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SLOPE FILTRATION OF QUASI-UNIPOTENT OVERCONVERGENT F-ISOCRYSTALS

by Nobuo TSUZUKI

1. Introduction.

Let X be a smooth curve over a perfect field k with a positive characteristic p. Let $\overline{X}$ and Z be the smooth compactification of X and the complement of X in $\overline{X}$, respectively. In [Cr2] R. Crew defined the notion of quasi-unipotent overconvergent (F-)isocrystals over X around Z and proved some expected properties, finiteness and duality for rigid cohomologies and the global monodromy theorem, of quasi-unipotent overconvergent (F-)isocrystals. However, the problem that what kinds of overconvergent (F-)isocrystals are quasi-unipotent is still open.

In this paper we study local properties of quasi-unipotent F-isocrystals. Let K be a complete valuation field with an absolute value $|\ |$ and let $\mathcal{R}$ be the Robba ring over K (2.2). The Robba ring is a ring of analytic functions on some annulus $\eta < |x| < 1$. We define φ-∇-modules over $\mathcal{R}$ by a free $\mathcal{R}$-module with a connection and Frobenius structures (3.2.1). A φ-∇-module is quasi-unipotent if and only if it is a successive extension of copies of the unit object as differential modules (4.1.1) after a finite etale extension. For φ-∇-modules over $\mathcal{R}$, we define a slope filtration for Frobenius structures (5.1.1). If a φ-∇-module has a slope filtration, then it is unique (5.1.5). We establish

THEOREM 5.2.1. — *A φ-∇-module over $\mathcal{R}$ is quasi-unipotent if and only if it has a slope filtration for Frobenius structures.*

Key words: Quasi-unipotent F-isocrystals – φ-∇-modules – Slope filtration.
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Let $\mathcal{M}$ be an overconvergent F-isocrystal on $\overline{X}$ around Z. $\mathcal{M}$ determines a φ-∇-module $i_s^*\mathcal{M}$ over a Robba ring for every closed point $s \in \overline{X}$ canonically. Then $\mathcal{M}$ is quasi-unipotent in the sense of Crew [Cr2, 10.1] if and only if $i_s^*\mathcal{M}$ is quasi-unipotent for any closed point $s \in X$ by (6.1.2) and (6.1.8).

The theorem above is useful since we have known finiteness of irregularities of φ-∇-modules with pure slopes [TN2]. So it implies finiteness of irregularities of quasi-unipotent φ-∇-modules in the sense of [TN2]. We will apply it to the global formula of Euler's number of quasi-unipotent overconvergent F-isocrystals in the future.

It is expected that any φ-∇-module over $\mathcal{R}$ is quasi-unipotent. If this holds, then any overconvergent F-isocrystal is quasi-unipotent (6.1). It is conjectured that an overconvergent F-isocrystal on a curve is quasi-unipotent if it has some geometric origin. (See [Cr2, 10.1].)

Now we explain the contents of this paper. In Section 2 we fix notations and prove some properties of the Robba ring $\mathcal{R}$. In Section 3 we define a φ-∇-module over $\mathcal{R}$. In Section 4 we define a quasi-unipotent φ-∇-module over $\mathcal{R}$ and prove that the category of quasi-unipotent φ-∇-modules over $\mathcal{R}$ is independent of the choice of Frobenius on $\mathcal{R}$. In Section 5 we define the slope filtration for Frobenius structures of φ-∇-modules over $\mathcal{R}$. We prove the existence of the slope filtration for quasi-unipotent φ-∇-modules over $\mathcal{R}$. In Section 6 we apply our local study to overconvergent F-isocrystals on a curve. We define a quasi-unipotent overconvergent F-isocrystal. The definition is a different form from that of Crew. Of course, the two definitions are equivalent to each other. We give some examples of quasi-unipotent overconvergent F-isocrystals.

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2. The Robba ring $\mathcal{R}$.

2.1. Let p be a prime number. Let k (resp. K) be a perfect field with characteristic p (a complete discrete valuation field of mixed characteristics $(0, p)$ with residue class field k). Fix an algebraic closure K^{alg} of K and denote by k^{alg} the residue class field of K^{alg}. Denote by $| \ |$ (resp. v_p) the absolute value (resp. the additive valuation) of K^{alg} which is normalized by $|p| = p^{-1}$ (resp. $v_p(p) = 1$).

For any valuation field L, we denote by O_L (resp. k_L, resp. L^{unr}, resp. m_L) the valuation ring of L (resp. the residue class field of L, resp. the maximum unramified subfield in the fixed algebraic closure of L whose residue class field is separable over k_L, resp. the maximal ideal of O_L).

Let $F = k((x))$ be the field of fraction of the ring of formal power series with k-coefficients. Fix an algebraic closure F^{alg} of k such that the residue class field of F^{alg} is k^{alg} and denote by F^{sep} the separable closure of F in F^{alg}.

For a matrix (a_{ij}) and for an application f (resp. for a norm N), define

$$f((a_{ij})) = (f(a_{ij})) \quad (\text{resp. } N((a_{ij})) = \sup_{i,j} N(a_{ij})).$$

2.2. For a complete field Ω with a non-Archimedean absolute value $|\ | : \Omega \to \mathbf{R}_{\geqq 0}$ and for an indeterminate x, we define several Ω-algebras as follows:

$$\mathcal{R}_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; \begin{array}{l} a_n \in \Omega, \sup_{n<0} |a_n|\xi^n < \infty \text{ for some } 0 < \xi < 1, \\ |a_n|\eta^n \to 0 \ (n \to +\infty) \text{ for any } 0 < \eta < 1 \end{array} \right\}$$

$$\mathcal{E}_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \;\middle|\; \begin{array}{l} a_n \in \Omega, \sup_{n} |a_n| < \infty, \\ |a_n| \to 0 \ (n \to -\infty) \end{array} \right\}$$

$$\mathcal{E}^{\dagger}_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \in \mathcal{R}_{x,\Omega} \;\middle|\; \sup_{n} |a_n| < \infty \right\}$$

$$S_{x,\Omega} = \Omega \bigotimes_{O_\Omega} O_\Omega[[x]].$$

Each ring is functorial in Ω. We have natural injections of Ω-algebras:

$$\begin{array}{ccc} & & \mathcal{R}_{x,\Omega} \\ & & \nearrow \\ S_{x,\Omega} \to \mathcal{E}^{\dagger}_{x,\Omega} & & \\ & & \searrow \\ & & \mathcal{E}_{x,\Omega}. \end{array}$$

We call the ring $\mathcal{R}_{x,\Omega}$ Robba ring over Ω and an element of $\mathcal{R}_{x,\Omega}$ is regarded as a function on some annulus $\xi < |x| < 1$ for some $\xi < 1$. We use the notations $\mathcal{R}, \mathcal{E}, \mathcal{E}^{\dagger}$ and S_K instead of $\mathcal{R}_{x,K}, \mathcal{E}_{x,K}, \mathcal{E}^{\dagger}_{x,K}$ and $S_{x,K}$ respectively if there is no ambiguity.

Remark 2.2.1. Our $\mathcal{R}_{x,\Omega}$ coincides with $\mathcal{R}_0(1)$ in [Ro, 2].

For formal Laurent power series $a = \sum a_n x^n$, we define $|a|_G \in \mathbf{R}_{\geqq 0} \cup \{\infty\}$ by $\sup_n |a_n|$. The field $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is a complete discrete valuation field (resp. a henselian discrete valuation field) under the absolute value $|\ |_G$. $|\ |_G$ is an extension of the absolute value $|\ |$ of K and the residue class field of $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is F by the natural projection. (See [Cr1, 4.2] [Ma, 3.2].) For a finite separable extension E over F in F^{sep}, denote by $\mathcal{E}_E$ (resp. $\mathcal{E}_E^\dagger$) the unique finite unramified extension of $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) with residue class field E in the fixed algebraic closure of $\mathcal{E}$.

LEMMA 2.2.2 ([Ma, 3.2]). — *Under the notation as above, $\mathcal{E}_E$ (resp. $\mathcal{E}_E^\dagger$) is isomorphic to $\mathcal{E}_{y,K_E}$ (resp. $\mathcal{E}_{y,K_E}^\dagger$) for any lifting y of a uniformizer of E. Here K_E is the unique finite unramified extension of K with residue class field k_E. Moreover the unique extension of the absolute value $|\ |_G$ of $\mathcal{E}$ on $\mathcal{E}_E$ coincides with the map $\sum b_n y^n \mapsto \sup_n |b_n|$.*

Let E be a finite separable extension of F and choose a lifting y of a uniformizer of E in $\mathcal{E}_E^\dagger$. Define a K algebra $\mathcal{R}_E$ by

$$\mathcal{R}_E = \mathcal{R}_{y,K_E}.$$

Since $x = x(y) \in \mathcal{E}_E^\dagger = \mathcal{E}_{y,K_E}^\dagger$, $\mathcal{R}$ is naturally included in $\mathcal{R}_E$.

LEMMA 2.2.3. — (1) *$\mathcal{R}_E$ is independent of the choice of the lifting of the uniformizer of E up to canonical isomorphism.*

(2) *$\mathcal{R}_E$ is free over $\mathcal{R}$ of degree $[E : F]$. Moreover, $\mathcal{R}_E \cong \mathcal{E}_E^\dagger \bigotimes_{\mathcal{E}^\dagger} \mathcal{R}$ and $\mathcal{E}^\dagger = \mathcal{R} \bigcap \mathcal{E}_E^\dagger$.*

Assume that the extension E/F is Galois and denote by $\mathrm{Gal}(E/F)$ the Galois group. Since $\mathcal{E}^\dagger$ is a henselian discrete valuation field, the Galois group $\mathrm{Gal}(\mathcal{E}_E^\dagger/\mathcal{E}^\dagger)$ is canonically isomorphic to $\mathrm{Gal}(E/F)$. The action of $\mathrm{Gal}(E/F)$ on $\mathcal{E}_E^\dagger$ extends naturally on $\mathcal{R}_E$. By [Se1, X.1.Prop.3] and Lemma (2.2.3) we have

LEMMA 2.2.4. — *Under the notation as above,*

(1) $H^0(\mathrm{Gal}(E/F), \mathcal{E}_E^\dagger) = \mathcal{E}^\dagger$ *and* $H^1(\mathrm{Gal}(E/F), GL_r(\mathcal{E}_E^\dagger)) = \{1\}$;

(2) $H^0(\mathrm{Gal}(E/F), \mathcal{R}_E) = \mathcal{R}$.

2.3. For formal Laurent power series $\sum a_n x^n$ of indeterminate x, we define an additive map $\delta_x = x\dfrac{d}{dx}$ by

$$\delta_x(\sum a_n x^n) = \sum n a_n x^n.$$

Then δ_x is a K-derivation on $\mathcal{R}$ (resp. $\mathcal{E}$, resp. $\mathcal{E}^\dagger$, resp. S_K).

Let R be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K. Define a free R-module ω_R of rank one by

$$\omega_R = R\frac{dx}{x}.$$

We define an additive map $d : R \to \omega_R$ by $d(a) = \delta_x(a)\dfrac{dx}{x}$ for $a \in R$. Then d is a K-derivation on R.

Let E be a finite separable extension of F and choose a lifting y of a uniformizer of E in $\mathcal{E}_E^\dagger$. Then the derivation δ_x extends uniquely on $\mathcal{R}_E$ and we also use the notation δ_x for this extension. We have the relation

$$\delta_x = \frac{x(y)}{\delta_y(x(y))}\delta_y,$$

where $x = x(y) \in \mathcal{E}_E^\dagger$ and δ_x commutes with the action of $\mathrm{Gal}(E/F)$ if E/F is Galois.

LEMMA 2.3.1. — *Under the notation as above, we have*

(1) $\ker(\delta_x : \mathcal{R}_E \to \mathcal{R}_E) = K_E$;

(2) $\operatorname{coker}(\delta_x : \mathcal{R}_E \to \mathcal{R}_E) \cong K_E\overline{\dfrac{x(y)}{\delta_y(x(y))}}$, *where* $\overline{\dfrac{x(y)}{\delta_y(x(y))}}$ *is the image of* $\dfrac{x(y)}{\delta_y(x(y))}$.

