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ON A VARIANT OF KAZHDAN’S PROPERTY (T) FOR SUBGROUPS OF SEMISIMPLE GROUPS

by M. B. BEKKA and N. LOUVET

1. Introduction and statement of the results.

Lubotzky and Zimmer considered in [LuZ] the following variant of Kazhdan’s property (T) for a locally compact group $G$. Let $\mathcal{R}$ be a set of equivalence classes of unitary representations of $G$ containing the trivial one dimensional representation $1_G$. Throughout this paper, all group representations will be assumed to be unitary and strongly continuous representations in non zero Hilbert spaces. The group $G$ is said to have property $(T; \mathcal{R})$ if $1_G$ is isolated in $\mathcal{R}$. For $\mathcal{R} = \hat{G}$, the unitary dual of $G$, this is just the celebrated Kazhdan’s property (T) (see [HaV] for an account on this property).

There are interesting examples of (discrete) groups that are not Kazhdan groups and that satisfy property $(T; \mathcal{R})$ for some natural classes of representations $\mathcal{R}$. For instance, $SL(2, \mathbb{Z})$ has property $(T; \mathcal{R})$ for the set $\mathcal{R}$ of all irreducible representations of $SL(2, \mathbb{Z})$ that factorize through a quotient by a congruence subgroup of $SL(2, \mathbb{Z})$. Another interesting example is the group $SL(2, \mathbb{Z}[1/p])$ which has the property $(T; \mathcal{R})$, $\mathcal{R}$ being the set of all finite dimensional representations of $SL(2, \mathbb{Z}[1/p])$ (see [LuZ]). Such isolation properties allow for instance the construction of expanders graphs (see [Lub]).

In [LuZ], other examples were discovered. They occur as follows. Let $\Gamma$ be a lattice in the direct product $G_1 \times G_2$ of two groups. Assume that $\Gamma$ projects densely on $G_1$ and that $G_2$ has property $(T)$. Then $\Gamma$ has $(T; \mathcal{R})$

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for the set $\mathcal{R}$ of all finite dimensional representations of $\Gamma$ (see below for some concrete examples of such $\Gamma$).

In this paper, we shall generalize the results in [LuZ]. It is interesting to note that our methods are elementary.

We fix, once and for all, a locally compact group $G$ and we let $H$ be a closed (not necessarily discrete) subgroup of $G$ and $N$ a closed normal subgroup of $G$. Throughout this paper, we shall make the following two assumptions:

(I) $HN$ is dense in $G$, and

(II) the homogeneous space $G/H$ has a finite invariant measure.

Denoting by $p$ the canonical projection $G \to G/N$, assumption (I) says that $p(H)$ is dense in $G/N$. If $\pi$ is an irreducible representation of $G/N$, it is clear that $(\pi \circ p)|_H$ is an irreducible representation of $H$ and that $\pi$ is determined (up to unitary equivalence) by $(\pi \circ p)|_H$. So, we may view $\widehat{G/N}$ as a subset of $\widehat{H}$. Our main result states that if an irreducible representation $\pi$ of $H$ is sufficiently close to the trivial representation $1_H$ and if $N$ has property (T), then $\pi$ is in $\widehat{G/N}$.

More precisely, the following holds:

**Theorem A.** — Let $G$, $H$, and $N$ satisfy the above assumptions (I) and (II). Assume moreover that $N$ has Kazhdan’s property (T). Let $U$ be a neighbourhood of $1_{G/N}$ in $\widehat{G/N}$. Then the set of all $(\pi \circ p)|_H$, $\pi \in U$, is a neighbourhood of $1_H$ in $\widehat{H}$.

In other words, $H$ has property $(T; \mathcal{R})$ with $\mathcal{R}$ the set of all irreducible representations of $H$ that do not factorize through $G/N$ together with $1_H$. Of course, in case $G$ has Kazhdan’s property (T), then taking $N = G$ shows that Theorem A is a generalization of the fact that property (T) is inherited by subgroups of cofinite volume. (see [HaV, Chapter 3, Theorem 4]).

