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THE BERGMAN KERNEL OF THE MINIMAL BALL AND APPLICATIONS

by K. OELJEKLAUS, P. PFLUG & E.H. YOUSSFI

1. Introduction.

Let $\mathbb{B}_*$ be the domain in $\mathbb{C}^n, n \geq 2$, defined by

$$\mathbb{B}_* := \{z \in \mathbb{C}^n : |z|^2 + |z \bullet z| < 1\},$$

where $z \bullet z := \sum_{j=1}^{n} z_j^2$. This is the ball of radius $\frac{\sqrt{2}}{2}$ with respect to the norm

$$N_*(z) := \sqrt{\frac{|z|^2 + |z \bullet z|}{2}}, \quad z \in \mathbb{C}^n.$$

The norm N_* was introduced by Hahn and Pflug [HP], and was shown to be the smallest norm in $\mathbb{C}^n$ that extends the euclidean norm in $\mathbb{R}^n$ under certain restrictions. The automorphism group of $\mathbb{B}_*$ is compact and its identity component is $\mathrm{Aut}^0_{\mathcal{O}}(\mathbb{B}_*) = S^1 \cdot SO(n, \mathbb{R})$, where the S^1-action is diagonal and the $SO(n, \mathbb{R})$-action is the matrix multiplication, see [K] or [OY]. This shows that for $n \geq 3$, the ball $\mathbb{B}_*$ is not biholomorphic to any Reinhardt domain. For $n = 2$, $\mathbb{B}_*$ is linearly biholomorphic to the Reinhardt triangle $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$.

The main purpose of this note is to establish the following

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THEOREM. — *The Bergman kernel of* $\mathbb{B}_*$ *is given by the formula*

$$K_{B_*}(z,w) = \frac{1}{n(n+1)V(B_*)} \frac{\sum_{j=0}^{[\frac{n}{2}]} \binom{n+1}{2j+1} X^{n-1-2j} Y^j \left(2nX - (n-2j)[X^2 - Y]\right)}{(X^2 - Y)^{n+1}},$$

where

$$X = 1 - < z, w >, \quad \textit{and} \quad Y = (z \bullet z)\overline{w \bullet w},$$

and $V(\mathbb{B}_*)$ *is the Lebesgue volume of* $\mathbb{B}_*$.

In particular, when $n = 2$, the Bergman kernel of $\mathbb{B}_*$ is

$$K_{\mathbb{B}_*}(z,w) = \frac{2}{\pi^2} \frac{3(1- < z,w >)^2(1+ < z,w >) + (z \bullet z)\overline{w \bullet w}(5 - 3 < z,w >)}{\left((1- < z,w >)^2 - (z \bullet z)\overline{w \bullet w}\right)^3}.$$

It should be noted that for $n = 2$ this formula can be obtained from the Bergman kernel of the above mentioned Reinhardt triangle whose Bergman kernel can be found in ([JP], p. 176).

Remark. — To the best of our knowledge, the domain $\mathbb{B}_*$ is the first bounded domain in $\mathbb{C}^n$ which is neither Reinhardt nor homogeneous, and for which we have an explicit formula for its Bergman kernel.

2. Preparatory results.

Let G be a semi-simple complex Lie group and K a maximal compact subgroup of G. Suppose that G acts irreducibly on a finite dimensional complex vector space E_Λ via a representation (Π_Λ, E_Λ) with dominant weight Λ and dominant vector v_Λ. Assume further that E_Λ is furnished with a K-invariant hermitian scalar product $[\cdot,\cdot]$. If $G = KAN$ is the Iwasawa decomposition of G, let ϱ denote the sum of the roots associated with the complex decomposition in the Lie algebra $\mathfrak{g}$ of G. If $\mathfrak{a}$ is the Lie algebra of A, then we have the following orthogonal decomposition with respect to the Cartan-Killing form

$$\mathfrak{a} = a_\Lambda \oplus a_\Lambda^\perp,$$

where a_Λ is the annihilator of Λ. If H_0 is the unique vector in $a_\Lambda^\perp$ such that $\Lambda(H_0) = 1$, we set

$$\sigma := 2\varrho(H_0). \tag{2.1}$$

Let $\mathbb{M}^*$ be the intersection of the G-orbit of v_Λ and the unit ball in E_Λ.

