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Slope filtration of quasi-unipotent overconvergent $F$-isocrystals


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1. Introduction.

Let $X$ be a smooth curve over a perfect field $k$ with a positive characteristic $p$. Let $\overline{X}$ and $Z$ be the smooth compactification of $X$ and the complement of $X$ in $\overline{X}$, respectively. In [Cr2] R. Crew defined the notion of quasi-unipotent overconvergent $(F)$-isocrystals over $X$ around $Z$ and proved some expected properties, finiteness and duality for rigid cohomologies and the global monodromy theorem, of quasi-unipotent overconvergent $(F)$-isocrystals. However, the problem that what kinds of overconvergent $(F)$-isocrystals are quasi-unipotent is still open.

In this paper we study local properties of quasi-unipotent $F$-isocrystals. Let $K$ be a complete valuation field with an absolute value $| |$ and let $\mathcal{R}$ be the Robba ring over $K$ (2.2). The Robba ring is a ring of analytic functions on some annulus $\eta < |x| < 1$. We define $\varphi$-$\nabla$-modules over $\mathcal{R}$ by a free $\mathcal{R}$-module with a connection and Frobenius structures (3.2.1). A $\varphi$-$\nabla$-module is quasi-unipotent if and only if it is a successive extension of copies of the unit object as differential modules (4.1.1) after a finite etale extension. For $\varphi$-$\nabla$-modules over $\mathcal{R}$, we define a slope filtration for Frobenius structures (5.1.1). If a $\varphi$-$\nabla$-module has a slope filtration, then it is unique (5.1.5). We establish

**Theorem 5.2.1.** — A $\varphi$-$\nabla$-module over $\mathcal{R}$ is quasi-unipotent if and only if it has a slope filtration for Frobenius structures.

**Key words:** Quasi-unipotent $F$-isocrystals – $\varphi$-$\nabla$-modules – Slope filtration.

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Let $\mathcal{M}$ be an overconvergent $F$-isocrystal on $\mathcal{X}$ around $Z$. $\mathcal{M}$ determines a $\varphi$-$\nabla$-module $i_s^* \mathcal{M}$ over a Robba ring for every closed point $s \in \mathcal{X}$ canonically. Then $\mathcal{M}$ is quasi-unipotent in the sense of Crew [Cr2, 10.1] if and only if $i_s^* \mathcal{M}$ is quasi-unipotent for any closed point $s \in \mathcal{X}$ by (6.1.2) and (6.1.8).

The theorem above is useful since we have known finiteness of irregularities of $\varphi$-$\nabla$-modules with pure slopes [TN2]. So it implies finiteness of irregularities of quasi-unipotent $\varphi$-$\nabla$-modules in the sense of [TN2]. We will apply it to the global formula of Euler's number of quasi-unipotent overconvergent $F$-isocrystals in the future.

It is expected that any $\varphi$-$\nabla$-module over $\mathcal{R}$ is quasi-unipotent. If this holds, then any overconvergent $F$-isocrystal is quasi-unipotent (6.1). It is conjectured that an overconvergent $F$-isocrystal on a curve is quasi-unipotent if it has some geometric origin. (See [Cr2, 10.1].)

Now we explain the contents of this paper. In Section 2 we fix notations and prove some properties of the Robba ring $\mathcal{R}$. In Section 3 we define a $\varphi$-$\nabla$-module over $\mathcal{R}$. In Section 4 we define a quasi-unipotent $\varphi$-$\nabla$-module over $\mathcal{R}$ and prove that the category of quasi-unipotent $\varphi$-$\nabla$-modules over $\mathcal{R}$ is independent of the choice of Frobenius on $\mathcal{R}$. In Section 5 we define the slope filtration for Frobenius structures of $\varphi$-$\nabla$-modules over $\mathcal{R}$. We prove the existence of the slope filtration for quasi-unipotent $\varphi$-$\nabla$-modules over $\mathcal{R}$. In Section 6 we apply our local study to overconvergent $F$-isocrystals on a curve. We define a quasi-unipotent overconvergent $F$-isocrystal. The definition is a different form from that of Crew. Of course, the two definitions are equivalent to each other. We give some examples of quasi-unipotent overconvergent $F$-isocrystals.

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2. The Robba ring $\mathcal{R}$.

2.1. Let $p$ be a prime number. Let $k$ (resp. $K$) be a perfect field with characteristic $p$ (a complete discrete valuation field of mixed characteristics $(0, p)$ with residue class field $k$). Fix an algebraic closure $K^\text{alg}$ of $K$ and denote by $k^\text{alg}$ the residue class field of $K^\text{alg}$. Denote by $| |$ (resp. $v_p$) the
absolute value (resp. the additive valuation) of $K^{\text{alg}}$ which is normalized by $|p| = p^{-1}$ (resp. $v_p(p) = 1$).

For any valuation field $L$, we denote by $O_L$ (resp. $k_L$, resp. $L^{\text{unr}}$, resp. $m_L$) the valuation ring of $L$ (resp. the residue class field of $L$, resp. the maximum unramified subfield in the fixed algebraic closure of $L$ whose residue class field is separable over $k_L$, resp. the maximal ideal of $O_L$).

Let $F = k((x))$ be the field of fraction of the ring of formal power series with $k$-coefficients. Fix an algebraic closure $F^{\text{alg}}$ of $k$ such that the residue class field of $F^{\text{alg}}$ is $k^{\text{alg}}$ and denote by $F^{\text{sep}}$ the separable closure of $F$ in $F^{\text{alg}}$.

For a matrix $(a_{ij})$ and for an application $f$ (resp. for a norm $N$), define

$$f((a_{ij})) = (f(a_{ij})) \quad \text{(resp. } N((a_{ij})) = \sup_{i,j} N(a_{ij})).$$

2.2. For a complete field $\Omega$ with a non-Archimedean absolute value $| | : \Omega \to \mathbb{R}_{\geq 0}$ and for an indeterminate $x$, we define several $\Omega$-algebras as follows:

$$\mathcal{R}_{x,\Omega} = \left\{ \sum_{n=\infty}^{\infty} a_n x^n \mid a_n \in \Omega, \sup_{n<0} |a_n| |\xi|^n < \infty \text{ for some } 0 < \xi < 1, \right\}$$

$$\mathcal{E}_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \mid a_n \in \Omega, \sup_{n<0} |a_n| < \infty, \right\}$$

$$\mathcal{E}^\dagger_{x,\Omega} = \left\{ \sum_{n=-\infty}^{\infty} a_n x^n \in \mathcal{R}_{x,\Omega} \mid \sup_{n} |a_n| < \infty \right\}$$

$$S_{x,\Omega} = \Omega \bigotimes_{O_\Omega} O_\Omega[[x]].$$

Each ring is functorial in $\Omega$. We have natural injections of $\Omega$-algebras:

$$\mathcal{R}_{x,\Omega} \supset S_{x,\Omega} \supset \mathcal{E}^\dagger_{x,\Omega} \supset \mathcal{E}_{x,\Omega}.$$
Remark 2.2.1. Our $\mathcal{R}_{\varpi, \Omega}$ coincides with $\mathcal{R}_0(1)$ in [Ro, 2].

For formal Laurent power series $a = \sum a_n x^n$, we define $|a|_G \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ by $\sup_n |a_n|$. The field $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is a complete discrete valuation field (resp. a henselian discrete valuation field) under the absolute value $| \cdot |_G$. $| \cdot |_G$ is an extension of the absolute value $| \cdot |$ of $K$ and the residue class field of $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is $F$ by the natural projection. (See [Cr1, 4.2] [Ma, 3.2].) For a finite separable extension $E$ over $F$ in $F^{\text{sep}}$, denote by $\mathcal{E}_E$ (resp. $\mathcal{E}_E^\dagger$) the unique finite unramified extension of $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) with residue class field $E$ in the fixed algebraic closure of $\mathcal{E}$.

Lemma 2.2.2 ([Ma, 3.2]). — Under the notation as above, $\mathcal{E}_E$ (resp. $\mathcal{E}_E^\dagger$) is isomorphic to $\mathcal{E}_{y,K_E}$ (resp. $\mathcal{E}_{y,K_E}^\dagger$) for any lifting $y$ of a uniformizer of $E$. Here $K_E$ is the unique finite unramified extension of $K$ with residue class field $k_E$. Moreover the unique extension of the absolute value $| \cdot |_G$ of $\mathcal{E}$ on $\mathcal{E}_E$ coincides with the map $\sum b_n y^n \mapsto \sup_n |b_n|$.

Let $E$ be a finite separable extension of $F$ and choose a lifting $y$ of a uniformizer of $E$ in $\mathcal{E}_E^\dagger$. Define a $K$ algebra $\mathcal{R}_E$ by

$$\mathcal{R}_E = \mathcal{R}_{y,K_E}.$$

Since $x = x(y) \in \mathcal{E}_E^\dagger = \mathcal{E}_{y,K_E}^\dagger$, $\mathcal{R}$ is naturally included in $\mathcal{R}_E$.

**Lemma 2.2.3.** — (1) $\mathcal{R}_E$ is independent of the choice of the lifting of the uniformizer of $E$ up to canonical isomorphism.

(2) $\mathcal{R}_E$ is free over $\mathcal{R}$ of degree $[E:F]$. Moreover, $\mathcal{R}_E \cong \mathcal{E}_E^\dagger \bigotimes_{\mathcal{E}_E^\dagger} \mathcal{R}$ and $\mathcal{E}^\dagger = \mathcal{R} \bigcap \mathcal{E}_E^\dagger$.

Assume that the extension $E/F$ is Galois and denote by $\text{Gal}(E/F)$ the Galois group. Since $\mathcal{E}^\dagger$ is a henselian discrete valuation field, the Galois group $\text{Gal}(\mathcal{E}_E^\dagger/\mathcal{E}^\dagger)$ is canonically isomorphic to $\text{Gal}(E/F)$. The action of $\text{Gal}(E/F)$ on $\mathcal{E}_E^\dagger$ extends naturally on $\mathcal{R}_E$. By [Se1, X.1.Prop.3] and Lemma (2.2.3) we have

**Lemma 2.2.4.** — Under the notation as above,

(1) $H^0(\text{Gal}(E/F), \mathcal{E}_E^\dagger) = \mathcal{E}^\dagger$ and $H^1(\text{Gal}(E/F), \mathcal{G}L_r(\mathcal{E}_E^\dagger)) = \{1\}$;

(2) $H^0(\text{Gal}(E/F), \mathcal{R}_E) = \mathcal{R}$. 
For formal Laurent power series $\sum a_n x^n$ of indeterminate $x$, we define an additive map $\delta_x = x \frac{d}{dx}$ by

$$\delta_x \left( \sum a_n x^n \right) = \sum n a_n x^n.$$ 

Then $\delta_x$ is a $K$-derivation on $\mathcal{R}$ (resp. $\mathcal{E}$, resp. $\mathcal{E}^\dagger$, resp. $S_K$).

Let $R$ be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^\dagger$ or $S_K$. Define a free $R$-module $\omega_R$ of rank one by

$$\omega_R = R \frac{dx}{x}.$$ 

We define an additive map $d : R \to \omega_R$ by $d(a) = \delta_x(a) \frac{dx}{x}$ for $a \in R$. Then $d$ is a $K$-derivation on $R$.

Let $E$ be a finite separable extension of $F$ and choose a lifting $y$ of a uniformizer of $E$ in $\mathcal{E}_E^\dagger$. Then the derivation $\delta_x$ extends uniquely on $\mathcal{R}_E$ and we also use the notation $\delta_x$ for this extension. We have the relation

$$\delta_x = \frac{x(y)}{\delta_y(x(y))} \delta_y,$$

where $x = x(y) \in \mathcal{E}_E^\dagger$ and $\delta_x$ commutes with the action of $\text{Gal}(E/F)$ if $E/F$ is Galois.

LEMMA 2.3.1. — Under the notation as above, we have

1. $\ker(\delta_x : \mathcal{R}_E \to \mathcal{R}_E) = K_E$;

2. $\text{coker}(\delta_x : \mathcal{R}_E \to \mathcal{R}_E) \cong K_E \frac{x(y)}{\delta_y(x(y))}$, where $\frac{x(y)}{\delta_y(x(y))}$ is the image of $\frac{x(y)}{\delta_y(x(y))}$.

Proof. — The assertion easily follows from the fact that $\frac{x(y)}{\delta_y(x(y))}$ is a unit in $\mathcal{R}_E$. \qed

2.4. Fix a power $q = p^a$ ($a \geq 1$) of $p$. Denote by $K_0$ the field of fraction of the Witt vector ring $W(k)$ and $\text{Frob}$ is the usual lifting of the $q$-th power map on $K_0$. We say that an automorphism $\sigma : K \to K$ is a Frobenius on $K$ if and only if $\sigma$ is a continuous lifting of the $q$-th power map on the residue class field $k$. Since $k$ is perfect, we have $\sigma|_{K_0} = \text{Frob}^a$. Note that, if $K$ has a Frobenius and if $L$ is an unramified extension of $K$, then the Frobenius $\sigma$ extends uniquely on $L$. 
For a Frobenius $\sigma$ on $K$, put $K^{\sigma=1} = \{u \in K \mid \sigma(u) = u\}$. One can easily see that $K^{\sigma=1}$ is finite over the field $\mathbb{Q}_p$ of $p$-adic integers.