Theorem A is a rigidity result in the following sense. It says that any irreducible representation of $H$ that is sufficiently close to the trivial representation $1_H$ extends to a representation of $G$. One may speculate whether this is always true for irreducible lattices in a (nontrivial) product of, say, simple Lie groups. Recall that the only simple Lie groups without property (T) are the ones that are locally isomorphic to $SO(n,1)$ or $SU(n,1)$.
Let $FD$ denote the subset of $\hat{H}$ consisting of the finite dimensional representations. Recall that a group $G$ is called minimally almost periodic if $1_G$ is the unique finite dimensional unitary irreducible representation of $G$. An immediate consequence of Theorem A is the following corollary, also noticed in [LuZ].

**Corollary.** — If $G/N$ is minimally almost periodic then $H$ has property $(T, FD)$.

Our Theorem A improves and makes more precise Theorem 2.2 in [LuZ] where the following was shown. Under the additional assumption that $G$ is a direct product $M \times N$, $H$ has property $(T; \mathcal{R})$ where $\mathcal{R}$ is the set of all $\pi \in \hat{H}$ such that, for all $k \geq 1$, no nontrivial subrepresentation of the symmetric power $S^k(\pi)$ factorize through $p : H \to M$. Of course, this is sufficient in order to deduce the corollary above.

Our proof of Theorem A is elementary. It is based on the following extension result which may be of independent interest.

**Lemma 1.** — Let $G$, $H$, and $N$ satisfy the above assumptions (I) and (II). Let $\pi$ be a unitary representation of $H$. Then the following are equivalent:

(i) $\pi$ contains a subrepresentation that factorizes through the canonical projection $p : G \to G/N$ i.e. there exists a representation $\sigma$ of $G/N$ so that $\pi$ contains $(\sigma \circ p)|_H$.

(ii) The induced representation $\text{Ind}_H^G \pi$ contains a non zero $N$-invariant vector.

For a representation $\pi$ of a locally compact group $H$, let $H^1(H, \pi)$ be the first cohomology group of $H$ with coefficients in $\pi$ (see [Gu1, Gu2]). It is well known that $H^1(H, \pi) = 0$ for any $\pi$ if (and only if) the group $H$ has Kazhdan's property (T) (see [HaV, Chapter 4, Theorem 7]).

Vershik and Karpushev [VeK, Theorem 2] showed that if $H^1(H, \pi) \neq 0$ for some irreducible representation $\pi$, then $\pi$ is infinitesimally small, that is, there exists a net $\pi_n$ in $\hat{H}$ such that $\lim \pi_n = 1_H$ and $\lim \pi_n = \pi$. This result has been conjectured by Guichardet [Gu2] and some partial results were previously obtained by Delorme [Del].
Using Theorem A and Vershik and Karpushev's result, we give an elementary proof of the following theorem, proved in [LuZ, Theorem 3.1] in the product case $G = M \times N$ (see also [BoW]).

**Theorem B.** — Let $G$, $H$ and $N$ be as in Theorem A and assume $G/N$ is minimally almost periodic. Then, for any finite dimensional irreducible unitary representation $\pi$ of $H$, we have

$$H^1(H, \pi) = 0.$$

Here are some examples of groups $H$ to which the above results apply. It seems that the only interesting examples occur when $G$ is locally isomorphic to a product $M \times N$ and $H$ is a lattice in $G$.

