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Extremal projectors in the semi-classical case


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EXTREMAL PROJECTORS
IN THE SEMI-CLASSICAL CASE

by Sophie CHEMLA

1. Introduction.

Let \( \mathfrak{g} \) be a complex semi-simple finite dimensional Lie algebra, \( \mathfrak{h} \) a Cartan subalgebra of \( \mathfrak{g} \) and \( \Delta \) the root system associated to \( \mathfrak{h} \). We will write \( \Delta^+ \) (respectively \( \Delta^- \)) for the set of positive (respectively negative) roots of \( \Delta \) and put \( \rho = \frac{1}{2} \sum_{\gamma \in \Delta^+} \gamma \). We will denote by \( B = (\alpha_1, \ldots, \alpha_l) \) the set of simple roots. Let \( \mathfrak{g}_{\gamma} \) be the root space associated to the root \( \gamma \). We put

\[
\mathfrak{n} = \bigoplus_{\gamma \in \Delta^+} \mathfrak{g}_{\gamma}, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{n}_- = \bigoplus_{\gamma \in \Delta^+} \mathfrak{g}_{-\gamma}.
\]

Let \( R(\mathfrak{h}) \) be the field of rational functions on \( \mathfrak{h}^* \). One introduces the algebra \( U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes R(\mathfrak{h}) \). Let us consider the generic Verma module \( \mathcal{V} = U'(\mathfrak{g}) \otimes S(\mathfrak{n}) \). Zhelobenko ([Z1]) showed that \( \mathcal{V}^n = R(\mathfrak{h})1_+ \) (where \( 1_+ = 1 + U'(\mathfrak{g})\mathfrak{n} \)). The decomposition \( \mathcal{V} = \mathcal{V}^- \mathcal{V} \oplus R(\mathfrak{h})1_+ \) defines a projector \( p \) onto \( R(\mathfrak{h})1_+ \) called the extremal projector. Inspired by a work of Asherova, Smirnov and Tolstoy ([AST]), Zhelobenko ([Z1]) showed that \( p \) factorizes into elementary projectors. Let \( (\gamma_1, \ldots, \gamma_m) \) be a normal ordering on the positive roots. Introduce the following notations:

\[
p_\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! f_{\alpha,k}^k} e_{-\alpha}^k e_{\alpha}^k
\]

\[
f_{\alpha,0} = 1,
\]

if \( k > 0, f_{\alpha,k} = (h_\alpha + \rho(h_\alpha) + 1) \ldots (h_\alpha + \rho(h_\alpha) + k) \)

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(e_\delta being the root vector associated to the root \delta and h_\delta the coroot). We have p = p_{\gamma_1} \ldots p_{\gamma_m} ([Z1]). Let w = s_1 \ldots s_j be a reduced decomposition of w \in W (with s_k = s_{\beta_k}, \beta_k a simple root). Put w_i = s_1 \ldots s_i. The roots \gamma_i = w_{i-1}(\beta_i) (w_0 = 1) are pairwise distinct and

$$\Delta_w = \{ \alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0 \} = \{ \gamma_1, \ldots, \gamma_j \}.$$ 

Put n_w = \bigoplus_{\alpha \in \Delta_w} g_\alpha. In [Z2], Zhelobenko gives an explicit description of V^{n_w}. We will establish similar results for the symmetric algebra (the so-called semi-classical case).

Let us consider the analytic manifold (g/n)^* and will endow it with the following coordinate system \(((e-\alpha)_{\alpha \in \Delta_+} (h_{\alpha_i})_{i \in [1,l]} \). We will call U_\delta the open subset of (g/n)^* defined by the equation \(h_\delta \neq 0 \). We define \(\Phi_\delta\) to be the following rational map of U_\delta:

$$\forall \lambda \in U_\delta, \ \Phi_\delta(\lambda) = \exp\left( \frac{e_{-\delta}(-\lambda)}{h_\delta(\lambda)} e_\delta \right) \cdot \lambda$$

where the dot denotes natural action of n on (g/n)^*. By composition, \(\Phi_\delta\) defines an algebra morphism of \(\mathcal{A}(U_\delta)\) which we call \(\pi_\delta\). We put

$$U_w = U_{\gamma_1} \cap \ldots \cap U_{\gamma_j}.$$ 

We will denote by \(\mathcal{P}(U_w)\) (respectively \(\mathcal{A}(U_w)\)) the set of regular functions (respectively analytic functions) on U_w and we will write \(\mathcal{P}(U_w)^{n_w}\) (respectively \(\mathcal{A}(U_w)^{n_w}\)) the set of invariant functions of \(\mathcal{P}(U_w)\) (respectively \(\mathcal{A}(U_w)\)) under the action of n_w. We prove the following result:

