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ON THE NORMS OF THE RANDOM WALKS ON PLANAR GRAPHS

by **Andrzej ŻUK**

1. Introduction.

Let us consider a connected planar graph X such that the degree of each vertex, *i.e.* the number of edges adjacent to the vertex, is finite and such that there are a finite number of vertices in any compact subset of the plane. Let us also suppose that there are no loops or multiple edges. On this graph we consider a random walk that goes from a vertex to one of its neighbors picked uniformly at random. We associate with this random walk a random walk operator M

$$Mf(q) = \frac{1}{N(q)} \sum_{p\sim q} f(p) \quad \text{for} \quad f \in \ell^2(X, N)$$

where $N(q)$ is the degree of vertex q and where $p \sim q$ means that $\{q, p\}$ is an edge, *i.e.* p and q are neighbors. The operator M is self-adjoint on the space $\ell^2(X, N)$.

In this article we will establish some upper bounds for $\|M\|$, the norm of this operator, acting on $\ell^2(X, N)$. If $P^n(q, p)$ is the probability of going from q to p in n steps, then we know (see [12]) that

$$\|M\| = \lim_{n\to\infty} \left(P^{2n}(q, q)\right)^{1/2n}.$$

The fact that the norm of M is strictly less than 1 is equivalent to a strong isoperimetric inequality (see [7]) and implies that the random

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walk is transient. In this paper we place local conditions on the planar graph which ensure that a global strong isoperimetric inequality holds. This is analogous to the following result in Riemannian geometry. If all the sectional curvatures of the simply connected complete Riemannian manifold are bounded above by a negative constant, then a strong isoperimetric inequality holds (see [11]). In general, local conditions on a graph are not sufficient to imply a strong isoperimetric inequality. This is one reason why this paper has been restricted to planar graphs.

Our main tool will be Euler's identity: for a given finite, connected, planar graph Z, let V_Z be the number of vertices, E_Z the number of edges and P_Z the number of polygons into which Z divides the plane. Then the following holds:

$$P_Z - E_Z + V_Z = 2.$$

1.1. Statement of results.

The results are as follows:

THEOREM 1. — *Let X be a planar graph, such that the degree of each vertex is at least $k \geq 7$. Then*

$$\|M\| \leq \frac{4 + 2\sqrt{k-5}}{k}.$$

Theorem 1 improves a result of Dodziuk [6], which states that $\|M\| < 1$ for triangulations of the plane such that the degree of each vertex is at least 7 and is uniformly bounded. Theorem 1 can be improved if we impose further conditions on the graph as follows:

THEOREM 2. — *Let X be a planar bipartite graph such that the degree of each vertex is at least $k \geq 5$. Then*

$$\|M\| \leq \frac{2 + 2\sqrt{k-3}}{k}.$$

The bound given in the above theorem can be improved if we suppose that the girth of the graph is at least 6.

THEOREM 3. — *Let X be a planar bipartite graph with girth at least* 6 *and such that the degree of each vertex is at least $k \geq 4$. Then*

$$\|M\| \leq \frac{\kappa(4\kappa^2 + k - 4)}{k(2\kappa^2 - 1)} < \frac{2}{\sqrt{k}}$$

where

$$\kappa = \frac{\sqrt{k + 2 + \sqrt{k^2 + 12k - 28}}}{2\sqrt{2}}.$$

For $k = 4$ we obtain $\|M\| < 0.92$.

Theorem 3 can be applied to the Cayley graph of surface groups in the standard presentation. The bound thus obtained improves the bounds established for this problem in [2], [4]. The upper bound for surface groups given in Theorem 3 was established in [14] using a different method.

The upper bounds in Theorems 1, 2 and 3 differ substantially from the other bounds obtained in the same context (see [6] and [10]). This is because those bounds were obtained by first proving a strong isoperimetric inequality and then using the Cheeger inequality [3], whilst we establish directly the upper bounds on the norm of the random walk operator (and a strong isoperimetric inequality follows from this).

Remarks.

1) In Theorem 1, when $k = 6$ the norm of the operator M can be equal to one. This is the case of the random walk on an equilateral triangulation of the plane.

2) In Theorem 2, when $k = 4$ the norm of the operator M can be equal to one. This is the case of the random walk on the square lattice in $\mathbb{Z}^2$.

3) In Theorem 3, when $k = 3$ the norm of the operator M can be equal to one. This is the case of the random walk on the honeycomb lattice.

4) Using the method presented here it is possible to slightly improve the bounds given in Theorems 1 and 2, but the expressions obtained, although elementary, are quite complicated. As these improvements are not substantial we shall not give the details in this paper.

We would like to compare our upper bounds with some lower bounds. In [15] the following lower bound is proved:

THEOREM 4. — *Let X be a graph, not necessarily planar, such that the degree of each vertex is at most k. Then*

$$\|M\| \geq \frac{2\sqrt{k-1}}{k}.$$

Remarks.

1) The norm of the random walk operator $\|M\|$ is equal to $\frac{2\sqrt{k-1}}{k}$ for the random walk on the homogeneous tree of degree k. In [8], Kesten proved this lower bound in the case of Cayley graphs.

2) The upper bounds given in Theorem 1 (and so in Theorems 2 and 3) are asymptotically the same as the lower bounds given in Theorem 4, *i.e.* their ratio tends to 1 as k tends to infinity. Already for $k = 4$ the lower bound $\frac{\sqrt{3}}{2} \approx 0.87$, is quite close to the upper bound 0.92 from Theorem 3.

We now present a very general condition under which the norm of the random walk operator on X is less than 1.

We suppose that at each vertex of the graph X there are at least 3 edges. As the graph X is planar this divides the plane into polygons (some of which may have an infinite number of edges).

Now we define a *hyperbolic polygon* of X. First of all, we say that infinite polygons are X-hyperbolic and let P be a finite polygon which has k_P edges. We say that the polygon P is X-hyperbolic if

$$\sum_{v \in P} \frac{1}{N(v)} < \frac{k_P - 2}{2}.$$

The above condition comes from hyperbolic geometry. It is well known that for given angles $\alpha_1, \dots, \alpha_k$ there exists a polygon in the hyperbolic plane with these angles, if and only if

$$\sum_{i=1}^{k} \alpha_i < \pi(k-2).$$

Examples.

1) All the graphs from Theorems 1, 2 and 3 divide the plane into hyperbolic polygons.

2) Each planar graph with a degree of each vertex at least 3 and a girth of at least 7 divides the plane into hyperbolic polygons.

THEOREM 5. — *If the planar graph X divides the plane into hyperbolic polygons, then*

$$\|M\| < 1.$$

Theorem 5 improves a result of Soardi [10], which states that the above theorem holds provided there are further restrictions on the graph, such as: all the polygons have the same number of edges and the degree of each vertex is uniformly bounded. After completion of this work we learned that a similar result was obtained independently by Woess [13].

