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TRANSVERSAL CRYSTALS OF FINITE LEVEL

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INTRODUCTION


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Key words: Griffiths transversality — Isogeny — Divided power of finite level — Differential operator — Frobenius structure.

introduces the notion of $T$-crystal ($T$ for transversal), which provides an excellent context to study this kind of questions. He uses it to prove a version of Mazur's theorem on the relation between the action of Frobenius and the Hodge filtration on crystalline cohomology which is valid for cohomology with coefficients in an $F$-crystal. As applications, he gets results about Newton and Hodge polygons (Katz conjecture) and degeneration of the Hodge spectral sequence. One of his key results shows that there is an equivalence between $F$-spans and $T$-crystals, provided we restrict to objects of width less than $p$.

In his letter to Illusie [B3], Berthelot develops the theory of crystals of level $m$. We use this new theory to extend Ogus' theorem to objects of width less than $p^{m+1}$: after defining $T$-$m$-crystals and $F$-$m$-spans, we show that one can identify $T$-$m$-crystals of width less than $p^{m+1}$ with a full subcategory of $F$-$m$-spans.

More precisely: let $S$ be a torsion free $p$-adic formal scheme, $S_0$ its reduction mod $p$ and $X$ a smooth $S_0$-scheme. A $T$-$m$-crystal on $X/S$ is a crystal $E$ of level $m$ with a filtration $\text{Fil}$ by submodules which after saturation (see Definition 1.1.6), behaves like a filtration by subcrystals. If $\Phi: X \to X'$ is the relative Frobenius of $X/S_0$, an $F$-$m$-span is a $p$-isogeny $\Phi: F^{m+1} E \to E'$ of $p$-torsion free $m$-crystals. We prove (Theorem 4.3.6) that if $(E, \text{Fil})$ is a $p$-torsion free $T$-$m$-crystal on $X/S$ such that $\text{Fil}^{p^{m+1}} \subset pE$, then there exists a unique $F$-$m$-span $\Phi: F^{m+1} E \to E'$ such that, up to saturation, $F^{m+1} \text{Fil}$ coincides with the filtration $M$ defined by $M^k = \Phi^{-1}(p^k E')$. This construction is functorial in $(E, \text{Fil})$ and the functor is fully faithful.

In order to prove this theorem, we consider a lifted situation: $X$ is a smooth formal $S$-scheme, $F_0$ is the relative Frobenius of $X_0$ over $S_0$, $F: X \to X'$ is a lifting of $F_0$ and we assume that there are coordinates $t_1, \ldots, t_d$ on $X$ and $X'$ such that $F(t_i) = t_i^p$. Then $T$-$m$-crystals correspond to Griffiths transversal $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules that are also transversal to the $m$-PD-ideal $(p)$ and $F$-$m$-spans correspond to $p$-isogenies of $\widehat{\mathcal{D}}^{(m)}_{X/S}$-modules. We prove the theorem in this local situation (Theorem 2.3.3 and Corollary 3.3.5).

Let us briefly describe the structure of this paper: in the first part, we recall Ogus' notion of transversality and Berthelot's notion of partial divided power structures as well as some properties of $p$-isogenies in this context. In the second part, we first recall Berthelot's theory of differential operators of finite level, we define Griffiths transversality for $\mathcal{D}^{(m)}$-modules
and we build the local version of our functor. In the third part, we define
and study $p$-$m$-curvature for $\mathcal{D}^{(m)}$-modules in characteristic $p$ and we use
this notion to prove the fullfaithfulness of our functor in a local situation.
In the fourth part, we recall Berthelot’s theory of $m$-crystals, we define $T$-
$m$-crystals and $F$-$m$-spans and we deduce our main theorem from its local
version. In the fifth and last part, we study the behavior of $T$-$m$-crystals
and $F$-$m$-spans when $m$ varies and use it to show that our results provide
some improvement on Ogus’ theory.