**Lemma 2.4.1** ([Cr1, 1.8]). — Let $\sigma$ be a Frobenius on $K$. Then there is a finite unramified extension $L$ of $K$ such that $L \cong L^{\sigma=1} \bigotimes_{L_0} (L^{\sigma=1}_{L_0})$ and that the unique extension $\sigma$ on $L$ is $\text{id}_{L^{\sigma=1}} \otimes \text{Frob}^a$. Assume furthermore that the residue class field $k$ is algebraically closed, then one can choose $L = K$.

**Proof.** — First we prove the assertion in the case where $k$ is algebraically closed. In this case there exists a uniformizer $\pi$ of $K$ which is algebraic over $\mathbb{Q}_p$. Then we have $K^{\sigma=1} \cong \mathbb{Q}_q(\pi)$ and $K \cong \mathbb{Q}_q(\pi) \bigotimes \mathbb{Q}_q$, where $\mathbb{Q}_q$ is the unique finite unramified extension of $\mathbb{Q}_p$ with residue class field $\mathbb{F}_q$ of $q$ elements. Now we prove the assertion in the case where $k$ is an arbitrary perfect field. Denote by $\widehat{K^{\text{unr}}}$ the $p$-adic completion of $K^{\text{unr}}$. Then $\sigma$ extends uniquely on $\widehat{K^{\text{unr}}}$. Put $L = K(\widehat{K^{\text{unr}}^{\sigma=1}})$ in $K^{\text{alg}}$. Then $L$ is finite over $K$ and is included in $\widehat{K^{\text{unr}}}$. Hence, $L$ is a desired extension of $K$. \hfill \Box

From now on to the end of this paper we assume that $K$ has a Frobenius $\sigma$.

We say a ring endomorphism $\sigma$ on $E$ (resp. $\mathcal{E}^\dagger$) is a Frobenius on $E$ (resp. $\mathcal{E}^\dagger$) if and only if it is the Frobenius $\sigma$ on $K$ and $\sigma(a) \equiv a^q (\text{mod } m_E)$ (resp. $\sigma(a) \equiv a^q (\text{mod } m_{\mathcal{E}^\dagger})$) for $a \in O_E$. (resp. $a \in O_{\mathcal{E}^\dagger}$). A Frobenius $\sigma$ on $E$ is that on $\mathcal{E}^\dagger$ if and only if $\sigma(x) \in \mathcal{E}^\dagger$. One can easily see that a Frobenius on $\mathcal{E}^\dagger$ extends naturally on $R$ by $\sum a_n x^n \mapsto \sum \sigma(a_n x^n)$ (adding coefficients in each term of $x^n$). We call this extension a Frobenius on $R$. We say a ring endomorphism $\sigma$ on $S_K$ is a Frobenius if and only if it is the Frobenius $\sigma$ on $E$ with $x^{-q} \sigma(x) \in S_K$.

For a Frobenius $\sigma$ on $E$, put

$$\mu = \mu(x, \sigma) = \frac{\delta_x(\sigma(x))}{\sigma(x)}.$$ 

Then $|\mu|_G < 1$. One can easily see that $\sigma$ is a Frobenius on $\mathcal{E}^\dagger$ (resp. $S_K$) if and only if $\mu \in \mathcal{E}^\dagger$ (resp. $\mu \in S_K$).

Let $R$ be either $\mathcal{R}$, $E$, $\mathcal{E}^\dagger$ or $S_K$ and let $\sigma$ be a Frobenius on $R$. 

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LEMMA 2.4.2. — If we regard $R$ as an $R$-module through the Frobenius $\sigma$, then $R$ is free of rank $q$.

Define $\sigma : \omega_R \to \omega_R$ by $a \frac{dx}{x} \mapsto \mu(\sigma(a)) \frac{dx}{x}$. Then the diagram below

$$
\begin{array}{c}
R \xrightarrow{d} \omega_R \\
\sigma \downarrow \quad \downarrow \sigma \\
R \xrightarrow{d} \omega_R
\end{array}
$$

is commutative. Equivalently, $\delta \circ \sigma = \mu \sigma \circ \delta$.

Let $E$ be a finite separable extension of $F$ and choose a lifting $y$ of a uniformizer of $E$ in $\mathcal{E}^+_E$. Then the Frobenius $\sigma$ on $R$ extends uniquely on $\mathcal{R}_E$ and we also use the same notation $\sigma$ for this extension. The Frobenius $\sigma$ commutes with the derivation $\delta_x$ (resp. the action of $\text{Gal}(E/F)$ if $E/F$ is Galois).

2.5. Fix a Frobenius $\sigma$ on $\mathcal{E}$ and put $\widetilde{\mathcal{E}} = K^{\sigma=1} \bigotimes_{(K^{\sigma=1})_0} W(F^{\text{alg}})$. Then there is a unique homomorphism

$$i_\sigma : \mathcal{E} \to \widetilde{\mathcal{E}}$$

such that (i) $|u|_G = |i_\sigma(u)|$ for $u \in \mathcal{E}$, where $| \cdot |$ is the unique valuation on $\widetilde{\mathcal{E}}$ which is the extension of that on $K$, (ii) the map on residue class field induced by $i_\sigma$ is the injection $F \subset F^{\text{alg}}$ and (iii) $i_\sigma(\sigma(u)) = (\text{id}_{A} \otimes \text{Frob}^\sigma)(i_\sigma(u))$. (See [TN1, 2.5.1].)

3. $\varphi$-$\nabla$-modules over $\mathcal{R}$.

Assume that the complete discrete valuation field $K$ has a Frobenius $\sigma$ from this section to the end of this paper.

3.1. Let $R$ be either $\mathcal{R}$, $\mathcal{E}$, $\mathcal{E}^1$ or $S_K$.

DEFINITION 3.1.1. — (1) A pair $(M, \nabla)$ is called a $\nabla$-module over $R$ if and only if it satisfies the conditions as follows:

(i) $M$ is a free $R$-module of finite rank.

(ii) $\nabla : M \to \omega_R \bigotimes_{R} M$ is a $K$-connection over $R$. 

(2) A morphism of $\nabla$-modules over $R$ is an $R$-linear homomorphism which commutes with connections.

(3) We denote by $\mathcal{M}_R^\nabla$ the category of $\nabla$-modules over $R$.

For a $\nabla$-module $M$ over $R$ and for a basis $\{e_1, e_2, \cdots, e_r\}$ of $M$, define a matrix $C_{M,e} \in M_r(R)$ by

$$\nabla(e_1, e_2, \cdots, e_r) = \frac{dx}{x} \otimes (e_1, e_2, \cdots, e_r)C_{M,e}.$$ 

The category $\mathcal{M}_R^\nabla$ is additive. We can define tensor products and duals for $\nabla$-modules by usual methods and, then, $(R, d)$ is the unit object of the category. We often use the notation $M$ instead of $(M, \nabla)$ for simplicity.

Since an $\mathcal{R}$-module of finite presentation with a connection is free over $\mathcal{R}$ by [Cr2, 6.1], we have

**Proposition 3.1.2.** — If $R = \mathcal{R}, \mathcal{E}$ or $\mathcal{E}^\dagger$, then the category $\mathcal{M}_R^\nabla$ is an abelian category.

Now fix a Frobenius $\sigma$ on $R$.

**Definition 3.1.3.** — (1) A pair $(M, \varphi)$ is called a $\varphi$-module over $R$ with respect to $\sigma$ if and only if it satisfies the conditions as follows:

(i) $M$ is a free $R$-module of finite rank;

(ii) $\varphi : M \rightarrow M$ is a $\sigma$-linear homomorphism such that the induced $R$-linear map

$$\varphi_\sigma : \sigma^*M \rightarrow M \quad a \otimes m \mapsto a\varphi(m)$$

is an isomorphism. Here $\sigma^*M$ is the scalar extension of $M$ by $\sigma$. We call $\varphi$ Frobenius.

(2) A morphism of $\varphi$-modules over $R$ is an $R$-linear homomorphism which commutes with Frobenius.

(3) We denote by $\mathcal{M}_R\Phi_{R,\sigma}$ the category of $\varphi$-modules over $R$ with respect to $\sigma$.

For a $\varphi$-module $M$ over $R$ and for a basis $\{e_1, e_2, \cdots, e_r\}$ of $M$, define a matrix $A_{M,e} \in M_r(R)$ by

$$\varphi(e_1, e_2, \cdots, e_r) = (e_1, e_2, \cdots, e_r)A_{M,e}.$$
The category $\mathbf{M}_{R,\sigma}^\varphi$ is additive. We can define tensor products and duals for $\varphi$-modules by usual methods and, then, $(R, \sigma)$ is the unit object. We often use the notation $M$ instead of $(M, \varphi)$ for simplicity.

**Proposition 3.1.4.** — If $R = \mathcal{E}, \mathcal{E}^\dagger$ or $S_K$, then the category $\mathbf{M}_{R,\sigma}^\varphi$ is an abelian category.

**Proof.** — In the case where $R = \mathcal{E}$ or $\mathcal{E}^\dagger$ the assertion is trivial. Let $R = S_K$. We have only to check that, for a morphism $\eta : M \rightarrow N$ of $\mathbf{M}_{S_K,\sigma}^\varphi$, the cokernel of $\eta$ is a free $S_K$-module, and then the rest is easy. Since $S_K$ is a principal ideal domain, the torsion submodules of the cokernel of $\eta$ is the form $\bigoplus S_K/(a_i)$ for some $a_i \in S_K$ with $|a_i|_G = 1$. Since $\sigma$ is flat by (2.4.2), the induced $S_K$-linear map $\sigma^*(\bigoplus S_K/(a_i)) \rightarrow \bigoplus S_K/(a_i)$ is isomorphic. However, we have

$$\dim_K \sigma^*(\bigoplus S_K/(a_i)) = \dim_K \bigoplus S_K/(\sigma(a_i)) = q \dim_K \bigoplus S_K/(a_i).$$

Hence, $N/\eta(M)$ is a free $S_K$-module. \qed

We recall the notion of slopes for Frobenius structures. Denote by the same notation $v_p$ the additive valuation of $\widetilde{\mathcal{E}}$ which is the unique extension of the valuation on $K$.

**Definition 3.1.5.** — (1) For an object $(M, \varphi)$ of $\mathbf{M}_{\mathcal{E},\sigma}$ (resp. $\mathbf{M}_{\mathcal{E}^\dagger,\sigma}$), we define the slopes of $(M, \varphi)$ by those of $(\mathcal{E} \bigotimes_R M, \varphi)$ as $\varphi$-spaces on $\widetilde{\mathcal{E}}$ (resp. by those of $(\mathcal{E} \bigotimes_R M, \varphi)$) which are measured using the valuation $-v_p$. Here $p^a = q$. We denote by $\text{Newton}(M)$ the Newton polygon of slopes of $M$.

(2) For an object $(M, \varphi)$ of $\mathbf{M}_{S_K,\sigma}$, we define the slopes of $M$ for the Frobenius structure at the generic point by those of $\mathcal{E} \bigotimes_R M$ and the slopes of $M$ for the Frobenius structure at the special point by those of $(K^{\text{unr}} \bigotimes_S M, \varphi)$ as $\varphi$-spaces on $K^{\text{unr}}$, where $S \rightarrow K$ (resp. $\varphi$) is the natural reduction modulo $x$ (resp. $\varphi$ modulo $xM$). We denote by $\text{Newton}_g(M)$ (resp. $\text{Newton}_s(M)$) the Newton polygon of slopes of $M$ at the generic point (resp. at the special point).

Since $\mathcal{E}$ is $p$-adically complete, we have
Proposition 3.1.6. — Let $M$ be an object of $\mathbf{M}_\varphi \mathcal{E}_\varphi$. Then there is an increasing filtration $\{S_\gamma M\}_{\gamma \in \mathbb{Q}}$ of $M$ such that each $S_\gamma M$ is an object of $\mathbf{M}_\varphi \mathcal{E}_\varphi$ and, for a sufficiently small positive rational number $\varepsilon << 1$, $S_\gamma M/S_{\gamma - \varepsilon} M$ is pure of slope $\gamma$.

By [Kal, 2.6.3] we have

Proposition 3.1.7. — Let $M$ be an object of $\mathbf{M}_\varphi \mathcal{E}_\varphi$. Assume that the Newton Polygon both at the generic point and at the special point coincide with each other, that is, $\text{Newton}_g(M) = \text{Newton}_s(M)$. Then there is an increasing filtration $\{S_\gamma M\}_{\gamma \in \mathbb{Q}}$ of $M$ such that each $S_\gamma M$ is an object of $\mathbf{M}_\varphi \mathcal{E}_\varphi$ and, for a sufficiently small positive rational number $\varepsilon << 1$, $S_\gamma M/S_{\gamma - \varepsilon} M$ is pure of slope $\gamma$ at both points.

3.2. Now we define $\varphi$-$\nabla$-modules over $R$.

Definition 3.2.1. — (1) A triple $(M, \varphi, \nabla)$ is called a $\varphi$-$\nabla$-module over $R$ with respect to $\sigma$ if and only if it satisfies the conditions as follows:

(i) $(M, \nabla)$ is a $\nabla$-module over $R$;

(ii) $(M, \varphi)$ is a $\varphi$-module over $R$ with respect to $\sigma$;

(iii) the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\nabla} & \omega_R \bigotimes_R M \\
\varphi \downarrow & & \downarrow \sigma \otimes \varphi \\
M & \xrightarrow{\nabla} & \omega_R \bigotimes_R M
\end{array}
$$

is commutative.