This article is organized as follows. In Section 2 we present the method we use. In Section 3 we prove the upper bounds given in Theorems 1, 2 and 3. Section 4 is devoted to the proof of the strong isoperimetric inequality for graphs dividing the plane into hyperbolic polygons.

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2. Statement of the method.

2.1. Gabber's lemma.

The upper bounds that we establish in this article are obtained using the following lemma by Gabber [5]:

LEMMA 1. — *Let us suppose that on the set of oriented edges of the graph X there is a positive function F such that :*

$$F((p,q)) = \frac{1}{F((q,p))}.$$

If for every vertex $q \in X$ one has

$$\frac{1}{N(q)} \sum_{p \sim q} F\big((p,q)\big) \leq k$$

then

$$\|M\| \leq k.$$

Proof. — Let us denote the scalar product on $\ell^2(X,N)$ by $\langle\cdot\,,\cdot\rangle$. Then for $f\in\ell^2(X,N)$

$$\begin{aligned}
\langle Mf,f\rangle &= \sum_{q\in X}\sum_{p\sim q} f(p)f(q)\\
&= \sum_{q\in X}\sum_{p\sim q} f(p)\sqrt{F((q,p))}\,\sqrt{F((p,q))}\,f(q)\\
&\le \sum_{q\in X}\sum_{p\sim q}\{\tfrac12 f^2(p)F((q,p))+\tfrac12 f^2(q)F((p,q))\}\\
&= \tfrac12\sum_{p\in X}\sum_{q\sim p} f^2(p)F((q,p))+\tfrac12\sum_{q\in X}\sum_{p\sim q} f^2(q)F((p,q))\\
&\le \tfrac12\sum_{p\in X} f^2(p)N(p)k+\tfrac12\sum_{q\in X} f^2(q)N(q)k = k\langle f,f\rangle.
\end{aligned}$$

Since M is a self-adjoint operator on $\ell^2(X,N)$ the lemma is proved. □

Remark. — Given a graph X there is always a function F as in Lemma 1, such that $k=\|M\|$. Indeed, there is always a positive function f, such that $Mf=\|M\|f$, and so we can define $F((p,q))=f(p)/f(q)$.

2.2. The structure of the graphs.

We will prove that the graphs from Theorems 1, 2 and 3 have a very simple structure. This will enable us to define the function F needed in Lemma 1.

Let us be more precise. Let X be any graph considered in Theorems 1, 2 or 3. For a vertex v let $|v|$ denote its distance from a given vertex e, *i.e.* the minimal number of edges connecting v to e. Let us subdivide the set of oriented edges (q,p) into *positive*, *neutral* and *negative* according to the value of the difference $|q|-|p|$. For a vertex $p\in X$ let $N_+(p)$ and $N_-(p)$ be the number of positive and negative edges starting at p.

Let us denote by V_n the set of vertices of X which are at distance n from e and by E_{n+1} the set of edges of X which connect V_n to V_{n+1}:

$$V_n=\{p\in X\,;\ |p|=n\},$$

$$E_{n+1}=\big\{\{q,p\}\,;\ q\sim p,\ (q\in V_n,\,p\in V_{n+1})\text{ or }(q\in V_{n+1},\,p\in V_n)\big\}.$$

The graph X is planar so it divides the plane into polygons (some of which may have an infinite number of sides). For any two vertices from V_n which belong to the same polygon and which are not connected by an edge, we add an edge connecting them. *A priori*, the graph X with added edges does not have to be planar, but it will be shown that it is planar.

For the proof we need the following definition:

DEFINITION 1. — *The sphere S_n of radius n is a graph with vertex set V_n and whose edges are the edges of X and the added edges S_n^+ which connect vertices from V_n.*

Figures 1a and 1b show the graphs X and $X \cup S_1$ where thick edges represent edges from X in S_1 and dotted edges represent added edges from S_1^+.

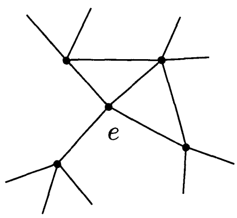

Figure 1a. Graph X

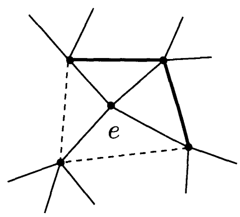

Figure 1b. Graph $X \cup S_1$

Then we prove

LEMMA 2. — *For all n, we have the following property :*

($\mathcal{P}_n$) *The sphere S_n is a cyclic graph and $X \cup S_1 \cup \ldots \cup S_n$ is a planar graph. More precisely we can draw X and the added edges $S_1^+, \ldots, S_n^+$ in such a way that there is no vertex p with $|p| > n$ inside the cycle S_n. Furthermore each vertex $v \in V_n$ has at most two neighbors that are closer to e, i.e. $N_-(v) \leq 2$.*

Proof of Lemma 2. — We prove property ($\mathcal{P}_n$) by induction.

1) ($\mathcal{P}_0$) is clearly satisfied.

2) Suppose that for some integer n, ($\mathcal{P}_n$) holds. Edges in E_{n+1} are cyclicly ordered since by induction the edges E_{n+1} are outside S_n and the vertices V_n are cyclicly ordered. With regard to this set we say that two edges are neighbors if there are no edges from E_{n+1} between them.

Figures 2a and 2b represent examples of pairs of neighboring edges. These pairs are represented by thick lines. Edges from S_n are represented by dotted lines.

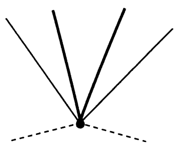

Figure 2a. Neighboring edges

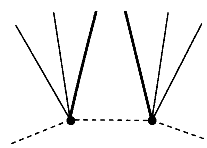

Figure 2b. Neighboring edges

PROPERTY 1. — *Suppose that* $(\mathcal{P}_n)$ *holds for some* n. *Consider edges* $\{p,w\}$ *and* $\{q,w\}$ *such that* $w \in V_{n+1}$ *and* $p,q \in V_n$. *Then these edges are neighbors and* $\{p,q\}$ *is in* S_n. *There are no vertices of* X *inside the triangle* $\{p,q,w\}$.

Proof of Property 1. — Let us consider vertices p and q which are in V_n and which have a common neighbor w that is in V_{n+1}. Let us consider the subgraph C of the graph X bounded by edges $\{p,w\}$, $\{q,w\}$ and the edges from S_n which connect p with q. Suppose that Property 1 is not true, *i.e.* that there are either interior vertices of C or vertices of C that belong to S_n which are different from p and q. Let x be the number of vertices of the first type and let y be the number of vertices of the second type. So for graph C we have:

$$V_C = 3 + x + y.$$

In order to finish the proof of Property 1 we need only to prove:

CLAIM 1. — *For graph* C, *we have* $x = y = 0$.