Proof. — The assertion easily follows from the fact that $\dfrac{x(y)}{\delta_y(x(y))}$ is a unit in $\mathcal{R}_E$. □

2.4. Fix a power $q = p^a$ ($a \geqslant 1$) of p. Denote by K_0 the field of fraction of the Witt vector ring $W(k)$ and Frob is the usual lifting of the q-th power map on K_0. We say that an automorphism $\sigma : K \to K$ is a Frobenius on K if and only if σ is a continuous lifting of the q-th power map on the residue class field k. Since k is perfect, we have $\sigma|_{K_0} = \mathrm{Frob}^a$. Note that, if K has a Frobenius and if L is an unramified extension of K, then the Frobenius σ extends uniquely on L.

For a Frobenius σ on K, put $K^{\sigma=1} = \{u \in K \mid \sigma(u) = u\}$. One can easily see that $K^{\sigma=1}$ is finite over the field $\mathbf{Q}_p$ of p-adic integers.

LEMMA 2.4.1 ([Cr1, 1.8]). — *Let σ be a Frobenius on K. Then there is a finite unramified extension L of K such that $L \cong L^{\sigma=1} \bigotimes_{(L^{\sigma=1})_0} L_0$ and that the unique extension σ on L is $\mathrm{id}_{L^{\sigma=1}} \otimes \mathrm{Frob}^a$. Assume furthermore that the residue class field k is algebraically closed, then one can choose $L = K$.*

Proof. — First we prove the assertion in the case where k is algebraically closed. In this case there exists a uniformizer π of K which is algebraic over $\mathbf{Q}_p$. Then we have $K^{\sigma=1} \cong \mathbf{Q}_q(\pi)$ and $K \cong \mathbf{Q}_q(\pi) \bigotimes_{\mathbf{Q}_q} K_0$, where $\mathbf{Q}_q$ is the unique finite unramified extension of $\mathbf{Q}_p$ with residue class field $\mathbf{F}_q$ of q elements. Now we prove the assertion in the case where k is an arbitrary perfect field. Denote by $\widehat{K^{\mathrm{unr}}}$ the p-adic completion of K^{unr}. Then σ extends uniquely on $\widehat{K^{\mathrm{unr}}}$. Put $L = K(\widehat{K^{\mathrm{unr}}}^{\sigma=1})$ in $\widehat{K^{\mathrm{alg}}}$. Then L is finite over K and is included in $\widehat{K^{\mathrm{unr}}}$. Hence, L is a desired extension of K. □

From now on to the end of this paper we assume that K has a Frobenius σ.

We say a ring endomorphism σ on $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is a Frobenius on $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) if and only if it is the Frobenius σ on K and $\sigma(a) \equiv a^q (\mathrm{mod}\ m_{\mathcal{E}})$ (resp. $\sigma(a) \equiv a^q (\mathrm{mod}\ m_{\mathcal{E}^\dagger})$) for $a \in O_{\mathcal{E}}$. (resp. $a \in O_{\mathcal{E}}^\dagger$). A Frobenius σ on $\mathcal{E}$ is that on $\mathcal{E}^\dagger$ if and only if $\sigma(x) \in \mathcal{E}^\dagger$. One can easily see that a Frobenius on $\mathcal{E}^\dagger$ extends naturally on $\mathcal{R}$ by $\sum a_n x^n \mapsto \sum \sigma(a_n x^n)$ (adding coefficients in each term of x^n). We call this extension a Frobenius on $\mathcal{R}$. We say a ring endomorphism σ on S_K is a Frobenius if and only if it is the Frobenius σ on $\mathcal{E}$ with $x^{-q}\sigma(x) \in S_K$.

For a Frobenius σ on $\mathcal{E}$, put

$$\mu = \mu(x, \sigma) = \frac{\delta_x(\sigma(x))}{\sigma(x)}.$$

Then $|\mu|_G < 1$. One can easily see that σ is a Frobenius on $\mathcal{E}^\dagger$ (resp. S_K) if and only if $\mu \in \mathcal{E}^\dagger$ (resp. $\mu \in S_K$).

Let R be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K and let σ be a Frobenius on R.

LEMMA 2.4.2. — *If we regard R as an R-module through the Frobenius σ, then R is free of rank q.*

Define $\sigma : \omega_R \to \omega_R$ by $a\dfrac{dx}{x} \mapsto \mu\sigma(a)\dfrac{dx}{x}$. Then the diagram below

$$\begin{array}{ccc} R & \xrightarrow{d} & \omega_R \\ \sigma\downarrow & & \downarrow\sigma \\ R & \xrightarrow[d]{} & \omega_R \end{array}$$

is commutative. Equivalently, $\delta \circ \sigma = \mu\sigma \circ \delta$.

Let E be a finite separable extension of F and choose a lifting y of a uniformizer of E in $\mathcal{E}_E^\dagger$. Then the Frobenius σ on R extends uniquely on $\mathcal{R}_E$ and we also use the same notation σ for this extension. The Frobenius σ commutes with the derivation δ_x (resp. the action of $\mathrm{Gal}(E/F)$ if E/F is Galois).

2.5. Fix a Frobenius σ on $\mathcal{E}$ and put $\widetilde{\mathcal{E}} = K^{\sigma=1} \bigotimes_{(K^{\sigma=1})_0} W(F^{\mathrm{alg}})$. Then there is a unique homomorphism

$$i_\sigma : \mathcal{E} \to \widetilde{\mathcal{E}}$$

such that (i) $|u|_G = |i_\sigma(u)|$ for $u \in \mathcal{E}$, where $|\ |$ is the unique valuation on $\widetilde{\mathcal{E}}$ which is the extension of that on K, (ii) the map on residue class field induced by i_σ is the injection $F \subset F^{\mathrm{alg}}$ and (iii) $i_\sigma(\sigma(u)) = (\mathrm{id}_\Lambda \otimes \mathrm{Frob}^a)(i_\sigma(u))$. (See [TN1, 2.5.1].)

3. φ-∇-modules over $\mathcal{R}$.

Assume that the complete discrete valuation field K has a Frobenius σ from this section to the end of this paper.

3.1. Let R be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K.

DEFINITION 3.1.1. — (1) A pair (M, ∇) is called a ∇-module over R if and only if it satisfies the conditions as follows:

(i) *M is a free R-module of finite rank.*

(ii) $\nabla : M \to \omega_R \bigotimes_R M$ *is a K-connection over R.*

(2) *A morphism of ∇-modules over R is an R-linear homomorphism which commutes with connections.*

(3) *We denote by $\underline{\mathbf{M}}_R^\nabla$ the category of ∇-modules over R.*

For a ∇-module M over R and for a basis $\{e_1, e_2, \cdots, e_r\}$ of M, define a matrix $C_{M,e} \in M_r(R)$ by

$$\nabla(e_1, e_2, \cdots, e_r) = \frac{dx}{x} \otimes (e_1, e_2, \cdots, e_r) C_{M,e}.$$

The category $\underline{\mathbf{M}}_R^\nabla$ is additive. We can define tensor products and duals for ∇-modules by usual methods and, then, (R, d) is the unit object of the category. We often use the notation M instead of (M, ∇) for simplicity.

Since an $\mathcal{R}$-module of finite presentation with a connection is free over $\mathcal{R}$ by [Cr2, 6.1], we have

PROPOSITION 3.1.2. — *If $R = \mathcal{R}, \mathcal{E}$ or $\mathcal{E}^\dagger$, then the category $\underline{\mathbf{M}}_R^\nabla$ is an abelian category.*

Now fix a Frobenius σ on R.

DEFINITION 3.1.3. — (1) *A pair (M, φ) is called a φ-module over R with respect to σ if and only if it satisfies the conditions as follows:*

(i) *M is a free R-module of finite rank;*

(ii) *$\varphi : M \to M$ is a σ-linear homomorphism such that the induced R-linear map*

$$\varphi_\sigma : \sigma^* M \to M \qquad a \otimes m \mapsto a\varphi(m)$$

is an isomorphism. Here $\sigma^* M$ is the scalar extension of M by σ. We call φ Frobenius.

(2) *A morphism of φ-modules over R is an R-linear homomorphism which commutes with Frobenius.*

(3) *We denote by $\underline{\mathbf{M\Phi}}_{R,\sigma}$ the category of φ-modules over R with respect to σ.*

For a φ-module M over R and for a basis $\{e_1, e_2, \cdots, e_r\}$ of M, define a matrix $A_{M,e} \in M_r(R)$ by

$$\varphi(e_1, e_2, \cdots, e_r) = (e_1, e_2, \cdots, e_r) A_{M,e}.$$

The category $\underline{\mathbf{M\Phi}}_{R,\sigma}$ is additive. We can define tensor products and duals for φ-modules by usual methods and, then, (R, σ) is the unit object. We often use the notation M instead of (M, φ) for simplicity.

PROPOSITION 3.1.4. — *If $R = \mathcal{E}, \mathcal{E}^\dagger$ or S_K, then the category $\underline{\mathbf{M\Phi}}_{R,\sigma}$ is an abelian category.*

Proof. — In the case where $R = \mathcal{E}$ or $\mathcal{E}^\dagger$ the assertion is trivial. Let $R = S_K$. We have only to check that, for a morphism $\eta : M \to N$ of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}$, the cokernel of η is a free S_K-module, and then the rest is easy. Since S_K is a principal ideal domain, the torsion submodules of the cokernel of η is the form $\oplus_i S_K/(a_i)$ for some $a_i \in S_K$ with $|a_i|_G = 1$. Since σ is flat by (2.4.2), the induced S_K-linear map $\sigma^*(\oplus_i S_K/(a_i)) \to \oplus_i S_K/(a_i)$ is isomorphic. However, we have

$$\dim_K \sigma^*\Big(\bigoplus_i S_K/(a_i)\Big) = \dim_K \bigoplus_i S_K/(\sigma(a_i)) = q \dim_K \bigoplus_i S_K/(a_i).$$

Hence, $N/\eta(M)$ is a free S_K-module. □

We recall the notion of slopes for Frobenius structures. Denote by the same notation v_p the additive valuation of $\widetilde{\mathcal{E}}$ which is the unique extension of the valuation on K.