In his work [Lo], Loeb proved that the Bergman kernel of the manifold $\mathbb{M}^*$ with an invariant form on $\mathbb{M}^*$ is given for $\zeta = \Pi_\Lambda(g_1)v_\Lambda, \eta = \Pi_\Lambda(g_2)v_\Lambda, g_1, g_2 \in G$, by

$$K_{\mathbb{M}^*}(\zeta,\eta) = \sum_{j=0}^{\infty}(2j+\sigma)T_\Lambda(j)[\zeta,\eta]^j, \tag{2.2}$$

where $T_\Lambda(j)$ is the dimension of the representation with dominant weight $j\Lambda$.

Here we consider the special case $G = SO(n+1,\mathbb{C})$ with its natural linear representation on the complex hermitian space $(\mathbb{C}^{n+1}, <\cdot,\cdot>)$, where Λ is the dominant weight associated to this representation and $v_\Lambda = \frac{\sqrt{2}}{2}(1,i,0,\cdots,0)$. The intersection of the G-orbit of v_Λ and the unit ball in $\mathbb{C}^{n+1}$ is $\mathbb{M}^* = \mathbb{M}\setminus\{0\}$, where

$$\mathbb{M} := \{z = (z_1,\cdots,z_{n+1}) \in \mathbb{C}^{n+1} : |z| < 1, z \bullet z = 0\}.$$

Then by formula (2.2), we see that the Bergman kernel of $\mathbb{M}^*$ with respect to an $SO(n+1,\mathbb{C})$-invariant form $\alpha(z)\wedge\overline{\alpha(z)}$ is given (up to a multiplicative constant) by

$$K_{\mathbb{M}^*}(\zeta,\eta) = \sum_{j=0}^{\infty}(2j+\sigma)T_\Lambda(j) < \zeta,\eta >^j, \tag{2.3}$$

for $\zeta,\eta \in \mathbb{M}^*$.

LEMMA 2.1. — *If $\alpha(z)$ is an $SO(n+1,\mathbb{C})$-invariant nonzero n-form on $\mathbb{M}^*$ (the invariant n-from α is unique up to a constant), then the Bergman kernel of $\mathbb{M}^*$ with respect to the invariant form $\alpha(z)\wedge\overline{\alpha(z)}$ is given (up to a multiplicative constant) by*

$$\begin{aligned} K_{\mathbb{M}^*}(\zeta,\eta) &= \frac{2(n+1) < \zeta,\eta > +2n-2}{(1- < \zeta,\eta >)^{n+1}} \\ &= \frac{4n}{(1- < \zeta,\eta >)^{n+1}} - \frac{2n+2}{(1- < \zeta,\eta >)^{n}}, \end{aligned}$$

for $\zeta,\eta \in \mathbb{M}^*$.

Proof. — Using the notations above, a calculation involving the Weyl character formula implies that

$$T_\Lambda(j) = \frac{n+2j-1}{n-1}\binom{n-2+j}{j}, \text{ for all positive integers } j.$$

See ([FH], pp. 267-315). In addition, some computing shows that $\sigma = 2n-2$. See ([FH], pp. 399-414). The lemma now follows from (2.3). □

LEMMA 2.2. — *The n-form on $(\mathbb{C} \setminus \{0\})^{n+1}$*

$$\widetilde{\alpha}(z) := \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1},$$

induces by restriction an $SO(n+1, \mathbb{C})$-invariant and holomorphic n-form α on $\mathbb{M}^*$.