This is proved separately for the graphs from Theorems 1, 2 and 3 in Section 3. □

Property 1 implies that for a vertex $v \in V_{n+1}$ we have $N_-(v) \leq 2$. So the second assertion of $(\mathcal{P}_{n+1})$ is proved for the vertices $v \in V_{n+1}$.

The neighboring edges have a common vertex or there is an edge or an added edge connecting their extremities which are in V_{n+1}. By Property 1 and $(\mathcal{P}_n)$ the graph with edges

$$\big\{\{p,q\}\ ;\ p \in V_{n+1} \text{ and } q \in V_{n+1} \text{ are extremities of neighboring edges in } E_{n+1}\big\}$$

is a cyclic graph. Moreover, we can draw the added edges or the edges connecting two neighboring edges in such a way that there are no vertices inside this circle which are at a distance greater than $n+1$ from e.

In order to finish the proof of $(\mathcal{P}_{n+1})$ we must show that there are no other edges in S_{n+1}. This is the content of Property 2 below:

PROPERTY 2. — *Suppose that $(\mathcal{P}_n)$ holds for some n. Consider an edge $\{p', q'\}$ in S_{n+1}. Then there exist vertices p and q in V_n such that the edges $\{p, p'\}$ and $\{q, q'\}$ are neighbors.*

Proof of Property 2. — Suppose the contrary, *i.e.* that there are vertices p and q in V_n which are not connected by an edge or added edge from S_n and which have neighbors p' and q' which are in V_{n+1} and which are connected by an edge or an added edge $\{p', q'\}$. Consider the subgraph D of the graph X bounded by edges $\{p, p'\}$, $\{p', q'\}$, $\{q', q\}$ and the edges from S_n which connect p with q. Let x be the number of interior vertices of D and let y be the number of vertices of D that belong to V_n which are different from p and q. So for graph D we have:

$$V_D = 4 + x + y.$$

In order to finish the proof of Property 2 we need only prove:

CLAIM 2. — *For graph D, we have $x = y = 0$.*

We prove this separately for the graphs from Theorems 1, 2 and 3 in Section 3. □

This completes the proof of Lemma 2. □

Lemma 2 enables us to divide the set of non-oriented edges of the graph X into three categories a, b and c.

Let us consider the edge $\{p, q\}$. If $|p| \neq |q|$, say $|p| > |q|$, the edge $\{p, q\}$ is of

- *type a* if $N_-(p) = 1$.
- *type b* if $N_-(p) = 2$.

We say that the edge $\{p, q\}$ is of

- *type c* if $|p| = |q|$.

The function F needed in Lemma 1 is defined as follows:

$$F((q,p)) = \xi(\{q,p\})^{|p|-|q|}, \tag{1}$$

where ξ is a function on the set of non-oriented edges depending only on the type of the edge, *i.e.* ξ takes three values.

In Section 3 we will prove that the configurations of edges around each vertex, with respect to the type and the orientation are very special. This will enable us to find ξ so that

$$\frac{1}{N(q)} \sum_{p \sim q} F((p,q))$$

is small for all $q \in X$.

3. Upper bounds.

In this section we prove the upper bounds given in Theorems 1, 2 and 3. To this end we will describe all possible configurations of edges around any vertex with respect to type and orientation. This will be done separately for graphs from Theorems 1, 2 and 3. The proofs use similar arguments in each case but the details are different.

For a vertex $v \in X$ let $N_t(v)$ be the number of edges of type t (where $t = a, b$ or c) at v and let $N_{+t}(v)$ and $N_{-t}(v)$ be the number of positive and negative edges of type t starting at v.

3.1. Graphs of degree of at least 7.

For the proof of Theorem 1 we need the following proposition.

PROPOSITION 1. — *Let X be a planar graph such that $N(v) \geq k \geq 7$ for each vertex $v \in X$. Then*

1) $N(e) = N_{+a}(e)$.

2) *For $v \neq e$, $N_{+b}(v) \leq 2$, $N_c(v) \leq 2$ and either*

(a) $N_-(v) = N_{-a}(v) = 1$ *or*

(b) $N_-(v) = N_{-b}(v) = 2$.

Proof of Proposition 1. — First of all we prove Lemma 2. It was shown in Section 2 that it is enough to prove Claims 1 and 2.

Proof of Claim 1. — In graph C, at each of x vertices there are at least k edges. At each of y vertices in V_n, we have supposed that there are at least $(k-2)$ edges in C. Also there are at least two edges at each of the vertices p, q and w. Hence

$$E_C \geq \frac{2\cdot 3 + kx + (k-2)y}{2}.$$

Each of the interior polygons of C has at least three edges and the exterior polygon has $y+3$ edges. So finally

$$3(P_C - 1) + (y+3) \leq 2E_C.$$

Together with Euler's identity this implies that:

$$\begin{aligned} 2 = P_C - E_C + V_C &\leq -\frac{1}{3}E_C - \frac{1}{3}y + V_C \\ &\leq -\frac{1}{3}\cdot\frac{2\cdot 3 + kx + (k-2)y}{2} - \frac{1}{3}y + 3 + x + y \\ &= 2 + \frac{6-k}{6}x + \frac{6-k}{6}y. \end{aligned}$$

As $k \geq 7$, it follows that $x = y = 0$. □

Proof of Claim 2. — In graph D, at each of x vertices there are at least k edges. At each of y vertices in V_n, we have supposed that there are at least $(k-2)$ edges in D. There are at least two edges at each of the vertices p and q and at the vertex p' there are two edges from the border of D. There may also be some interior edges at the vertex p'. So let us say that there are at least ε interior edges at p', where $\varepsilon = 0$ or 1. The same holds for q' and we say that at vertex q' there are at least ε' interior edges, where $\varepsilon' = 0$ or 1. Hence

$$E_D \geq \frac{2\cdot 4 + \varepsilon + \varepsilon' + kx + (k-2)y}{2}.$$

The exterior polygon has $4+y$ edges.

Let us investigate how many edges the interior polygon which includes $\{p', q'\}$ as one of its edges has. We have supposed that there is no edge connecting p and q and this means that if there are no interior edges at either p' or q' the polygon has at least five edges. We have supposed that neither edge $\{p', q\}$ nor $\{q', p\}$ exist. This means that if there is at least one interior edge at p' or q' but not at both, then the polygon has at least four edges. Finally, if there are interior edges at both p' and q' the polygon has at least three edges. So the interior polygon which includes $\{p', q'\}$ as one of its edges has at least $5 - \varepsilon - \varepsilon'$ edges.