DEFINITION 3.1.5. — (1) *For an object (M, φ) of $\underline{\mathbf{M\Phi}}_{\mathcal{E},\sigma}$ (resp. $\underline{\mathbf{M\Phi}}_{\mathcal{E}^\dagger,\sigma}$), we define the slopes of (M, φ) by those of $(\widetilde{\mathcal{E}} \bigotimes_R M, \varphi)$ as φ-spaces on $\widetilde{\mathcal{E}}$ (resp. by those of $(\mathcal{E} \bigotimes_{\mathcal{E}^\dagger} M, \varphi)$) which are measured using the valuation $\frac{1}{a} v_p$. Here $p^a = q$. We denote by Newton(M) the Newton polygon of slopes of M.*

(2) *For an object (M, φ) of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}$, we define the slopes of M for the Frobenius structure at the generic point by those of $\mathcal{E} \bigotimes_{S_K} M$ and the slopes of M for the Frobenius structure at the special point by those of $(\widehat{K^{\mathrm{unr}}} \bigotimes_S M, \overline{\varphi})$ as φ-spaces on $\widehat{K^{\mathrm{unr}}}$, where $S \to K$ (resp. $\overline{\varphi}$) is the natural reduction modulo x (resp. φ modulo xM). We denote by Newton$_\eta(M)$ (resp. Newton$_s(M)$) the Newton polygon of slopes of M at the generic point (resp. at the special point).*

Since $\mathcal{E}$ is p-adically complete, we have

PROPOSITION 3.1.6. — *Let M be an object of $\underline{\mathbf{M}\Phi}_{\mathcal{E},\sigma}$. Then there is an increasing filtration $\{S_\gamma M\}_{\gamma\in\mathbf{Q}}$ of M such that each $S_\gamma M$ is an object of $\underline{\mathbf{M}\Phi}_{\mathcal{E},\sigma}$ and, for a sufficiently small positive rational number $\epsilon << 1$, $S_\gamma M/S_{\gamma-\epsilon}M$ is pure of slope γ.*

By [Ka1, 2.6.3] we have

PROPOSITION 3.1.7. — *Let M be an object of $\underline{\mathbf{M}\Phi}_{S_K,\sigma}$. Assume that the Newton Polygon both at the generic point and at the special point coincide with each other, that is, $\mathrm{Newton}_\eta(M) = \mathrm{Newton}_s(M)$. Then there is an increasing filtration $\{S_\gamma M\}_{\gamma\in\mathbf{Q}}$ of M such that each $S_\gamma M$ is an object of $\underline{\mathbf{M}\Phi}_{S_K,\sigma}$ and, for a sufficiently small positive rational number $\epsilon << 1$, $S_\gamma M/S_{\gamma-\epsilon}M$ is pure of slope γ at both points.*

3.2. Now we define φ-∇-modules over R.

DEFINITION 3.2.1. — (1) *A triple (M,φ,∇) is called a φ-∇-module over R with respect to σ if and only if it satisfies the conditions as follows:*

(i) *(M,∇) is a ∇-module over R;*

(ii) *(M,φ) is a φ-module over R with respect to σ;*

(iii) *the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & \omega_R \bigotimes_R M \\ \varphi\downarrow & & \downarrow \sigma\otimes\varphi \\ M & \xrightarrow[\nabla]{} & \omega_R \bigotimes_R M \end{array}$$

is commutative.

(2) *A morphism of φ-modules over R is an R-linear homomorphism which commutes with connections and Frobenius.*

(3) *We denote by $\underline{\mathbf{M}\Phi}^{\nabla}_{R,\sigma}$ the category of φ-∇-modules over R with respect to σ.*

For a φ-∇-module M and for a basis $\{e_1, e_2, \cdots, e_r\}$, the condition (3.2.1)(1)(iii) is equivalent to the relation

$$\delta_x(A_{M,e}) + C_{M,e}A_{M,e} = \mu(x,\sigma)A_{M,e}\sigma(C_{M,e}). \tag{3.2.2}$$

We can define tensor products and duals for φ-∇-modules by usual methods and, then, (R,σ,d) is the unit object of the category. We often use the notation M instead of (M,φ,∇) for simplicity.

By Proposition (3.1.2) and Proposition (3.1.4) we have

THEOREM 3.2.3. — *The category $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma}$ is an abelian category with tensor products and duals.*

By the extension of scalar there are natural functors

$$\mathcal{C}_{S_K} \to \mathcal{C}_{\mathcal{E}^\dagger} \begin{array}{l} \nearrow \ \mathcal{C}_{\mathcal{R}} \\ \searrow \ \mathcal{C}_{\mathcal{E}} \end{array}$$

of categories, where $\mathcal{C}$ is either $\underline{\mathbf{M}}^{\nabla}$, $\underline{\mathbf{M\Phi}}$ or $\underline{\mathbf{M\Phi}}^{\nabla}_{,\sigma}$. For an object M of $\mathcal{C}_{\mathcal{R}}$, a sub $\mathcal{E}^\dagger$-module (resp. a sub S_K-module, resp. a sub K-space) L is an $\mathcal{E}^\dagger$-lattice (an S_K-lattice, a K-lattice) if and only if $M \cong \mathcal{R} \bigotimes_{\mathcal{E}^\dagger} L$ (resp. $M \cong \mathcal{R} \bigotimes_{S_K} L$, resp. $M \cong \mathcal{R} \bigotimes_{K} L$) and $(L, \varphi|_L, \nabla|_L)$ belongs to $\mathcal{C}_{\mathcal{E}^\dagger}$ (resp. $(L, \varphi|_L, \nabla|_L)$ belongs to $\mathcal{C}_{S_K}$, resp. L is stable under φ and ∇).

3.3. In this subsection we define inverse images and direct images of φ-∇-modules.

Let $f : F \to E$ be a finite separable extension in F^{sep} and let R_F be either $\mathcal{R}_F(= \mathcal{R})$, $\mathcal{E}_F(= \mathcal{E})$ or $\mathcal{E}^\dagger_F(= \mathcal{E}^\dagger)$. Then the extension f determines a unique finite and flat extension R_E over R_F and denote by the same notation f the extension $R_F \to R_E$. Fix a Frobenius σ on R_F. Then σ extends on R_E and $\omega_{R_E} \cong R_E \bigotimes_R \omega_R$.

Let $\mathcal{C}$ be either the category $\underline{\mathbf{M}}^{\nabla}$, $\underline{\mathbf{M\Phi}}_{\sigma}$ or $\underline{\mathbf{M\Phi}}^{\nabla}_{\sigma}$. Define an inverse image functor

$$f^* : \mathcal{C}_{R_F} \to \mathcal{C}_{R_E}$$

as follows. For an object M of $\mathcal{C}_{R_F}$, put $f^*M = (M_E, \varphi_E, \nabla_E)$ to be

$$\begin{aligned} M_E &= R_E \bigotimes_R M \\ \varphi_E &= \sigma \otimes \varphi \\ \nabla_E &= d \otimes \mathrm{id}_M + \mathrm{id}_{R_E} \otimes \nabla. \end{aligned}$$

One can easily check that f^*M is an object of $\mathcal{C}_{R_E}$. By the definition f^* is faithful and exact.

Define a direct image functor

$$f_* : \mathcal{C}_{R_E} \to \mathcal{C}_{R_F}$$

as follows. For an object M of $\mathcal{C}_{R_E}$, put $f_*M = (M_F, \varphi_F, \nabla_F)$ to be

$$\begin{aligned} M_F &= M \text{ (we regard it as an } R\text{-module)} \\ \varphi_F &= \varphi \\ \nabla_F &= \nabla : M_F \to \omega_{R_E} \bigotimes_{R_E} M \cong \omega_R \bigotimes_R M_F. \end{aligned}$$

LEMMA 3.3.1. — *For an object M of $\mathcal{C}_{R_E}$, f_*M belongs to $\mathcal{C}_{R_F}$.*

Proof. — It is sufficient to check that the natural map from $\sigma^*(M_F)$ (a pull back by $\sigma : R_F \to R_F$) to σ^*M (a pull back by $\sigma : R_E \to R_E$) is bijective. Since M is free over $\mathcal{R}_E$, it is enough to prove that the natural map $\sigma^*((\mathcal{R}_E)_F) \to \sigma^*\mathcal{R}_E$ is bijective. The following Lemma (3.3.2) implies the assertion by (2.2.3).

LEMMA 3.3.2. — *Under the notation as above, the natural map $\sigma^*((\mathcal{E}_E^\dagger)_F) \to \sigma^*\mathcal{E}_E^\dagger$ is bijective.*

Proof. — Denote by σ_q the q-th power map. Consider the perfections both of F and E, and dimensions over F, then $\sigma_q^*(E_F) \to \sigma_q^*(E)$ is injective, hence bijective. The assertion holds by Nakayama's Lemma. □

We show some properties of inverse images and direct images.

LEMMA 3.3.3. — *Let $f : F \to E_1$ and $g : E_1 \to E_2$ be finite separable extensions over F in F^{sep}. Then, we have $(gf)^* = g^*f^*$ and $(gf)_* = f_*g_*$.*

PROPOSITION 3.3.4. — (1) *The functor f^* (resp. f_*) commutes with natural functors $\mathcal{C}_{\mathcal{E}^\dagger} \to \mathcal{C}_{\mathcal{R}}$ and $\mathcal{C}_{\mathcal{E}^\dagger} \to \mathcal{C}_{\mathcal{E}}$.*

(2) *The functor f^* preserves tensor products and duals.*

(3) *f_* is a right adjoint of f^* and f^* is a left adjoint of f_*.*

We study the behavior of Newton polygons of φ-modules under an inverse image functor (resp. a direct image functor). By the definition of Newton polygon we have

PROPOSITION 3.3.5. — *Let R_F be either $\mathcal{E}_F$ or $\mathcal{E}_F^\dagger$. The Newton polygon of φ-modules is preserved by the inverse image functor f^*. In other words, we have*

$$\mathrm{Newton}(f^*M) = \mathrm{Newton}(M)$$

for any object M of $\underline{\mathbf{M\Phi}}_{R_F}$.

PROPOSITION 3.3.6. — *Let R_F be either $\mathcal{E}_F$ or $\mathcal{E}_F^\dagger$. For an object M of $\underline{\mathbf{M\Phi}}_{R_E,\sigma}$, the Newton polygon* $\mathrm{Newton}(f_*M)$ *of f_*M is $[E:F]$ times* $\mathrm{Newton}(M)$. *In other words, the rank of the slope γ-part of f_*M is $[E:F]$ times the rank of the slope γ-part of M.*

Proof. — One may assume that the extension E over F is Galois by (3.3.5). If we denote by M_τ a scalar extension of M by an $\mathcal{R}_F$-embedding $\tau : R_E \to \widetilde{\mathcal{E}}$, then we have

$$\widetilde{\mathcal{E}} \bigotimes_{R_F} f_*M \cong \bigoplus_{\tau \in \mathrm{Hom}\mathcal{R}_F(\mathcal{R}_E, \widetilde{\mathcal{E}})} M_\tau$$

as φ-modules over $\widetilde{\mathcal{E}}$. Since the action of Galois commutes with Frobenius, we obtain the assertion. □

3.4. Let R be either $\mathcal{E}$, $\mathcal{E}^\dagger$ or S_K. Let M be an object of $\underline{\mathbf{M}}_R^\nabla$ and $\{e_1, e_2, \cdots, e_r\}$ a basis of M. For an element $m = a_1e_1 + \cdots + a_re_r$, define

$$||m||_{M,e} = \max_i |a_i|_G.$$

Then $||\ ||_{M,e}$ is a norm on M which is compatible with the norm $|\ |_G$ of R. The topology which is determined by the norm $||\ ||_{M,e}$ is independent of the choice of the basis of M.

Define a K-linear map $\nabla^{[n]} : M \to M$ by

$$\nabla^{[0]} = \mathrm{id}_M \quad \text{and} \quad \nabla^{[n+1]} = \Big(\nabla(x\frac{d}{dx}) - n\Big)\nabla^{[n]}.$$

for any non-negative integer n. Here the map $\nabla\Big(x\dfrac{d}{dx}\Big)$ is defined by $\nabla(m) = \dfrac{dx}{x} \otimes \nabla\Big(x\dfrac{d}{dx}\Big)(m)$ for $m \in M$. By Leibniz's rules we have

LEMMA 3.4.1. — $\nabla^{[n]}(am) = \sum_{i+j=n} \dfrac{n!}{i!j!}\delta^{[i]}(a)\nabla^{[j]}(m)$ *for $a \in R$, $m \in M$.*

Let M be an object of $\underline{\mathbf{M}}_R^{\nabla}$. Consider the conditions (C) and (OC) as follows:

$$\text{(C)} \qquad \left\| \frac{1}{n!} \nabla^{[n]}(m) \right\|_{M,e} \eta^n \to 0 \quad (n \to \infty)$$

for any $m \in M$ and any number $0 < \eta < 1$;

$$\text{(OC)} \qquad \sum_{n=0}^{\infty} \frac{w^n}{n!} \nabla^{[n]}(m) \text{ converges in } M$$

for any $m \in M$ and for any $w \in R$ with $|w|_G < 1$. If $R = \mathcal{E}$ and S_K, the condition (C) implies (OC) since R is complete in the p-adic topology. In the case of $\mathcal{E}^\dagger$, however, the condition (OC) is delicate since $\mathcal{E}^\dagger$ is not complete.