**Theorem.** — The algebra morphism \(\pi_w = \pi_{\gamma_1} \circ \ldots \circ \pi_{\gamma_j}\) does not depend on the reduced expression of w. It establishes an isomorphism between

$$\mathcal{C}_w = \left\{ f \in \mathcal{A}(U_w) \mid \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots = \frac{\partial f}{\partial e_{-\gamma_j}} = 0 \right\}$$

and \(\mathcal{A}(U_w)^{n_w}\). Moreover \(\pi_w\) sends \(\mathcal{C}_w \cap \mathcal{P}(U_w)\) onto \(\mathcal{P}(U_w)^{n_w}\).

Let N_w be the connected simply connected group whose Lie algebra is n_w. The main ingredient of the proof will be the choice of a point in each N_w-orbit lying in U_w in accordance with the following proposition:

**Proposition.** — Let \(\lambda\) be in U_w. The point \(\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}(\lambda)\) is the unique point of the orbit N_w \cdot \lambda whose coordinates \(e_{-\gamma_1}, \ldots, e_{-\gamma_j}\) vanish.
In the appendix, we shall give a factorization for the extremal projector of the Virasoro algebra in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering.

Notations. — Along all this article \( \mathfrak{g} \) will denote a complex semi-simple finite dimensional Lie algebra and \( \mathfrak{h}, \Delta, \Delta^+, \Delta^-, n, n_- \), \( B = (\alpha_1, \ldots, \alpha_i) \) will be as above. Denote by \( W \) the Weyl group associated to these choices and \( \overline{w} \) its longest element. Let \( \gamma \) be an element of \( \Delta^+ \) and let \( h_\gamma \) be the unique element of \( [\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}] \) such that \( \gamma(h_\gamma) = 2 \). If \( e_\gamma \) is in \( \mathfrak{g}_\gamma \), then there exists a unique \( e_{-\gamma} \) such that \((h_\gamma, e_\gamma, e_{-\gamma})\) is a \( \text{sl}(2) \)-triple. If \( \alpha \) and \( \beta \) are two roots, we set \([e_\alpha, e_\beta] = C_{\alpha, \beta} e_{\alpha+\beta}\) with the convention that \( C_{\alpha, \beta} \) is zero if \( \alpha + \beta \) is not a root.

The ordering \((\gamma_1, \ldots, \gamma_m)\) on the positive roots is normal if any composite root is located between its components. Thus for all positive roots \( \gamma_i, \gamma_j, \gamma_k \), the equality \( \gamma_k = \gamma_i + \gamma_j \) implies \( i \leq k \leq j \) or \( j \leq k \leq i \). There is a one to one correspondence between normal orderings and reduced expression of \( \overline{w} \) ([Z2]). Let us recall it. Denote by \( s_\beta \) the reflexion with respect to a simple root \( \beta \). If \( \overline{w} = s_1 \ldots s_m \), then \((\beta_1, s_1(\beta_2), \ldots, s_1 \ldots s_{i-1}(\beta_i), \ldots, s_1 \ldots s_{m-1}(\beta_m))\) are in normal ordering.

If \( V \) is a vector space, \( S(V) \) will be the symmetric algebra of \( V \). Lastly, if \( P \) is in \( S(V) \), \( S(V)_P \) will be the localization of \( S(V) \) with respect to \( \{P^n \mid n \in \mathbb{N}\} \).

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2. Extremal equations in \((\mathfrak{g}/n)^*\).

We consider \((\mathfrak{g}/n)^*\) as an analytic manifold. We endow it with the following coordinate system \( \left((e_\alpha)_\alpha \in \Delta^+, (h_\alpha)_i \in [1, d]\right) \). If \( \delta \) is a positive root, we will denote by \( U_\delta \) the open subset of \((\mathfrak{g}/n)^*\) defined by the equation \( h_\delta \neq 0 \). If \( U \) is an open subset for the Zariski topology, we will write \( \mathcal{A}(U) \) for the algebra of analytic functions on \( U \) and \( \mathcal{P}(U) \) for the algebra of regular functions on \( U \). We will define \( \Phi_\delta \) to be the following rational map
of $U_{\delta}$

$$\forall \lambda \in U_{\delta}, \quad \Phi_{\delta}(\lambda) = \exp \left( \frac{e_{-\delta}(\lambda)}{h_{\delta}(\lambda)} e_{\delta} \right) \cdot \lambda.$$ 