(2) A morphism of $\varphi$-modules over $R$ is an $R$-linear homomorphism which commutes with connections and Frobenius.

(3) We denote by $\mathbf{M}^\nabla_{\varphi, \sigma}$ the category of $\varphi$-$\nabla$-modules over $R$ with respect to $\sigma$.

For a $\varphi$-$\nabla$-module $M$ and for a basis $\{e_1, e_2, \cdots, e_r\}$, the condition (3.2.1)(1)(iii) is equivalent to the relation

(3.2.2) \[ \delta_x(A_{M,e}) + C_{M,e}A_{M,e} = \mu(x, \sigma)A_{M,e}\sigma(C_{M,e}). \]

We can define tensor products and duals for $\varphi$-$\nabla$-modules by usual methods and, then, $(R, \sigma, d)$ is the unit object of the category. We often use the notation $M$ instead of $(M, \varphi, \nabla)$ for simplicity.
By Proposition (3.1.2) and Proposition (3.1.4) we have

**Theorem 3.2.3.** — The category $\mathbf{M}^{\mathcal{V}}_{\mathcal{R},\sigma}$ is an abelian category with tensor products and duals.

By the extension of scalar there are natural functors

$$C_{\mathcal{R}}$$

$$C_{\mathcal{E}^\dagger}$$

$$C_{\mathcal{E}}$$

of categories, where $C$ is either $\mathbf{M}^{\mathcal{V}}$, $\mathbf{M\Phi}$ or $\mathbf{M\Phi}^{\mathcal{V}}$. For an object $M$ of $C_{\mathcal{R}}$, a sub $\mathcal{E}^\dagger$-module (resp. a sub $S_K$-module, resp. a sub $K$-space) $L$ is an $\mathcal{E}^\dagger$-lattice (an $S_K$-lattice, a $K$-lattice) if and only if $M \cong \mathcal{R} \bigotimes_{\mathcal{E}^\dagger} L$ (resp. $M \cong \mathcal{R} \bigotimes_{S_K} L$, resp. $M \cong \mathcal{R} \bigotimes_{K} L$) and $(L, \varphi|_L, \nabla|_L)$ belongs to $C_{\mathcal{E}^\dagger}$ (resp. $(L, \varphi|_L, \nabla|_L)$ belongs to $C_{S_K}$, resp. $L$ is stable under $\varphi$ and $\nabla$).

**3.3.** In this subsection we define inverse images and direct images of $\varphi$-$\nabla$-modules.

Let $f : F \to E$ be a finite separable extension in $F^{sep}$ and let $R_F$ be either $\mathcal{R}_F(= \mathcal{R})$, $\mathcal{E}_F(= \mathcal{E})$ or $\mathcal{E}_F^\dagger(= \mathcal{E}^\dagger)$. Then the extension $f$ determines a unique finite and flat extension $R_E$ over $R_F$ and denote by the same notation $f$ the extension $R_F \to R_E$. Fix a Frobenius $\sigma$ on $R_F$. Then $\sigma$ extends on $R_E$ and $\omega_{R_E} \cong R_E \bigotimes_R \omega_R$.

Let $C$ be either the category $\mathbf{M}^{\mathcal{V}}$, $\mathbf{M\Phi}_{\sigma}$ or $\mathbf{M\Phi}^{\mathcal{V}}_{\sigma}$. Define an inverse image functor

$$f^* : C_{R_F} \to C_{R_E}$$

as follows. For an object $M$ of $C_{R_F}$, put $f^* M = (M_E, \varphi_E, \nabla_E)$ to be

$$M_E = R_E \bigotimes_R M$$

$$\varphi_E = \sigma \otimes \varphi$$

$$\nabla_E = d \otimes \text{id}_M + \text{id}_{R_E} \otimes \nabla.$$

One can easily check that $f^* M$ is an object of $C_{R_E}$. By the definition $f^*$ is faithful and exact.
Define a direct image functor

\[ f_* : \mathcal{C}_{R_E} \to \mathcal{C}_{R_F} \]

as follows. For an object \( M \) of \( \mathcal{C}_{R_E} \), put \( f_*M = (M_F, \varphi_F, \nabla_F) \) to be

\[
M_F = M \quad \text{(we regard it as an \( R \)-module)}
\]
\[
\varphi_F = \varphi
\]
\[
\nabla_F = \nabla : M_F \to \omega_{R_E} \bigotimes_{R} M \cong \omega_{R} \bigotimes_{R} M_F.
\]

**Lemma 3.3.1.** — For an object \( M \) of \( \mathcal{C}_{R_E} \), \( f_*M \) belongs to \( \mathcal{C}_{R_F} \).

**Proof.** — It is sufficient to check that the natural map from \( \sigma^*(M_F) \) (a pull back by \( \sigma : R_F \to R_F \)) to \( \sigma^*M \) (a pull back by \( \sigma : R_E \to R_E \)) is bijective. Since \( M \) is free over \( R_E \), it is enough to prove that the natural map \( \sigma^*((\mathcal{E}_E^\dagger)_F) \to \sigma^*\mathcal{E}_E \) is bijective. The following Lemma (3.3.2) implies the assertion by (2.2.3).

**Lemma 3.3.2.** — Under the notation as above, the natural map \( \sigma^*((\mathcal{E}_E^\dagger)_F) \to \sigma^*\mathcal{E}_E^\dagger \) is bijective.

**Proof.** — Denote by \( \sigma_q \) the \( q \)-th power map. Consider the perfections both of \( F \) and \( E \), and dimensions over \( F \), then \( \sigma_q^*(E_F) \to \sigma_q^*(E) \) is injective, hence bijective. The assertion holds by Nakayama's Lemma.

We show some properties of inverse images and direct images.

**Lemma 3.3.3.** — Let \( f : F \to E_1 \) and \( g : E_1 \to E_2 \) be finite separable extensions over \( F \) in \( F^{\text{sep}} \). Then, we have \( (gf)^* = g^*f^* \) and \( (gf)_* = f_*g_* \).

**Proposition 3.3.4.** — (1) The functor \( f^* \) (resp. \( f_* \)) commutes with natural functors \( \mathcal{C}_{\mathcal{E}^\dagger} \to \mathcal{C}_R \) and \( \mathcal{C}_{E^\dagger} \to \mathcal{C}_E \).

(2) The functor \( f^* \) preserves tensor products and duals.

(3) \( f_* \) is a right adjoint of \( f^* \) and \( f^* \) is a left adjoint of \( f_* \).

We study the behavior of Newton polygons of \( \varphi \)-modules under an inverse image functor (resp. a direct image functor). By the definition of Newton polygon we have
Proposition 3.3.5. — Let $R_F$ be either $E_F$ or $E_F^\dagger$. The Newton polygon of $\varphi$-modules is preserved by the inverse image functor $f^*$. In other words, we have
\[
\text{Newton}(f^*M) = \text{Newton}(M)
\]
for any object $M$ of $\mathbf{M}_{R_F}^\varphi$.

Proposition 3.3.6. — Let $R_F$ be either $E_F$ or $E_F^\dagger$. For an object $M$ of $\mathbf{M}_{R_E,\sigma}$, the Newton polygon $\text{Newton}(f_*M)$ of $f_*M$ is $[E : F]$ times $\text{Newton}(M)$. In other words, the rank of the slope $\gamma$-part of $f_*M$ is $[E : F]$ times the rank of the slope $\gamma$-part of $M$.

Proof. — One may assume that the extension $E$ over $F$ is Galois by (3.3.5). If we denote by $M_\tau$ a scalar extension of $M$ by an $\mathcal{R}_F$-embedding $\tau : R_E \to \tilde{E}$, then we have
\[
\tilde{E} \otimes_{R_F} f_*M \cong \bigoplus_{\tau \in \text{Hom}_{\mathcal{R}_F}(R_E, \tilde{E})} M_\tau
\]
as $\varphi$-modules over $\tilde{E}$. Since the action of Galois commutes with Frobenius, we obtain the assertion. \qed

3.4. Let $R$ be either $E$, $E^\dagger$ or $S_K$. Let $M$ be an object of $\mathbf{M}_{R}^\varphi$ and \{e_1, e_2, \ldots, e_r\} a basis of $M$. For an element $m = a_1e_1 + \cdots + a_re_r$, define
\[
||m||_{M,e} = \max_i |a_i|_G.
\]
Then $|| ||_{M,e}$ is a norm on $M$ which is compatible with the norm $| |_G$ of $R$. The topology which is determined by the norm $|| ||_{M,e}$ is independent of the choice of the basis of $M$.

Define a $K$-linear map $\nabla^{[n]} : M \to M$ by
\[
\nabla^{[0]} = \text{id}_M \quad \text{and} \quad \nabla^{[n+1]} = \left(\nabla\left(x \frac{d}{dx}\right) - n\right)\nabla^{[n]}.
\]
for any non-negative integer $n$. Here the map $\nabla\left(x \frac{d}{dx}\right)$ is defined by $\nabla(m) = \frac{dx}{x} \otimes \nabla\left(x \frac{d}{dx}\right)(m)$ for $m \in M$. By Leibniz’s rules we have

Lemma 3.4.1. — $\nabla^{[n]}(am) = \sum_{i+j=n} \frac{n!}{i!j!}\delta^{[i]}(a)\nabla^{[j]}(m)$ for $a \in R$, $m \in M$. 


Let $M$ be an object of $\mathbf{M}^{\nabla}_{R}$. Consider the conditions (C) and (OC) as follows:

\[(C) \quad \left\| \frac{1}{m!} \nabla^{[n]}(m) \right\|_{M,e} \eta^n \to 0 \quad (n \to \infty)\]

for any $m \in M$ and any number $0 < \eta < 1$;

\[(OC) \quad \sum_{n=0}^{\infty} \frac{w^n}{n!} \nabla^{[n]}(m) \text{ converges in } M\]

for any $m \in M$ and for any $w \in R$ with $|w|_G < 1$. If $R = E$ and $S_K$, the condition (C) implies (OC) since $R$ is complete in the $p$-adic topology. In the case of $E^{\dagger}$, however, the condition (OC) is delicate since $E^{\dagger}$ is not complete.

**Proposition 3.4.2.** — Any object $M$ of $\mathbf{M}^{\nabla}_{R,\sigma}$ satisfies the condition (C).

**Proof.** — Fix a positive integer $k$ with $\eta < p^{-1/(p^k(p-1))}$. By (3.4.1) we have only to prove the condition (C) for one basis of $M$. Choose a basis $\{e_1, e_2, \ldots, e_r\}$ of $M$ such that $|C|_G \leq p^{-(p^k-1)/(p-1)}$, where we denote $C = C_{M,e}$. We can choose such a basis after changing a basis by $(e_1, e_2, \ldots, e_r) \mapsto (e_1, e_2, \ldots, e_r)A\sigma(A) \cdots \sigma^n(A)$ for a sufficiently large $n$, where $A = A_{M,e}$. Define matrices $C^{[n]} \in M_{r}(R)$ by $\nabla^{[n]}(e_1, e_2, \ldots, e_r) = (e_1, e_2, \ldots, e_r)C^{[n]}$. Since $|C^{[n+1]} - (\delta_x(C^{[n]}) - nC^{[n]}))|_G \leq |C^{[n]}|_G p^{-(p^k-1)/(p-1)}$, one can easily check that $|C^{[n]}|_G \leq p^{-(i+1)(p^k-1)/(p-1)}$ for $n = ip^k + j$ ($i \geq 0, 0 < j \leq p^k$). Note that $v_p(n!) < n/(p-1)$ for any positive integer $n$. Since

\[(i+1)(p^k-1)/(p-1) + n/(p^k(p-1)) - v_p(n!)
\]\
\[= ((p^k-1)/(p-1) - v_p(j!)) + (i/(p-1) - v_p(i!)) + j/(p^k(p-1)) > 0,\]

we have $|C^{[n+1]}/n!|_G \eta^n \to 0$ if $n \to \infty$. ∎
SLOPE FILTRATION OF F-ISOCRYSTALS

COROLLARY 3.4.3. — The connection of objects in $\mathbf{M}_R^V_{\Phi R,\sigma}$ is topologically nilpotent.

Define a map $\alpha_N : \mathcal{E} \rightarrow \mathbb{R}$ by

$$\alpha_N\left(\sum_{n \leq N} a_n x^n\right) = \sup_{n \leq N} |a_n|$$

for any integer $N$. Note that (i) $a \in \mathcal{E}^+$ if and only if $\alpha_N(a) \leq c\xi^{-N}$ for any integer $N$ for some $c > 0$ and $0 < \xi < 1$ and (ii) if $\alpha_N(a) \leq c_a\xi^{-N}$ and $\alpha_N(b) \leq c_b\xi^{-N}$, then $\alpha_N(ab) \leq c_ac_b\xi^{-N}$.

PROPOSITION 3.4.4. — Any object $M$ of $\mathbf{M}_R^V_{\Phi E,\sigma}$ satisfies the condition (OC).