Proof. — Let $A \in SO(n+1, \mathbb{C})$, $z \in \mathbb{M}^*$ and set $w = Az$. Denote by A_{jk} the $n \times n$ matrix obtained from A by deleting the jth row and the kth column. Since $A \in SO(n+1, \mathbb{C})$, Cramer's rule gives that

$$a_{jk} = (-1)^{k+j} \det A_{jk}. \tag{2.4}$$

Note also that for $z \in \mathbb{M}^*$ and $z_j \neq 0$, then

$$dz_j = -\sum_{l \neq j} \frac{z_l}{z_j} dz_l \quad \text{on } T_z\mathbb{M}^*, \tag{2.5}$$

where $T_z\mathbb{M}^*$ denotes the tangent space of $\mathbb{M}^*$ at the point z. Denote by $A^*\alpha$ the pull-back of α. Then

$$\begin{aligned}(A^*\alpha)(z) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\ &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} \sum_{k=1}^{n+1} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \cdots \wedge dz_{n+1} \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{j=1}^{n+1} \frac{(-1)^{k+j}}{w_j} \det A_{jk} dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1}\end{aligned}$$

$$\text{by (2.4)} \quad = \sum_{k=1}^{n+1}(-1)^{k-1}\sum_{j=1}^{n+1}\frac{a_{jk}}{w_j}dz_1 \wedge \cdots \wedge \widehat{dz_k} \wedge \cdots \wedge dz_{n+1}$$

$$\text{by (2.5)} \quad = \sum_{k=1}^{n+1}(-1)^{k}\sum_{j=1}^{n+1}\frac{a_{jk}}{w_j}dz_1 \wedge \cdots \wedge dz_{j-1} \wedge \widehat{dz_j} \wedge \Big(\sum_{l\neq j}\frac{z_l}{z_j}dz_l\Big)$$

$$\wedge\, dz_{j+1} \wedge \cdots \widehat{dz_k} \wedge \cdots \wedge dz_{n+1}$$

$$= \sum_{k=1}^{n+1}(-1)^{k-1}\sum_{j=1}^{n+1}(-1)^{j-k}\frac{a_{jk}}{z_j w_j}z_k dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1}$$

$$= \sum_{j=1}^{n+1}\left(\frac{(-1)^{j-1}}{w_j z_j}\sum_{k=1}^{n+1}a_{jk}z_k\right)dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1}$$

$$= \sum_{j=1}^{n+1}(-1)^{j-1}\frac{1}{z_j}dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} = \alpha(z).$$

That the restriction of α to $\mathbb{M}^*$ is holomorphic can be seen by evaluating the form α on the n-fold exterior power of the tangent space. □

3. Proper holomorphic mappings from $\mathbb{M}$ into $\mathbb{C}^n$.

Consider the projection $pr : \mathbb{C}^{n+1} \to \mathbb{C}^n$ defined by

$$pr(z_1, \cdots, z_{n+1}) := (z_1, \cdots, z_n).$$

The restriction $F := pr|_{\mathbb{M}}$ of pr to $\mathbb{M}$ gives a proper holomorphic mapping of degree 2 from $\mathbb{M}$ onto $\mathbb{B}_*$. Let W be the branching locus of F and V the image of W under F. Denote by φ and ψ the two local inverses of F defined for $z = (z_1, \cdots, z_n) \in \mathbb{B}_* \setminus V$ by

$$\varphi(z) = (z, i\sqrt{z \bullet z})$$
$$\psi(z) = (z, -i\sqrt{z \bullet z}).$$

LEMMA 3.1. — *If $\varphi := (\varphi_1, \cdots, \varphi_{n+1})$ and $\psi := (\psi_1, \cdots, \psi_{n+1})$ are the local inverses of F defined on $\mathbb{B}_* \setminus V$, then*

$$\varphi^*(\alpha) = \frac{1+n}{i\sqrt{z \bullet z}}(-1)^n dz_1 \wedge \cdots \wedge dz_n \tag{3.1}$$

$$\psi^*(\alpha) = \frac{1+n}{-i\sqrt{z \bullet z}}(-1)^n dz_1 \wedge \cdots \wedge dz_n. \tag{3.2}$$

Proof. — The pull back of α under φ is

$$\begin{aligned}\varphi^*(\alpha) &= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{w_j} dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{n+1} \\ &= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{w_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge dz_n \wedge dw_{n+1} + \frac{(-1)^n}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_n.\end{aligned}$$