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Conventions. — We let $p$ be a non-zero prime and $m \in \mathbb{N}$. All formal
schemes are $p$-adic formal schemes. All schemes are locally killed by some
power of $p$ and might hence be considered as formal schemes. Also, all
PD-structures are compatible with $p$. We will use the subindex $0$ to indicate
reduction mod $p$. We will adopt the standard multiindex notation, and if
$k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we will write $|k| = k_1 + \cdots + k_d$.

1. PRELIMINARIES

1.1. Transversal filtrations.

We briefly recall the notion of a transversal module from [O2]. We
call transversal what Ogus calls $G$-transversal and almost transversal what
he calls $G'$-transversal. Let us first fix some terminology and notations:

1.1.1. Definition. — Let $A$ be a ring (in a topos). A module filtration
Fil on an $A$-module $M$ is a decreasing filtration by submodules $\text{Fil}^k$
such that there exists an integer $a$ such that $\text{Fil}^a = M$. It is called effective if
we can take $a = 0$. In general, if we set $\text{Fil}[r]^k := \text{Fil}^{k+r}$, we see that $\text{Fil}[a]$
is an effective filtration on $M$. If $\varphi : (\mathcal{J}, A') \to (\mathcal{J}, A)$ is a morphism of
ringed sites, $(M, \text{Fil})$ is a filtered $A$-module and $\text{Fil}_\varphi^k$ denotes the image
of $\varphi^* \text{Fil}^k$ in $\varphi^* M$, then $\varphi^*(M, \text{Fil}) := (\varphi^* M, \text{Fil} \varphi)$ is called the inverse
image of $(M, \text{Fil})$. 
In this article, in order to simplify the notations, we will only consider effective nitrations.

1.1.2. Definition. — A ring nitrination on a ring $A$ is a module nitrination $I^{(*)}$ such that $I^{(k)} I^{(\ell)} \subset I^{(k+\ell)}$. If $(A, I^{(*)})$ is a filtered ring, we set $I := I^{(1)}$ and we say that a filtered module $(M, \Fil)$ has width at most $w$ (with respect to $I$) if there exists an integer $a$ such that $\Fil^a = M$ and $\Fil^{a+w+1} \subset IM$. A filtered ringed site $(\mathcal{T}, A, I^{(*)})$ is a site endowed with a filtered ring. A morphism of filtered ringed sites

$$\varphi : (\mathcal{T}', A', I'^{(*)}) \to (\mathcal{T}, A, I^{(*)})$$

is a morphism of ringed sites such that $\varphi^* I^{(k)}$ maps into $I'^{(k)}$ for all $k$.

1.1.3. Definition. — A filtered module $(M, \Fil)$ in a filtered ringed site $(\mathcal{T}, A, I^{(*)})$ is transversal (a $T$-module for short) if it satisfies

$$IM \cap \Fil^k = I \Fil^{k-1} + I^{(2)} \Fil^{k-2} + I^{(3)} \Fil^{k-3} + \ldots$$

for all $k$. It is almost transversal if

$$IM \cap \Fil^k \subset I \Fil^{k-1} + I^{(2)} \Fil^{k-2} + I^{(3)} \Fil^{k-3} + \ldots$$

for all $k$ and saturated if $I^{(k)} \Fil^\ell \subset \Fil^{\ell+k}$ for all $k, \ell$.

Since there will sometimes be several ring nitrations involved, we will, if necessary, say (almost) transversal to $I^{(*)}$ and saturated with respect to $I^{(*)}$. If $I^{(k)} = I^k$ for all $k$, we will just say (almost) transversal to $I$ and saturated with respect to $I$.

1.1.4. Example. — A filtered module $(M, \Fil)$ in a ringed site $(\mathcal{T}, A)$ is transversal to an ideal $I$ of $A$ if and only if it satisfies $IM \cap \Fil^k = I \Fil^{k-1}$ for all $k$.