Proof. — Keep the notation as in the proof of (3.4.2). By (3.4.1) we have only to prove the condition (OC) for one basis of $M$. Choose a positive integer $k$, a basis $\{e_1, e_2, \ldots, e_r\}$ of $M$ and a real number $0 < \xi < 1$ such that $\alpha_N(w) < p^{-1/(p^k(p-1))}\min\{\xi^{-N}, 1\}$ and $\alpha_N(C) \leq p^{-k(p-1)/(p-1)}\min\{\xi^{-N}, 1\}$ for any integer $N$. Then one can easily check that $\alpha_N(C[n]) \leq p^{-(i+1)(p^k(p-1)/(p-1))}\min\{\xi^{-N}, 1\}$ for $n = ip^k + j$ ($i \geq 0, 0 < j \leq p^k$). By the calculation of valuations as in the proof of (3.4.2) we have $\alpha_N(C[n]w^n/n!) \leq \min\{\xi^{-N}, 1\}$. Since $\sum_{n=0}^{\infty} C[n]w^n/n!$ is convergent in $M_r(\mathcal{E})$ by (3.4.2), $\sum_{n=0}^{\infty} C[n]w^n/n!$ is convergent in $M_r(\mathcal{E}^+)$.

Let $\sigma_1$ and $\sigma_2$ be Frobenius on $R$. For an object $M$ of $\mathbf{M}_R^V_{\Phi E,\sigma}$, define an $R$-linear homomorphism

$$\epsilon_{\sigma_1, \sigma_2} : \sigma_1^* M \rightarrow \sigma_2^* M$$

by

$$\epsilon_{\sigma_1, \sigma_2}(a \otimes m) = a \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\sigma_1(x)}{\sigma_2(x)} - 1 \right)^n \otimes \nabla^n(m).$$

Since one knows the identity

$$\sigma_1(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\sigma_1(x)}{\sigma_2(x)} - 1 \right)^n \sigma_2(\delta[n](a))$$

for any $a \in \mathcal{E}$, the map $\epsilon_{\sigma_1, \sigma_2}$ is well-defined and continuous by (3.4.2) and (resp. (3.4.3)). By easy calculations we have
LEMMA 3.4.5. — Let \( \sigma_1, \sigma_2, \sigma_3 \) be Frobenius on \( R \). Then

\begin{enumerate}[(i)]
  \item \( \epsilon_{\sigma_1,\sigma_1} = \text{id} \);
  \item \( \epsilon_{\sigma_1,\sigma_3} = \epsilon_{\sigma_1,\sigma_2} \epsilon_{\sigma_2,\sigma_3} \).
\end{enumerate}

Define a functor

\[ \tilde{\epsilon}_{\sigma_1,\sigma_2} : \mathbf{M}^\vee_{\Phi_R,\sigma_2} \to \mathbf{M}^\vee_{\Phi_R,\sigma_1} \]

by

\[ (M, \varphi, \nabla) \mapsto (M, \varphi \sigma_2 \circ \epsilon_{\sigma_1,\sigma_2}|_{\otimes M}, \nabla). \]

LEMMA 3.4.6. — Under the notation as above, the triple \((M, \varphi \sigma_2 \circ \epsilon_{\sigma_1,\sigma_2}|_{\otimes M}, \nabla)\) is an object of \( \mathbf{M}^\vee_{\Phi_R,\sigma_1} \).

Proof. — Put \( \varphi_1 = \varphi \sigma_2 \circ \epsilon_{\sigma_1,\sigma_2}|_{\otimes M} \). By (3.4.5) \( \epsilon_{\sigma_1,\sigma_2} \) is isomorphic, hence \((\varphi_1)_{\sigma_1}\) is isomorphic. An easy calculation implies the commutative of \( \varphi_1 \) and \( \nabla \).

LEMMA 3.4.7. — Let \( \sigma_1, \sigma_2, \sigma_3 \) be Frobenius on \( R \). Then

\begin{enumerate}[(i)]
  \item \( \tilde{\epsilon}_{\sigma_1,\sigma_1} = \text{id} \);
  \item \( \tilde{\epsilon}_{\sigma_1,\sigma_3} = \tilde{\epsilon}_{\sigma_1,\sigma_2} \tilde{\epsilon}_{\sigma_2,\sigma_3} \).
\end{enumerate}

LEMMA 3.4.8. — (1) The functor \( \tilde{\epsilon}_{\sigma_1,\sigma_2} \) commutes with tensor products and duals.

(2) For a finite separable extension \( f : F \to E \) in \( F^{\text{sep}} \), the functor \( \tilde{\epsilon}_{\sigma_1,\sigma_2} \) commutes with \( f^* \) and \( f_* \).

PROPOSITION 3.4.9. — Let \( \sigma_1 \) and \( \sigma_2 \) be Frobenius on \( R \) and let \( M \) be an object of \( \mathbf{M}^\vee_{\Phi_R,\sigma_2} \). Then the slopes of \( M \) for Frobenius structures coincide with those of \( \tilde{\epsilon}_{\sigma_1,\sigma_2}(M) \). In other words,

\[
\begin{align*}
\text{Newton}(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \text{Newton}(M) \\
\text{(resp. Newton}_\eta(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \text{Newton}_\eta(M) \\
\text{Newton}_s(\tilde{\epsilon}_{\sigma_1,\sigma_2}(M)) & = \text{Newton}_s(M)
\end{align*}
\]

if \( R = \mathcal{E} \) or \( \mathcal{E}^\dagger \) (resp. if \( R = S_K \)).

Proof. — We have only to prove the assertion in the case where \( R = \mathcal{E} \) and \( M \) is pure of slopes 0 by (3.1.6). We can choose a suitable basis of \( M \).
with $A_{M,e} \in GL_r(O_{E})$ and $\epsilon_{\sigma_1,\sigma_2}(e_i) \equiv e_i \pmod{m_{E}}$. Therefore, we have the assertion. □

Now we have obtained

**Theorem 3.4.10.** — The category $M_{\Phi_{R,\sigma}}^\nabla$ is independent of the choice of Frobenius up to canonical equivalence.

## 4. Quasi-unipotent $\varphi$-$\nabla$-modules.

### 4.1. Fix a Frobenius $\varphi$ on $R$. We define quasi-unipotent $\varphi$-$\nabla$-modules.

**Definition 4.1.1.** — (1) A $\nabla$-module $M$ (resp. a $\varphi$-$\nabla$-module $M$) over $R$ is unipotent if and only if $M$ is a successive extension of the unit object $(\mathcal{O}_R, d)$ (resp. $(M, \nabla)$) is a unipotent $\nabla$-module).

(2) A $\nabla$-module $M$ (resp. a $\varphi$-$\nabla$-module $M$) over $R$ is quasi-unipotent if and only if there exists a finite separable extension $f : F \to E$ such that the inverse image $f^*M$ is unipotent.

(3) We denote by $M_{R}^{\nabla, qu}$ (resp. $M_{\Phi_{R,\sigma}}^{\nabla, qu}$) the full subcategory of $M_{R}^{\nabla}$ (resp. $M_{\Phi_{R,\sigma}}^{\nabla}$) whose objects consist of quasi-unipotent $\nabla$-modules (resp. $\varphi$-$\nabla$-modules).

By the standard arguments we have

**Proposition 4.1.2.** — (1) Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence in $M_{R}^{\nabla}$ (resp. $M_{\Phi_{R,\sigma}}^{\nabla}$). $M_2$ is quasi-unipotent if and only if both $M_1$ and $M_3$ are quasi-unipotent.

(2) The category $M_{R}^{\nabla, qu}$ (resp. $M_{\Phi_{R,\sigma}}^{\nabla, qu}$) is an abelian subcategory of $M_{R}^{\nabla}$ (resp. $M_{\Phi_{R,\sigma}}^{\nabla}$) with tensor products and duals.

**Proposition 4.1.3.** — Let $f : F \to E$ be a finite separable extension in $F^{sep}$.

(1) Let $M$ be an object of $M_{R}^{\nabla}$ (resp. $M_{\Phi_{R,\sigma}}^{\nabla}$). $M$ is quasi-unipotent if and only if $f^*M$ is quasi-unipotent.

(2) Let $M$ be an object of $M_{R,E}^{\nabla}$ (resp. $M_{\Phi_{R,E,\sigma}}^{\nabla}$). $M$ is quasi-unipotent if and only if $f_*M$ is quasi-unipotent.
Proof. — The assertion on inverse images is easy. In the case of direct images we may assume that the extension \( E \) is Galois over \( F \) by (1) and (4.1.2). For \( \tau \in \text{Gal}(E/F) \), denote by \( M_\tau \) the \( \nabla \)-module (resp. \( \varphi \nabla \)-module) whose \( \mathcal{R}_E \)-action is defined by \((a, m) \mapsto \tau(a)m \) for \( a \in \mathcal{R}_E \) and \( m \in M \).

Then \( f^* f_* M \cong \bigoplus_{\tau \in \text{Gal}(E/F)} M_\tau \). The assertion (2) easily follows from the isomorphism.

Example 4.1.4. — (1) Any \( \varphi \nabla \)-module \( M \) over \( \mathcal{R} \) of rank one is quasi-unipotent. Indeed, if we fix a base \( e \) of \( M \), then \( A_{M,e} \in \mathcal{R}^\times = (\mathcal{E}^\dagger)^\times \).

By the relation (3.2.2) we have \( C_{M,e} \in \mathcal{E}^\dagger \). Hence, \( M \) has an \( \mathcal{E}^\dagger \)-lattice and it is quasi-unipotent by [Cr1, 4.11] (or (2) below).

(2) Any \( \varphi \nabla \)-module over \( \mathcal{R} \) which has an etale \( \mathcal{E}^\dagger \)-lattice is quasi-unipotent [TN1, 4.2.6]. ("Etale" means that all slopes of Frobenius are 0.)

4.2. We show some properties of unipotent \( \varphi \nabla \)-modules.

Proposition 4.2.1. — (1) An object in \( \mathbf{M}_{\mathcal{R},a}^{\mathcal{E}^\dagger, \varphi^u} \) has an \( \mathcal{E}^\dagger \)-lattice.

(2) Assume that \( \sigma \) is Frobenius on \( S_K \). An object of \( \mathbf{M}_{\mathcal{R},a}^{\mathcal{E}^\dagger} \) is unipotent if and only if it has an \( S_K \)-lattice.

Remark 4.2.2. — The \( \mathcal{E}^\dagger \)-lattice (resp. the \( S_K \)-lattice) is not unique in Proposition (4.2.1).

Proposition (4.2.1)(1) (resp. (2)) follows from Lemma (4.2.5) (resp. Lemmas (4.2.6) and (4.2.7)) below.

Put \( u \in (\mathcal{E}^\dagger)^\times \) to be \( \sigma(x) = x^q \) for the Frobenius \( \sigma \). Then \( |u-1|_\sigma < 1 \) and one can define \( \log(u) \) in \( \mathcal{E}^\dagger \). If \( \sigma \) is a Frobenius on \( S_K \), then \( \log(u) \) belongs to \( S_K \). Note that \( \mu = \mu(x, \sigma) = \frac{\delta_x(\sigma(x))}{\sigma(x)} = q + \frac{\delta_x(u)}{u} \) and \( \delta_x(\log(u)) = \frac{\delta_x(u)}{u} \).

Lemma 4.2.3. — Let \( C_1 = \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \) (resp. \( C_2 = \begin{pmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \)) be a matrix of degree \( r_1 \) (resp. \( r_2 \)). A matrix \( Q \in M_{r_1, r_2}(\mathcal{R}) \) (resp. \( Q \in M_{r_1, r_2}(K[[x]]) \)) satisfies the relation

\[ \delta_x(Q) + C_1 Q = \mu Q C_2 \]
if and only if

\[
Q = \begin{cases}
0 \cdots 0 & \alpha_1 \alpha_2 \cdots \alpha_r \\
& \cdots \cdots \cdots \\
& \cdots \cdots \cdots \\
0 & \alpha_1 \alpha_2 \cdots \alpha_r \\
& \cdots \cdots \cdots \\
q^{r_1-1}\alpha_1 & q^{r_1-2}\alpha_2 \\
& \cdots \cdots \cdots \\
& q^{r_2-2}\alpha_2 & q^{r_2-1}\alpha_1 \\
& \cdots \cdots \cdots \\
0 & 0
\end{cases}
\]

\[
\text{if } r_1 \leq r_2
\]

\[
\text{if } r_1 \geq r_2
\]

with \( \alpha_1 = \beta_1, \alpha_2 = \beta_1 \log(u) + \beta_2, \ldots, \alpha_r = \frac{\beta_1}{(r-1)!} \log^{r-1}(u) + \frac{\beta_2}{(r-2)!} \log^{r-2}(u) + \cdots + \beta_r \) for some \( \beta_i \in K \).

Proof. — We use Lemma (2.3.1) to show the assertion. Assume that \( Q = (q_{i,j}) \) is a solution of the differential equation above.

First we prove that \( q_{r_1,j} = 0 \) (1 \( \leq j < r_2 \)) and \( q_{r_1,r_2} \) is contained in \( K \). Since \( \delta_x(q_{r_1,1}) = 0 \), \( q_{r_1,1} \) is contained in \( K \). Then the identity \( \delta_x(q_{r_1,2}) = \mu q_{r_1,1} \) implies that \( q_{r_1,1} = 0 \) and \( q_{r_1,2} \) is contained in \( K \). Repeating these, we proved the assertion.