All other polygons have at least three edges. So finally

$$3(P_D - 2) + (y + 4) + (5 - \varepsilon - \varepsilon') \leq 2E_D.$$

With Euler's identity, this implies that:

$$\begin{aligned} 2 = P_D - E_D + V_D &\leq -\frac{1}{3}E_D - 1 + \frac{\varepsilon + \varepsilon'}{3} + V_E \\ &\leq -\frac{1}{3} \cdot \frac{2 \cdot 4 + \varepsilon + \varepsilon' + kx + (k-2)y}{2} - 1 + \frac{\varepsilon + \varepsilon'}{3} + 4 + x + y \\ &= 2 + \frac{\varepsilon + \varepsilon' - 2}{6} + \frac{6-k}{6}x + \frac{6-k}{6}y. \end{aligned}$$

As $k \geq 7$ and $\varepsilon + \varepsilon' \leq 2$, it follows that $x = y = 0$. □

Thus Lemma 2 is proved. In order to finish the proof of Proposition 1 we need only show the following:

PROPERTY 3. — *For any vertex $p \in X$, we have $N_{+b}(p) \leq 2$.*

Proof of Property 3. — Suppose the contrary, *i.e.* that there is an n and vertex p in V_n which has neighbors v, v' and v'' in V_{n+1} such that the edges $\{p, v\}$, $\{p, v'\}$ and $\{p, v''\}$ are of type b. We can suppose that for instance the edge $\{p, v'\}$ is between the edges $\{p, v\}$ and $\{p, v''\}$. So v' has a neighbor $p' \neq p$ in V_n. By Lemma 2 this means that the edge $\{v', p'\}$ has to cross one of the edges $\{p, v\}$ or $\{p, v''\}$. As X is planar, this gives the desired contradiction. □

This ends the proof of Proposition 1. □

Proof of Theorem 1. — We now show how Lemma 1 and Proposition 1 yield Theorem 1. The function F needed in Lemma 1 is defined by (1) and the value of ξ is given by

- $\xi(\{p, q\}) = \sqrt{k-5}$ if the edge $\{p, q\}$ is of type a,
- $\xi(\{p, q\}) = 1$ if the edge $\{p, q\}$ is of type b or c.

For all the situations mentioned in Proposition 1 we have:

1) $$\frac{1}{N(e)}\sum_{p\sim e}F((p,e))=\frac{1}{\sqrt{k-5}}<\frac{4+2\sqrt{k-5}}{k},$$

2) (a) $$\begin{aligned}\frac{1}{N(q)}\sum_{p\sim q}F((p,q))&=\frac{N_{+b}(q)\cdot 1+N_c(q)\cdot 1+\sqrt{k-5}+N_{+a}(q)/\sqrt{k-5}}{N_{+b}(q)+N_c(q)+1+N_{+a}(q)}\\&\le\frac{2+2+\sqrt{k-5}+(k-5)/\sqrt{k-5}}{2+2+1+(k-5)}=\frac{4+2\sqrt{k-5}}{k},\end{aligned}$$

2) (b) $$\begin{aligned}\frac{1}{N(q)}\sum_{p\sim q}F((p,q))&=\frac{N_{+b}(q)\cdot 1+N_c(q)\cdot 1+2+N_{+a}(q)/\sqrt{k-5}}{N_{+b}(q)+N_c(q)+2+N_{+a}(q)}\\&\le\frac{2+2+2+(k-6)/\sqrt{k-5}}{2+2+2+(k-6)}<\frac{4+2\sqrt{k-5}}{k}.\end{aligned}$$

This ends the proof of Theorem 1. □

3.2. Bipartite graphs.

Now we prove Theorem 2. In order to do this we need the following:

PROPOSITION 2. — *Let X be a bipartite planar graph such that $N(v)\ge k\ge 5$ for each vertex $v\in X$. Then*

1) $N(e)=N_{+a}(e)$.

2) *For $v\ne e$ $N_{+b}(v)\le 2$ and either*

(a) $N_-(v)=N_{-a}(v)=1$ *or*

(b) $N_-(v)=N_{-b}(v)=2$.

Proof of Proposition 2. — First of all we will prove Lemma 2. As was shown in Section 1 it is enough to prove Claims 1 and 2.

Proof of Claim 1. — In graph C, at each of the x vertices there are at least k edges. At each of the y vertices in V_n we have supposed that there are at least $(k-2)$ edges, plus 2 edges from S_n. There are at least two edges at each of the vertices p, q and w. Hence

$$E_C \geq \frac{2 \cdot 3 + kx + ky}{2}.$$

As the graph X is bipartite, each polygon of C that does not have an edge in common with S_n has at least 4 edges. There are at most $y+1$ polygons in the interior of C that have edges in common with S_n and each of these has at least three edges. The exterior polygon has $y+3$ edges. So finally,

$$4(P_C - (y+2)) + 3(y+1) + (y+3) \leq 2E_C.$$

With Euler's identity, we have the following:

$$\begin{aligned} 2 = P_C - E_C + V_C &\leq -\frac{1}{2}E_C + \frac{1}{2} + V_C \\ &\leq -\frac{1}{2} \cdot \frac{2 \cdot 3 + kx + ky}{2} + \frac{1}{2} + 3 + x + y = 2 + \frac{4-k}{4}x + \frac{4-k}{4}y. \end{aligned}$$

As $k \geq 5$, this implies that $x = y = 0$. □

Proof of Claim 2. — In graph D, at each of the x vertices there are at least k edges. At each of the y vertices in V_n we have supposed that there are at least $(k-2)$ edges, plus two edges from S_n. There are at least two edges at each of the vertices p and q. At the vertex p' there are two edges from the border of D. There may also be some interior edges at the vertex p'. So let us say that there are at least ε interior edges at p', where $\varepsilon = 0$ or 1. The same holds for q' and we say that at vertex q' there are at least ε' interior edges, where $\varepsilon' = 0$ or 1. Hence

$$E_D \geq \frac{2 \cdot 4 + \varepsilon + \varepsilon' + kx + ky}{2}.$$

As the graph X is bipartite, each of its polygons has at least four edges. So each of the polygons of D, that has no edge in common with $S_n \cup S_{n+1}$, has at least four edges. The exterior polygon has $4+y$ edges and there are at most $y+1$ interior polygons with edges in common with S_n and these have at least three edges. Finally, there is one interior polygon whose border contains the edge $\{p', q'\}$. We need to investigate how many edges that polygon has. We have supposed that there is no edge connecting p and q. This means that if there is no interior edge at p' or at q' this polygon has at least five edges. We have supposed that neither edge $\{p', q\}$ nor $\{q', p\}$ exist. This means that if there is at least one interior edge at p' or at q'

but not at both, this polygon has at least four edges. If there are interior edges at both p' and q' this polygon has at least three edges. So the interior polygon which includes $\{p', q'\}$ as one of its edges has at least $5 - \varepsilon - \varepsilon'$ edges. So finally,

$$4(P_D - (y+3)) + 3(y+1) + (5 - \varepsilon - \varepsilon') + (y+4) \leq 2E_D.$$