(1) As it is well known, $H = SL(n, \mathbb{Q})$ is, via diagonal embedding, a lattice in

$$SL(n, A) = SL(n, \mathbb{R}) \times SL(n, A_f),$$

where $A$ is the ring of adeles of $\mathbb{Q}$ and $A_f$ the subring of finite adeles. By the Strong Approximation Theorem (see, e.g., [Hum], 14.3), $SL(n, \mathbb{Q})$ is dense in $SL(n, A_f)$. The dual space of $SL(n, A)$ is, as a topological space, a restricted product of the dual spaces of the factors $SL(n, \mathbb{Q}_p)$, $p \in P = \{\text{primes in } \mathbb{N}\} \cup \{\infty\}$ (see [Gu2, Corollary 11]). When $n \geq 3$, $SL(n, \mathbb{R})$ has property (T) and Theorem A gives a neat description of the topology in the neighborhood of $1_{SL(n, \mathbb{Q})}$. The above can be generalized to $H = G(\mathbb{Q})$, the rational points of a connected simple algebraic group $G$ defined over $\mathbb{Q}$. The group $G(\mathbb{Q})$ is a lattice in

$$G(A) = G(\mathbb{R}) \times G(A_f),$$

and when $G$ is simply connected and $G(\mathbb{R})$ is non compact, it projects densely into $G(A_f)$ (see [Bor, 5.6. Theorem] and [Mar, Chap. II, (6.8) Theorem]).

(2) Let $G = SO(q)$ be the subgroup of $SL(n + 1)$ preserving the quadratic form $q(x) = \sum_{i=1}^{n} x_i^2 - x_{n+1}^2$. For $n \geq 2$, the group $G(\mathbb{R}) \cong SO(n, 1)$ has real rank one. If $p \equiv 1 \pmod{4}$, the equation $x^2 + 1 = 0$ has a solution in $\mathbb{Q}_p$. This implies that the $\mathbb{Q}_p$-rank of $G(\mathbb{Q}_p)$ is at least two.
and $G(\mathbb{Q}_p)$ has Kazhdan's property. $H = G(\mathbb{Z}[1/p])$ is an irreducible lattice in $G(\mathbb{R}) \times G(\mathbb{Q}_p)$.

Note that in case $n \geq 4$ the equation $x^2 + y^2 + 1 = 0$ has a solution in $\mathbb{Q}_p$ for any prime $p$, and the $\mathbb{Q}_p$-rank of $G(\mathbb{Q}_p)$ is at least two.

(3) For $n \geq 2$, let $q$ be the quadratic form
\[ q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 + \sqrt{2}x_{n+2}^2. \]

Let $K$ be the number field $\mathbb{Q}(\sqrt{2})$, and let $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ be the ring of integers of $K$. Let $G$ be the subgroup of $SL(n+2)$ preserving the form $q$. Then $G$ is defined over $K$, the group $G(\mathbb{R})$ is isomorphic to $SO(n+1,1)$ and the real rank of $G(\mathbb{R})$ is one.

Now, let $\sigma$ be the automorphism of $K$ with $\sigma(\sqrt{2}) = -\sqrt{2}$ and denote by $G^\sigma$ the subgroup of $SL(n+2)$ preserving the form
\[ \sigma q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - \sqrt{2}x_{n+2}^2. \]

Then $G^\sigma(\mathbb{R}) \cong SO(n,2)$ has property (T) as it has real rank 2. Take $H = G(\mathcal{O})$. Embedded in $G^\sigma(\mathbb{R}) \times G(\mathbb{R})$ by means of
\[ G(\mathcal{O}) \to G^\sigma(\mathbb{R}) \times G(\mathbb{R}) \]
\[ g \to (g^\sigma, g), \]
$G(\mathcal{O})$ an irreducible lattice in this product.

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2. Proof of Theorem A.

Recall the following formula, valid for any representations $\sigma$ of $G$ and $\pi$ of a closed subgroup $H$,
\[ \text{Ind}_{H}^{G}(\sigma|_{H} \otimes \pi) \cong \sigma \otimes \text{Ind}_{H}^{G}\pi, \]
(see, e.g., [Fe1, Lemma 4.2]).