Secondly we prove that \( q_{i,1} = 0 \) (2 \( \leq i \)) and \( q_{1,1} \) is contained in \( K \). Assume that \( q_{i+1,1} = \cdots = q_{r_2,1} = 0 \). Since \( \delta_x(q_{i,1}) + q_{i+1,1} = 0 \), \( q_{i,1} \) is contained in \( K \). So the assertion follows from \( \delta_x(q_{i-1,1}) + q_{i,1} = 0 \).

Thirdly we prove that, if \( q_{i,n+1} \) is a linear combination of \( 1, \log(u), \log^2(u), \ldots \) over \( K \) and if \( q^{-i+1}q_{i,n+1} \) does not depend on \( i \) when \( n \) is fixed, then \( q_{i,n+1+i} \) is a linear combination of \( 1, \log(u), \log^2(u), \ldots \) over \( K \) and \( q^{-i+1}q_{i,n+1+i} \) is independent on \( i \). The former assertion holds by the equation \( \delta_x(q_{i,j}) + q_{i+1,j} = \mu q_{i,j-1} \) (\( i < r_1, j > 1 \)) and \( \mu = q + \frac{\delta_x(u)}{u} \) and by two assertions above. Moreover \( q^{-i+1}q_{i,n+1+i} \) does not depend on \( i \) up to constant terms. (When \( q_{i,1} \) (resp. \( q_{r_1,j} \)) appears, \( q^{-i+1}q_{i,n+1+i} = 0 \) and \( q^{i-1}q_{i,n+1+i} \) does not depend on \( i \) up to constant terms.) Since

\[
\delta_x(q_{i,n+1+(i+1)}) = \mu q_{i,n+1+i} + q_{i+1,n+1+(i+1)} + q_{i,n+1+i}(u)
\]

the constant term must vanish. Hence, the later assertion also holds.
Finally we have got the relation $\delta_x(q_i, r_2) = \mu q_i, r_2 - q_{i+1, r_2} = \frac{\delta_x(u)}{u} q_i, r_2 - 1$. Therefore, $Q$ has a form as in the assertion. The converse can be easily checked.

Let $f : F \to E$ be a finite separable extension in $F^{\text{sep}}$. Denote by $x$ (resp. $y$) a lift of uniformizer of $F$ (resp. $E$) in $\mathcal{E}_F$ (resp. $\mathcal{E}_E$). Using similar arguments as in Lemma (4.2.3) and by Lemma (2.3.1) we obtain

**Lemma 4.2.4.** — Under the notation as above, let $C_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$ (resp. $C_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$) be a matrix of degree $r_1$ (resp. $r_2$). A matrix $Q \in M_{r_1, r_2}(\mathcal{R}_E)$ satisfies the differential equation

$$\delta_x(Q) + C_1 Q = Q C_2$$

for the derivation $\delta_x = x \frac{d}{dx}$ if and only if

$$Q = \begin{cases} \begin{pmatrix} 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{r_1} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \alpha_2 & \alpha_1 \end{pmatrix} & \text{if } r_1 \leq r_2 \\ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{r_2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \alpha_2 & \alpha_1 \\ 0 & 0 & \cdots & \alpha_1 \end{pmatrix} & \text{if } r_1 \geq r_2 \end{cases}$$

for some $\alpha_i \in K_E$.

**Corollary 4.2.5.** — (1) Under the notation as above, assume furthermore that $M$ is a unipotent $\nabla$-module over $\mathcal{R}_E$. Then there is a basis $\{e_1, e_2, \cdots, e_r\}$ of $M$ such that, if we define a matrix $C_{M,e,x} \in M_r(\mathcal{R}_E)$ by
\( \nabla(e_1, e_2, \cdots, e_r) = \frac{dx}{x} \otimes (e_1, e_2, \cdots, e_r)C_{M,e,x} \)

\[
C_{M,e,x} = \begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_s \end{pmatrix}
\text{ with } C_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & 0 \end{pmatrix}.
\]

Moreover, if \( M \) has a \( \sigma \)-linear homomorphism \( \varphi : M \to M \) which is compatible with the connection and if \( L_E \) is an \( E^t_E \)-subspace which is generated by \( \{e_1, e_2, \cdots, e_r\} \), then \( L_E \) is stable under \( \varphi \).

(2) Let \( M \) be an object of \( M_{\mathbb{Q}_p}^{\phi, q} \) and let \( f : F \to E \) be a finite separable extension in \( F^{\text{sep}} \) such that \( f^*M \) is unipotent. If \( \{e_1, e_2, \cdots, e_r\} \) is a basis of \( f^*M \) as in (1) and if we denote by \( L_E \) the \( E^t_E \)-subspace which is generated by \( \{e_1, e_2, \cdots, e_r\} \), then \( L_E \) is stable under the action of \( \text{Gal}(E/F) \).

Proof. — (1) We use induction on \( r \). Let \( \{e_1, e_2, \cdots, e_{r-1}, e'\} \) be a basis of \( M \) such that \( C_{M,e',x} = \begin{pmatrix} C_{11} & \, \, \, C_{12} \\ 0 & \, \, \, 0 \end{pmatrix} \) with \( C_{11} \) as in the assertion and some \( C_{12} \in \mathbb{R}^{r-1} \). Using (2.3.1), one can get a matrix of type \( Q = \begin{pmatrix} 1 & Q_{12} \\ 0 & 1 \end{pmatrix} \) with \( Q_{12} \in \mathbb{R}^{r-1} \) such that \( \{e_1, e_2, \cdots, e_{r-1}, e'\}Q \) is the desired basis. Let \( \{e_1, e_2, \cdots, e_r\} \) be a basis as in the former assertion. Then we have \( \delta_x(A_{M,e} + C_{M,e,x}A_{M,e}) = (x, \sigma)A_{M,e}C_{M,e,x} \) by the commutativity of Frobenius and connection. By (4.2.3) there is a matrix \( A_x \in GL_r(E^t) \) which satisfies the relation \( \delta_x(A_x) + C_{M,e,x}A_x = (x, \sigma)A_xC_{M,e,x} \). Hence we have

\[ \delta_x(A_{M,e}A_x^{-1}) + C_{M,e,x}A_{M,e}A_x^{-1} = A_{M,e}A_x^{-1}C_{M,e,x} \]

and \( A_{M,e}A_x^{-1} \in GL_r(K_E) \) by (4.2.4). The assertion (2) easily follows from the commutativity of the Galois action and the connection and by (4.2.4).

\[ \square \]

Let \( M \) be an object in \( \mathbf{M}_{\mathbb{Q}_p}^\phi \). Put \( \overline{M} = M/xM \) (resp. \( N_M = \nabla\left( x \frac{d}{dx} \right) \) to be the induced \( K \)-linear map). By the relation (3.2.2) we have

**Lemma 4.2.6.** — For any object \( M \) of \( \mathbf{M}_{\mathbb{Q}_p}^\phi \), the \( K \)-linear map \( N_M \) is nilpotent.
Lemma 4.2.7. — Let $M$ be an object of $\mathbf{M}_\Phi^\nabla_{\mathcal{K},\sigma}$ and let $\{e_1, e_2, \cdots, e_r\}$ be a basis of $M$. Put $C_0$ to be the representation matrix of the $K$-linear map $N_M$ for the basis $\{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_r\}$. Then there exists a solution $Q \in 1_r + xM_r(K[[x]])$ of the system of linear differential equations

$$\delta_x(Q) + C_{M,e}Q = QC_0$$

such that $Q$ belongs to $GL_r(\mathcal{R})$.

Proof. — Since all proper values of $C_0$ are 0 (4.2.6), one can uniquely solve the system of differential equation above in $M_r(K[[x]])$ with $Q \pmod{xK[[x]]} = 1_r$. Put $A_0 = Q^{-1}A\sigma(Q)$. Then the pair $(A_0, C_0)$ satisfies the relation (3.2.2.). Hence, $A_0$ is contained in $GL_r(S_K)$ by (4.2.3). If we denote by $\gamma$ the radius of convergence of $Q$, then $0 < \gamma \leq 1$ and the radius of convergence of $\sigma(Q)$ is $\gamma^q$. By the relation $Q A_0 = A\sigma(Q)$ we have

$$\min\{\gamma, 1\} = \min\{\gamma^q, 1\}.$$ 

Hence, $\gamma = 1$ and $Q$ is contained in $M_r(\mathcal{R})$. Consider the dual object $M^\vee$ of $M$ and the dual basis $\{e^\vee_1, e^\vee_2, \cdots, e^\vee_r\}$. Then there is a matrix $Q^\vee \in M_r(K[[x]]) \cap M_r(\mathcal{R})$ with $Q^\vee \pmod{xK[[x]]} = 1_r$ and $\delta_x(Q^\vee) - tC_{M,e}Q^\vee = -Q^{\vee t}C_0$. So we have

$$\delta_x(Q^\vee Q) + C_0Q^\vee Q = Q^\vee QC_0.$$ 

Therefore $Q$ is invertible by (4.2.4). \[\square\]

4.3. Let $K'$ be an extension of $K$ which is complete under the extension of the valuation of $K$ and put $\mathcal{R}' = \mathcal{R}_{K',x}$ to be an extension of $\mathcal{R}$. Denote by $g_{K'/K}^* : \mathbf{M}_\Phi^\nabla_{\mathcal{K}} \rightarrow \mathbf{M}_\Phi^\nabla_{\mathcal{K}'}$, the natural functor which is defined by the scalar extension. If the Frobenius $\sigma$ on $K$ extends on $K'$, then the Frobenius $\sigma$ on $\mathcal{R}$ extends on $\mathcal{R}_{K'}$. (The extension of the Frobenius on $\mathcal{R}_{K'}$ is uniquely determined by the extension of the Frobenius on $K'$.) In this case there is a natural functor $g_{K'/K}^* : \mathbf{M}_\Phi^\nabla_{\mathcal{K}} \rightarrow \mathbf{M}_\Phi^\nabla_{\mathcal{K}'}$.

Proposition 4.3.1. — Under the notation as above, let $\sigma$ be a Frobenius on $\mathcal{R}$ and let $M$ be an object of $M^\nabla_{\mathcal{R},qu}$. Then there exists a finite extension $K'$ over $K$ and a positive integer $d$ such that the Frobenius $\sigma$ on $K$ extends on $K'$ and that $g_{K'/K}^* M$ has a Frobenius structure with respect to $\sigma^d$. In other words, there exists a $\sigma^d$-linear homomorphism $\varphi_d : M \rightarrow M$ such that the triple $(\mathcal{R}_{K'} \bigotimes_{\mathcal{R}} M, \varphi_d, \nabla)$ is an object of $\mathbf{M}_\Phi^\nabla_{\mathcal{R}_{K'},\sigma^d}$. 


Proof. — Let $f : F \to E$ be a finite Galois extension in $F^{\text{sep}}$ such that $f^*M$ is unipotent. Let $\{\rho_\lambda\}$ be the finite set of all irreducible representations of $\text{Gal}(E/F)$ in $\mathbb{Q}_p^{\text{alg}}$. Choose a finite extension $K'$ over $K$ and a positive integer $d$ such that (1) $K'$ contains all eigenvalues of $\rho_\lambda$, (2) $\sigma$ extends on $K'$ and (3) $\sigma^d \circ \rho_\lambda = \rho_\lambda$. We can choose such $K'$ and $d$ by (2.4.1). Replacing $K$, $q$ and $\sigma$ into $K'$, $q^d$ and $\sigma^d$, we may assume that all eigenvalues of $\rho_\lambda$ are contained in $K$ and $\sigma \circ \rho_\lambda = \rho_\lambda$.

Let $\{e_1, e_2, \cdots, e_r\}$ be a basis of $R_E \otimes R M$ such that $C_{E, e} \in M_r(K)$ (4.2.5) and denote by $L_E$ (resp. $\Gamma_E$) the $E^t$-subspace (resp. the $K$-subspace) of $R_E \otimes R M$ which is generated by $\{e_1, e_2, \cdots, e_r\}$. We prove that there exists a Frobenius structure $\varphi$ on $f^*M$ which commutes with the action of $\text{Gal}(E/F)$. By (4.2.4) $\Gamma_E$ is stable under the action of $\text{Gal}(E/F)$. By the assumption and Schur’s Lemma $\Gamma_E$ is a direct sum of $\Gamma_{E, \lambda}$ such that the Galois group $\text{Gal}(E/F)$ acts on $\Gamma_{E, \lambda}$ via $\rho_\lambda$ and that $\nabla \left( \frac{d}{dx} \right)(\Gamma_{E, \lambda}) \subset \Gamma_{E, \lambda}$. So it is enough to prove the existence of Frobenius structure on $R_E \otimes \Gamma_{E, \lambda}$ which commutes with the Galois action. Since $C_{f^*M, e}$ is nilpotent and the Galois action commutes with the nilpotent endomorphism $\nabla|_{\Gamma_{E, \lambda}}$, one can choose a basis $\{e_1^{\lambda}, \cdots, e_{r_\lambda}^{\lambda}, \cdots, e_{r_\lambda}^{\lambda}\}$ of $\Gamma_{E, \lambda}$ such that $\{e_{ij}^{\lambda}\}_{1 \leq j \leq r_\lambda}$ is a basis of the irreducible component on which $\text{Gal}(E/F)$ acts via $\rho_\lambda$ and that the differential structure is given by a direct sum of the type $C_{E, \lambda} = \begin{pmatrix} 0_{r_\lambda} & 1_{r_\lambda} \\ & \ddots & \ddots \\ & & 0_{r_\lambda} & 1_{r_\lambda} \\ 0 & & & 0_{r_\lambda} \end{pmatrix}$ by Schur’s Lemma. Here $r_\lambda$ is the degree of $\rho_\lambda$. Hence, there exists a Frobenius structure $\varphi$ which commutes with the Galois action by (4.2.3) and the condition (3) above in this proof. Of course, $L_E$ is stable under $\varphi$. Put $L = L_E^{\text{Gal}(E/F)}$ to be the Galois invariant part. Then $(L, \nabla|_L)$ is an $\mathcal{E}^t$-lattice of $M$ and $L$ is stable under $\varphi$.