With Euler's identity, this implies that:

$$\begin{aligned} 2 = P_D - E_D + V_D &\leq -\frac{1}{2}E_D + \frac{\varepsilon + \varepsilon'}{4} + V_D \\ &\leq -\frac{1}{2} \cdot \frac{2 \cdot 4 + \varepsilon + \varepsilon' + kx + ky}{2} + \frac{\varepsilon + \varepsilon'}{4} + 4 + x + y \\ &= 2 + \frac{4-k}{4}x + \frac{4-k}{4}y. \end{aligned}$$

As $k \geq 5$, this implies that $x = y = 0$. □

Thus Lemma 2 is proved. In order to finish the proof of Proposition 2 we need only prove Property 3. However the proof of this property is the same as for the previous case. This ends the proof of Proposition 2. □

Proof of Theorem 2. — We are now in a position to show how Lemma 1 and Proposition 2 yield Theorem 2. For the function ξ defining F by (1) we impose the following values:

- $\xi(\{p,q\}) = \sqrt{k-3}$ if the edge $\{p,q\}$ is of type a,
- $\xi(\{p,q\}) = 1$ if the edge $\{p,q\}$ is of type b.

For every situation given in Proposition 2 we obtain

1) $$\frac{1}{N(e)} \sum_{p \sim e} F((p,e)) = \frac{1}{\sqrt{k-3}} \leq \frac{2 + 2\sqrt{k-3}}{k},$$

2) (a) $$\begin{aligned} &\frac{1}{N(q)} \sum_{p \sim q} F((p,q)) \\ &\quad = \frac{N_{+b}(q) \cdot 1 + \sqrt{k-3} + N_{+a}(q)/\sqrt{k-3}}{N_{+b}(q) + 1 + N_{+a}(q)} \\ &\quad \leq \frac{2 + \sqrt{k-3} + k - 3/\sqrt{k-3}}{2 + 1 + (k-3)} = \frac{2 + 2\sqrt{k-3}}{k}, \end{aligned}$$

2) (b)
$$\frac{1}{N(q)}\sum_{p\sim q}F((p,q)) = \frac{N_{+b}(q)\cdot 1+2+N_{+a}(q)/\sqrt{k-3}}{N_{+b}(q)+2+N_{+a}(q)} \leq \frac{2+2+k-4/\sqrt{k-3}}{2+2+(k-4)} < \frac{2+2\sqrt{k-3}}{k}.$$

This ends the proof of Theorem 2. □

3.3. Bipartite graphs with a large girth.

We now want to prove Theorem 3. To do this, we need the following:

PROPOSITION 3. — *Let X be a bipartite planar graph whose girth is at least 6 and such that $N(v) \geq k \geq 4$ for each vertex $v \in X$. Then*

1) $N(e) = N_{+a}(e)$.

For $x \neq e$ either

2) $N_-(v) = N_{-a}(v) = 1$ and $N_{+b}(v) \leq 1$

or

3) $N_-(v) = N_{-b}(v) = 2$ and $N_+(v) = N_{+a}(v)$.

Proof of Proposition 3. — First of all we prove Lemma 2. As was shown in Section 1 it is enough to prove Claims 1 and 2.

Proof of Claim 1. — In graph C, at each of the x vertices there are at least k edges. At each of the y vertices in V_n we have supposed that there are at least $(k-2)$ edges, plus two edges from S_n. Let ε_p, ε_q and ε_w be the number of interior edges of C at the vertices p, q and w respectively. So the number of edges at these three vertices is equal to $6+\varepsilon_p+\varepsilon_q+\varepsilon_w$. Hence

$$E_C \geq \frac{6+\varepsilon_p+\varepsilon_q+\varepsilon_w+kx+ky}{2}.$$

As the graph X is bipartite and its girth is at least 6, each polygon of C that does not have an edge in common with S_n has at least six edges. There are at most $y+1$ polygons in the interior of C that have edges in common with the S_n and they each have at least three edges. The exterior polygon has $y+3$ edges. Thus

$$6(P_C-(y+2))+3(y+1)+(y+3) \leq 2E_C.$$

With Euler's identity this implies that

$$\begin{aligned}2 = P_C - E_C + V_C &\le -\frac{2}{3}E_C + 1 + \frac{1}{3}y + V_C \\ &\le -\frac{2}{3} \cdot \frac{6 + \varepsilon_p + \varepsilon_q + \varepsilon_w + kx + ky}{2} + \frac{1}{3}y + 3 + x + y \\ &= 2 + \frac{3-k}{3}x + \frac{4-k}{3}y - \frac{\varepsilon_p + \varepsilon_q + \varepsilon_w}{3}.\end{aligned}$$

As $k \ge 4$, it follows that $x = 0$ and $\varepsilon_p = \varepsilon_q = \varepsilon_w = 0$. This implies that the interior edges of C are only between the vertices in V_n. But if $y \ne 0$, as X is bipartite, condition $(\mathcal{P}_n)$ implies that at each of y vertices there are edges from E_{n+1} inside C, which gives a contradiction. Thus $y = 0$. □

Proof of Claim 2. — In graph D, at each of the x vertices there are at least k edges. At each of the y vertices in V_n we have supposed that there are at least $(k-2)$ edges, plus two edges from S_n. There are at least two edges from the border of D at each of the vertices p, q, p' and q'. There may also be some interior edges at these vertices. So let us say that there are at least ε_p interior edges at p, where $\varepsilon_p = 0$ or 1 and let us define ε_q, $\varepsilon_{p'}$ and $\varepsilon_{q'}$ similarly. Hence

$$E_D \ge \frac{2 \cdot 4 + \varepsilon_p + \varepsilon_q + \varepsilon_{p'} + \varepsilon_{q'} + kx + ky}{2}.$$