By composition, $\Phi_{\delta}$ defines an algebra morphism of $A(U_{\delta})$ which we call $\pi_{\delta}$. We will denote by $X_{\delta}$ the natural action of $e_{\delta}$ on $A(U_{\delta})$. Remark that $X_{\delta}$ is a derivation. If $f$ is in $P(U_{\delta})$, we have

$$\pi_{\delta}(f) = \sum_{k=0}^{\infty} (-1)^k \frac{e_{-\delta}^k}{k! h_{\delta}^k} X_{\delta}^k \cdot f$$

where $e_{-\delta}$ denotes the multiplication by $e_{-\delta}$. The operator $\pi_{\delta}$ is the commutative analog of the Zhelobenko’s elementary projector.

Let $w = s_1 \ldots s_j$ be a reduced decomposition of $w \in W$ (with $s_k = s_{\beta_k}, \beta_k \in B$). Put $w_i = s_1 \ldots s_i$. The roots $\gamma_i = w_{i-1}(\beta_i) \quad (w_0 = 1)$ are pairwise distinct and

$$\Delta_w = \{ \alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0 \} = \{ \gamma_1, \ldots, \gamma_j \}.$$ 

An ordering in $\Delta_w$ is called normal if it coincides with the initial segment of some normal ordering in $\Delta^+$ (that is compatible with one of the reduced expression of $\overline{w}$). Note that $(\gamma_1, \ldots, \gamma_j)$ is a normal ordering of $\Delta_w$. Put

$$U_w = \bigcap_{\delta \in \Delta_w} U_{\delta}.$$ 

We have

$$P(U_w) = \left( \frac{S(g)}{S(g) n} \right)_{h_{\gamma_1} \ldots h_{\gamma_j}} = S \left( \frac{g}{n} \right)_{h_{\gamma_1} \ldots h_{\gamma_j}}.$$ 

We will denote by $N_w$ the connected and simply connected group whose Lie algebra is $n_w = \bigoplus_{\alpha \in \Delta_w} g_\alpha$. We will start by proving the following proposition.

**Proposition 2.1.** — Let $\lambda$ be in $U_w$. The point $\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}(\lambda)$ is the unique point of the orbit $N_w \cdot \lambda$ whose coordinates $e_{-\gamma_1}, \ldots, e_{-\gamma_j}$ vanish. In particular $\Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}$ does not depend on the normal ordering on $\Delta_w$.

**Proof of Proposition 2.1.** — Complete $(\gamma_1, \ldots, \gamma_j)$ into a normal ordering on the positive roots $(\gamma_1, \ldots, \gamma_m)$. $g/\mathfrak{n}$ is endowed with the basis $(e_{-\gamma_1}, \ldots, e_{-\gamma_m}, h_{\alpha_1}, \ldots, h_{\alpha_l})$. Let $(e^*_{-\gamma_1}, \ldots, e^*_{-\gamma_m}, h^*_{\alpha_1}, \ldots, h^*_{\alpha_l})$ be the dual basis. We will often identify the point $a_{\gamma_1} e^*_{-\gamma_1} + \ldots + a_{\gamma_m} e^*_{-\gamma_m} + b_1 h^*_{\alpha_1} + \ldots + b_l h^*_{\alpha_l}$ with the point $a_{\gamma_1} + \ldots + a_{\gamma_m} e_{-\gamma_1} + \ldots + e_{-\gamma_m} + b_1 h_{\alpha_1} + \ldots + b_l h_{\alpha_l}$ in $N_w$.
... + b_l h_{\alpha_l}^* \text{ with its coordinates } (a_{\gamma_1}, \ldots, a_{\gamma_m}, b_1, \ldots, b_l). \text{ Let us see that there is a unique point in } N_w : \lambda \text{ whose coordinates } e_{-\gamma_1}, \ldots, e_{-\gamma_j} \text{ vanish. Assume that there are two such points } f = (0, \ldots, 0, a_{\gamma_{j+1}}, \ldots, a_{\gamma_m}, b_1, \ldots, b_l) \text{ and } f' = (0, \ldots, 0, a'_{\gamma_{j+1}}, \ldots, a'_{\gamma_m}, b'_1, \ldots, b'_l). \text{ Then there exist complex numbers } (t_1, \ldots, t_j) \text{ such that } \exp(t_1 e_{\gamma_1} + \ldots + t_j e_{\gamma_j}) \cdot f = f'. \text{ One can show easily the following equalities:}

\[ e_{\gamma_1} \cdot e_{-\gamma_k} = -C_{\gamma_1,-\gamma_k} e_{-\gamma_1-\gamma_k} \]
\[ e_{\gamma_1} \cdot h_{\alpha_i}^* = -h_{-\alpha_i}^*(h_{-\gamma_1}) e_{-\gamma_1}. \]