From this proposition we know that, if one want to study some properties of quasi-unipotent $\nabla$-modules, then it is enough to work on $\varphi \cdot \nabla$-modules.

4.4. Let $\sigma_1$ and $\sigma_2$ be Frobenius on $R$. Define a functor

$$e_{\sigma_1, \sigma_2}^{\nabla_{qu}} : M_{\Phi_{\mathcal{R}, \sigma_2}} \to M_{\Phi_{\mathcal{R}, \sigma_1}}$$
as follows. For an object $M$ of $\mathbf{M}^\mathbf{\Phi}_{\mathcal{R},\sigma_2}^{\mathcal{V},\mathcal{Q}_{\mathcal{U}}}$ and for an $\mathcal{E}^\mathcal{I}$-lattice $L$ of $M$ (4.2.1), put

$$\tilde{\epsilon}_{\sigma_1,\sigma_2}^{\mathcal{Q}_{\mathcal{U}}} (M) = \mathcal{R} \bigotimes_{\mathcal{E}^\mathcal{I}} \tilde{\epsilon}_{\sigma_1,\sigma_2}(L).$$

(See the definition of $\tilde{\epsilon}_{\sigma_1,\sigma_2}$ in (3.4).)

**Lemma 4.4.1.** — The construction of the functor $\tilde{\epsilon}_{\sigma_1,\sigma_2}^{\mathcal{Q}_{\mathcal{U}}}(M)$ is independent of the choice of $\mathcal{E}^\mathcal{I}$-lattices.

**Proof.** — Let $L^\lambda$ (resp. $\{e^\lambda_1, e^\lambda_2, \cdots, e^\lambda_r\}$) be an $\mathcal{E}^\mathcal{I}$-lattice of an object $M$ of $\mathbf{M}^\mathbf{\Phi}_{\mathcal{R},\sigma_2}^{\mathcal{V},\mathcal{Q}_{\mathcal{U}}}$ (resp. a basis of $L^\lambda$) ($\lambda = \alpha, \beta$). Denote by $\epsilon_{\sigma_1,\sigma_2}^{\lambda,\mathcal{Q}_{\mathcal{U}}}$ the map which is defined using $L^\lambda$ ($\lambda = \alpha, \beta$). Define a matrix $Q \in GL_r(\mathbb{R})$ by $(e^\alpha_1, e^\alpha_2, \cdots, e^\alpha_r) = (e^\beta_1, e^\beta_2, \cdots, e^\beta_r)Q$ and put a matrix $\Omega^\lambda$ to be $\epsilon_{\sigma_1,\sigma_2}^{\lambda,\mathcal{Q}_{\mathcal{U}}}(1 \otimes (e^\lambda_1, e^\lambda_2, \cdots, e^\lambda_r)) = (1 \otimes (e^\lambda_1, e^\lambda_2, \cdots, e^\lambda_r))\Omega^\lambda$. It is enough to prove that the diagram

$$\begin{array}{ccc}
\sigma_1^* M & \xrightarrow{\epsilon_{\sigma_1,\sigma_2}^{\alpha,\mathcal{Q}_{\mathcal{U}}}} & \sigma_2^* M \\
\downarrow & & \downarrow \\
\sigma_1^* M & \rightarrow & \sigma_2^* M
\end{array}$$

is commutative. In other words, we have only to prove $\sigma_2(Q)\Omega^\alpha = \Omega^\beta\sigma_1(Q)$.

Assume that $A_{M,e^\lambda,\sigma_1}, C_{M,e^\lambda}$ ($\lambda = \alpha, \beta$ and $i = 1, 2$) and $Q$ are convergent and $\sigma_1$ (resp. $\sigma_2$) is defined on the annulus $\gamma \leq |x| < 1$ for some $\gamma < 1$. Define a $K$-algebra

$$\mathcal{E}(\gamma) = \left\{ \sum_{n=\infty}^\infty a_n x^n \mid a_n \in K, |a_n| \gamma^n \text{ is bounded, } \right\}.$$ 

Then $\mathcal{E}(\gamma)$ is complete under the norm $|\sum_{n=\infty}^\infty a_n x^n|_{\gamma} = \sup_n |a_n| \gamma^n$ and $\sigma_i$ ($i = 1, 2$) induces a map on $\mathcal{E}(\gamma)$. The pair $(A_{M,e^\lambda,\sigma_1}, C_{M,e^\lambda})$ ($\lambda = \alpha, \beta$ and $i = 1, 2$) define an $\mathcal{E}(\gamma)$ module $L^\lambda_i(\gamma)$ with a connection and a Frobenius structure with respect to $\sigma_i$ ($i = 1, 2$). Since $Q$ is contained in $GL_n(\mathcal{E}(\gamma))$, $L^\lambda_i(\gamma)$ is isomorphic to $L^i_\beta(\gamma)$ ($i = 1, 2$). By the similar arguments as in (3.4) we can define a similar map of $\epsilon_{\sigma_1,\sigma_2}$ for $\mathcal{E}(\gamma)$ and the matrix $\Omega^\lambda$ is the representative matrix of this map for the basis $\{e^\lambda_1, e^\lambda_2, \cdots, e^\lambda_r\}$. Therefore, we have $\sigma_2(Q)\Omega^\alpha = \Omega^\beta\sigma_1(Q)$. 

**Lemma 4.4.2.** — Let $\sigma_1$, $\sigma_2$ and $\sigma_3$ be Frobenius on $\mathcal{R}$. Then we have

(i) $\tilde{\epsilon}_{\sigma_1,\sigma_1} = \text{id}$;

(ii) $\tilde{\epsilon}_{\sigma_1,\sigma_3} = \tilde{\epsilon}_{\sigma_1,\sigma_2} \tilde{\epsilon}_{\sigma_2,\sigma_3}$. 

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Theorem 4.4.3. — The category $\Phi_{\mathcal{R}, \sigma}^\nabla$ is independent of the choice of Frobenius on $\mathcal{R}$ via the functor $\mathcal{F}_{\sigma_1, \sigma_2}^{\nabla}$.

Remark 4.4.4. — The author does not know whether the category $\Phi_{\mathcal{R}, \sigma}^\nabla$ is independent of the choice of Frobenius on $\mathcal{R}$ or not. But it is expected that the natural functor $\Phi_{\mathcal{R}, \sigma}^\nabla \to \Phi_{\mathcal{R}, \sigma}^\nabla$ is an equivalence.

5. Slope filtration for Frobenius structures.

In this section we define a slope filtration for Frobenius structures and prove that a $\varphi\nabla$-module over $\mathcal{R}$ is quasi-unipotent if and only if it has a slope filtration.

5.1. Fix a Frobenius $\sigma$ on $\mathcal{R}$.

Definition 5.1.1. — Let $M$ be an object of $\Phi_{\mathcal{R}, \sigma}^\nabla$. An increasing filtration $\{S_\gamma M\}_{\gamma \in \mathbb{Q}}$ of $M$ is a slope filtration for Frobenius structures if and only if it satisfies the condition as follows:

(i) $S_\gamma M$ is a sub $\varphi\nabla$-module of $M$ over $\mathcal{R}$;

(ii) $S_\gamma M = 0$ ($\gamma < 0$) and $S_\gamma M = M$ ($\gamma > 0$);

(iii) for a sufficiently small positive rational number $\epsilon$, there exists an $\mathcal{E}\nabla$-lattice $L_\gamma$ of $S_\gamma M/S_{\gamma-\epsilon} M$ which is pure of slope $\gamma$.

Proposition 5.1.2. — If $L$ is an object of $\Phi_{\mathcal{R}, \sigma}^\nabla$ pure of slope $\gamma$, then there are a finite separable extension $f : F \to E$ and a basis $\{e_1, e_2, \ldots, e_r\}$ of $f^* M$ such that $C_{f^* M, e} = 0$.

Proof. — Replacing $(M, \varphi, \nabla)$ into $(M, a\varphi^d, \nabla)$ for a suitable positive integer $d$ and $a \in K$, we may assume $\gamma = 0$. The assertion follows [TN2, 4.2.6].

Proposition 5.1.3. — Let $\eta : M_1 \to M_2$ be a morphism of $\Phi_{\mathcal{R}, \sigma}^\nabla$. Assume that both $M_1$ and $M_2$ have a slope filtration $S_\gamma M_i$ ($i = 1, 2$) for Frobenius structures. Then $\eta$ is strict for filtrations, that is, $\eta(S_\gamma M_1) = \eta(M_1) \cap S_\gamma M_2$ for any $\gamma \in \mathbb{Q}$.

Proposition (5.1.3) follows from Lemma (5.1.4) below.
LEMMA 5.1.4. — Let $M_1$ (resp. $M_2$) be an object of $\mathbf{M}^{\Phi_{\mathcal{R},\sigma}}$ with an $\mathcal{E}^\dagger$-lattice $L_1$ (resp. $L_2$) pure of slope $\gamma_1$ (resp. $\gamma_2$).

(1) If $\gamma_1 \neq \gamma_2$, then there is no nontrivial morphism from $M_1$ to $M_2$.

(2) If $\gamma_1 = \gamma_2$, then any morphism $\eta_1 : M_1 \to M_2$ preserves the $\mathcal{E}^\dagger$-lattice, that is, $\eta(L_1) = \eta(M_1) \cap L_2$.

Proof. — (1) Since $\text{Hom}_{\mathbf{M}^{\Phi_{\mathcal{R},\sigma}}}(M_1, M_2) \cong \text{Hom}_{\mathbf{M}^{\Phi_{\mathcal{R},\sigma}}}((\mathcal{R}, M_1^\gamma \otimes M_2)$, we have only to prove the assertion in the case where $M_1 = \mathcal{R}$ and $M_2$ is an arbitrary object with $\mathcal{E}$-lattice $L$ pure of slopes $\gamma$. There exist a finite separable extension $f : F \to E$ in $F^{\text{sep}}$ and an element $A \in \text{GL}_r(K)$ such that $M$ is isomorphic to $((\mathcal{R}_E)^r, A\sigma, d)$ by (5.1.2). One can easily see that there is no morphism from the unit object to $f^*M$ if $\gamma \neq 0$.

The assertion (2) follows (2.2.3) and (5.1.2).

COROLLARY 5.1.5. — A slope filtration for Frobenius structures of an object of $\mathbf{M}^{\Phi_{\mathcal{R},\sigma}}$ is unique.

5.2. We state one of our main local theorems.

THEOREM 5.2.1. — Let $M$ be an object of $\mathbf{M}^{\Phi_{\mathcal{R},\sigma}}$. $M$ is quasi-unipotent if and only if $M$ has a slope filtration $\{S_{\gamma}M\}_{\gamma \in \mathbb{Q}}$ for Frobenius structures.

Proof. — It is enough to prove the assertion in the case where $\sigma(x) = x^q$ by (3.4.9), (3.4.10) and (4.4.3). Let $f : F \to E$ be a finite separable extension in $F^{\text{sep}}$ such that $f^*M$ is unipotent. Then there exists a $\text{Gal}(E/F)$-stable $K$-lattice $\Gamma_E$ of $f^*M$. In fact, choose a basis $\{e_1, e_2, \ldots, e_r\}$ of $f^*M$ as in (4.2.5) and put $\Gamma_E$ to be a $K_E$-subspace of $f^*M$ which is generated by $\{e_1, e_2, \ldots, e_r\}$. Here $K_E$ is the finite unramified extension with residue class field $k_E$. Then $\Gamma_E$ is stable under the Frobenius structure $\varphi$ and the action $\text{Gal}(E/F)$ by (4.2.4) and (4.2.5), that is, $\nabla|_{\Gamma_E} \circ \varphi|_{\Gamma_E} = q\varphi|_{\Gamma_E} \circ \nabla|_{\Gamma_E}$. By the theory of $\varphi$-spaces with a nilpotent structure over a complete discrete valuation field we have a slope filtration $\{S_{\gamma}\Gamma_E\}$ for the Frobenius structure $\varphi|_{\Gamma_E}$ of $\Gamma_E$ which is compatible with the nilpotent operator $\nabla|_{\Gamma_E}$. Moreover the theory of slopes implies that the filtration $\{S_{\gamma}\Gamma_E\}$ is compatible with the action of $\text{Gal}(E/F)$ since $\varphi|_{\Gamma_E}$ commutes with the action of $\text{Gal}(E/F)$. Define a filtration $\{S_{\gamma}M\}$ of
\[ S^\gamma M = \mathcal{R} \bigotimes_{E^\dagger} (\mathcal{E}_E^\dagger \bigotimes_{K_E} S^\gamma \Gamma_E)^{\text{Gal}(E/F)}. \]

\( \{S^\gamma M\} \) is a slope filtration for Frobenius structures of \( M \) by (2.2.4) and (3.3.5). The converse follows from (5.1.2).