But for $1 \leq j \leq n$

$$\begin{aligned}&\frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge d\varphi_{n+1} \\ &= \frac{(-1)^{j-1}}{z_j} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge \Big(- \sum_{k=1}^{n} \frac{z_k}{w_{n+1}} dz_k \Big) \\ &= \frac{(-1)^{j}}{w_{n+1}} dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n \wedge dz_j \\ &= \frac{(-1)^{j+n-j}}{w_{n+1}} dz_1 \wedge \cdots \wedge dz_j \wedge \cdots \wedge dz_n \\ &= (-1)^n \frac{dz_1 \wedge \cdots \wedge dz_n}{w_{n+1}}.\end{aligned}$$

Thus

$$\begin{aligned}\varphi^*(\alpha) &= \left(\frac{(-1)^n}{w_{n+1}} + (-1)^n \frac{n}{w_{n+1}} \right) dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{w_{n+1}} (-1)^n dz_1 \wedge \cdots \wedge dz_n \\ &= \frac{1+n}{i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.\end{aligned}$$

Similarily one has that

$$\psi^*(\alpha) = \frac{1+n}{-i\sqrt{z \bullet z}} (-1)^n dz_1 \wedge \cdots \wedge dz_n.$$

□

If $P_{\mathbb{M}^*}$ denotes the Bergman projection of $\mathbb{M}^*$ with respect to the volume form $\alpha(z) \wedge \overline{\alpha(z)}$, and if $P_{\mathbb{B}_*}$ denotes the Bergman projection of $\mathbb{B}_*$, then we have the following transformation rule

LEMMA 3.2. — *If $\varphi := (\varphi_1, \cdots, \varphi_{n+1})$ and $\psi := (\psi_1, \cdots, \psi_{n+1})$ are the local inverses of F defined locally on $\mathbb{B}_* \setminus V$, then*

$$P_{\mathbb{M}^*}(z_{n+1}(h \circ F))(z) = z_{n+1}((P_{\mathbb{B}_*} h) \circ F)(z)$$

for all $h \in L^2(\mathbb{B}_*)$, where V is the image of the branching locus of F.

Proof. — First observe that the lemma holds for all holomorphic functions $h \in L^2(\mathbb{B}_*)$. Indeed, by virtue of Lemma 3.1 we have that

$$\begin{aligned}
\int_{\mathbb{M}^*} & |z_{n+1}(h \circ F)(z)|^2 \alpha(z) \wedge \overline{\alpha(z)} \\
&= \int_{\mathbb{M}^* \setminus W} |(z_{n+1}(h \circ F))(z)|^2 \alpha(z) \wedge \overline{\alpha(z)} \\
&= \int_{\mathbb{B}_* \setminus V} |\varphi_{n+1}(w) h(w)|^2 \varphi^*(\alpha)(w) \wedge \varphi^*(\overline{\alpha})(w) \\
&\quad + \int_{\mathbb{B}_* \setminus V} |\psi_{n+1}(w) h(w)|^2 \psi^*(\alpha)(w) \wedge \psi^*(\overline{\alpha})(w) \\
&= 2(n+1)^2 \int_{\mathbb{B}_* \setminus V} |h(w)|^2 \, dv(w) < +\infty.
\end{aligned}$$

Thus $z_{n+1}(h \circ F)(z) \in L^2(\mathbb{M}^*, \alpha(z) \wedge \overline{\alpha(z)})$.

Next let $f \in L^2(\mathbb{M}^*, \alpha(z) \wedge \overline{\alpha(z)})$ be a holomorphic function, and let g be an element of the space $\mathcal{C}_0^\infty(\mathbb{B}_* \setminus V)$ of all C^∞-function with compact support in $\mathbb{B}_* \setminus V$. Then

$$\begin{aligned}
&\int_{\mathbb{M}^*} f(z) \overline{z_{n+1} \left(\frac{\partial g}{\partial w_j} \circ F \right)(z)} \alpha(z) \wedge \overline{\alpha(z)} \\
&= (n+1)^2 \left[\int_{\mathbb{B}_*} \frac{(f \circ \varphi)(w)}{\varphi_{n+1}(w)} \overline{\frac{\partial g}{\partial w_j}(w)} \, dv(w) + \int_{\mathbb{B}_*} \frac{(f \circ \psi)(w)}{\psi_{n+1}(w)} \overline{\frac{\partial g}{\partial w_j}(w)} \, dv(w) \right]
\end{aligned}$$

so that by integration by parts we obtain that

$$P_{\mathbb{M}} \left(z_{n+1} (\frac{\partial g}{\partial w_j} \circ F) \right) = 0, \quad \text{for all } \ j = 1, \cdots, n.$$