Proof of Lemma 1. — Suppose the representation $\pi$ of $H$ contains a subrepresentation of the form
\[ (\sigma \circ \rho)|_{H} \]
where $\sigma$ is a representation of $G/N$, and $\rho$ denote the canonical projection of $G$ onto $G/N$.

Then the induced $\text{Ind}_H^G \pi$ contains the representation
\[
\text{Ind}_H^G \left( (\sigma \circ \rho)|_H \right) = (\sigma \circ \rho) \otimes \text{Ind}_H^G 1_H.
\]

Since $G/H$ carries a finite invariant measure, $\text{Ind}_H^G 1_H$ has invariant vectors. Therefore, $\text{Ind}_H^G \pi$ contains the representation $\sigma \circ \rho$. Restricting to $N$ gives condition (ii) of the lemma.

The proof of the converse is much more involved. Let $\pi$ be a representation of $H$, with space $\mathcal{H}_\pi$, such that $\text{Ind}_H^G \pi$ has a $N$-invariant vector $\xi$ of norm one, that is, a measurable map
\[
\xi : G \to \mathcal{H}_\pi
\]
such that

1. for all $h \in H$, $\xi(xh) = \pi(h^{-1}) \cdot \xi(x)$ for almost all $x \in G$,
2. for all $n \in N$, $\xi(n^{-1}x) = \xi(x)$ for almost all $x \in G$,
3. 
\[
\int_{G/H} \|\xi(\dot{x})\|^2 d\dot{x} = 1
\]

where $\dot{x} = xH$ and $d\dot{x}$ denotes the $G$-invariant measure on $G/H$.

Let $\rho$ denote the induced representation $\text{Ind}_H^G \pi$ with space $\mathcal{H}_\rho$. Using a well known smoothing procedure, we are going to construct a continuous $N$-invariant vector in $\mathcal{H}_\rho$ as follows. For every compact neighbourhood $U$ of the identity in $G$, fix a continuous non negative function $\varphi_U$ on $G$ with support in $U$ and $\int_G \varphi_U(g) dg = 1$.

Consider the map
\[
\xi_U : G \to \mathcal{H}_\pi
\]
defined by
\[
\xi_U(x) = \int_G \varphi_U(xg) \xi(g^{-1}) dg.
\]

Then the following holds:

(a) $\xi_U$ is continuous on $G$. 

To see this, observe first that the function
\[ G \to \mathbb{R}, \quad g \to \|\xi(g)\| \]
is integrable over any compact subset \( K \) of \( G \). Indeed, let the left Haar measure \( dh \) on \( H \) be so normalized that \( dg = dh dg \) holds. Then, denoting by \( \chi_K \) the characteristic function of \( K \), one has
\[
\int_K \|\xi(g)\| \, dg = \int_G \|\xi(g)\| \chi_K(g) \, dg
\]
\[
= \int_{G/H} \left( \int_H \|\xi(gh)\| \chi_K(gh) \, dh \right) \, dg
\]
\[
= \int_{G/H} \|\xi(\dot{g})\| \left( \int_H \chi_K(gh) \, dh \right) \, dg
\]
\[
= \int_{G/H} \|\xi(\dot{g})\| \mu_H(H \cap g^{-1}K) \, dg,
\]
where \( \mu_H(H \cap g^{-1}K) \) denotes the \( H \)-measure of \( H \cap g^{-1}K \) and depends only on \( \dot{g} \). Observe that \( H \cap g^{-1}K \) is non empty if and only if \( \dot{g} = \dot{k} \) for some \( k \) in \( K \). Hence, \( \mu_H(H \cap g^{-1}K) \leq \mu_H(H \cap K^{-1}K) \). Note that \( \mu_H(H \cap K^{-1}K) < \infty \) since \( H \cap K^{-1}K \) is compact. Thus, by Cauchy–Schwarz inequality,
\[
\int_K \|\xi(g)\| \, dg \leq \mu_H(H \cap K^{-1}K) \int_{G/H} \|\xi(\dot{g})\| \, d\dot{g}
\]
\[
\leq \mu_H(H \cap K^{-1}K) \sqrt{\text{vol}(G/H)} \left( \int_{G/H} \|\xi(\dot{g})\|^2 \, d\dot{g} \right)^{1/2}
\]
\[
< \infty.
\]
Now, fix a compact neighbourhood \( V \) of the group unit \( e \) of \( G \), let \( x \) be in \( G \) and \( y \) in \( Vx \). Denote by \( K \) the compact set \( x^{-1}(U \cup V^{-1}U) \). Since the support of \( \varphi_U \) is contained in \( U \), one has
\[
\|\xi_U(x) - \xi_U(y)\| \leq \int_G \|\varphi_U(xg)\xi(g^{-1}) - \varphi_U(yg)\xi(g^{-1})\| \, dg
\]
\[
= \int_{x^{-1}(U \cup V^{-1}U)} \Delta(g^{-1}) |\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})| \|\xi(g)\| \, dg
\]
\[
\leq \sup_{g \in K} \Delta(g^{-1}) |\varphi_U(xg^{-1}) - \varphi_U(yg^{-1})| \int_K \|\xi(g)\| \, dg,
\]
where $\Delta$ denotes the modular function on $G$. Let $\varepsilon > 0$. As $\varphi_U$ is uniformly continuous, there exists some neighbourhood $W$ of $e$ contained in $V$ such that