PROPOSITION 3.4.2. — *Any object M of $\underline{\mathbf{M\Phi}}_{R,\sigma}^{\nabla}$ satisfies the condition (C).*

Proof. — Fix a positive integer k with $\eta < p^{-1/(p^k(p-1))}$. By (3.4.1) we have only to prove the condition (C) for one basis of M. Choose a basis $\{e_1, e_2, \cdots, e_r\}$ of M such that $|C|_G \leqslant p^{-(p^k-1)/(p-1)}$, where we denote $C = C_{M,e}$. We can choose such a basis after changing a basis by $(e_1, e_2, \cdots, e_r) \mapsto (e_1, e_2, \cdots, e_r)A\sigma(A)\cdots\sigma^n(A)$ for a sufficiently large n, where $A = A_{M,e}$. Define matrixes $C^{[n]} \in M_r(R)$ by $\nabla^{[n]}(e_1, e_2, \cdots, e_r) = (e_1, e_2, \cdots, e_r)C^{[n]}$. Since $|C^{[n+1]} - (\delta_x(C^{[n]}) - nC^{[n]})|_G \leqslant |C^{[n]}|_G p^{-(p^k-1)/(p-1)}$, one can easily check that $|C^{[n]}|_G \leqslant p^{-(i+1)(p^k-1)/(p-1)}$ for $n = ip^k + j$ $(i \geqslant 0, 0 < j \leqslant p^k)$. Note that $v_p(n!) < n/(p-1)$ for any positive integer n. Since

$$\begin{aligned}
&(i+1)(p^k-1)/(p-1) + n/(p^k(p-1)) - v_p(n!) \\
&\quad = ((p^k-1)/(p-1) - v_p(j!)) + (i/(p-1) - v_p(i!)) + j/(p^k(p-1)) > 0,
\end{aligned}$$

we have $|C^{[n+1]}/n!|_G \eta^n \to 0$ if $n \to \infty$. □

COROLLARY 3.4.3. — *The connection of objects in* $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma}$ *is topologically nilpotent.*

Define a map $\alpha_N : \mathcal{E} \to \mathbf{R}$ by

$$\alpha_N(\sum a_n x^n) = \sup_{n \leqslant N} |a_n|$$

for any integer N. Note that (i) $a \in \mathcal{E}^\dagger$ if and only if $\alpha_N(a) \leqslant c\xi^{-N}$ for any integer N for some $c > 0$ and $0 < \xi < 1$ and (ii) if $\alpha_N(a) \leqslant c_a\xi^{-N}$ and $\alpha_N(b) \leqslant c_b\xi^{-N}$, then $\alpha_N(ab) \leqslant c_a c_b \xi^{-N}$

PROPOSITION 3.4.4. — *Any object* M *of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^\dagger,\sigma}$ *satisfies the condition (OC).*

Proof. — Keep the notation as in the proof of (3.4.2). By (3.4.1) we have only to prove the condition (OC) for one basis of M. Choose a positive integer k, a basis $\{e_1, e_2, \cdots, e_r\}$ of M and a real number $0 < \xi < 1$ such that $\alpha_N(w) < p^{-1/(p^k(p-1))}\min\{\xi^{-N}, 1\}$ and $\alpha_N(C) \leqslant p^{-(p^k-1)/(p-1)}\min\{\xi^{-N}, 1\}$ for any integer N. Then one can easily check that $\alpha_N(C^{[n]}) \leqslant p^{-(i+1)(p^k-1)/(p-1)}\min\{\xi^{-N}, 1\}$ for $n = ip^k + j$ $(i \geqslant 0, 0 < j \leqslant p^k)$. By the calculation of valuations as in the proof of (3.4.2) we have $\alpha_N(C^{[n]}w^n/n!) \leqslant \min\{\xi^{-N}, 1\}$. Since $\sum_{n=0}^{\infty} C^{[n]}w^n/n!$ is convergent in $M_r(\mathcal{E})$ by (3.4.2), $\sum_{n=0}^{\infty} C^{[n]}w^n/n!$ is convergent in $M_r(\mathcal{E}^\dagger)$. □

Let σ_1 and σ_2 be Frobenius on R. For an object M of $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_2}$, define an R-linear homomorphism

$$\epsilon_{\sigma_1,\sigma_2} : \sigma_1^* M \to \sigma_2^* M$$

by

$$\epsilon_{\sigma_1,\sigma_2}(a \otimes m) = a \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\sigma_1(x)}{\sigma_2(x)} - 1\right)^n \otimes \nabla^{[n]}(m).$$

Since one knows the identity

$$\sigma_1(a) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\sigma_1(x)}{\sigma_2(x)} - 1\right)^n \sigma_2(\delta^{[n]}(a))$$

for any $a \in \mathcal{E}$, the map $\epsilon_{\sigma_1,\sigma_2}$ is well-defined and continuous by (3.4.2) and (resp. (3.4.3)). By easy calculations we have

LEMMA 3.4.5. — *Let σ_1, σ_2 and σ_3 be Frobenius on R. Then*

(i) $\epsilon_{\sigma_1,\sigma_1} = \mathrm{id}$;

(ii) $\epsilon_{\sigma_1,\sigma_3} = \epsilon_{\sigma_1,\sigma_2}\epsilon_{\sigma_2,\sigma_3}$.

Define a functor

$$\tilde{\epsilon}_{\sigma_1,\sigma_2} : \underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_2} \to \underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_1}$$

by

$$(M, \varphi, \nabla) \mapsto (M, \varphi_{\sigma_2} \circ \epsilon_{\sigma_1,\sigma_2}|_{1\otimes M}, \nabla).$$

LEMMA 3.4.6. — *Under the notation as above, the triple $(M, \varphi_{\sigma_2} \circ \epsilon_{\sigma_1,\sigma_2}|_{1\otimes M}, \nabla)$ is an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_1}$.*

Proof. — Put $\varphi_1 = \varphi_{\sigma_2} \circ \epsilon_{\sigma_1,\sigma_2}|_{1\otimes M}$. By (3.4.5) $\epsilon_{\sigma_1,\sigma_2}$ is isomorphic, hence $(\varphi_1)_{\sigma_1}$ is isomorphic. An easy calculation implies the commutative of φ_1 and ∇. □

LEMMA 3.4.7. — *Let σ_1, σ_2 and σ_3 be Frobenius on R. Then*

(i) $\tilde{\epsilon}_{\sigma_1,\sigma_1} = \mathrm{id}$;

(ii) $\tilde{\epsilon}_{\sigma_1,\sigma_3} = \tilde{\epsilon}_{\sigma_1,\sigma_2}\tilde{\epsilon}_{\sigma_2,\sigma_3}$.

LEMMA 3.4.8. — (1) *The functor $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ commutes with tensor products and duals.*

(2) *For a finite separable extension $f : F \to E$ in F^{sep}, the functor $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ commutes with f^* and f_*.*

PROPOSITION 3.4.9. — *Let σ_1 and σ_2 be Frobenius on R and let M be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma_2}$. Then the slopes of M for Frobenius structures coincide with those of $\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)$. In other words,*

$$\begin{array}{rl} \mathrm{Newton}(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \mathit{Newton}(M) \\ (\mathit{resp.}\ \mathrm{Newton}_\eta(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \mathrm{Newton}_\eta(M) \\ \mathrm{Newton}_s(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \mathrm{Newton}_s(M)) \end{array}$$

if $R = \mathcal{E}$ or $\mathcal{E}^\dagger$ (resp. if $R = S_K$).

Proof. — We have only to prove the assertion in the case where $R = \mathcal{E}$ and M is pure of slopes 0 by (3.1.6). We can choose a suitable basis of M

with $A_{M,e} \in GL_r(O_{\mathcal{E}})$ and $\epsilon_{\sigma_1,\sigma_2}(e_i) \equiv e_i \pmod{m_{\mathcal{E}}}$. Therefore, we have the assertion. □

Now we have obtained

THEOREM 3.4.10. — *The category* $\underline{\mathbf{M\Phi}}^{\nabla}_{R,\sigma}$ *is independent of the choice of Frobenius up to canonical equivalence.*

4. Quasi-unipotent φ-∇-modules.

4.1. Fix a Frobenius φ on $\mathcal{R}$. We define quasi-unipotent φ-∇-modules.

DEFINITION 4.1.1. — (1) A ∇-module M (resp. a φ-∇-module M) over $\mathcal{R}$ is unipotent if and only if M is a successive extension of the unit object $(\mathcal{R}, d)$ (resp. (M, ∇) is a unipotent ∇-module).

(2) A ∇-module M (resp. a φ-∇-module M) over $\mathcal{R}$ is quasi-unipotent if and only if there exists a finite separable extension $f : F \to E$ such that the inverse image f^*M is unipotent.

(3) We denote by $\underline{\mathbf{M}}^{\nabla,qu}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma}$) the full subcategory of $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ (resp. $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$) whose objects consist of quasi-unipotent ∇-modules (resp. φ-∇-modules).

By the standard arguments we have

PROPOSITION 4.1.2. — (1) *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence in $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ *(resp.* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$*).* M_2 *is quasi-unipotent if and only if both* M_1 *and* M_3 *are quasi-unipotent.*

(2) *The category* $\underline{\mathbf{M}}^{\nabla,qu}_{\mathcal{R}}$ *(resp.* $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma}$*) is an abelian subcategory of* $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ *(resp.* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$*) with tensor products and duals.*

PROPOSITION 4.1.3. — *Let* $f : F \to E$ *be a finite separable extension in* F^{sep}.

(1) *Let* M *be an object of* $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}}$ *(resp.* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$*).* M *is quasi-unipotent if and only if* f^*M *is quasi-unipotent.*

(2) *Let* M *be an object of* $\underline{\mathbf{M}}^{\nabla}_{\mathcal{R}_E}$ *(resp.* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}_E,\sigma}$*).* M *is quasi-unipotent if and only if* f_*M *is quasi-unipotent.*

Proof. — The assertion on inverse images is easy. In the case of direct images we may assume that the extension E is Galois over F by (1) and (4.1.2). For $\tau \in \mathrm{Gal}(E/F)$, denote by M_τ the ∇-module (resp. φ-∇-module) whose $\mathcal{R}_E$-action is defined by $(a, m) \mapsto \tau(a)m$ for $a \in \mathcal{R}_E$ and $m \in M$. Then $f^* f_* M \cong \bigoplus_{\tau \in \mathrm{Gal}(E/F)} M_\tau$. The assertion (2) easily follows from the isomorphism. □

Example 4.1.4. — (1) Any φ-∇-module M over $\mathcal{R}$ of rank one is quasi-unipotent. Indeed, if we fix a base e of M, then $A_{M,e} \in \mathcal{R}^\times = (\mathcal{E}^\dagger)^\times$. By the relation (3.2.2) we have $C_{M,e} \in \mathcal{E}^\dagger$. Hence, M has an $\mathcal{E}^\dagger$-lattice and it is quasi-unipotent by [Cr1, 4.11] (or (2) below).

(2) Any φ-∇-module over $\mathcal{R}$ which has an etale $\mathcal{E}^\dagger$-lattice is quasi-unipotent [TN1, 4.2.6]. ("Etale" means that all slopes of Frobenius are 0.)

4.2. We show some properties of unipotent φ-∇-modules.

PROPOSITION 4.2.1. — (1) *An object in* $\underline{\mathbf{M}\Phi}^{\nabla, qu}_{\mathcal{R},\sigma}$ *has an* $\mathcal{E}^\dagger$*-lattice.*

(2) *Assume that* σ *is Frobenius on* S_K. *An object of* $\underline{\mathbf{M}\Phi}^{\nabla}_{\mathcal{R},\sigma}$ *is unipotent if and only if it has an* S_K*-lattice.*

Remark 4.2.2. — The $\mathcal{E}^\dagger$-lattice (resp. the S_K-lattice) is not unique in Proposition (4.2.1).

Proposition (4.2.1)(1) (resp. (2)) follows from Lemma (4.2.5) (resp. Lemmas (4.2.6) and (4.2.7)) below.