As the graph X is bipartite and its girth is at least 6, each of the polygons of D with no edges in common with $S_n \cup S_{n+1}$ has at least six edges. The exterior polygon has $4 + y$ edges. If $\varepsilon_p = 0$, the polygon that includes as one of its edges the edge from S_n attached to the vertex p, has at least four edges. So let us say that this polygon has at least $(4 - \varepsilon_p)$ edges. The same holds for the polygon which includes as one of its edges, the edge from S_n which contains q. The other at most $(y-1)$ interior polygons, which have edges in common with S_n, have at least three edges. Finally there is one interior polygon which includes the edge $\{p', q'\}$. Let us investigate how many edges this polygon has: we have supposed that no edge connects p and q. This means that if there is no interior edge at either p' or q' this polygon has at least five edges. We have supposed that neither edge $\{p', q\}$ nor $\{q', p\}$ exist. This means that if there is at least one interior edge at p' or q' but at only one of them, this polygon has at least four edges and if there are interior edges at both p' and q' this polygon has at least three edges. So the interior polygon which includes $\{p', q'\}$ as one of its edges has at least $5 - \varepsilon_{p'} - \varepsilon_{q'}$ edges. So finally

$$6(P_D - (y+3)) + 3(y-1) + (y+4) + (4 - \varepsilon_p) + (4 - \varepsilon_q) + (5 - \varepsilon_{p'} - \varepsilon_{q'}) \le 2E_D.$$

With Euler's identity, this implies that

$$
\begin{aligned}
2 = P_D - E_D + V_D &\leq -\frac{2}{3}E_D + \frac{\varepsilon_p + \varepsilon_q + \varepsilon_{p'} + \varepsilon_{q'}}{6} + \frac{2}{3} + \frac{1}{3}y + V_D \\
&\leq -\frac{2}{3} \cdot \frac{2\cdot 4 + \varepsilon_p + \varepsilon_q + \varepsilon_{p'} + \varepsilon_{q'} + kx + ky}{2} \\
&\qquad + \frac{\varepsilon_p + \varepsilon_q + \varepsilon_{p'} + \varepsilon_{q'}}{6} + \frac{2}{3} + \frac{1}{3}y + 4 + x + y \\
&= 2 - \frac{\varepsilon_p + \varepsilon_q + \varepsilon_{p'} + \varepsilon_{q'}}{6} + \frac{3-k}{3}x + \frac{4-k}{3}y.
\end{aligned}
$$

As $k \geq 4$ it follows that $x = 0$ and $\varepsilon_p = \varepsilon_q = \varepsilon_{p'} = \varepsilon_{q'} = 0$. This means that interior edges of D lie only between vertices which are in V_n. But if $y \neq 0$, as X is bipartite, condition $(\mathcal{P}_n)$ implies that at each of y vertices there are edges from E_{n+1} inside D, which gives the contradiction. Thus $y = 0$. □

Thus Lemma 2 is proved. In order to finish the proof of Proposition 3 we need only prove

PROPERTY 4. — *For any vertex $p \in X$ the following holds :*

1) *If $N_-(p) = 1$ then $N_b(p) \leq 1$.*

2) *If $N_-(p) = 2$ then $N_b(p) = 2$.*

Proof of Property 4. — Suppose that 1) is not true, *i.e.* that there is a vertex p in V_n which has neighbors v, v' in V_{n+1} and for whom the edges $\{p, v\}$, $\{p, v'\}$ are of type b (see Fig. 3a). This means that vertices v, v' have neighbors w, $w' \neq p$ in V_{n+1} and vertex p has a neighbor q in V_{n-1}. By Lemma 2, vertex q has at least two neighbors in V_n. By Lemma 2 and the planarity of X, either w or w' must be a neighbor of q. This contradicts the fact that the girth of X is at least 6.

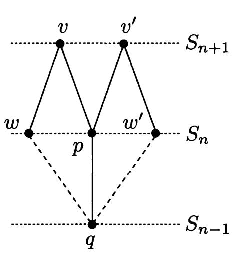

Figure 3a

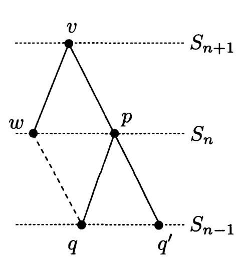

Figure 3b

Suppose that 2) is not true, *i.e.* that there is a vertex p in V_n which has neighbors q, q' in V_{n-1} and a neighbor v in V_{n+1} such that the edge $\{v,p\}$ is of type b (see Fig. 3b). This means that vertex v has a neighbor $w \neq p$ in V_n. By Lemma 2 and the planarity of X, w must be a neighbor of either q or q'. This contradicts the fact that the girth of X is at least 6. □

This ends the proof of Proposition 3. □

Proof of Theorem 3. — We are now in a position to show how Lemma 1 and Proposition 3 yield Theorem 3. The function F needed in Lemma 1 is given by (1) and for ξ we impose the following values:

- $\xi(\{p,q\}) = 2\kappa - \kappa^{-1}$ if the edge $\{p,q\}$ is of type a,
- $\xi(\{(p,q\}) = \kappa$ if the edge $\{p,q\}$ is of type b.

For every situation given in Proposition 3 we have:

1) $$\frac{1}{N(e)}\sum_{p\sim e} F((p,e)) = \frac{1}{2\kappa-\kappa^{-1}} < \frac{\kappa(4\kappa^2+k-4)}{k(2\kappa^2-1)},$$

2) $$\begin{aligned}\frac{1}{N(q)}&\sum_{p\sim q} F((p,q))\\ &= \frac{(2\kappa-\frac{1}{\kappa}) + N_{+b}(q)\cdot\frac{1}{\kappa} + N_{+a}(q)\frac{1}{2\kappa-\kappa^{-1}}}{1+N_{+b}(q)+N_{+a}(q)}\\ &\leq \frac{(2\kappa-\frac{1}{\kappa}) + \frac{1}{\kappa} + (k-2)\frac{1}{2\kappa-\kappa^{-1}}}{1+1+(k-2)} = \frac{\kappa(4\kappa^2+k-4)}{k(2\kappa^2-1)},\end{aligned}$$

3) $$\begin{aligned}\frac{1}{N(q)}\sum_{p\sim q} F((p,q)) &= \frac{2\kappa + N_{+a}(q)\frac{1}{2\kappa-\kappa^{-1}}}{2+N_{+a}(q)}\\ &\leq \frac{2\kappa+(k-2)\frac{1}{2\kappa-\kappa^{-1}}}{2+(k-2)} = \frac{\kappa(4\kappa^2+k-4)}{k(2\kappa^2-1)}.\end{aligned}$$

This ends the proof of Theorem 3. □

4. Hyperbolic polygons.

We now use the so-called strong isoperimetric inequality to prove Theorem 5.