From these equalities, one deduces easily that the term in \( e_{-\gamma_1}^* \) of \( \exp(t_1 e_{\gamma_1} + \ldots + t_j e_{\gamma_j}) \cdot (0, \ldots, 0, a_{\gamma_{j+1}}, \ldots, a_{\gamma_m}, b_1, \ldots, b_l) \) is \(-t_1 f(h_{\gamma_1}).\) As \( f \) is in \( U_w, \) we get \( t_1 = 0. \) We reproduce the same reasoning to show that \( t_2, t_3, \ldots, t_j \) are zero. So that we have proved that the two points \( f \) and \( f' \) coincide. It is not difficult to deduce from the normal ordering property that \( \Phi_{\gamma_l} \) sends the point \((x_{\gamma_1}, \ldots, x_{\gamma_m}, y_1, \ldots, y_l)\) to a point \((x'_{\gamma_1}, \ldots, x'_{\gamma_{l-1}}, 0, x'_{\gamma_{l+1}}, \ldots, x'_{\gamma_m}, y_1, \ldots, y_l)\) and that it sends the point \((0, \ldots, 0, x_{\gamma_{l+1}}, \ldots, x_{\gamma_m}, y_1, \ldots, y_l)\) to a point \((0, \ldots, 0, x'_{\gamma_{l+1}}, \ldots, x'_{\gamma_m}, y_1, \ldots, y_l)\). So that \( \Phi_{\gamma_l} \Phi_{\gamma_{l-1}} \ldots \Phi_{\gamma_1} (\lambda) \) is the unique point of \( N_w : \lambda \) whose coordinates \( e_{-\gamma_1}, \ldots, e_{-\gamma_j} \) vanish. This finishes the proof of Proposition 2.1.

As a consequence of the previous proposition, we may write \( \Phi_w \) for the operator \( \Phi_{\gamma_j} \Phi_{\gamma_{j-1}} \ldots \Phi_{\gamma_1}. \) The algebra homomorphism defined by \( \Phi_w \) on \( \mathcal{A}(U_w) \) will be denoted by \( \pi_w. \) Using Proposition 2.1, we will give a geometric proof of the following result.

**Theorem 2.2.** — 1) If \( \overline{\mathcal{A}}_w \) denotes the linear hull of \((e_{-\alpha})_{\alpha \in \Delta_w},\) one has \( \ker \pi_w = \overline{\mathcal{A}}_w \mathcal{A}(U_w).\)

2) The operator \( \pi_w \) is the projector onto \( \mathcal{A}(U_w)^{\overline{w}} \) with kernel \( \overline{\mathcal{A}}_w \mathcal{A}(U_w) \) and its restriction to \( \mathcal{P}(U_w) \) is the projector onto \( \mathcal{P}(U_w)^{\overline{w}} \) with kernel \( \overline{\mathcal{A}}_w \mathcal{P}(U_w).\)

3) The operator \( \pi_w \) establishes an isomorphism \( \Pi_w \) between

\[ \mathcal{C}_w = \left\{ f \in \mathcal{A}(U_w) \mid \frac{\partial f}{\partial e_{-\gamma_1}} = \ldots = \frac{\partial f}{\partial e_{-\gamma_j}} = 0 \right\} \]

and \( \mathcal{A}(U_w)^{\overline{w}}.\) Moreover \( \Pi_w \) sends \( \mathcal{C}_w \cap \mathcal{P}(U_w) \) onto \( \mathcal{P}(U_w)^{\overline{w}}. \) If \( f \) is in \( \mathcal{A}(U_w)^{\overline{w}}, \) \( \Pi_w^{-1}(f) \) is the restriction of \( f \) to the subvariety of equations \( e_{-\gamma_1} = \ldots = e_{-\gamma_j} = 0.\)
Proof of Theorem 2.2. — From the previous proposition, the inclusion \( \bar{\pi}_w A(U_w) \subset \text{Ker}\pi_w \) is clear. Moreover, a standard reasoning shows that 
\[ A(U_w) = C_w \oplus \bar{\pi}_w A(U_w). \]

Then one sees easily that \( \text{Ker}\pi_w \cap C = \{0\} \). So that we have \( \bar{\pi}_w A(U_w) = \text{Ker}\pi_w \).