Put $u \in (\mathcal{E}^\dagger)^\times$ to be $\sigma(x) = x^q u$ for the Frobenius σ. Then $|u-1|_G < 1$ and one can define $\log(u)$ in $\mathcal{E}^\dagger$. If σ is a Frobenius on S_K, then $\log(u)$ belongs to S_K. Note that $\mu = \mu(x,\sigma) = \dfrac{\delta_x(\sigma(x))}{\sigma(x)} = q + \dfrac{\delta_x(u)}{u}$ and $\delta_x(\log(u)) = \dfrac{\delta_x(u)}{u}$.

LEMMA 4.2.3. — *Let* $C_1 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$ *(resp.* $C_2 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$*) be a matrix of degree* r_1 *(resp.* r_2*). A matrix* $Q \in M_{r_1,r_2}(\mathcal{R})$ *(resp.* $Q \in M_{r_1,r_2}(K[[x]])$*) satisfies the relation*

$$\delta_x(Q) + C_1 Q = \mu Q C_2$$

if and only if

$$Q=\begin{cases}\begin{pmatrix}0 & \cdots & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{r_1}\\ & & & & \ddots & \ddots & \vdots\\ & & & & & \ddots & q^{r_1-2}\alpha_2\\ 0 & & & & & & q^{r_1-1}\alpha_1\end{pmatrix} & \text{if } r_1\leqslant r_2\\ \begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{r_2}\\ 0 & \ddots & \ddots & \vdots\\ & \ddots & \ddots & q^{r_2-2}\alpha_2\\ & & \ddots & q^{r_2-1}\alpha_1\\ 0 & & & 0\end{pmatrix} & \text{if } r_1\geqslant r_2\end{cases}$$

with $\alpha_1 = \beta_1, \alpha_2 = \beta_1\log(u)+\beta_2,\cdots,\alpha_r = \dfrac{\beta_1}{(r-1)!}\log^{r-1}(u) + \dfrac{\beta_2}{(r-2)!}\log^{r-2}(u)+\cdots+\beta_r$ *for some* $\beta_i\in K$.

Proof. — We use Lemma (2.3.1) to show the assertion. Assume that $Q=(q_{i,j})$ is a solution of the differential equation above.

First we prove that $q_{r_1,j}=0$ $(1\leqslant j<r_2)$ and q_{r_1,r_2} is contained in K. Since $\delta_x(q_{r_1,1})=0$, $q_{r_1,1}$ is contained in K. Then the identity $\delta_x(q_{r_1,2})=\mu q_{r_1,1}$ implies that $q_{r_1,1}=0$ and $q_{r_1,2}$ is contained in K. Repeating these, we proved the assertion.

Secondly we prove that $q_{i,1}=0$ $(2\leqslant i)$ and $q_{1,1}$ is contained in K. Assume that $q_{i+1,1}=\cdots=q_{r_2,1}=0$. Since $\delta_x(q_{i,1})+q_{i+1,1}=0$, $q_{i,1}$ is contained in K. So the assertion follows from $\delta_x(q_{i-1,1})+q_{i,1}=0$.

Thirdly we prove that, if $q_{i,n+i}$ is a linear combination of $1,\log(u)$, $\log^2(u),\cdots$ over K and if $q^{-i+1}q_{i,n+i}$ does not depend on i when n is fixed, then $q_{i,n+1+i}$ is a linear combination of $1,\log(u),\log^2(u),\cdots$ over K and $q^{-i+1}q_{i,n+1+i}$ is independent on i. The former assertion holds by the equation $\delta_x(q_{i,j})+q_{i+1,j}=\mu q_{i,j-1}$ $(i<r_1, j>1)$ and $\mu = q+\dfrac{\delta_x(u)}{u}$ and by two assertions above. Moreover $q^{-i+1}q_{i,n+1+i}$ does not depend on i up to constant terms. (When $q_{i,1}$ (resp. $q_{r_1,j}$) appears, $q^{-i+1}q_{i,n+1+i}=0$ and $q^{i-1}q_{i,n+1+i}$ does not depend on i up to constant terms.) Since

$$\begin{aligned}\delta_x(q_{i,n+1+(i+1)}) &= \mu q_{i,n+1+i} - q_{i+1,n+1+(i+1)}\\ &= \text{constant term } + \frac{\delta_x(u)}{u}q_{i,n+1+i},\end{aligned}$$

the constant term must vanish. Hence, the later assertion also holds.

Finally we have got the relation $\delta_x(q_{i,r_2}) = \mu q_{i,r_2-1} - q_{i+1,r_2} = \dfrac{\delta_x(u)}{u} q_{i,r_2-1}$. Therefore, Q has a form as in the assertion. The converse can be easily checked. □

Let $f : F \to E$ be a finite separable extension in F^{sep}. Denote by x (resp. y) a lift of uniformizer *of* F (resp. E) in $\mathcal{E}^\dagger = \mathcal{E}^\dagger_F$ (resp. $\mathcal{E}^\dagger_E$)). Using similar arguments as in Lemma (4.2.3) and by Lemma (2.3.1) we obtain

LEMMA 4.2.4. — *Under the notation as above, let* $C_1 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$ *(resp.* $C_2 = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}$*) be a matrix of degree* r_1 *(resp.* r_2*). A matrix* $Q \in M_{r_1,r_2}(\mathcal{R}_E)$ *satisfies the differential equation*

$$\delta_x(Q) + C_1 Q = Q C_2$$

for the derivation $\delta_x = x\dfrac{d}{dx}$ *if and only if*

$$Q = \begin{cases} \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{r_1} \\ & & & & \ddots & \ddots & \vdots \\ & & & & & \ddots & \alpha_2 \\ \mathbf{0} & & & & & & \alpha_1 \end{pmatrix} & \text{if } r_1 \leqslant r_2 \\ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{r_2} \\ 0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \alpha_2 \\ & & \ddots & \alpha_1 \\ \mathbf{0} & & & 0 \end{pmatrix} & \text{if } r_1 \geqq r_2 \end{cases}$$

for some $\alpha_i \in K_E$.

COROLLARY 4.2.5. — (1) *Under the notation as above, assume furthermore that* M *is a unipotent* ∇*-module over* $\mathcal{R}_E$*. Then there is a basis* $\{e_1, e_2, \cdots, e_r\}$ *of* M *such that, if we define a matrix* $C_{M,e,x} \in M_r(\mathcal{R}_E)$ *by*

$$\nabla(e_1, e_2, \cdots, e_r) = \frac{dx}{x} \otimes (e_1, e_2, \cdots, e_r) C_{M,e,x},$$

$$C_{M,e,x} = \begin{pmatrix} C_1 & & & \mathbf{0} \\ & C_2 & & \\ & & \ddots & \\ \mathbf{0} & & & C_s \end{pmatrix} \quad \text{with} \quad C_i = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}.$$

Moreover, if M has a σ-linear homomorphism $\varphi : M \to M$ which is compatible with the connection and if L_E is an $\mathcal{E}_E^\dagger$-subspace which is generated by $\{e_1, e_2, \cdots, e_r\}$, then L_E is stable under φ.

(2) Let M be an object of $M_{\mathcal{R}}^{\nabla, qu}$ and let $f : F \to E$ be a finite separable extension in F^{sep} such that f^*M is unipotent. If $\{e_1, e_2, \cdots, e_r\}$ is a basis of f^*M as in (1) and if we denote by L_E the $\mathcal{E}_E^\dagger$-subspace which is generated by $\{e_1, e_2, \cdots, e_r\}$, then L_E is stable under the action of $\mathrm{Gal}(E/F)$.

Proof. — (1) We use induction on r. Let $\{e_1, e_2, \cdots, e_{r-1}, e'\}$ be a basis of M such that $C_{M,e',x} = \begin{pmatrix} C_{11} & C_{12} \\ 0 & 0 \end{pmatrix}$ with C_{11} as in the assertion and some $C_{12} \in \mathcal{R}^{r-1}$. Using (2.3.1), one can get a matrix of type $Q = \begin{pmatrix} 1 & Q_{12} \\ 0 & 1 \end{pmatrix}$ with $Q_{12} \in \mathcal{R}^{r-1}$ such that $(e_1, e_2, \cdots, e_{r-1}, e')Q$ is the desired basis. Let $\{e_1, e_2, \cdots, e_r\}$ be a basis as in the former assertion. Then we have $\delta_x(A_{M,e}) + C_{M,e,x}A_{M,e} = \mu(x, \sigma)A_{M,e}C_{M,e,x}$ by the commutativity of Frobenius and connection. By (4.2.3) there is a matrix $A_x \in GL_r(\mathcal{E}^\dagger)$ which satisfies the relation $\delta_x(A_x) + C_{M,e,x}A_x = \mu(x, \sigma)A_xC_{M,e,x}$. Hence we have

$$\delta_x(A_{M,e}{A_x}^{-1}) + C_{M,e,x}A_{M,e}{A_x}^{-1} = A_{M,e}{A_x}^{-1}C_{M,e,x}$$

and $A_{M,e}{A_x}^{-1} \in GL_r(K_E)$ by (4.2.4). The assertion (2) easily follows from the commutativity of the Galois action and the connection and by (4.2.4). □

Let M be an object in $\underline{\mathbf{M}}_{S_K}^{\nabla}$. Put $\overline{M} = M/xM$ (resp. $N_M = \overline{\nabla\left(x\frac{d}{dx}\right)}$ to be the induced K-linear map). By the relation (3.2.2) we have

LEMMA 4.2.6. — *For any object M of $\underline{\mathbf{M\Phi}}_{S_K,\sigma}^{\nabla}$, the K-linear map N_M is nilpotent.*

LEMMA 4.2.7. — *Let M be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{S_K,\sigma}$ and let $\{e_1, e_2, \cdots, e_r\}$ be a basis of M. Put C_0 to be the representation matrix of the K-linear map N_M for the basis $\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_r\}$. Then there exists a solution $Q \in 1_r + xM_r(K[[x]])$ of the system of linear differential equations*

$$\delta_x(Q) + C_{M,e}Q = QC_0$$

such that Q belongs to $GL_r(\mathcal{R})$.

Proof. — Since all proper values of C_0 are 0 (4.2.6), one can uniquely solve the system of differential equation above in $M_r(K[[x]])$ with $Q \pmod{xK[[x]]} = 1_r$. Put $A_0 = Q^{-1}A\sigma(Q)$. Then the pair (A_0, C_0) satisfies the relation (3.2.2.). Hence, A_0 is contained in $GL_r(S_K)$ by (4.2.3). If we denote by γ the radius of convergence of Q, then $0 < \gamma \leqslant 1$ and the radius of convergence of $\sigma(Q)$ is γ^q. By the relation $QA_0 = A\sigma(Q)$ we have

$$\min\{\gamma, 1\} = \min\{\gamma^q, 1\}.$$

Hence, $\gamma = 1$ and Q is contained in $M_r(\mathcal{R})$. Consider the dual object $M^\vee$ of M and the dual basis $\{e^\vee{}_1, e^\vee{}_2, \cdots, e^\vee{}_r\}$. Then there is a matrix $Q^\vee \in M_r(K[[x]]) \bigcap M_r(\mathcal{R})$ with $Q^\vee \pmod{xK[[x]]} = 1_r$ and $\delta_x(Q^\vee) - {}^tC_{M,e}Q^\vee = -Q^{\vee t}C_0$. So we have

$$\delta_x(Q^\vee Q) + C_0Q^\vee Q = Q^\vee QC_0.$$

Therefore Q is invertible by (4.2.4). □

4.3. Let K' be an extension of K which is complete under the extension of the valuation of K and put $\mathcal{R}_{K'} = \mathcal{R}_{K',x}$ to be an extension of $\mathcal{R}$. Denote by $g^*_{K'/K} : \underline{\mathbf{M}}^{\nabla}_{\mathcal{R}} \to \underline{\mathbf{M}}^{\nabla}_{\mathcal{R}_{K'}}$ the natural functor which is defined by the scalar extension. If the Frobenius σ on K extends on K', then the Frobenius σ on $\mathcal{R}$ extends on $\mathcal{R}_{K'}$. (The extension of the Frobenius on $\mathcal{R}_{K'}$ is uniquely determined by the extension of the Frobenius on K'.) In this case there is a natural functor $g^*_{K'/K} : \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}} \to \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}_{K'}}$.