**Remark 5.2.2.** — In Theorem (5.2.1) the slope filtration \( \{S^\gamma M\} \) of \( M \) is split as \( \varphi \)-modules (not as \( \nabla \)-modules) over \( \mathcal{R} \) if we choose a Frobenius \( \sigma(x) = x^q \), because the filtration \( \{S^\gamma \Gamma_E\} \) of \( \Gamma_E \) over \( K_E \) is split as \( \varphi \)-\( \text{Gal}(E/F) \)-modules in the above proof. In general cases the slope filtration is not always split as \( \varphi \)-modules.

**6. Quasi-unipotent overconvergent \( F \)-isocrystals on a curve.**

In this section we give a definition of quasi-unipotent overconvergent \( F \)-isocrystals on a curve and apply our local study to them. We use some results on overconvergent \( F \)-isocrystals on curves from [Bel], [Be2], [Be3] and [Cr1].

**6.1.** Let \( k \) (resp. \( K \)) be a perfect field of positive characteristic \( p \) (resp. a complete discrete valuation field with the residue class field \( k \) and with a Frobenius \( \sigma \)). Let \( X \) be a smooth curve over \( \text{Spec} \ k \) which is geometrically connected. For a closed point \( s \in X \), denote by \( k(s) \) (resp. \( K(s) \)) the residue class field at \( s \) (resp. the finite unramified extension of \( K \) with the residue class field \( k(s) \)).

Let \( U \) be a dense open subscheme of \( X \) and put \( Z = X - U \). Fix a closed point \( s \in X \) and denote by \( \mathcal{X} \) a formal scheme over \( \text{Spf} \ OK \) which is a lifting of \( X/\text{Spec} \ k \) and formally smooth around \( s \). Choose a section \( x \in \Gamma(O_{\mathcal{X}}) \) which is a lifting of a local parameter of \( O_X \) at \( s \). Since \( \mathcal{X}/\text{Spf} \ OK \) is formally smooth at \( s \), the completion of \( O_{\mathcal{X}} \) at \( s \) is isomorphic to \( O_{K(s)}[[x]] \). Put \( \mathcal{R}_s \) (resp. \( E_s \), resp. \( E^\dagger_s \), resp. \( S_{K(s)} \)) to be \( \mathcal{R}_{x,K(s)} \), (resp. \( E_{x,K(s)} \), resp. \( E^\dagger_{x,K(s)} \), resp. \( K \bigotimes_{O_{K(s)}} O_{K(s)}[[x]] \)). Therefore, we have an injective homomorphism

\[ i_s : \Gamma(O_{[U]}) \to E_s \ (x \mapsto x) \]
of $K$-algebras. The map $i_s$ is independent of the choice of the lifting of parameter via the natural isomorphism $\mathcal{E}^\dagger_{x,K(s)} \cong \mathcal{E}^\dagger_{x',K(s)}$ for any parameter $x'$. Especially, if $s \in U$, then $i_s(\Gamma(O_{\mathcal{U}})) \subset S_{K(s)}$. By [Cr1, 4.7.] we have

**Lemma 6.1.1.** Assume that $X$ is affine and $U = X - \{s\}$. Under the notation as above, we have

$$i_s(\Gamma(O_{\mathcal{U}})) = \operatorname{Im}(i_s) \cap S_{K(s)};$$

$$i_s(\Gamma(j^\dagger O_{\mathcal{U}})) = \operatorname{Im}(i_s) \cap \mathcal{E}^\dagger_s,$$

where $j : U \to X_{\text{an}}$.

By the construction, $i_s\left(\frac{d}{dx}(u)\right) = \delta_s (i_s(u))$ for any section $u \in \Gamma(O_{\mathcal{U}})$. If $\sigma : O_{\mathcal{U}} \to O_{\mathcal{U}}$ is a lifting of $q$-th power map on $O_U$ ($q = p^a$) which is an extension of the Frobenius $\sigma$ on $K$, then $\sigma$ extends on $\mathcal{E}_s$ (resp. $S_{K(s)}$ if $s \in U$). We call the extension $\sigma$ a Frobenius on $O_{\mathcal{U}}$.

Denote by $\text{Isoc}^\dagger(U, X/K)$ (resp. $\text{F}^a\text{-Isoc}^\dagger(U, X/K)$) the abelian category of overconvergent isocrystals on $U/K$ around $Z$ (resp. the category of overconvergent $\text{F}^a$-isocrystals on $U/K$ around $Z$) [Be3, (2.2.10)]. By the natural extension $i_{\mathcal{R}_s} : \Gamma(j^\dagger O_{\mathcal{U}}) \to \mathcal{R}_s$ of scalar there is a functor

$$i_{\mathcal{R}_s}^* : \text{Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{R}_s},$$

which is factored via the natural functor $i_{\mathcal{E}_s}^* : \text{Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{E}_s}$ (resp. $i_{S_{K(s)}}^* : \text{Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{S_{K(s)}}$ if $s \in U$). For any Frobenius $\sigma$ on $O_{\mathcal{U}}$, we also have a natural functor

$$i_{\mathcal{R}_s,\sigma}^* : \text{F}^a\text{-Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{R}_s,\sigma},$$

which is factored via the natural functor $i_{\mathcal{E}_s,\sigma}^* : \text{F}^a\text{-Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{E}_s,\sigma}$ (resp. $i_{S_{K(s)},\sigma}^* : \text{F}^a\text{-Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{S_{K(s)},\sigma}$ if $s \in U$). One can easily see that the functor $i_{\mathcal{R}_s}^*$ (resp. $i_{\mathcal{R}_s,\sigma}^*$) is independent of all choices up to canonical transformations. One can also see that the functor $i_{\mathcal{R}_s,\sigma}^*$ is independent of the choice of Frobenius $\sigma$ up to the functor $\mathcal{E}_{\sigma_1,\sigma_2}$ by the definition of $F$-isocrystals, Proposition (3.4.10) and Lemma (4.3.1).

Now we define a quasi-unipotent overconvergent isocrystal. Our definition differs from that in [Cr2, 10.11], but we will prove that our definition is equivalent to Crew's one in Theorem (6.1.6).

**Definition 6.1.2.** (1) An object $\mathcal{M}$ of $\text{Isoc}^\dagger(U, X/K)$ (resp. $\text{F}^a\text{-Isoc}^\dagger(U, X/K)$)

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| NOBUO TSUZUKI | of $K$-algebras. The map $i_s$ is independent of the choice of the lifting of parameter via the natural isomorphism $\mathcal{E}^\dagger_{x,K(s)} \cong \mathcal{E}^\dagger_{x',K(s)}$ for any parameter $x'$. Especially, if $s \in U$, then $i_s(\Gamma(O_{\mathcal{U}})) \subset S_{K(s)}$. By [Cr1, 4.7.] we have

**Lemma 6.1.1.** — Assume that $X$ is affine and $U = X - \{s\}$. Under the notation as above, we have

$$i_s(\Gamma(O_{\mathcal{U}})) = \operatorname{Im}(i_s) \cap S_{K(s)};$$

$$i_s(\Gamma(j^\dagger O_{\mathcal{U}})) = \operatorname{Im}(i_s) \cap \mathcal{E}^\dagger_s,$$

where $j : U \to X_{\text{an}}$.

By the construction, $i_s\left(\frac{d}{dx}(u)\right) = \delta_s (i_s(u))$ for any section $u \in \Gamma(O_{\mathcal{U}})$. If $\sigma : O_{\mathcal{U}} \to O_{\mathcal{U}}$ is a lifting of $q$-th power map on $O_U$ ($q = p^a$) which is an extension of the Frobenius $\sigma$ on $K$, then $\sigma$ extends on $\mathcal{E}_s$ (resp. $S_{K(s)}$ if $s \in U$). We call the extension $\sigma$ a Frobenius on $O_{\mathcal{U}}$.

Denote by $\text{Isoc}^\dagger(U, X/K)$ (resp. $\text{F}^a\text{-Isoc}^\dagger(U, X/K)$) the abelian category of overconvergent isocrystals on $U/K$ around $Z$ (resp. the category of overconvergent $\text{F}^a$-isocrystals on $U/K$ around $Z$) [Be3, (2.2.10)]. By the natural extension $i_{\mathcal{R}_s} : \Gamma(j^\dagger O_{\mathcal{U}}) \to \mathcal{R}_s$ of scalar there is a functor

$$i_{\mathcal{R}_s}^* : \text{Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{R}_s},$$

which is factored via the natural functor $i_{\mathcal{E}_s}^* : \text{Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{E}_s}$ (resp. $i_{S_{K(s)}}^* : \text{Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{S_{K(s)}}$ if $s \in U$). For any Frobenius $\sigma$ on $O_{\mathcal{U}}$, we also have a natural functor

$$i_{\mathcal{R}_s,\sigma}^* : \text{F}^a\text{-Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{R}_s,\sigma},$$

which is factored via the natural functor $i_{\mathcal{E}_s,\sigma}^* : \text{F}^a\text{-Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{\mathcal{E}_s,\sigma}$ (resp. $i_{S_{K(s)},\sigma}^* : \text{F}^a\text{-Isoc}^\dagger(U, X/K) \to \mathcal{M}^\mathcal{V}_{S_{K(s)},\sigma}$ if $s \in U$). One can easily see that the functor $i_{\mathcal{R}_s}^*$ (resp. $i_{\mathcal{R}_s,\sigma}^*$) is independent of all choices up to canonical transformations. One can also see that the functor $i_{\mathcal{R}_s,\sigma}^*$ is independent of the choice of Frobenius $\sigma$ up to the functor $\mathcal{E}_{\sigma_1,\sigma_2}$ by the definition of $F$-isocrystals, Proposition (3.4.10) and Lemma (4.3.1).

Now we define a quasi-unipotent overconvergent isocrystal. Our definition differs from that in [Cr2, 10.11], but we will prove that our definition is equivalent to Crew's one in Theorem (6.1.6).

**Definition 6.1.2.** (1) An object $\mathcal{M}$ of $\text{Isoc}^\dagger(U, X/K)$ (resp. $\text{F}^a\text{-Isoc}^\dagger(U, X/K)$)
(U, X/K) is unipotent at a closed point s ∈ X if and only if \( i_{\mathcal{M}}^* \mathcal{M} \) is unipotent. An object \( \mathcal{M} \) of \( \text{Isoc}^\dagger(U, X/K) \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K) \)) is unipotent if and only if \( \mathcal{M} \) is unipotent at any closed point on \( X \).

(2) An object \( \mathcal{M} \) of \( \text{Isoc}^\dagger(U, X/K) \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K) \)) is quasi-unipotent at a closed point \( s \in X \) if and only if \( i_{\mathcal{M}}^* \mathcal{M} \) is quasi-unipotent. An object \( \mathcal{M} \) of \( \text{Isoc}^\dagger(U, X/K) \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K) \)) is quasi-unipotent if and only if \( \mathcal{M} \) is quasi-unipotent at any closed point on \( X \). Denote by \( \text{Isoc}^\dagger(U, X/K)^{\text{qu}} \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K)^{\text{qu}} \)) the full subcategory of \( \text{Isoc}^\dagger(U, X/K) \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K) \)) which consists of quasi-unipotent objects.

**Proposition 6.1.3.** — The category \( \text{Isoc}^\dagger(U, X/K)^{\text{qu}} \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K)^{\text{qu}} \)) is an abelian subcategory of \( \text{Isoc}^\dagger(U, X/K) \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K) \)) which is closed under subquotients, tensor products and duals.

Let \( i : Y \subset X \) (resp. \( V \subset U \)) be a non-empty open subscheme and put \( Z_Y = Y - V \). Denote by \( i^\dagger : \text{Isoc}^\dagger(U, X/K) \to \text{Isoc}^\dagger(V, Y/K) \) (resp. \( i^\dagger : \text{F}^a\text{-Isoc}^\dagger(U, X/K) \to \text{F}^a\text{-Isoc}^\dagger(V, Y/K) \)) the natural inverse image functor which is induced by \( i \). By the definition we have

**Proposition 6.1.4.** — Under the notation as above, let \( \mathcal{M} \) be an object of \( \text{Isoc}^\dagger(U, X/K) \) (resp. \( \text{F}^a\text{-Isoc}^\dagger(U, X/K) \)). If \( \mathcal{M} \) is unipotent (resp. quasi-unipotent), then \( i^\dagger \mathcal{M} \) is so. Assume furthermore that \( Y = X \), then \( \mathcal{M} \) is unipotent (resp. quasi-unipotent) if and only if \( i^\dagger \mathcal{M} \) is so.