Since the space

$$\mathcal{H} := \left\{ \frac{\partial g}{\partial w_j} : g \in \mathcal{C}_0^\infty(\mathbb{B}_* \setminus V) \right\}$$

is dense in the orthogonal complement in $L^2(\mathbb{B}_*)$ of the subspace $L_h^2(\mathbb{B}_*)$ of all square integrable holomorphic functions on $\mathbb{B}_*$, the lemma follows. □

LEMMA 3.3. — *If φ and ψ are as before, then*

$$z_{n+1} K_{\mathbb{B}_*}(F(z), w) = (n+1)^2 \left[\frac{K_{\mathbb{M}^*}(z, \varphi(w))}{\overline{\varphi_{n+1}(w)}} + \frac{K_{\mathbb{M}^*}(z, \psi(w))}{\overline{\psi_{n+1}(w)}} \right], \qquad z \in \mathbb{M}^*, w \in \mathbb{B}_*.$$

Proof. — Let $w \in \mathbb{B}_* \setminus V$ and let $r > 0$ be chosen so small that $w + r\Delta^n \subset \mathbb{B}_* \setminus V$, where Δ is the unit disc in $\mathbb{C}$. By Remark 6.1.4 in [JP], there is a $\mathcal{C}^\infty$-function $u : \mathbb{C}^n \to [0, +\infty)$ with compact support in $w + r\Delta^n$ such that

$$P_{\mathbb{B}_*} u = K_{\mathbb{B}_*}(\cdot, w).$$

By virtue of Lemma 3.2 we see that for $z \in \mathbb{M}^*$,

$$\begin{aligned} z_{n+1} K_{\mathbb{B}_*}(F(z), w) &= z_{n+1}(P_{\mathbb{B}_*} u)(F(z)) = P_{\mathbb{M}^*}(z_{n+1}(u \circ F)(z)) \\ &= \int_{\mathbb{M}^*} \zeta_{n+1}(u \circ F)(\zeta) K_{\mathbb{M}^*}(z, \zeta)\, \alpha(\zeta) \wedge \overline{\alpha(\zeta)} \\ &= (n+1)^2 \int_{\mathbb{B}_*} u(\eta) \left[\frac{K_{\mathbb{M}^*}(z, \varphi(\eta))}{\overline{\varphi_{n+1}(\eta)}} + \frac{K_{\mathbb{M}^*}(z, \psi(\eta))}{\overline{\psi_{n+1}(\eta)}} \right] dv(\eta) \\ &= (n+1)^2 \left(\frac{K_{\mathbb{M}^*}(z, \varphi(w))}{\overline{\varphi_{n+1}(w)}} + \frac{K_{\mathbb{M}^*}(z, \psi(w))}{\overline{\psi_{n+1}(w)}} \right), \end{aligned}$$

and the lemma is proved. □

4. Proof of the main result.

Proof of the theorem. — For $z, w \in \mathbb{B}_* \setminus V$, set

$$\begin{cases} s := 1 - \langle z, w \rangle, & t := \varphi_{n+1}(z)\overline{\varphi_{n+1}(w)} \\ x := \langle z, w \rangle + t & \text{and} \quad y := \langle z, w \rangle - t. \end{cases}$$

