be the inverse of the determinant bundle on $M_{\text{SL}_N}$.

Remark 4.6. — Notice that in this case, $M_{\text{SL}_N} = \mathbb{P}^{N-1}$ and that the determinant bundle is just $\mathcal{O}(1)$ (see [Tu], Theorem 7 for instance).

Lemma 4.7. — The line bundle $\pi^*(L)$ is ample.

Proof. — One can assume that $G$ is semisimple. Let $q$ be the natural morphism

$$q : \Gamma(T) \otimes_{\mathbb{Z}} X \to M_G.$$
It is enough to prove that $q^*(L)$ is ample. Let us choose a basis of $\Gamma(L)$ identifying $\Gamma(T) \otimes X$ with $X^l$ ($l$ is the rank of $G$). Let $\gamma : G_m \to T$ be a non trivial element in $\Gamma(T)$. Let $q_\gamma : X \to M_G$ be the morphism defined by $\gamma$. One can assume that $\gamma(z) = \text{diag}(z^{\gamma_1}, \ldots, z^{\gamma_l})$ for $z \in \mathbb{C}^*$ (with $\sum \gamma_i = 0$). Then (see Remark 4.4),

$$q_\gamma^*(L) = \Theta \sum_i \gamma_i^2$$

which is ample because $\sum \gamma_i^2 > 0$ (recall that $\gamma$ is non trivial). The rank-$N$ vector bundle bundle parameterized by $(x_1, \ldots, x_l)$ is

$$\bigoplus_i \mathcal{O} \left( \sum_{\gamma \in \gamma_i} (x_\gamma - x) \right).$$

By additivity of the determinant bundle, $q^*(L)$ is of the form

$$\bigotimes_{1 \leq i \leq l} \Theta^{b_i} \text{ with } b_i > 0$$

and therefore is ample.

The fibers of $\pi$ are therefore finite, and the proper morphism $\pi$ is finite.

4.8. Let $\pi^{-1}(0)$ be the fiber of $\pi$ at the trivial bundle $G_X$. Let us first prove that $\pi^{-1}(0)$ is set-theoretically reduced to $[0]$, the class $W.O$. Let us first prove the following general result.

4.9. Let us consider the following situation: let $p : X \to S$ be a proper morphism such that $\mathcal{O}_S = p_* \mathcal{O}_X$ is an isomorphism. Assume that $p$ has a section $\sigma : S \to X$. Let $A \subset B$ be a reductive subgroup of a linear group $B$.

LEMMA 4.10. — Let $\alpha$ be an $A$-bundle trivial along $\sigma$. Then, if the associated $B$-bundle $\beta = \alpha(B)$ is trivial, the $A$-bundle $\alpha$ is so.

Proof. — Let $s$ be the section of $\beta/A$ defined by $\alpha$. Because $\beta/A$ is affine over $S$, the section $s$ factors through $p$ in a section $\tilde{s}$. Because $\alpha$ is trivial along $\sigma$, the section $\tilde{s}$ comes from a section of the restriction to $\sigma$ of the trivial bundle $\beta$ and can be lifted to a section $s'$ of $\beta$. The section $s' \mod A$ of $\beta/A$ is equal to $s$ and defines a trivialization of $\alpha = s^* \beta$.

4.11. Choose an embedding $G$ in a product $G' = \prod_i \text{GL}_{n_i}$ of linear groups such that $Z_0(G) \subset Z_0(G')$ ($Z_0$ denotes the neutral component). Let $T'$ be a maximal torus of $G'$ containing $T$. Let $f : M_G \to M_{G'}$ be a natural morphism (see [Ra2], Corollary of Theorem 7.1). Let $E$ be a $T$-bundle such that $E \in \pi^{-1}(0)$ and let $E'$ be the corresponding $T'$-bundle.
Because $f(E(G)) = [E'(G')]$, the semistable bundle is equivalent to the trivial bundle and is therefore trivial (a direct sum of line bundles of degree 0 is equivalent to the trivial bundle if and only if all summands are trivial). Applying the preceding lemma with $\alpha = E, A = T, B = G'$ and $\mathcal{X} = X$ for instance, one gets that $E$ is trivial.

4.12. It remains to show that $\pi$ is étale at the origin: this will follow from the fact that the completion of $\pi$ at the origin can be identified to the completion at the origin of the Chevalley isomorphism $t/W \cong \mathfrak{g}/G$.