Let \( f : Y \to X \) be a finite morphism of smooth curves over \( \text{Spec} \ k \) and put \( U_Y = Y \times_X U \) and \( Z_Y = Y \times_X Z \). Assume that the restriction \( f_U : U_Y \to U \) of \( f \) is finite and etale. Since one can choose a lifting \( Y \) of \( Y \) such that \( [U_Y] \to [U] \) is finite etale and \( j^! O_Y[1] \) is finite of degree \( \text{deg}(f) \) over \( j^! O_X[1] \) locally at \( s \), one can define the inverse image functor (resp. the direct image functor)

\[
\begin{align*}
    f^* : \text{Isoc}^\dagger(U, X/K) &\to \text{Isoc}^\dagger(U_Y, Y/K) \\
    (\text{resp. } f_* : \text{Isoc}^\dagger(U_Y, Y/K) &\to \text{Isoc}^\dagger(U, X/K))
\end{align*}
\]

by \( f^* \mathcal{M} = \bigotimes_{f^{-1}j^! O_{X[1]} \to f_* j^! O_{Y[1]}} f^{-1} \mathcal{M} \) (resp. the restriction \( j^! O_{X[1]} \to f_* j^! O_{Y[1]} \) of scalar). One can also define the inverse image functor \( f^* \) and the direct image functor \( f_* \) for \( F \)-isocrystals. Let \( t \in Y \) be a closed point with
Choose a formally lifting $y$ over $\text{Sp}/\text{OK}$ of $V/\text{Spec} \ k$ which is formally smooth around $t$, a lifting $f : Y \to X$ over $\text{Sp}/\text{OK}$ of $f : Y \to X$, a section $y \in \Gamma(O_Y)$ which is a lifting of a local parameter at $t$. Such lifting $f$ always exists locally on $X$ and our arguments below work well on this situation. Then $f$ induces an injection $f : \mathcal{R}_s \to \mathcal{R}_t$ of $K$-algebras and we have natural commutative diagrams

$$
\begin{array}{ccc}
\text{Isoc}_U(U, X/K) & \xrightarrow{f^*} & \text{Isoc}_U(U_Y, Y/K) \\
\downarrow i_{\mathcal{R}_s}^* & & \downarrow i_{\mathcal{R}_t}^* \\
\mathcal{M}\mathcal{R}_s & \xrightarrow{f^*} & \mathcal{M}\mathcal{R}_t
\end{array}
$$

and

$$
\begin{array}{ccc}
\text{Isoc}_U(U_Y, Y/K) & \xrightarrow{f^*} & \text{Isoc}_U(U, X/K) \\
\downarrow i_{\mathcal{R}_t}^* & & \downarrow i_{\mathcal{R}_s}^* \\
\mathcal{M}\mathcal{R}_t & \xrightarrow{f^*} & \mathcal{M}\mathcal{R}_s
\end{array}
$$

If $\sigma$ is a Frobenius on $O_{U[r]}$, then $\sigma$ extends uniquely on $O_{U_Y[r]}$ since $f_U$ is etale. We also have commutative diagrams for $F$-isocrystals as in above diagrams. By Proposition (4.1.3) and (6.1.3) we have

**Proposition 6.1.5.** — Under the notation as above,

1. an object $M$ of $\text{Isoc}_U(U, X/K)$ (resp. $F^a\text{-Isoc}_U(U, X/K)$) is quasi-unipotent if and only if $f^* M$ is quasi-unipotent;

2. an object $M$ of $\text{Isoc}_U(U_Y, Y/K)$ (resp. $F^a\text{-Isoc}_U(U_Y, Y/K)$) is quasi-unipotent if and only if $f^* M$ is quasi-unipotent.

Now we compare Crew's definition to ours.

**Theorem 6.1.6.** — Let $M$ be an object of $\text{Isoc}_U(U, X/K)$ (resp. $F^a\text{-Isoc}_U(U, X/K)$. $M$ is quasi-unipotent if and only if there is a finite morphism $f : Y \to X$ of smooth curves over $\text{Spec} \ k$ and a nonempty open subscheme $i : V \to U$ such that $f_V : V_Y \to V$ is etale and that $f_V^* i^! M$ is unipotent.

**Proof.** — Assume that $M$ is quasi-unipotent. Denote by $K(X)$ the field of rational functions of $X$. Since $Z$ is a finite set, there is a finite separable extension $L$ of $K(X)$ such that, for any point $s \in Z$ and for any place $t$ of $L$ above $s$, $f_{t \to s}^* i_{\mathcal{R}_s}^! M$ is unipotent over $\mathcal{R}_t (= \mathcal{R}_{L_t})$. Here $K(X)_s$ (resp. $L_t$) is completion of $K(X)$ (resp. $L$) at $s$ (resp. $t$) and $f_{t \to s} : K(X)_s \to L_t$ is a structure map. Define a smooth curve $Y$ over
k by the normalization of X in L. Since L is separable over K(X), the natural morphism f : Y → X is generically etale. Therefore we obtain the assertion by (4.1.3). The converse follows from (4.1.3).

Remark 6.1.7. — Matsuda pointed out that, either if X is affine or if the number of geometric points in X – U is greater than 1, then one can choose a finite covering Y of X such that U_Y is etale over U in Theorem 6.1.6 by [Ka2, 2.1.6].

6.2. We give some examples of quasi-unipotent overconvergent F-isocrystals. By Proposition (4.2.1) we have

PROPOSITION 6.2.1. — A convergent F-isocrystal on X/K is quasi-unipotent.

DEFINITION 6.2.2. — Let M be an object of F_a-Isoc^+(U, X/K). An increasing filtration \( \{S_\gamma M\}_{\gamma \in \mathbb{Q}} \) of M is a slope filtration for Frobenius structures if and only if it satisfies the conditions as follows:

(i) \( S_\gamma M \) is a subobject of \( M \) in \( F_a-\text{Isoc}^+(U, X/K) \);

(ii) \( S_\gamma M = 0 \) (\( \gamma << 0 \)) and \( S_\gamma M = M \) (\( \gamma >> 0 \));

(iii) for a Frobenius \( \sigma \) on \( j^! O_{U,1} \), \( \{i_{R_s}^* S_\gamma M\}_{\gamma} \) is a slope filtration for Frobenius structures of \( i_{R_s}^* M \) of \( M^{\phi}_{R_s,\sigma} \) at any point \( s \in X \).

The condition (iii) above is independent of the choice of Frobenius by Proposition (3.4.9). By Theorem (5.2.1) we have

PROPOSITION 6.2.3. — If an object M of F_a-Isoc^+(U, X/K) has a slope filtration for Frobenius structures, then M is quasi-unipotent.

COROLLARY 6.2.4 ([Cr1, 4.12]). — An overconvergent F_a-isocrystal on U/K around Z of rank one is quasi-unipotent.

COROLLARY 6.2.5. — A unit-root overconvergent F_a-isocrystal on U/K around Z is quasi-unipotent.

Example 6.2.6. — Let \( p \) be an odd prime. Let \( k = \mathbb{F}_p \), \( K = \mathbb{Q}_p(\pi) \) with \( \pi^{p-1} = -p \) and \( \sigma \) be a continuous lifting of \( p \)-th power map on \( K \) with \( \sigma(\pi) = \pi \). Put \( X = \mathbb{P}^1_k \) (resp. \( U = \mathcal{O}m_k \), resp. \( Z = \{0, \infty\} \)) and \( X = \mathbb{P}^1 \) over \( \text{Spf } O_K \) with a coordinate \( x \). In [Dw] B. Dwork constructed the Bessel
overconvergent $F$-isocrystal $\mathcal{M}$ on $U/K$ around $Z$. $\mathcal{M}$ is of rank 2 and is defined by the following differential and Frobenius structures:

$$\nabla(e_1, e_2) = dx \otimes (e_1, e_2) \begin{pmatrix} 0 & -x^{-1} \\ -\pi^2 & 0 \end{pmatrix}$$

$$\varphi(e_1, e_2) = (e_1, e_2) \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

on the strict neighbourhood $|x| \leq \gamma$ for some $\gamma > 1$ of $|U[\mathcal{X}]$ with

$$\left( \begin{array}{cc} a_1(0) & a_2(0) \\ a_3(0) & a_4(0) \end{array} \right) = \left( \begin{array}{cc} 1 & * \\ 0 & p \end{array} \right), \quad \left( \begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array} \right) \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \quad (\text{mod} \ \pi)$$

and

$$\det \left( \begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array} \right) = p.$$ 

CLAIM. — $\mathcal{M}$ is quasi-unipotent.

By Proposition (4.2.1) $\mathcal{M}$ is unipotent on any closed point $s \in X - \{\infty\}$. Now we discuss the quasi-unipotency of $\mathcal{M}$ at $\infty$ following the arguments of [Dw, Section 8]. We change the coordinate $x$ into $x^{-1}$ and denote by $F = k((x))$ the completion of the field of fractions of the local ring $O_{X,\infty}$ at the infinity. Define a tamely ramified extension $E = k((y))$ over $F$ with $4y^2 = x$ and choose a lifting $y$ of the parameter of $\mathcal{R}_E$ with $4y^2 = x$. Then the differential structure of $i_{\infty, *}\mathcal{M}$ over $\mathcal{R}_E$ is given by

$$\nabla(e_1, e_2) = \frac{dy}{y} \otimes (e_1, e_2) \begin{pmatrix} 0 & 2 \\ 0 & -\pi^2 y^{-2} \end{pmatrix}.$$ 

If $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a solution of the differential equation $\delta_y \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) + \begin{pmatrix} 0 & 2 \\ 0 & -\pi^2 y^{-2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$, then $z_1$ satisfies the differential equation $\delta_y^2(z_1) = \pi^2 y^{-2} z_1$. Consider the formal solution $z_1 = y^{\frac{1}{2}} u_+(y) \exp(\pm \pi y^{-1})$. Then $u_\pm = u_\pm(y)$ satisfies the differential equation:

$$4y^2 \delta_y^2(u_\pm) + 4(y \mp 2\pi) \delta_y(u_\pm) + xy_\pm = 0.$$

By easy calculations we have

$$u_\pm = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{(2n-1)!!}{(8\pi)^n} y^n,$$

where $(2n - 1)!! = 1 \times 3 \times \cdots \times (2n - 1)$, and $u_\pm$ is convergent on the unit disk $|y| < 1$. Put a matrix

$$Q = \begin{pmatrix} u_+ & u_- \\ \delta_y(u_+) + \left( \frac{1}{2} - \pi y^{-1} \right) u_+ & \delta_y(u_-) + \left( \frac{1}{2} + \pi y^{-1} \right) u_- \end{pmatrix}.$$
Since $\delta_y(\det Q) = -\det Q$, we have $\det Q = 2\pi y^{-1}$ and $Q \in \text{GL}_2(\mathcal{R}_F)$. Change the basis $(e_1, e_2)$ into $(e_+, e_-) = (e_1, e_2)Q$. By our construction we have

$$\nabla(e_+, e_-) = \frac{dy}{y} \otimes (e_+, e_-)C \quad \text{with} \quad C = \begin{pmatrix} -\frac{1}{2} + \pi y^{-1} & 0 \\ 0 & -\frac{1}{2} - \pi y^{-1} \end{pmatrix}. $$

Put a matrix $A = A_{\mathcal{M}, e_{\pm}}$. Note that $\sigma(y) = 2^{p-1}y^p$, and the pair $(A, C)$ satisfies the relation $\delta_y(A) + CA = pA\sigma(C)$. Since $\exp(2\pi y^{-1})$ is not contained in $\mathcal{R}_F$, we have

$$A = \begin{pmatrix} \alpha_y^{-\frac{p-1}{2}} \exp(\pi(y^{-1} - \sigma(y^{-1})) & 0 \\ 0 & \alpha_y^{-\frac{p-1}{2}} \exp(-\pi(y^{-1} - \sigma(y^{-1})) \end{pmatrix}$$

for some $\alpha_+, \alpha_- \in K^\times$ with $\alpha_+\alpha_- = 2^{1-p}$. Hence, $\mathcal{M}$ is quasi-unipotent at $\infty$ by the example (4.1.4). Finally we determine slopes of $\mathcal{M}$ at $\infty$. Since $\tau(y) = -y$ for the nontrivial element $\tau$ in $\text{Gal}(E/F)$, $e_+ + e_-$ and $ye_+ - ye_-$ is a basis of $t_{\infty}^*\mathcal{M}$ over $\mathcal{R}_F$. By the commutativity between the Galois action and the Frobenius structure we have

$$\varphi(e_+ + e_-) = b_1(e_+ + e_-) + b_2(ye_+ - ye_-) \quad \text{with} \quad b_1, b_2 \in \mathcal{R}_F.$$ 

On the other hand we have

$$\varphi(e_+ + e_-) = \alpha_+ y^{-\frac{p-1}{2}} \exp(\pi(y^{-1} - \sigma(y^{-1}))) e_+ + \alpha_- y^{-\frac{p-1}{2}} \exp(-\pi(y^{-1} - \sigma(y^{-1}))) e_-.$$ 

Comparing both identities, we obtain $v_p(\alpha_+) = v_p(\alpha_-) = \frac{1}{2}$ for $\alpha_+\alpha_- = 2^{1-p}$. Therefore, all slopes of $\mathcal{M}$ at $\infty$ are $\frac{1}{2}$ by Proposition (3.3.5).

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