Then using the notations in the main theorem we have $X = s$ and $Y = t^2$. By Lemma 2.1 we see that for some positive constant C we have

$$\begin{aligned} K_{\mathbb{M}^*}(\varphi(z), \varphi(w)) &= C \left(\frac{4n}{(1-x)^{n+1}} - \frac{2n+2}{(1-x)^n} \right) \\ K_{\mathbb{M}^*}(\varphi(z), \psi(w)) &= C \left(\frac{4n}{(1-y)^{n+1}} - \frac{2n+2}{(1-y)^n} \right), \end{aligned}$$

so that by Lemma 3.3 we obtain that

$$\begin{aligned} K_{\mathbb{B}_*}(z, w) &= 4C(n+1)^2 \frac{f(x) - f(y)}{x - y}, \quad \text{where} \\ f(u) &= \frac{2n}{(1-u)^{n+1}} - \frac{n+1}{(1-u)^n}. \end{aligned}$$

On the other hand,

$$\frac{f(x) - f(y)}{x - y} = n \frac{(s+t)^{n+1} - (s-t)^{n+1}}{t(s^2 - t^2)^{n+1}} - \frac{n+1}{2} \frac{(s+t)^n - (s-t)^n}{t(s^2 - t^2)^n},$$

and

$$\begin{aligned}\frac{(s+t)^{n+1}-(s-t)^{n+1}}{t} &= \frac{(s+t)^{n+1}-s^{n+1}}{t}+\frac{(s-t)^{n+1}-s^{n+1}}{-t} \\ &= \sum_{j=1}^{n+1}\binom{n+1}{j}s^{n+1-j}t^{j-1} \\ &\quad +\sum_{j=1}^{n+1}\binom{n+1}{j}s^{n+1-j}(-t)^{j-1} \\ &= \sum_{j=1}^{n+1}\binom{n+1}{j}s^{n+1-j}[t^{j-1}+(-t)^{j-1}] \\ &= \sum_{k=0}^{n}\binom{n+1}{k+1}s^{n-k}[t^k+(-t)^k] \\ &= 2\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-2k}t^{2k}.\end{aligned}$$

Similarily we have that

$$\frac{(s+t)^n-(s-t)^n}{t} = 2\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}s^{n-1-2k}t^{2k}.$$

Therefore,

$$\begin{aligned}\frac{f(x)-f(y)}{x-y} = &\frac{2n\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-2k}t^{2k}}{(s^2-t^2)^{n+1}} \\ &-(n+1)\frac{\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n}{2k+1}s^{n-1-2k}t^{2k}}{(s^2-t^2)^n}.\end{aligned}$$

But $\binom{n}{2k+1}=\frac{n-2k}{n+1}\binom{n+1}{2k+1}$. Thus

$$\begin{aligned}\frac{f(x)-f(y)}{x-y} = &\frac{2n\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-2k}t^{2k}}{(s^2-t^2)^{n+1}} \\ &-\frac{\sum_{k=0}^{[\frac{n-1}{2}]}(n-2k)\binom{n+1}{2k+1}s^{n-1-2k}t^{2k}(s^2-t^2)}{(s^2-t^2)^{n+1}} \\ = &\frac{\sum_{k=0}^{[\frac{n}{2}]}\binom{n+1}{2k+1}s^{n-1-2k}t^{2k}[2ns-(n-2k)(s^2-t^2)]}{(s^2-t^2)^{n+1}}.\end{aligned}$$

It follows that

$$K_{\mathbb{B}_*}(z,w)$$
$$= 4C(n+1)^2 \frac{\sum_{k=0}^{[\frac{n}{2}]} \binom{n+1}{2k+1} X^{n-1-2k} Y^k [2nX - (n-2k)(X^2-Y)]}{(X^2-Y)^{n+1}},$$

where X and Y are as in the statement of the theorem. To compute the constant we use the formula

$$1 = \int_{\mathbb{B}_*} K_{\mathbb{B}_*}(0,w)dV(w).$$

□

5. Applications.

THEOREM 5.1. — *Let $D \subset \mathbf{C}^n$ be a pseudoconvex domain with C^2-boundary and let $f : D \to \mathbb{B}_*$ be a proper holomorphic mapping. Then $A_f \subset V(f)$ where*

$$A = A_f := \{z \in D : f(z) \bullet f(z) = 0\}, V(f) := \{z \in D : \det f'(z) = 0\}.$$