**Lemma 4.13.** — The morphism $\pi : (\Gamma(T) \otimes \text{Pic}^0(X))/W \to M_G$ is étale at the origin.

**Proof.** — Let’s briefly recall how to construct the moduli space $M_G$ (see [Ra1], [BLS]), or better of an affine neighborhood $M$ of the trivial bundle $X \times G$ as a GIT quotient $Y/H$ of a smooth affine scheme $Y$ by some reductive group $H$ (with Lie algebra $\mathfrak{h}$). One choose first a faithful representation $G \hookrightarrow \text{GL}_n$ inducing an embedding $\Gamma(T) \otimes \text{Pic}^0(X) \hookrightarrow (\text{Pic}^0(X))^n$. For $m$ big enough, one knows that the canonical morphism

$$\iota_P : H^0(P(C^n) \otimes \mathcal{O}(mx)) \otimes \mathcal{O} \to P(C^n) \otimes \mathcal{O}(mx)$$

is surjective for all semistable bundles $P$ and that $H^0(\iota_P)$ is bijective. Let $\chi$ be the Euler characteristic of some $P(C^n) \otimes \mathcal{O}(mx)$. By the theory of Hilbert schemes, the pairs $(P, \iota)$ where $P$ is a semistable $G$-bundle and $\iota$ an isomorphism

$$H^0(P(C^n) \otimes \mathcal{O}(mx)) \cong \mathbb{C}^\chi$$

are parameterized by a smooth scheme $Y$ and $M_G$ is a GIT quotient of this scheme by $H = \text{GL}_\chi$ (see [BLS]). Notice that the stabilizer of the “trivial pair” is $G$ itself.

Let $U$ be the universal $T$-bundle on $X \times (\Gamma(T) \otimes \text{Pic}^0(X))$. Let us chose a trivialization of the vector bundle $R\Gamma(L \boxtimes \mathcal{O}(mx))$ on some symmetric affine neighborhood $S^0$ of 0 in $\text{Pic}^0(X)$. Therefore, the direct image of $U(C^n) \boxtimes \mathcal{O}(m)$ is trivial on $S = (S^0)^n \cap \Gamma(T) \otimes \text{Pic}^0(X)$ and the trivialization is $W$-equivariant. The induced morphism $\pi : S \to M_G = Y/H$ is therefore induced by a $W$-equivariant morphism $S \to Y$ mapping 0 to $y$. Notice that the orbit $H.y$ is closed. By considering some $H$-invariant affine open neighborhood of $y$, one can assume that $Y$ is affine (the quotient $Y/H$ is now a neighborhood of $H.y$ in $M_G$).
Let’s consider the following commutative diagram:

\[
\begin{array}{ccc}
C[Y]_+ & \rightarrow & C[S]_+ \\
\downarrow & & \downarrow \\
V = (T^*_y Y)/\mathfrak{h} & \xrightarrow{k} & T^*_0 S
\end{array}
\]

where \(C[Y]_+\) (resp. \(C[S]_+)\) denotes the maximal ideal of \(y\) (resp. 0). The transpose of \(k\) is the tangent map of \(S \rightarrow \mathcal{M}_G\) from \(S\) to the stack of \(G\)-bundles on \(X\), namely the Kodaira-Spencer map

\[
k : t = t \otimes H^1(X, \mathcal{O}_X) = T_0 S \rightarrow (T_y Y)/\mathfrak{h} = g \otimes H^1(X, \mathcal{O}) = g.
\]

**Lemma 4.14.** — The Kodaira-Spencer map \(k\) is the canonical inclusion \(t \hookrightarrow g\).

**Proof.** — By functoriality, one is reduced to the case where \(G = \text{GL}_n\) and \(T\) is the torus of invertible diagonal matrices. Consider the one parameter subgroup of differential \(aE_{i,i}\) for some integer \(a\) (\(E_{i,i}\) is the standard diagonal rank 1 matrix). If \((\lambda_{\alpha,\beta})\) is a Cech-cocycle representing \(\lambda \in H^1(\mathcal{O})\), the derivative

\[
\frac{\partial \pi}{\partial (\gamma \otimes \lambda)}(0)
\]

is defined by the vector bundle on \(X[e]/(e^2 = 0)\) with cocycle \(1 + a\epsilon \lambda_{\alpha,\beta} E_{i,i}\). In other words,

\[
\frac{\partial \pi}{\partial (\gamma \otimes \lambda)}(0) = \lambda d\gamma,
\]

which proves the lemma. \(\square\)

Notice that \(k\) is \(N(G, T)\)-equivariant. By Luna’s results ([Lu]), one obtains an étale slice of \(Y \rightarrow Y/H\) as follows.