PROPOSITION 4.3.1. — *Under the notation as above, let σ be a Frobenius on $\mathcal{R}$ and let M be an object of $M^{\nabla,qu}_{\mathcal{R}}$. Then there exists a finite extension K' over K and a positive integer d such that the Frobenius σ on K extends on K' and that $g^*_{K'/K}M$ has a Frobenius structure with respect to σ^d. In other words, there exists a σ^d-linear homomorphism $\varphi_d : M \to M$ such that the triple $(\mathcal{R}_{K'} \bigotimes_{\mathcal{R}} M, \varphi_d, \nabla)$ is an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R}_{K'},\sigma^d}$.*

Proof. — Let $f : F \to E$ be a finite Galois extension in F^{sep} such that f^*M is unipotent. Let $\{\rho_\lambda\}$ be the finite set of all irreducible representations of $\mathrm{Gal}(E/F)$ in $\mathbf{Q}_p{}^{\mathrm{alg}}$. Choose a finite extension K' over K and a positive integer d such that (1) K' contains all eigenvalues of ρ_λ, (2) σ extends on K' and (3) $\sigma^d \circ \rho_\lambda = \rho_\lambda$. We can choose such K' and d by (2.4.1). Replacing K, q and σ into K', q^d and σ^d, we may assume that all eigenvalues of ρ_λ are contained in K and $\sigma \circ \rho_\lambda = \rho_\lambda$.

Let $\{e_1, e_2, \cdots, e_r\}$ be a basis of $\mathcal{R}_E \bigotimes_{\mathcal{R}} M$ such that $C_{M,e} \in M_r(K)$ (4.2.5) and denote by L_E (resp. Γ_E) the $\mathcal{E}_E^\dagger$-subspace (resp. the K-subspace) of $\mathcal{R}_E \bigotimes_{\mathcal{R}} M$ which is generated by $\{e_1, e_2, \cdots, e_r\}$. We prove that there exists a Frobenius structure φ on f^*M which commutes with the action of $\mathrm{Gal}(E/F)$. By (4.2.4) Γ_E is stable under the action of $\mathrm{Gal}(E/F)$. By the assumption and Schur's Lemma Γ_E is a direct sum of $\Gamma_{E,\lambda}$ such that the Galois group $\mathrm{Gal}(E/F)$ acts on $\Gamma_{E,\lambda}$ via ρ_λ and that $\nabla\left(x\frac{d}{dx}\right)(\Gamma_{E,\lambda}) \subset \Gamma_{E,\lambda}$. So it is enough to prove the existence of Frobenius structure on $\mathcal{R}_E \bigotimes_K \Gamma_{E,\lambda}$ which commutes with the Galois action. Since $C_{f^*M,e}$ is nilpotent and the Galois action commutes with the nilpotent endomorphism $\nabla|_{\Gamma_{E,\lambda}}$, one can choose a basis $\{e_{11}^\lambda, \cdots, e_{1r_\lambda}^\lambda, \cdots, e_{tr_\lambda}^\lambda\}$ of $\Gamma_{E,\lambda}$ such that $\{e_{ij}^\lambda\}_{1 \leqslant j \leqslant r_\lambda}$ is a basis of the irreducible component on which $\mathrm{Gal}(E/F)$ acts via ρ_λ and that the differential structure is given by a direct sum of the type
$$C_{M,e^\lambda} = \begin{pmatrix} 0_{r_\lambda} & 1_{r_\lambda} & & \mathbf{0} \\ & \ddots & \ddots & \\ & & 0_{r_\lambda} & 1_{r_\lambda} \\ \mathbf{0} & & & 0_{r_\lambda} \end{pmatrix}$$
by Schur's Lemma. Here r_λ is the degree of ρ_λ. Hence, there exists a Frobenius structure φ which commutes with the Galois action by (4.2.3) and the condition (3) above in this proof. Of course, L_E is stable under φ. Put $L = L_E^{\mathrm{Gal}(E/F)}$ to be the Galois invariant part. Then $(L, \nabla|_L)$ is an $\mathcal{E}^\dagger$-lattice of M and L is stable under φ. □

From this proposition we know that, if one want to study some properties of quasi-unipotent ∇-modules, then it is enough to work on φ-∇-modules.

4.4. Let σ_1 and σ_2 be Frobenius on $\mathcal{R}$. Define a functor
$$\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2} : \underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_2} \to \underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_1}$$

as follows. For an object M of $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_2}$ and for an $\mathcal{E}^\dagger$-lattice L of M (4.2.1), put

$$\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2}(M) = \mathcal{R}\bigotimes_{\mathcal{E}^\dagger}\tilde{\epsilon}_{\sigma_1,\sigma_2}(L).$$

(See the definition of $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ in (3.4).)

LEMMA 4.4.1. — *The construction of the functor $\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2}(M)$ is independent of the choice of $\mathcal{E}^\dagger$-lattices.*

Proof. — Let L^λ (resp. $\{e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r\}$) be an $\mathcal{E}^\dagger$-lattice of an object M of $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma_2}$ (resp. a basis of L^λ) ($\lambda = \alpha, \beta$). Denote by $\epsilon^{\lambda,qu}_{\sigma_1,\sigma_2}$ the map which is defined using L^λ ($\lambda = \alpha, \beta$). Define a matrix $Q \in GL_r(\mathcal{R})$ by $(e^\alpha{}_1, e^\alpha{}_2, \cdots, e^\alpha{}_r) = (e^\beta{}_1, e^\beta{}_2, \cdots, e^\beta{}_r)Q$ and put a matrix Ω^λ to be $\epsilon^{\lambda,qu}_{\sigma_1,\sigma_2}(1 \otimes (e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r)) = (1 \otimes (e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r))\Omega_\lambda$. It is enough to prove that the diagram

$$\begin{array}{ccc} \sigma_1^* M & \overset{\epsilon^{\alpha,qu}_{\sigma_1,\sigma_2}}{\longrightarrow} & \sigma_2^* M \\ \| & & \| \\ \sigma_1^* M & \underset{\epsilon^{\beta,qu}_{\sigma_1,\sigma_2}}{\longrightarrow} & \sigma_2^* M \end{array}$$

is commutative. In other words, we have only to prove $\sigma_2(Q)\Omega^\alpha = \Omega^\beta\sigma_1(Q)$.

Assume that $A_{M,e^\lambda,\sigma_i}, C_{M,e^\lambda}$ ($\lambda = \alpha, \beta$ and $i = 1, 2$) and Q are convergent and σ_1 (resp. σ_2) is defined on the annulus $\gamma \leqslant |x| < 1$ for some $\gamma < 1$. Define a K-algebra

$$\mathcal{E}(\gamma) = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \mid \begin{array}{l} a_n \in K, |a_n|\gamma^n \text{ is bounded,} \\ |a_n|\gamma^n \to 0 \ (n \to -\infty) \end{array} \right\}.$$

Then $\mathcal{E}(\gamma)$ is complete under the norm $|\sum a_n x^n|_\gamma = \sup_n |a_n|\gamma^n$ and σ_i ($i = 1, 2$) induces a map on $\mathcal{E}(\gamma)$. The pair $(A_{M,e^\lambda,\sigma_i}, C_{M,e^\lambda})$ ($\lambda = \alpha, \beta$ and $i = 1, 2$) define an $\mathcal{E}(\gamma)$ module $L_i^\lambda(\gamma)$ with a connection and a Frobenius structure with respect to σ_i ($i = 1, 2$). Since Q is contained in $GL_n(\mathcal{E}(\gamma))$, $L_i^\alpha(\gamma)$ is isomorphic to $L_i^\beta(\gamma)$ ($i = 1, 2$). By the similar arguments as in (3.4) we can define a similar map of $\epsilon_{\sigma_1,\sigma_2}$ for $\mathcal{E}(\gamma)$ and the matrix Ω_λ is the representative matrix of this map for the basis $\{e^\lambda{}_1, e^\lambda{}_2, \cdots, e^\lambda{}_r\}$. Therefore, we have $\sigma_2(Q)\Omega_\alpha = \Omega_\beta\sigma_1(Q)$. □

LEMMA 4.4.2. — *Let σ_1, σ_2 and σ_3 be Frobenius on $\mathcal{R}$. Then we have*

(i) $\tilde{\epsilon}_{\sigma_1,\sigma_1} = \mathrm{id}$;

(ii) $\tilde{\epsilon}_{\sigma_1,\sigma_3} = \tilde{\epsilon}_{\sigma_1,\sigma_2}\tilde{\epsilon}_{\sigma_2,\sigma_3}$.

THEOREM 4.4.3. — *The category* $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma}$ *is independent of the choice of Frobenius on* $\mathcal{R}$ *via the functor* $\tilde{\epsilon}^{qu}_{\sigma_1,\sigma_2}$.

Remark 4.4.4. — The author does not know whether the category $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ is independent of the choice of Frobenius on $\mathcal{R}$ or not. But it is expected that the natural functor $\underline{\mathbf{M\Phi}}^{\nabla,qu}_{\mathcal{R},\sigma} \to \underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ is an equivalence.

5. Slope filtration for Frobenius structures.

In this section we define a slope filtration for Frobenius structures and prove that a φ-∇-module over $\mathcal{R}$ is quasi-unipotent if and only if it has a slope filtration.

5.1. Fix a Frobenius σ on $\mathcal{R}$.

DEFINITION 5.1.1. — *Let M be an object of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$. *An increasing filtration* $\{S_\gamma M\}_{\gamma\in\mathbf{Q}}$ *of M is a slope filtration for Frobenius structures if and only if it satisfies the condition as follows:*

(i) *$S_\gamma M$ is a sub φ-∇-module of M over $\mathcal{R}$;*

(ii) *$S_\gamma M = 0$ $(\gamma << 0)$ and $S_\gamma M = M$ $(\gamma >> 0)$;*

(iii) *for a sufficiently small positive rational number ϵ, there exists an $\mathcal{E}^\dagger$-lattice L_γ of $S_\gamma M/S_{\gamma-\epsilon}M$ which is pure of slope γ.*

PROPOSITION 5.1.2. — *If L is an object of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{E}^\dagger,\sigma}$ *pure of slope γ, then there are a finite separable extension $f : F \to E$ and a basis $\{e_1, e_2, \cdots, e_r\}$ of f^*M such that $C_{f^*M,e} = 0$.*

Proof. — Replacing (M,φ,∇) into $(M, a\varphi^d, \nabla)$ for a suitable positive integer d and $a \in K$, we may assume $\gamma = 0$. The assertion follows [TN2, 4.2.6]. □

PROPOSITION 5.1.3. — *Let $\eta : M_1 \to M_2$ be a morphism of* $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$. *Assume that both M_1 and M_2 have a slope filtration $S_\gamma M_i$ $(i = 1, 2)$ for Frobenius structures. Then η is strict for filtrations, that is, $\eta(S_\gamma M_1) = \eta(M_1) \bigcap S_\gamma M_2$ for any $\gamma \in \mathbf{Q}$.*

Proposition (5.1.3) follows from Lemma (5.1.4) below.