Proof. — Observe that A is an analytic subset of D. Assume that there exists a point $a \in A$ with $\det f'(a) \neq 0$. To get a contradiction it suffices to show that:

$$\text{if } z^\nu \in A, z^\nu \to z^0 \in \partial D, \text{ then } \det f'(z^\nu) \to 0.$$

We choose a ball $B(z^0,s)$, $0 < \eta < 1$ so that $\eta(n+2) > n+1$, and a defining function r of $D \cap B(z^0,s)$ such that $\widetilde{r} := -(-r)^\eta$ is plurisubharmonic on $D \cap B(z^0,s)$; this can be achieved using a result of Diederich-Fornaess [DF]. Moreover, we may assume that $f(z^\nu) \to w^0 \in \mathbb{H} \cap \partial\mathbb{B}_*$, where $\mathbb{H} := \{\zeta \in \mathbb{C}^n : \zeta \bullet \zeta = 0\}$.

Assuming that f is a mapping with multiplicity m, we know by Pinchuk [Pi2] that

$$mK_D(z,z) \geq |\det f'(z)|^2 K_{\mathbb{B}_*}(f(z),f(z)), \ z \in D.$$

It is well known that $K_D(z,z) \leq C_1 \mathrm{dist}\,(z,\partial D)^{-(n+1)}$, $z \in D$. Hence we get

$$|\det f'(z)|^2 \leq C_2 \left(K_{\mathbb{B}_*}(f(z),f(z))\right)^{-1} \mathrm{dist}\,(z,\partial D)^{-(n+1)}.$$

Now we apply the theorem to obtain that

$$|\det f'(z^\nu)|^2 \le C_3(1-|f(z^\nu)|^2)^{n+2}/\operatorname{dist}(z^\nu,\partial D)^{n+1},\ \nu >> 1.$$

Fix $s' < s$ and define on D the following function:

$$v(z) := \begin{cases} \max\{\widetilde{r}(z), |z-z^0|^2 - s'^2\} \text{ if } z \in D \cap \overline{B(z^0,s')}, \\ |z-z^0|^2 - s'^2, \text{ if } z \in D \setminus B(z^0,s'). \end{cases}$$

It is clear that v is plurisubharmonic on D and that $v(z) = \widetilde{r}(z)$ for $z \in D \cap B(z^0, s'')$, $0 < s'' < s'$ sufficiently small.

For $w \in \mathbb{B}_*$ we put $\rho(w) := \max\{v(z) : z \in D, f(z) = w\}$. Obviously, ρ is plurisubharmonic on $\mathbb{B}_*$. In particular, for $\nu >> 1$ we have $\rho(f(z^\nu)) \ge v(z^\nu) = \widetilde{r}(z^\nu)$.

Exploiting that $\mathbb{B}_*$ is balanced and the Hopf-Lemma on $\mathbb{H} \cap \mathbb{B}_*$ leads to the following estimate: $\rho(f(z^\nu)) \le C_4(|f(z^\nu)|^2 - 1)$, $\nu >> 1$; $C_4 > 0$ independent of z^ν. Therefore

$$|\det f'(z^\nu)|^2 \le C_5(-r(z^\nu))^{\eta(n+2)}/\operatorname{dist}(z^\nu,\partial D)^{n+1} \to 0, \text{ if } \nu \to \infty,$$

which leads to that contradiction we mentioned at the beginning of the proof. □

COROLLARY 5.2. — *There are no unbranched proper holomorphic mappings from D onto $\mathbb{B}_*$ for any bounded pseudoconvex domain with a C^2- boundary; in particular, such a D is never biholomorphically equivalent to $\mathbb{B}_*$.*

Moreover, if D is assumed to be strongly pseudoconvex we get even more:

THEOREM 5.3. — *Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2-boundary. If $f : D \to \mathbb{B}_*$ is a proper holomorphic mapping, then $A = V(f)$.*