$$|\varphi_U(g) - \varphi_U(zg)| < \varepsilon$$

for all $z$ in $W$ and all $g$ in $G$. Hence, for all $y$ in $W x$

$$\|\xi_U(x) - \xi_U(y)\| \leq C\varepsilon,$$

where $C = \sup_{g \in K} \Delta(g^{-1}) \int_K \|\xi(g)\| dg$ is a constant depending only on $x, U$, and $V$. Thus, $\xi_U$ is continuous.

(b) $\xi_U$ belongs to the space $H_\rho$ of the induced representation $\rho = \text{Ind}_H^G \pi$. Indeed, let $x$ in $G$ and $h$ in $H$. According to (1) above, for almost all $g \in G$,

$$\xi(g^{-1}xh) = \pi(h^{-1}) \cdot \xi(g^{-1}x)$$

and hence

$$\xi_U(xh) = \int_G \varphi_U(g) \xi(g^{-1}xh) dg$$

$$= \int_G \varphi_U(g) \pi(h^{-1}) \cdot \xi(g^{-1}x) dg$$

$$= \pi(h^{-1}) \cdot \int_G \varphi_U(g) \xi(g^{-1}x) dg$$

$$= \pi(h^{-1}) \cdot \xi_U(x).$$

Moreover,

$$\langle \xi_U(x), \xi_U(x) \rangle \leq \int_G \int_G \varphi_U(g) \varphi_U(g') | \langle \xi(g^{-1}x), \xi(g'^{-1}x) \rangle | dg dg'$$

$$\leq \int_G \int_G \varphi_U(g) \varphi_U(g') \| \xi(g^{-1}x) \| \| \xi(g'^{-1}x) \| dg dg',$$

and hence, using Cauchy–Schwarz inequality in the space $L^2(G/H)$,

$$\|\xi_U\|^2 \leq \int_G \varphi_U(g) \int_G \varphi_U(g') \int_{G/H} \| \xi(g^{-1}x) \| \| \xi(g'^{-1}x) \| d\delta dg dg'$$

$$\leq \int_G \varphi_U(g) \int_G \varphi_U(g') \| \rho(g) \| \| \rho(g') \| \xi \| dg dg'$$

$$\leq \left( \int_G \varphi_U(g) \| \rho(g) \| dg \right)^2 \| \xi \|^2.$$
(c) \( \xi_U \) is \( N \)-invariant in \( \mathcal{H}_\rho \).