LEMMA 5.1.4. — *Let M_1 (resp. M_2) be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ with an $\mathcal{E}^{\dagger}$-lattice L_1 (resp. L_2) pure of slope γ_1 (resp. γ_2).*

(1) *If $\gamma_1 \neq \gamma_2$, then there is no nontrivial morphism from M_1 to M_2.*

(2) *If $\gamma_1 = \gamma_2$, then any morphism $\eta_1 : M_1 \to M_2$ preserves the $\mathcal{E}^{\dagger}$-lattice, that is, $\eta(L_1) = \eta(M_1) \bigcap L_2$.*

Proof. — (1) Since $\mathrm{Hom}_{\mathbf{M\Phi}\mathcal{R},\sigma}(M_1, M_2) \cong \mathrm{Hom}_{\mathbf{M\Phi}\mathcal{R},\sigma}(\mathcal{R}, M_1^{\vee} \otimes M_2)$, we have only to prove the assertion in the case where $M_1 = \mathcal{R}$ and M_2 is an arbitrary M with $\mathcal{E}^{\dagger}$-lattice L pure of slopes γ. There exist a finite separable extension $f : F \to E$ in F^{sep} and an element $A \in GL_r(K)$ such that M is isomorphic to $((\mathcal{R}_E)^r, A\sigma, d)$ by (5.1.2). One can easily see that there is no morphism from the unit object to f^*M if $\gamma \neq 0$.

The assertion (2) follows (2.2.3) and (5.1.2). □

COROLLARY 5.1.5. — *A slope filtration for Frobenius structures of an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$ is unique.*

5.2. We state one of our main local theorems.

THEOREM 5.2.1. — *Let M be an object of $\underline{\mathbf{M\Phi}}^{\nabla}_{\mathcal{R},\sigma}$. M is quasi-unipotent if and only if M has a slope filtration $\{S_\gamma M\}_{\gamma \in \mathbf{Q}}$ for Frobenius structures.*

Proof. — It is enough to prove the assertion in the case where $\sigma(x) = x^q$ by (3.4.9), (3.4.10) and (4.4.3). Let $f : F \to E$ be a finite separable extension in F^{sep} such that f^*M is unipotent. Then there exists a $\mathrm{Gal}(E/F)$-stable K-lattice Γ_E of f^*M. In fact, choose a basis $\{e_1, e_2, \cdots, e_r\}$ of f^*M as in (4.2.5) and put Γ_E to be a K_E-subspace of f^*M which is generated by $\{e_1, e_2, \cdots, e_r\}$. Here K_E is the finite unramified extension with residue class field k_E. Then Γ_E is stable under the Frobenius structure φ and the action $\mathrm{Gal}(E/F)$ by (4.2.4) and (4.2.5), that is, $\nabla|_{\Gamma_E} \circ \varphi|_{\Gamma_E} = q\varphi|_{\Gamma_E} \circ \nabla|_{\Gamma_E}$. By the theory of φ-spaces with a nilpotent structure over a complete discrete valuation field we have a slope filtration $\{S_\gamma \Gamma_E\}$ for the Frobenius structure $\varphi|_{\Gamma_E}$ of Γ_E which is compatible with the nilpotent operator $\nabla|_{\Gamma_E}$. Moreover the theory of slopes implies that the filtration $\{S_\gamma \Gamma_E\}$ is compatible with the action of $\mathrm{Gal}(E/F)$ since $\varphi|_{\Gamma_E}$ commutes with the action of $\mathrm{Gal}(E/F)$. Define a filtration $\{S_\gamma M\}$ of

M by

$$S_\gamma M = \mathcal{R} \bigotimes_{\mathcal{E}^\dagger} (\mathcal{E}_E^\dagger \bigotimes_{K_E} S_\gamma \Gamma_E)^{\mathrm{Gal}(E/F)}.$$

$\{S_\gamma M\}$ is a slope filtration for Frobenius structures of M by (2.2.4) and (3.3.5). The converse follows from (5.1.2). □

Remark 5.2.2. — In Theorem (5.2.1) the slope filtration $\{S_\gamma M\}$ of M is split as φ-modules (not as ∇-modules) over $\mathcal{R}$ if we choose a Frobenius $\sigma(x) = x^q$, because the filtration $\{S_\gamma \Gamma_E\}$ of Γ_E over K_E is split as φ-$\mathrm{Gal}(E/F)$-modules in the above proof. In general cases the slope filtration is not always split as φ-modules.

6. Quasi-unipotent overconvergent F-isocrystals on a curve.

In this section we give a definition of quasi-unipotent overconvergent F-isocrystals on a curve and apply our local study to them. We use some results on overconvergent F-isocrystals on curves from [Be1], [Be2], [Be3] and [Cr1].

6.1. Let k (resp. K) be a perfect field of positive characteristic p (resp. a complete discrete valuation field with the residue class field k and with a Frobenius σ). Let X be a smooth curve over Spec k which is geometrically connected. For a closed point $s \in X$, denote by $k(s)$ (resp. $K(s)$) the residue class field at s (resp. the finite unramified extension of K with the residue class field $k(s)$).

Let U be a dense open subscheme of X and put $Z = X - U$. Fix a closed point $s \in X$ and denote by $\mathcal{X}$ a formal scheme over Spf O_K which is a lifting of $X/\mathrm{Spec}\ k$ and formally smooth around x. Choose a section $x \in \Gamma(O_{\mathcal{X}})$ which is a lifting of a local parameter of O_X at s. Since $\mathcal{X}/\mathrm{Spf}\ O_K$ is formally smooth at s, the completion of $O_{\mathcal{X}}$ at s is isomorphic to $O_{K(s)}[[x]]$. Put $\mathcal{R}_s$ (resp. $\mathcal{E}_s$, resp. $\mathcal{E}_s^\dagger$, resp. $S_{K(s)}$) to be $\mathcal{R}_{x,K(s)}$, (resp. $\mathcal{E}_{x,K(s)}$, resp. $\mathcal{E}^\dagger_{x,K(s)}$, resp. $K \bigotimes_{O_K} O_{K(s)}[[x]]$). Therefore, we have an injective homomorphism

$$i_s : \Gamma(O_{]U[}) \to \mathcal{E}_s \quad (x \mapsto x)$$

of K-algebras. The map i_s is independent of the choice of the lifting of parameter via the natural isomorphism $\mathcal{E}^\dagger_{x,K(s)} \cong \mathcal{E}^\dagger_{x',K(s)}$ for any parameter x'. Especially, if $s \in U$, then $i_s(\Gamma(O_{]U[})) \subset S_{K(s)}$. By [Cr1, 4.7.] we have

LEMMA 6.1.1. — *Assume that X is affine and $U = X - \{s\}$. Under the notation as above, we have*

$$i_s(\Gamma(O_{]X[})) = \mathrm{Im}(i_s) \bigcap S_{K(s)};$$
$$i_s(\Gamma(j^\dagger O_{]X[})) = \mathrm{Im}(i_s) \bigcap \mathcal{E}^\dagger_s,$$

where $j :]U[\to \mathcal{X}^{an}$.

By the construction, $i_s\Big(x\dfrac{d}{dx}(u)\Big) = \delta_x(i_s(u))$ for any section $u \in \Gamma(O_{]U[})$. If $\sigma : O_{]U[} \to O_{]U[}$ is a lifting of q-th power map on O_U ($q = p^a$) which is an extension of the Frobenius σ on K, then σ extends on $\mathcal{E}_s$ (resp. $S_{K(s)}$ if $s \in U$). We call the extension σ a Frobenius on $O_{]U[}$.

Denote by $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) the abelian category of overconvergent isocrystals on U/K around Z (resp. the category of overconvergent F^a-isocrystals on U/K around Z) [Be3, (2.2.10)]. By the natural extension $i_{\mathcal{R}_s} : \Gamma(j^\dagger O_{]X[}) \to \mathcal{R}_s$ of scalar there is a functor

$$i^*_{\mathcal{R}_s} : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M}}^\nabla_{\mathcal{R}_s}$$

which is factored via the natural functor $i^*_{\mathcal{E}^\dagger_s} : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M}}^\nabla_{\mathcal{E}^\dagger_s}$ (resp. $i^*_{S_{K(s)}} : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M}}^\nabla_{S_{K(s)}}$ if $s \in U$). For any Frobenius σ on $O_{]X[}$, we also have a natural functor

$$i^*_{\mathcal{R}_s,\sigma} : F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M\Phi}}^\nabla_{\mathcal{R}_s,\sigma}$$

which is factored via the natural functor $i^*_{\mathcal{E}^\dagger_s,\sigma} : F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M\Phi}}^\nabla_{\mathcal{E}^\dagger_s,\sigma}$ (resp. $i^*_{S_{K(s)},\sigma} : F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathbf{M\Phi}}^\nabla_{S_{K(s)},\sigma}$ if $s \in U$). One can easily see that the functor $i^*_{\mathcal{R}_s}$ (resp. $i^*_{\mathcal{R}_s,\sigma}$) is independent of all choices up to canonical transformations. One can also see that the functor $i^*_{\mathcal{R}_s,\sigma}$ is independent of the choice of Frobenius σ up to the functor $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ by the definition of F-isocrystals, Proposition (3.4.10) and Lemma (4.3.1).

Now we define a quasi-unipotent overconvergent isocrystal. Our definition differs from that in [Cr2, 10.11], but we will prove that our definition is equivalent to Crew's one in Theorem (6.1.6).

DEFINITION 6.1.2. — (1) *An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger$*

$(U, X/K)$ is unipotent at a closed point $s \in X$ if and only if $i^*_{\mathcal{R}_s}\mathcal{M}$ is unipotent. An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is unipotent if and only if $\mathcal{M}$ is unipotent at any closed point on X.

(2) An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is quasi-unipotent at a closed point $s \in X$ if and only if $i^*_{\mathcal{R}_s}\mathcal{M}$ is quasi-unipotent. An object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is quasi-unipotent if and only if $\mathcal{M}$ is quasi-unipotent at any closed point on X. Denote by $\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$) the full subcategory of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) which consists of quasi-unipotent objects.

PROPOSITION 6.1.3. — *The category $\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)^{qu}$) is an abelian subcategory of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) which is closed under subquotients, tensor products and duals.*

Let $\iota : Y \subset X$ (resp. $V \subset U$) be a non-empty open subscheme and put $Z_Y = Y - V$. Denote by $\iota^\dagger : \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathrm{Isoc}}^\dagger(V, Y/K)$ (resp. $\iota^\dagger : F^a$-$\underline{\mathrm{Isoc}}^\dagger(U, X/K) \to F^a$-$\underline{\mathrm{Isoc}}^\dagger(V, Y/K)$) the natural inverse image functor which is induced by ι. By the definition we have

PROPOSITION 6.1.4. — *Under the notation as above, let $\mathcal{M}$ be an object of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$). If $\mathcal{M}$ is unipotent (resp. quasi-unipotent), then $\iota^\dagger\mathcal{M}$ is so. Assume furthermore that $Y = X$, then $\mathcal{M}$ is unipotent (resp. quasi-unipotent) if and only if $\iota^\dagger\mathcal{M}$ is so.*

Let $f : Y \to X$ be a finite morphism of smooth curves over Spec k and put $U_Y = Y \times_X U$ and $Z_Y = Y \times_X Z$. Assume that the restriction $f_U : U_Y \to U$ of f is finite and etale. Since one can choose a lifting $\mathcal{Y}$ of Y such that $]U_Y[\to]U[$ is finite etale and $j^\dagger\mathcal{O}_{]Y[}$ is finite of degree $\deg(f)$ over $j^\dagger\mathcal{O}_{]X[}$ locally at s, one can define the inverse image functor (resp. the direct image functor)

$$\begin{aligned} f^* &: \underline{\mathrm{Isoc}}^\dagger(U, X/K) \to \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) \\ (\text{resp.}\quad f_* &: \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) \to \underline{\mathrm{Isoc}}^\dagger(U, X/K)) \end{aligned}$$

by $f^*\mathcal{M} = j^\dagger\mathcal{O}_{]Y[} \bigotimes_{f^{-1}j^\dagger\mathcal{O}_{]X[}} f^{-1}\mathcal{M}$ (resp. the restriction $j^\dagger\mathcal{O}_{]X[} \to f_*j^\dagger\mathcal{O}_{]Y[}$ of scalar). One can also define the inverse image functor f^* and the direct image functor f_* for F-isocrystals. Let $t \in Y$ be a closed point with