One choose an \(H_y = \text{Aut}_G(X \times G) = G\)-invariant section \(\sigma\) of \(C[Y]_+ \rightarrow (T^*_y Y)/\mathfrak{h}\) and the induced morphism \(Y \rightarrow V\) identifies étale locally \(Y/H\) and \(V/H_y\). The group \(N(G, T)\) being reductive and \(^t k\) being surjective, one pick an invariant section \(\tau\) of \(^t k : g^* \rightarrow t^*\) which defines a morphism (still denoted by \(\tau\))

\[
t^* \rightarrow C[Y]_+ \rightarrow C[S]_+
\]
which is $W$-equivariant. This is a $W$-equivariant section of $C[S]^H_+ \to T_0^*S$ and therefore defines an étale slice of $S \to S/W$. Shrinking $S$ and $Y$ if necessary, one obtains from the diagram (4.1) the commutative diagram

$$
\begin{array}{ccc}
C[Y]^H_+ & \xrightarrow{\pi} & C[S]^W_+ \\
S(\sigma) & \downarrow & S(\tau) \\
(S g^*)^G & \xrightarrow{(S t^*)^W} & (S t^*)^W
\end{array}
$$

where $S(\sigma)$ and $S(\tau)$ are étale. By Chevalley’s theorem, $\pi$ is therefore étale at the origin. \hfill \Box

4.15. The morphism $\pi$ is therefore a finite morphism between normal varieties and is of degree 1. We have proved the

THEOREM 4.16. — The morphism

$$\pi : [\Gamma(T) \otimes_Z X]/W \to M_G$$

is an isomorphism.

4.17. Assume that $G$ is simple and simply connected. Let $\theta$ be the longest root and $\alpha_i, i = 1, \ldots, l$ the basis of the root system $\Phi(B, G)$. The coroot $\theta^\vee$ is a sum

$$\theta^\vee = \sum_i g_i \alpha_i^\vee$$

where $\alpha_i^\vee$ is the coroot of $\alpha$. Then Looijenga [Lo] has proved that $[\Gamma(T) \otimes_Z X]/W$ is the weighted projective space $P(1, g_1, \ldots, g_l)$.

Remark 4.18. — The proof in [Lo] is not correct. See [BS] for a more general result and hints for a complete proof.

4.19. It is interesting to remark ([D], remarques 1.8) that $\mathcal{O}(l)$ is locally free if and only if $l$ is a multiple of $\text{lcm}(g_i)$ although it is reflexive (Lemme 4.1 of loc. cit.). In particular, $M_G$ is locally factorial if and only if $\text{lcm}(g_i) = 1$, condition which is equivalent to $G$ special in the sense of Serre (look at the table of [Bo]). If one notice that $\text{lcm}(g_i)$ is also the minimal Dynkin index of the representations of $G$ (see [LS]), this funny characterization of special groups in terms of $M_G$ is the version in the genus one case of Proposition 13.2 of [BLS] (which deals with the genus $> 1$). In all the cases, one has the formula

$$\dim H^0(M_G, \mathcal{O}(l)) = \text{card}(P_l)$$
where $P_i$ is the number of dominant weights $w$ such that $< \theta^\vee, w > \leq l$, as predicted by the Verlinde formula (see [Be]).

4.20. Let us explain briefly the link between the theorem of Narasimhan and Seshadri and our description of $M_G$. Suppose that $G$ is semisimple with maximal compact subgroup $K$. The theorem of Narasimhan and Seshadri says that the complex points of $M_G$ are parameterized by equivalence classes of pairs of elements of $K$ which commutes ($K$ acting on these pairs diagonally through the adjoint action). Suppose further that $G$ is simply connected. Then such a class has a representative in $T_R \times T_R$ (where $T_R$ is the maximal torus of $K$). Suppose that $X(C)$ is a complex torus $C/Z \oplus Z\tau$ of period $\tau$ in the Poincaré upper half plane. The complex structure $(a, b) \rightarrow a - \tau b$ on $R \times R$ induces a complex structure on $T_R \times T_R$ which is naturally the maximal torus $T$ of $G$. We get a diagram

\[
\begin{array}{ccc}
\Gamma(T) \otimes C & \xrightarrow{\sim} & T_R \times T_R \\
\downarrow & & \downarrow \\
\Gamma(T) \otimes X(C) & \xrightarrow{\pi} & M_G(C)
\end{array}
\]

One checks easily that this diagram commutes.

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