Indeed, for \( n \in N, x \in G \) and \( \eta \in \mathcal{H}_\rho \)
\[
\int_{G/H} \langle \xi_U(n\dot{x}), \eta(\dot{x}) \rangle \, d\dot{x} \\
= \int_{G/H} \left( \int_{G} \varphi_U(g) \xi(g^{-1}n\dot{x}) dg, \eta(\dot{x}) \right) \, d\dot{x} \\
= \int_{G} \varphi_U(g) \left( \int_{G/H} \langle \xi((g^{-1}ng)g^{-1}\dot{x}), \eta(\dot{x}) \rangle \, d\dot{x} \right) \, dg \\
= \int_{G} \varphi_U(g) \left( \int_{G/H} \langle \rho(g)\rho(g^{-1}ng)\xi(\dot{x}), \eta(\dot{x}) \rangle \, d\dot{x} \right) \, dg \\
= \int_{G} \varphi_U(g) \left( \int_{G/H} \langle \rho(g)\xi(\dot{x}), \eta(\dot{x}) \rangle \, d\dot{x} \right) \, dg \\
= \int_{G/H} \int_{G} \varphi_U(g) \langle \xi(g^{-1}\dot{x}), \eta(\dot{x}) \rangle \, d\dot{x} \, dg \\
= \int_{G/H} \langle \xi_U(\dot{x}), \eta(\dot{x}) \rangle \, d\dot{x}
\]
as \( N \) is a normal subgroup and \( \xi \) is \( N \)-invariant.

(d) \( \xi_U \) is non zero, for \( U \) sufficiently small. This is clear since \( \|\xi_U - \xi\| \to 0 \) when \( U \to \{e\} \).

Now, for any \( c_1, \ldots, c_n \in \mathbb{C}, g_1, \ldots, g_n \in G \), \( k \in N, h \in H \), using the continuity of \( \xi_U \), we have
\[
\sum_{i=1}^{n} c_i \xi_U(g_ihk) = \sum_{i=1}^{n} c_i \xi_U((g_ih)k(g_ih)^{-1}g_ih) \\
= \sum_{i=1}^{n} c_i \rho( (g_ih)k(g_ih)^{-1}) \xi_U(g_ih) \\
= \sum_{i=1}^{n} c_i \xi_U(g_ih)
\]
and
\[
\left\| \sum_{i=1}^{n} c_i \xi_U(g_ih) \right\| = \| \pi(h^{-1}) \sum_{i=1}^{n} c_i \xi_U(g_i) \| = \left\| \sum_{i=1}^{n} c_i \xi_U(g_i) \right\|.
\]
Therefore, by density of $\mathcal{H}_N$ in $G$ and, again, by continuity of $\xi_U,$

$$\left\| \sum_{i=1}^n c_i \xi_U(g_i g) \right\| = \left\| \sum_{i=1}^n c_i \xi_U(g_i) \right\|$$

for all $g$ in $G.$

Let $\mathcal{W}_\pi$ be the (non zero) closed subspace of $\mathcal{H}_\pi$ generated by $\xi_U(G).$ Then, for any $g$ in $G,$

$$\mathcal{W}_\pi \longrightarrow \mathcal{W}_\pi$$

$$\sum_{i=1}^n c_i \xi_U(g_i) \mapsto \sum_{i=1}^n c_i \xi_U(g_i g^{-1})$$

is a unitary operator depending only on the class of $g$ in $G/N.$ This defines a unitary representation $\sigma$ of $G/N.$ As $\xi_U$ is continuous, $\sigma$ is continuous. Since

$$\sigma \circ p(h) \xi_U(g) = \xi_U(gh^{-1}) = \pi(h) \xi_U(g),$$

it is clear that

$$\sigma \circ p(h) = \pi(h)$$

on $\mathcal{W}_\pi,$ for all $h$ in $H.$

We shall need the following lemma.