4.1. The strong isoperimetric inequality.

For a finite subset A of vertices of X, let us define the boundary ∂A as the set of the edges of X with only one vertex in A, *i.e.*

$$\partial A = \big\{\{q,p\}\ ;\ q \sim p,\ (q \in A,\ p \notin A) \text{ or } (q \notin A,\ p \in A)\big\}.$$

We define the *measure* $N(A)$ of A and the *measure* $|\partial A|$ of its boundary ∂A as follows:

$$N(A) = \sum_{p \in A} N(p), \quad |\partial A| = \#\big\{\{p,q\}\ ;\ \{p,q\} \in \partial A\big\}.$$

We say that the graph X satisfies a *strong isoperimetric inequality* if there exists $\delta > 0$ such that for every finite subset A of vertices of X:

$$N(A) \leq \delta |\partial A|.$$

As was shown in [7], a strong isoperimetric inequality is equivalent to

$$\|M\| < 1.$$

So in order to prove Theorem 5 we need only prove the following:

PROPOSITION 4. — *Let X be a planar graph which divides the plane into hyperbolic polygons. Then for any finite subset A of the vertices of X we have :*

$$N(A) \leq 6006\,|\partial A|.$$

Proof of Proposition 4. — First of all, note that because at each vertex there are at least three edges, it is possible to prove a slightly stronger hyperbolicity condition for polygons, *i.e.*

LEMMA 3. — *If the polygon P is hyperbolic then*

$$\sum_{p \in P} \frac{1}{N(p)} < \frac{(1 - \frac{1}{1000})k_P - 2}{2}.$$

Proof. — This is clear if $k_P \geq 7$. For $k_P = 3, \ldots, 6$ there are only a finite number of cases to consider. □

Secondly, let us note that:

LEMMA 4. — *If all polygons of the graph X are hyperbolic, then X is proper, i.e. each edge belongs to two polygons and each vertex belongs to as many edges as there are polygons containing it.*

Proof. — Suppose the contrary, *i.e.* that there is a polygon, with a graph inside and that this graph is attached to the polygon by only one edge or one vertex. The graph inside must be finite. This means that amongst all such graphs there is a minimal one Y and this is a proper graph. Let o be the vertex of Y by which Y is attached to the polygon. Let us attach two edges to o and glue together seven copies of Y using these edges, so that the added edges will form a 7-gon. Thus we construct a finite graph Y' which is proper and for whom all its polygons are hyperbolic (see Fig. 4).

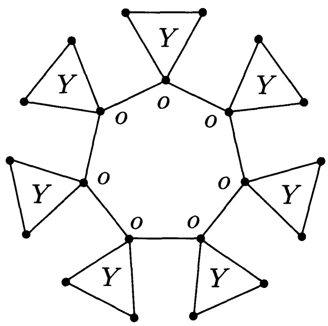

Figure 4. The construction of Y'

If we sum the hyperbolicity condition

$$\sum_{p\in P} \frac{1}{N(p)} - \frac{k_P}{2} + 1 < 0$$

over all the polygons P of Y', we obtain:

$$V_{Y'} - E_{Y'} + P_{Y'} < 0,$$

which contradicts Euler's identity. □

Let us now consider a finite subset A of vertices of X. The presentation of the proof will be easier if we consider a graph $\widetilde{A}$ which consists of the vertices in A and the edges of X which have both extremities in A.

If $\widetilde{A}$ has a cycle of edges, let us add all the edges which are inside this cycle to $\widetilde{A}$ and let B be the graph so obtained. Thus

$$|\partial B| \leq |\partial \widetilde{A}|, \quad N(B) \geq N(\widetilde{A}).$$

It is enough then to prove

$$N(B) \leq 6006\,|\partial B|.$$

The graph B can have many connected components. As it is enough to prove a strong isoperimetric inequality for each connected component, we can suppose that B is connected.

Let B_1 be a union of those edges of B which belong to some cycle of edges of B and let B_2 be a union of the other edges of B. Let $B_1^{(1)}, \ldots, B_1^{(k)}$ be connected components of B_1. We will prove

LEMMA 5. — *Let $B_1^{(i)}$ be a finite, connected subgraph of X such that each of its edges belongs to a cycle of edges from $B_1^{(i)}$. Then*

$$N(B_1^{(i)}) \leq 2001\,\big|\partial B_1^{(i)}\big|, \quad \big|\partial B_1^{(i)}\big| \geq 3. \tag{2}$$

Proof. — Let z be the number of vertices of $B_1^{(i)}$ which belong only to polygons in $B_1^{(i)}$. Let s be the number of vertices of $B_1^{(i)}$ such that for each of them, all polygons which contain it, apart from one, belong to $B_1^{(i)}$, *i.e.* each of the s vertices belongs only to edges from $B_1^{(i)}$. And finally let r be the number of vertices of $B_1^{(i)}$ which belong to at least two polygons which are not in $B_1^{(i)}$, *i.e.* each of the r vertices belongs to at least one edge which is not in $B_1^{(i)}$. So

$$V_{B_1^{(i)}} = r + s + z, \quad \big|\partial B_1^{(i)}\big| \geq r.$$

For any polygon P in $B_1^{(i)}$ we have:

$$\sum_{p\in P} \frac{1}{N(p)} - \frac{k_P}{2} + 1 < -\frac{1}{2000} k_P.$$

Let us sum this inequality over all polygons in $B_1^{(i)}$:

$$\sum_{P\in B_1^{(i)}} \Big(\sum_{p\in P} \frac{1}{N(p)} - \frac{k_P}{2} + 1\Big) < -\sum_{P\in B_1^{(i)}} \frac{1}{2000} k_P. \tag{3}$$

For the right hand side of (3) we have

$$-\sum_{P\in B_1^{(i)}} \frac{1}{2000} k_P = -\frac{1}{1000}\left(k_{B_1^{(i)}} + \frac{k_{\partial B_1^{(i)}}}{2}\right) \leq -\frac{1}{2000}\left(k_{B_1^{(i)}} + k_{\partial B_1^{(i)}}\right),$$

where $k_{B_1^{(i)}}$ is the number of edges which belong to two polygons in $B_1^{(i)}$ and $k_{\partial B_1^{(i)}}$ is the number of edges which belong to only one polygon in $B_1^{(i)}$.

Now let us analyze the left hand side of (3). First of all

$$\sum_{P\in B_1^{(i)}} 1$$

is of course the number of polygons in $B_1^{(i)}$.

On the other hand

$$\sum_{P\in B_1^{(i)}} \frac{k_P}{2} = k_{B_1^{(i)}} + \frac{k_{\partial B_1^{(i)}}}{2} = k_{B_1^{(i)}} + k_{\partial B_1^{(i)}} - \frac{r+s}{2},$$

because $k_{\partial B_1^{(i)}} = r + s$.

If p is any of z vertices then

$$\sum_{\substack{P;p\in P\\ P\in B_1^{(i)}}} \frac{1}{N(p)} = 1.$$

If p is any of s vertices then

$$\sum_{\substack{P;p\in P\\ P\in B_1^{(i)}}} \frac{1}{N(p)} > \frac{2}{3}$$

because for each of the b vertices, all but one of the polygons containing it are in $B_1^{(i)}$.