Proof. — Assume the inclusion $V(f) \subset A$ is not correct. Then, by the maximum principle, there is a sequence $z^\nu \in V(f)$, $z^\nu \to z^0 \in \partial D$ such that $|f(z^\nu) \bullet f(z^\nu)| > C > 0$. Without loss of generality we assume that $f(z^\nu) \to w^0$. Since $|w^0 \bullet w^0| > 0$ we conclude that w^0 is a strongly pseudoconvex boundary point of $\mathbb{B}_*$. By Theorem 3 of [Ber] there is a neighborhood $U = U(z^0)$ such that f extends to a continuous mapping on $U \cap \bar{D}$. Then using Theorem 3′ of [Pi1] we obtain that $f \in C^1(\bar{D} \cap U)$. Finally using Theorem 1 of [Pi2] we finally get the contradiction to the fact that $z^0 \in \overline{V(f)}$. □

We recall that a bounded domain Ω is said to satisfy condition (Q) if the Bergman projection of Ω maps $C_0^\infty(\Omega)$ into the space $\mathcal{O}(\overline{\Omega})$ of all holomorphic functions on a neighborhood of $\overline{\Omega}$. It was proved recently in [Th] that Ω satisfies condition Q is if and only if that for every compact subset L of Ω, there is an open neighborhood $U = U(L)$ of $\overline{\Omega}$ such that the Bergman kernel $K_\Omega(z, w)$ of Ω extends to be holomorphic on U as a function of z for each $w \in L$, and K_Ω is continuous on $U \times L$.

LEMMA 5.4. — *The ball $\mathbb{B}_*$ satisfies condition (Q).*

Proof. — For $z, w \in \mathbb{B}_*$, we have

$$\begin{aligned}\left|(1- < z,w >)^2 - (z \bullet z)\overline{(w \bullet w)}\right| &\geq |1- < z,w > |^2 - |z \bullet z||w \bullet w| \\ &\geq (1 - |z||w|)^2 - |z \bullet z||w \bullet w| \\ &\geq \left(1 - |z||w| - \sqrt{|z \bullet z|}\sqrt{|w \bullet w|}\right)^2 \\ &\geq \left(1 - \sqrt{|z|^2 + |z \bullet z|}\sqrt{|w|^2 + |w \bullet w|}\right)^2\end{aligned}$$

where the last inequality holds because of Cauchy-Schwarz's inequality. Therefore for some positive constant C we have

$$|K_{\mathbb{B}_*}(z,w)| \leq \frac{C}{\left(1 - \sqrt{|z|^2 + |z \bullet z|}\sqrt{|w|^2 + |w \bullet w|}\right)^{2n+4}}, \text{ for all } z, w \in \mathbb{B}_*.$$

This shows that $\mathbb{B}_*$ satisfies condition (Q). □

THEOREM 5.5. — *Let $D \subset \mathbb{C}^n$ be an arbitrary bounded circular domain which contains the origin.*

(1) *If $f : \mathbb{B}_* \to D$ is a proper holomorphic mapping, then f extends holomorphically to a neighborhood of $\overline{\mathbb{B}_*}$.*

(2) *If D is smooth then there is no proper holomorphic mapping from $\mathbb{B}_*$ into D.*

Proof. — Since, by Lemma 5.4, $\mathbb{B}_*$ satisfies condition Q, part (1) of the theorem becomes a consequence of Theorem 2 of [Bel]. To see that part (2) of theorem holds, it is enough to notice that if there is proper holomorphic mapping $f : \mathbb{B}_* \to D$, and if ϱ is a defining function of D, then $\varrho \circ f$ is a defining function for $\mathbb{B}_*$, which will imply that $\mathbb{B}_*$ is smooth and thus leads to a contradiction. □

THEOREM 5.6. — *Let L be a compact subset of $\mathbb{B}_*$ and let ζ be a boundary point of $\mathbb{B}_*$. Then every holomorphic function f in a* *neighborhood of L is the uniform limit of functions in the complex span of the functions*

$$\frac{\partial}{\partial\zeta^\beta}K_{\mathbb{B}_*}(.,\zeta),\quad \beta\in\mathbb{N}_0^n.$$

Proof. — Since $\mathbb{B}_*$ is a Runge domain and satisfies condition (Q) the proposition follows from Theorem 2.5 of [Th]. □

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