**Lemma 2.** — Let $G$, $N$ and $H$ be as in Lemma 1. Let $\rho$ be the quasi-regular representation of $G$ on $L^2(G/H).$ The $N$-invariant functions in $L^2(G/H)$ are constant.

**Proof.** — Lemma 2 amounts to saying that the action of $N$ (by left multiplication) on the homogeneous space $G/H$ is ergodic. By Moore's duality theorem, ergodicity of the $N$-action on $G/H$ is equivalent to ergodicity of the action of $H$ on $G/N$ by left multiplication (see [Zim, Corollary 2. 2. 3]). Density of $HN$ in $G$ implies that the subgroup $p(H)$ is dense in the group $G/N,$ and this is equivalent with ergodicity of the action of $H$ on $G/N$ (see [Zim, Lemma 2. 2. 13]).

We shall frequently use Fell's inner hull-kernel topology on the set $\text{Rep}(G)$ of all equivalence classes of unitary representations of a locally compact group $G.$ This topology is defined as follows. For $\pi$ in $\text{Rep}(G),$...
\( \varepsilon > 0, \) a compact subset \( K \) of \( G, \) and positive definite functions \( \varphi_1, \ldots, \varphi_n \) associated with \( \pi, \) let \( W(\varphi_1, \ldots, \varphi_n; K; \varepsilon; \pi) \) be the set of all \( \rho \) in \( \text{Rep}(G) \) such that there exists \( \psi_1, \ldots, \psi_n, \) each of which is a sum of positive definite functions associated with \( \rho, \) for which

\[
|\varphi_i(x) - \psi_i(x)| < \varepsilon \quad \forall i = 1, \ldots, n \quad \forall x \in K.
\]

The subsets \( W(\varphi_1, \ldots, \varphi_n; K; \varepsilon; \pi) \) form a basis of neighbourhoods of \( \pi \) (see [Fel, Section 2]). This topology may also be described in terms of weak containment. Recall that \( \pi \) is weakly contained in a set \( S \) of representations of \( G \) if every positive definite function associated with \( \pi \) is the limit, uniformly on compact subsets of \( G, \) of sums of positive definite functions associated with representations from \( S. \) It is clear that a net \( \pi_n \) of unitary representations of \( G \) converges to \( \pi \) if and only if, for every subnet \( \pi_{n'} \) of \( \pi_n, \) \( \pi \) is weakly contained in the set \( \{ \pi_{n'} \}. \) Restricted to \( \hat{G}, \) this is just the usual Fell-Jacobson topology on \( \hat{G} \) (see also [Dix, Chap.18]). We are now in position to prove Theorem A.

\textbf{Proof of Theorem A.} — Let \( \pi_n \) be a net of irreducible representations of \( H \) converging to \( 1_H \) in \( \hat{H}. \) Then, by continuity of inducing (see [Fel, Theorem 4.1]),

\[
\text{Ind}_H^G \pi_n \to \text{Ind}_H^G 1_H
\]

in \( \text{Rep}(G). \) Since \( H \) has finite covolume, \( 1_G \) is contained in \( \text{Ind}_H^G 1_H \) and this implies

\[
\text{Ind}_H^G \pi_n \to 1_G.
\]

As \( N \) has Kazhdan's property, we may assume that \( \text{Ind}_H^G \pi_n \) has \( N \)-invariant vectors for all \( n. \) Hence, by Lemma 1, there are (irreducible) representations \( \sigma_n \) of \( G/N \) such that \( \pi_n = (\sigma_n \circ p)|_H \) where \( p : G \to G/N \) is the canonical projection. The proof will be finished if we show that

\[
\sigma_n \circ p \to 1_G
\]

in \( \hat{G}. \)