Let us now show that \( \text{Im}\pi_w = A(U_w)^n_w \) and that \( \pi_w \) is a projector. Let \( \alpha \) be in \( \Delta_w \). For any \( f \) in \( A(U_w) \) and any \( \lambda \) in \( U_w \), we have
\[
(X_\alpha \circ \pi_w)(f)(\lambda) = \frac{d}{dt} f(\Phi_{\gamma_1} \ldots \Phi_{\gamma_i} \exp(-te_\alpha)\lambda)|_{t=0}.
\]

But for any \( t, \Phi_{\gamma_j} \ldots \Phi_{\gamma_i} \exp(-te_\alpha)\lambda \) is the unique point of \( N_w \cdot \lambda \) whose coordinates \( e_{-\gamma_1}, \ldots, e_{-\gamma_j} \) vanish. So that \( X_\alpha \circ \pi_w = 0 \). We have thus proved the inclusion \( \text{Im}\pi_w \subset A(U_w)^{n_w} \). Now it is clear that \( \pi_w \) is a projector: check that \( \pi_w \circ \pi_w = \pi_w \) on coordinates using the formula (\( * \)). The reverse inclusion \( A(U_w)^{n_w} \subset \text{Im}\pi_w \) will be a consequence of the following lemma.

**Lemma 2.3.** — Let \( k \) be in \([1,j]\) and let \( f \) be in \( A(U_w) \). If \( X_{\gamma_k} f = 0 \), then \( \pi_{\gamma_k} f = f \).

**Proof of Lemma 2.3.** — We first remark that \( (\pi_{\gamma_k} (e_{-\gamma_1}), \ldots, \\
\pi_{\gamma_k} (e_{-\gamma_{k-1}}), e_{-\gamma_k}, \pi_{\gamma_k} (e_{-\gamma_{k+1}}), \ldots, \\
\pi_{\gamma_k} (e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l}) \) is a coordinate system in \( U_w \). Indeed, one may see by induction that for any \( i \leq k - 1 \) (respectively \( i \geq k + 1 \)), \( e_{-\gamma_i} \) may be expressed as a regular function of \( (\pi_{\gamma_k} (e_{-\gamma_1}), \ldots, \\
\pi_{\gamma_k} (e_{-\gamma_i}), h_{\alpha_1}, \ldots, h_{\alpha_l}) \) (respectively \( (\pi_{\gamma_k} (e_{-\gamma_i}), \ldots, \\
\pi_{\gamma_k} (e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l}) \)). We put \( (\epsilon_1, \ldots, \epsilon_{m+l}) = \\
(\pi_{\gamma_k} (e_{-\gamma_1}), \ldots, \\
\pi_{\gamma_k} (e_{-\gamma_i}), e_{-\gamma_k}, \pi_{\gamma_k} (e_{-\gamma_{k+1}}), \ldots, \\
\pi_{\gamma_k} (e_{-\gamma_m}), h_{\alpha_1}, \ldots, h_{\alpha_l}) \).

In these coordinates, we have \( X_{\gamma_k} = h_{\gamma_k} \partial_{\epsilon_k} \). So that if \( X_{\gamma_k} f = 0 \), then \( f \) does not depend on \( \epsilon_k \) and it becomes clear that there exists \( g \) such that \( f = \pi_{\gamma_k} g \). As \( \pi_{\gamma_k} \) is a projector, we have \( \pi_{\gamma_k} f = \pi_{\gamma_k} \pi_{\gamma_k} g = \pi_{\gamma_k} g \), which finishes the proof of the lemma.

It is clear from the proof that \( \pi_w \) sends \( C_w \cap P(U_w) \) onto \( P(U_w)^n_w \).

In particular \( \pi_w |_{P(U_w)} \) is the projector onto \( S(h)_{\gamma_1} \ldots h_{\gamma_m} \) with kernel \( n_\pi P(U_w) \). By analogy to Asherova, Tolstoy, Smirnov and Zhelobenko’s work, we will call it the extremal projector.

Proposition 2.1 gives a geometric interpretation of the projector \( \pi_w \).