$f(t) = s$. Choose a formally lifting $\mathcal{Y}$ over Spf O_K of Y/Spec k which is formally smooth around t, a lifting $f : \mathcal{Y} \to \mathcal{X}$ over Spf O_K of $f : Y \to X$, a section $y \in \Gamma(O_{\mathcal{Y}})$ which is a lifting of a local parameter at t. Such lifting f always exists locally on $\mathcal{X}$ and our arguments below work well on this situation. Then f induces an injection $f : \mathcal{R}_s \to \mathcal{R}_t$ of K-algebras and we have natural commutative diagrams

$$\begin{array}{ccc} \underline{\mathrm{Isoc}}^\dagger(U, X/K) & \xrightarrow{f^*} & \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) \\ i^*_{\mathcal{R}_s} \downarrow & & \downarrow i^*_{\mathcal{R}_t} \\ \underline{\mathbf{M}}^\nabla_{\mathcal{R}_s} & \xrightarrow[f^*]{} & \underline{\mathbf{M}}^\nabla_{\mathcal{R}_t} \end{array}$$

and

$$\begin{array}{ccc} \underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K) & \xrightarrow{f_*} & \underline{\mathrm{Isoc}}^\dagger(U, X/K) \\ i^*_{\mathcal{R}_t} \downarrow & & \downarrow i^*_{\mathcal{R}_s} \\ \underline{\mathbf{M}}^\nabla_{\mathcal{R}_t} & \xrightarrow[f_*]{} & \underline{\mathbf{M}}^\nabla_{\mathcal{R}_s}. \end{array}$$

If σ is a Frobenius on $O_{]U[}$, then σ extends uniquely on $O_{]U_Y[}$ since f_U is etale. We also have commutative diagrams for F-isocrystals as in above diagrams. By Proposition (4.1.3) and (6.1.3) we have

PROPOSITION 6.1.5. — *Under the notation as above,*

(1) *an object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U, X/K)$) is quasi-unipotent if and only if $f^*\mathcal{M}$ is quasi-unipotent;*

(2) *an object $\mathcal{M}$ of $\underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger(U_Y, Y/K)$) is quasi-unipotent if and only if $f_*\mathcal{M}$ is quasi-unipotent.*

Now we compare Crew's definition to ours.

THEOREM 6.1.6. — *Let $\mathcal{M}$ be an object of $\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ (resp. F^a-$\underline{\mathrm{Isoc}}^\dagger$ $(U, X/K)$. $\mathcal{M}$ is quasi-unipotent if and only if there is a finite morphism $f : Y \to X$ of smooth curves over* Spec k *and a nonempty open subscheme $\iota : V \to U$ such that $f_V : V_Y \to V$ is etale and that $f_V^*\iota^\dagger\mathcal{M}$ is unipotent.*

Proof. — Assume that $\mathcal{M}$ is quasi-unipotent. Denote by $K(X)$ the field of rational functions of X. Since Z is a finite set, there is a finite separable extension L of $K(X)$ such that, for any point $s \in Z$ and for any place t of L above s, $f^*_{t\mapsto s} i^*_{\mathcal{R}_s}\mathcal{M}$ is unipotent over $\mathcal{R}_t(= \mathcal{R}_{L_t})$. Here $K(X)_s$ (resp. L_t) is completion of $K(X)$ (resp. L) at s (resp. t) and $f_{t\mapsto s} : K(X)_s \to L_t$ is a structure map. Define a smooth curve Y over

k by the normalization of X in L. Since L is separable over $K(X)$, the natural morphism $f : Y \to X$ is generically etale. Therefore we obtain the assertion by (4.1.3). The converse follows from (4.1.3). □

Remark 6.1.7. — Matsuda pointed out that, either if X is affine or if the number of geometric points in $X - U$ is greater than 1, then one can choose a finite covering Y of X such that U_Y is etale over U in Theorem 6.1.6 by [Ka2, 2.1.6].

6.2. We give some examples of quasi-unipotent overconvergent F-isocrystals. By Proposition (4.2.1) we have

PROPOSITION 6.2.1. — *A convergent F-isocrystal on X/K is quasi-unipotent.*

DEFINITION 6.2.2. — *Let $\mathcal{M}$ be an object of $F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K)$. An increasing filtration $\{S_\gamma\mathcal{M}\}_{\gamma\in\mathbf{Q}}$ of M is a slope filtration for Frobenius structures if and only if it satisfies the conditions as follows:*

(i) *$S_\gamma\mathcal{M}$ is a subobject of $\mathcal{M}$ in $F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K)$;*

(ii) *$S_\gamma\mathcal{M} = 0$ ($\gamma << 0$) and $S_\gamma\mathcal{M} = \mathcal{M}$ ($\gamma >> 0$);*

(iii) *for a Frobenius σ on $j^\dagger O_{]U[}$, $\{i^*_{\mathcal{R}_s} S_\gamma\mathcal{M}\}_\gamma$ is a slope filtration for Frobenius structures of $i^*_{\mathcal{R}_s}\mathcal{M}$ of $\underline{\mathbf{M\Phi}}^\nabla_{\mathcal{R}_s,\sigma}$ at any point $s \in X$.*

The condition (iii) above is independent of the choice of Frobenius by Proposition (3.4.9). By Theorem (5.2.1) we have

PROPOSITION 6.2.3. — *If an object $\mathcal{M}$ of $F^a\text{-}\underline{\mathrm{Isoc}}^\dagger(U, X/K)$ has a slope filtration for Frobenius structures, then $\mathcal{M}$ is quasi-unipotent.*

COROLLARY 6.2.4 ([Cr1, 4.12]). — *An overconvergent F^a-isocrystal on U/K around Z of rank one is quasi-unipotent.*

COROLLARY 6.2.5. — *A unit-root overconvergent F^a-isocrystal on U/K around Z is quasi-unipotent.*

Example 6.2.6. — Let p be an odd prime. Let $k = \mathbf{F}_p$, $K = \mathbf{Q}_p(\pi)$ with $\pi^{p-1} = -p$ and σ be a continuous lifting of p-th power map on K with $\sigma(\pi) = \pi$. Put $X = \mathbb{P}^1_k$ (resp. $U = \mathfrak{G}\mathrm{m}_k$, resp. $Z = \{0, \infty\}$) and $\mathcal{X} = \widehat{\mathbb{P}}^1$ over $\mathrm{Spf}\, O_K$ with a coordinate x. In [Dw] B. Dwork constructed the Bessel overconvergent F-isocrystal $\mathcal{M}$ on U/K around Z. $\mathcal{M}$ is of rank 2 and is defined by the following differential and Frobenius structures:

$$\nabla(e_1, e_2) = dx \otimes (e_1, e_2) \begin{pmatrix} 0 & -x^{-1} \\ -\pi^2 & 0 \end{pmatrix}$$

$$\varphi(e_1, e_2) = (e_1, e_2) \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

on the strict neighbourhood $|x| \leqq \gamma$ for some $\gamma > 1$ of $]U[_{\mathcal{X}}$ with $\begin{pmatrix} a_1(0) & a_2(0) \\ a_3(0) & a_4(0) \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & p \end{pmatrix}$, $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (mod π) and $\det \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = p$.

Claim. — *$\mathcal{M}$ is quasi-unipotent.*

By Proposition (4.2.1) $\mathcal{M}$ is unipotent on any closed point $s \in X - \{\infty\}$. Now we discuss the quasi-unipotency of $\mathcal{M}$ at ∞ following the arguments of [Dw, Section 8]. We change the coordinate x into x^{-1} and denote by $F = k((x))$ the completion of the field of fractions of the local ring $O_{X\infty}$ at the infinity. Define a tamely ramified extension $E = k((y))$ over F with $4y^2 = x$ and choose a lifting y of the parameter of $\mathcal{R}_E$ with $4y^2 = x$. Then the differential structure of $i_\infty^* \mathcal{M}$ over $\mathcal{R}_E$ is given by

$$\nabla(e_1, e_2) = \frac{dy}{y} \otimes (e_1, e_2) \begin{pmatrix} 0 & 2 \\ 2^{-1}\pi^2 y^{-2} & 0 \end{pmatrix}.$$

If $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a solution of the differential equation $\delta_y \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 2^{-1}\pi^2 y^{-2} & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$, then z_1 satisfies the differential equation $\delta_y^2(z_1) = \pi^2 y^{-2} z_1$. Consider the formal solution $z_1 = y^{\frac{1}{2}} u_\pm(y) \exp(\pm\pi y^{-1})$. Then $u_\pm = u_\pm(y)$ satisfies the differential equation:

$$4y\delta_y^2(u_\pm) + 4(y \mp 2\pi)\delta_y(u_\pm) + xu_\pm = 0.$$

By easy calculations we have

$$u_\pm = 1 + \sum_{n=1}^{\infty} (\pm 1)^n \frac{((2n-1)!!)^2}{(8\pi)^n n!} y^n,$$

where $(2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$, and $u_\pm$ is convergent on the unit disk $|y| < 1$. Put a matrix

$$Q = \begin{pmatrix} u_+ & u_- \\ \delta_y(u_+) + (\frac{1}{2} - \pi y^{-1})u_+ & \delta_y(u_-) + (\frac{1}{2} + \pi y^{-1})u_- \end{pmatrix}.$$

Since $\delta_y(\det Q) = -\det Q$, we have $\det Q = 2\pi y^{-1}$ and $Q \in GL_2(\mathcal{R}_E)$. Change the basis (e_1, e_2) into $(e_+, e_-) = (e_1, e_2)Q$. By our construction we have

$$\nabla(e_+, e_-) = \frac{dy}{y} \otimes (e_+, e_-)C \quad \text{with } C = \begin{pmatrix} -\frac{1}{2} + \pi y^{-1} & 0 \\ 0 & -\frac{1}{2} - \pi y^{-1} \end{pmatrix}.$$

Put a matrix $A = A_{i_\infty^* \mathcal{M}, e_\pm}$. Note that $\sigma(y) = 2^{p-1} y^p$, and the pair (A, C) satisfies the relation $\delta_y(A) + CA = pA\sigma(C)$. Since $\exp(2\pi y^{-1})$ is not contained in $\mathcal{R}_E$, we have

$$A = \begin{pmatrix} \alpha_+ y^{-\frac{p-1}{2}} \exp(\pi(y^{-1} - \sigma(y^{-1})) & 0 \\ 0 & \alpha_- y^{-\frac{p-1}{2}} \exp(-\pi(y^{-1} - \sigma(y^{-1}))) \end{pmatrix}$$

for some $\alpha_+, \alpha_- \in K^\times$ with $\alpha_+\alpha_- = 2^{1-p}p$. Hence, $\mathcal{M}$ is quasi-unipotent at ∞ by the example (4.1.4). Finally we determine slopes of $\mathcal{M}$ at ∞. Since $\tau(y) = -y$ for the nontrivial element τ in $\mathrm{Gal}(E/F)$, $e_+ + e_-$ and $ye_+ - ye_-$ is a basis of $i_\infty^* \mathcal{M}$ over $\mathcal{R}_F$. By the commutativity between the Galois action and the Frobenius structure we have

$$\varphi(e_+ + e_-) = b_1(e_+ + e_-) + b_2(ye_+ - ye_-) \quad \text{with } b_1, b_2 \in \mathcal{R}_F.$$

On the other hand we have

$$\varphi(e_+ + e_-) = \alpha_+ y^{\frac{-p-1}{2}} \exp(\pi(y^{-1} - \sigma(y^{-1})))e_+ + \alpha_- y^{\frac{-p-1}{2}} \exp(-\pi(y^{-1} - \sigma(y^{-1})))e_-.$$

Comparing both identities, we obtain $v_p(\alpha_+) = v_p(\alpha_-) = \dfrac{1}{2}$ for $\alpha_+\alpha_- = 2^{1-p}p$. Therefore, all slopes of $\mathcal{M}$ at ∞ are $\dfrac{1}{2}$ by Proposition (3.3.5).

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