Since \( H \) has finite covolume, one has

\[
(*) \text{Ind}_H^G \pi_n = \text{Ind}_H^G (\sigma_n \circ p)|_H = (\sigma_n \circ p) \otimes \rho = (\sigma_n \circ p) \oplus ((\sigma_n \circ p) \otimes \rho^0),
\]

where \( \rho = \text{Ind}_H^G \) and \( \rho^0 \) is the restriction of \( \rho \) to the orthogonal of the constants in \( L^2(G/N). \)
Now, the restriction to $N$ of $(\sigma_n \circ \rho) \otimes \rho^0$ is a multiple of $\rho^0|_N$. Since $N$ has property $(T)$, Lemma 2 above implies that $\rho^0|_N$ does not weakly contain the trivial representation $1_N$. So, $(\sigma_n \circ \rho) \otimes \rho^0$ cannot converge to $1_G$. As

$$\text{Ind}_H^G \pi_n \to 1_G,$$

we conclude from (*) that

$$\sigma_n \circ \rho \to 1_G.$$

\[\Box\]

3. Proof of Theorem B.

The finite dimensional representation $\pi$ decomposes as a finite sum

$$\pi = \sum_{i=1}^n \pi_i$$

of irreducible subrepresentations $\pi_i$. Since, in an obvious way,

$$H^1(H, \pi) \cong \bigoplus_{i=1}^n H^1(H, \pi_i),$$

we may assume that $\pi$ is irreducible.

We first deal with the case where $\pi$ is the trivial representation $1_H$. Then, $H^1(H, \pi)$ is the group of all (continuous) homomorphisms from $H$ to the additive group of the complex numbers $\mathbb{C}$. Let $[H, H]$ denote the closure of the commutator subgroup of $H$. The corollary to Theorem A implies that the dual group of the abelian group $H/[H, H]$ is discrete, since the trivial character is isolated. So, by duality theory, $H/[H, H]$ is compact. Hence, $H^1(H, 1_H) = 0$.

Suppose now that $\pi \neq 1_H$ and assume, by contradiction, that

$$H^1(H, \pi) \neq 0.$$ 

Then, by Vershik-Karpushev theorem (see [VeK, Theorem 2]) there exists a net $\pi_n$ in $\hat{H}$ such that

$$\pi_n \to \pi \quad \text{and} \quad \pi_n \to 1_H.$$ 

By Theorem A, we may assume that $\pi_n = (\sigma_n \circ \rho)|_H$ for irreducible representations $\sigma_n$ of $G/N$. 


Let $\bar{\pi}$ denote the conjugate representation of $\pi$. Then, by continuity of tensoring (see [Fe2, Theorem 1]),
\[ \pi_n \otimes \bar{\pi} \to \pi \otimes \bar{\pi}. \]
Hence,
\[ (\sigma_n \circ p) \otimes \text{Ind}_H^G \bar{\pi} = \text{Ind}_H^G (\pi_n \otimes \bar{\pi}) \to \text{Ind}_H^G (\pi \otimes \bar{\pi}). \]
Restricting to $N$, this implies that $\text{Ind}_H^G \bar{\pi}|_N$ weakly contains the representation $\text{Ind}_H^G (\pi \otimes \bar{\pi})|_N$.

Since $\pi$ is finite dimensional, it is well known that $\pi \otimes \bar{\pi}$ has an invariant vector. Therefore, as $H$ has finite covolume, $\text{Ind}_H^G \bar{\pi}|_N$ weakly contains $1_N$. We conclude that $\text{Ind}_H^G \bar{\pi}$ has $N$-invariant vectors. But then, Lemma 1 implies that $\pi$ factorizes to a representation $\sigma$ of $G/N$, thus, $\pi = (\sigma \circ p)|_H$. As $G/N$ is minimally almost periodic, this forces $\sigma = 1_{G/N}$ and hence $\pi = 1_H$, a contradiction. \hfill \Box

\section*{BIBLIOGRAPHY}


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