So together

$$\sum_{p\in B_1^{(i)}} \sum_{\substack{P;p\in P\\ P\in B_1^{(i)}}} \frac{1}{N(p)} \geq \frac{2}{3}s + z.$$

For the left hand side of (3), because of Euler's identity we have

$$\begin{aligned}\sum_{P\in B_1^{(i)}}\Big(\sum_{p\in P}\frac{1}{N(p)}-\frac{k_P}{2}+1\Big) &\geq z+\frac{2}{3}s-(k_{B_1^{(i)}}+k_{\partial B_1^{(i)}})+\frac{r+s}{2}+\sum_{P\in B_1^{(i)}}1\\ &\geq (z+r+s)-(k_{B_1^{(i)}}+k_{\partial B_1^{(i)}})+\Big(\sum_{P\in B_1^{(i)}}1\Big)-\frac{r}{2}\\ &> -\frac{r}{2}.\end{aligned}$$

So from (3) we now have

$$1000r \geq k_{B_1^{(i)}}+k_{\partial B_1^{(i)}}.$$

So finally

$$N\big(B_1^{(i)}\big)=2\big(k_{B_1^{(i)}}+k_{\partial B_1^{(i)}}\big)+\big|\partial B_1^{(i)}\big|\leq 2000\,r+\big|\partial B_1^{(i)}\big|\leq 2001\,\big|\partial B_1^{(i)}\big|,$$

which ends the proof of (2).

We now show that

$$\big|\partial B_1^{(i)}\big|\geq 3.$$

If this were not true then $|\partial B_1^{(i)}|=0,1$ or 2. Let us add one or two edges to the graph $B_1^{(i)}$ so that $|\partial B_1^{(i)}|=2$. Let us glue together seven copies of $B_1^{(i)}$. We will do this using two edges from $\partial B_1^{(i)}$ in such a way that these edges will form a 7-gon. We obtain then a finite planar graph W which divides the plane into hyperbolic polygons (see Fig. 4). As a finite graph has no boundary, one has $|\partial W|=0$. Thus the inequality (2) applied to this graph gives the required contradiction.

This ends the proof of Lemma 5. □

Now we show that Lemma 5 is enough for us to conclude the same for B in general. Let us contract each component $B_1^{(1)},\dots,B_1^{(k)}$ of B_1 to a single vertex b_i. By construction we obtain a tree T from B in a graph which has at least three edges at each vertex. The tree T considered with the edges of its boundary does not have to be a tree, but the measure of its boundary is the same as if it were a tree. Thus one can apply the well known strong isoperimetric inequality, that holds for any subset of a tree which has at least three edges at each vertex, to the subset T:

$$3\,|\partial T| \geq N(T).$$

By the construction one has

$$\sum_{i=1}^{k} |\partial B_1^{(i)}| \leq N(T), \quad N(B) \leq \sum_{i=1}^{k} N(B_1^{(i)}) + N(T), \quad |\partial B| = |\partial T|.$$

So

$$\begin{aligned} N(B) &\leq \sum_{i=1}^{k} N(B_1^{(i)}) + N(T) \leq 2001 \sum_{i=1}^{k} |\partial B_1^{(i)}| + 3\,|\partial T| \\ &\leq 2001\, N(T) + 3\,|\partial T| \leq 6006\,|\partial T| = 6006\,|\partial B|. \end{aligned}$$

This ends the proof of Proposition 4. □

5. Final remarks.

Lemma 2, which we established for the graphs from Theorems 1, 2 and 3, has some other corollaries, which we would like to state now.

COROLLARY 1. — *Let X be a graph from Theorems* 1, 2 *or* 3 *such that the number of edges of the finite polygons into which X divides the plane is uniformly bounded. Then X is hyperbolic in the sense of Gromov.*

By a theorem of Ancona [1] this implies that the random walk on these graphs converges almost surely to the hyperbolic boundary of X (see also [12]).

Proof of Corollary 1. — As was proved in [9], it is enough to show that all bigons are thin. Let us consider two segments of geodesics which have a common beginning (suppose this is e) and a common end. Let $d(n)$ be the number of vertices in V_n which are strictly between two segments of geodesics. As the number of edges of all finite polygons in X is uniformly bounded, in order to prove that all bigons are thin, we need only show that $d(n)$ is uniformly bounded.

For the graphs from Theorems 1 and 2, Lemma 2 ensures that each vertex in V_n has at least three neighbors which are in V_{n+1}. This implies that if $d(n)$ is at least 2 for some n, then it cannot decrease and so the two segments cannot have a common end. So we have proved that for graphs from Theorems 1 and 2, $d(n)$ is bounded by 1.

For the graphs from Theorem 3, the fact that each vertex in V_n has at least two neighbors in V_n (Lemma 2) and the fact that the girth is at least 6, enables us to conclude that if $d(n) \geq 3$, then $d(n+2)$ is at least 3. Thus for two segments of geodesics which have a common beginning either $d(n)$ is bounded by 2 or they cannot have a common end. □

Another almost immediate corollary is:

COROLLARY 2. — *Using three colors, it is possible to paint the polygons into which the graphs from Theorems* 1, 2 *and* 3 *divide the plane, so that polygons which have a common edge have different colors.*

As we remarked after Theorem 4, upper bounds from Theorems 1, 2 and 3 are asymptotically the same as the lower bound from Theorem 4 which is the norm of the random walk on a homogeneous tree. This has a geometric explanation. Namely one can prove:

COROLLARY 3. — *Let X be planar graph, such that degree of each vertex is at least $k \geq 7$. Then X contains a subgraph which is a homogeneous tree of degree $(k-4)$.*

Proof. — The construction of the tree is based on Proposition 1. The tree can be obtained in the following way. First of all we remove all the edges of type c. Then if at any vertex there are two edges of type b which are negatively oriented, we arbitrarily remove one of them. After this procedure, at any vertex different from e there will be one edge which is negatively oriented and the rest of the edges are positively oriented. Thus we obtain a tree. At each vertex we have removed at most two edges of type c, at most one edge of type b which is negatively oriented and at most two edges of type b which are positively oriented. Thus the degree of each vertex of the tree is at least $(k-4)$. □

For graphs from Theorem 2 one can prove:

COROLLARY 4. — *Let X be planar bipartite graph, such that degree of each vertex is at least $k \geq 5$. Then X contains a subgraph which is a homogeneous tree of degree $(k-2)$.*

Proof. — This is a consequence of Proposition 2 and the proof is similar to the proof of Corollary 3. □

And for graphs from Theorem 3 one can prove:

COROLLARY 5. — *Let X be planar bi-partite graph with girth at least 6 and such that degree of each vertex is at least $k \geq 4$. Then X contains a subgraph which is a homogeneous tree of degree $(k-1)$.*

Proof. — This is a consequence of Proposition 3 and the proof is similar to the proof of Corollary 3. □

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