In this section, we shall give a factorization of the Virasoro algebra extremal projector in the semi-classical case. Note that the non commutative case is still open. It is very different from the semi-simple case because the Virasoro algebra does not admit any normal ordering. Recall that the Virasoro algebra $\text{Vir}$ is the infinite dimensional Lie algebra generated by $\{e_i \mid i \in \mathbb{Z}\} \cup \{c\}$ with commutation rules

$$[e_i, e_j] = (j - i) e_{i+j} + \frac{(j^3 - j)}{12} \delta_{i+j,0} c, \quad [e_i, c] = 0.$$ 

$\text{Vir}$ admits the following triangular decomposition:

$$\text{Vir} = \text{Vir}_+ \oplus \text{Vir}_0 \oplus \text{Vir}_-$$

where

$$\text{Vir}_+ = \bigoplus_{i \geq 1} C e_i, \quad \text{Vir}_0 = Ce_0 \oplus Cc, \quad \text{Vir}_- = \bigoplus_{i \leq -1} C e_i.$$ 

We will also use the notation

$$\text{Vir}_{r,+} = \bigoplus_{i \geq r} C e_i \quad \text{and} \quad \text{Vir}_{r,-} = \bigoplus_{i \leq -r} C e_i.$$ 

$\text{Vir}_{r,+}$ and $\text{Vir}_{r,-}$ are Lie subalgebras of $\text{Vir}.$

Let $R(\text{Vir}_0)$ be the field of fractions of $S(\text{Vir}_0).$ We introduce the algebra

$$S'(\text{Vir}) = S(\text{Vir}) \otimes_{S(\text{Vir}_0)} R(\text{Vir}_0) = S' (\text{Vir}/\text{Vir}_-).$$ 

There is a natural action of $\text{Vir}_-$ on $S' (\text{Vir}/\text{Vir}_-).$ Through this action, for any negative $i,$ $e_i$ defines a derivation $X_i$ of $S' \left( \frac{\text{Vir}}{\text{Vir}_-} \right).$ Set

$$T_r = \left( \frac{S'(\text{Vir})}{S'(\text{Vir})\text{Vir}_-} \right)^{\text{Vir}_{r,-}}.$$ 

The result and the proof of the following lemma is left to the reader.

**Lemma 3.1.**

$$T_r = \bigoplus_{k_1, \ldots, k_{r-1} \in \mathbb{N}} R(\text{Vir}_0) e_1^{k_1} \cdots e_{r-1}^{k_{r-1}}.$$
As a consequence of Lemma 3.1, we have the following decomposition:

\[ S' (\text{Vir/Vir}_-) = T_r \oplus \text{Vir}_{r,+} S' (\text{Vir/Vir}_-) \]

The proof of the next lemma is an easy computation.

**Lemma 3.2.** — For any \( i > 1 \), the operator

\[
\pi_i = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \left( 2ie_0 + \frac{(i^3 - i)c}{12} \right)} k e_i^k X_{-i}^k
\]

is an algebra morphism and satisfies the relations

\[ X_{-i} \circ \pi_i = 0 \text{ and } \pi_i \circ e_i = 0 \]

(where \( e_i \) denotes multiplication by \( e_i \)).

It is not hard to see that the operator \( \Pi_r = \prod_{i=r}^{\infty} \pi_i \) is well defined.

Actually \( \prod_{i=r}^{\infty} \pi_i(e_1^{a_1} \ldots e_k^{a_k}) = \prod_{r \leq i \leq k} \pi_i(e_1^{a_1} \ldots e_k^{a_k}) = 0 \) (by Lemma 3.2).

**Theorem 3.3.** — The operator \( \Pi_r \) satisfies the relations

\[ \forall i \geq r, \ X_{-i} \circ \Pi_k = 0, \ \Pi_k \circ e_i = 0. \]

It is the projector onto \( T_r \) with kernel \( \text{Vir}_{r,+} S' \left( \frac{\text{Vir}}{\text{Vir}_-} \right) \).

In particular, \( \Pi_1 \) is the extremal projector.

**Proof of Theorem 3.3.** — The relations of the theorem are easy to check and they prove that \( \Pi_r \) is a projector. To prove that the kernel of \( \Pi_r \) is \( \text{Vir}_{r,+} \), we proceed as in the semi-simple case. The inclusion \( \text{Im} \Pi_r \subset T_r \) is a consequence of the theorem. To prove the reverse inclusion, remark that if \( x \) is in \( T_r \), then \( \Pi_r x = x \), so that \( x \) is in \( \text{Im} \Pi_r \).
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