1.1.5. Remark. — A filtered module is transversal if and only if it is almost transversal and saturated.

Starting from any almost transversal filtration, there exists a natural process that turns it into a transversal one:

1.1.6. Definition. — If $(M, \Fil)$ is a filtered module on a filtered ringed site $(\mathcal{T}, A, I^{(*)})$, we set

$$\overline{\Fil}^k = \Fil^k + I \Fil^{k-1} + I^{(2)} \Fil^{k-2} + I^{(3)} \Fil^{k-3} + \ldots$$

We call $(M, \overline{\Fil})$ the saturation of $(M, \Fil)$. 
1.1.7. Proposition (see [O2], 2.3.1).

(i) The filtration \( \overline{\text{Fil}} \) is the finest filtration on \( M \) that is saturated and coarser than the given one.

(ii) If \( (M, \text{Fil}) \) is almost transversal, then its saturation is transversal.

This saturation process is specially useful in view of the following result:

1.1.8. Proposition (see [O2], 2.2.1). — Let

\[
\varphi : (\mathcal{T}', A', I'^{(*)}) \longrightarrow (\mathcal{T}, A, I^{(*)})
\]

be a morphism of filtered ringed sites such that the natural map \( \varphi^{-1} A/I \to A'/I' \) is flat. If \( (M, \text{Fil}) \) is an almost transversal module, then so is \( \varphi^*(M, \text{Fil}) \).

1.2. \( p \)-isogenies.

We introduce the \( m \)-PD-filtration \( (p, \{ \}) \) and we describe transversality with respect to this filtration in terms of \( p \)-isogenies.

1.2.1. Definition. — If \( A \) is a \( \mathbb{Z}(p) \)-algebra and \( M, M' \) two \( p \)-torsion free \( A \)-modules, a \( p \)-isogeny \( \Phi : M \to M' \) of width at most \( w \) is an injective homomorphism \( \Phi : M \to M' \otimes \mathbb{Q} \) of \( A \)-modules such that there exists an integer \( a \) such that \( p^{a+w} M' \subset \Phi(M) \subset p^a M' \). It is called effective if one can take \( a = 0 \). In general, if we set \( \Phi[r] = p^{-r} \Phi \), we see that \( \Phi[a] \) is effective.

As we do for filtrations, we will only consider effective \( p \)-isogenies.

Transversality with respect to \( p \), meaning to the ideal \( (p) \), has a very nice interpretation in terms of \( p \)-isogenies:

1.2.2. Proposition (see [O2], 5.1.2). — The functor \( \Phi \mapsto (M, \text{Fil}) \), where \( \text{Fil}^k = \Phi^{-1}(p^k M') \), is an equivalence from the category of \( p \)-isogenies of width at most \( w \) onto the category of filtered modules transversal to \( p \) of width at most \( w \).

Actually, the filtration that will naturally appear in the sequel is not \( (p)^k \) but the \( m \)-PD-filtration defined below (and generalized in Definition 1.3.4).

1.2.3. Definition. — For \( k = qp^m + r \) with \( 0 \leq r < p^m \), we let \( p^{\{k\}} := p^k/q! \). The \( m \)-PD-filtration \( (p)^{\{k\}} \) on a \( \mathbb{Z}_p \)-algebra \( A \) is the finest
ring filtration such that \( p^{(k)} \in (p)^{(k)} \). We will also write \( (p, \{ \} ) \) for this filtration.

In the sequel, we will also need the notion of modified binomial coefficients. Let us recall what they are:

1.2.4. Definition. — If \( k' \) and \( k'' \in \mathbb{N}^d \), and

\[
\begin{align*}
    k' &= q' p m + r', \quad 0 \leq r' < pm, \\
    k'' &= q'' p m + r'', \quad 0 \leq r'' < pm, \\
    k &= k' + k'' = q p m + r, \quad 0 \leq r < pm,
\end{align*}
\]

one sets:

\[
\left\{ \frac{k}{k'} \right\} := \frac{q!}{q'! q''!} \in \mathbb{N} \quad \text{and} \quad \left( \frac{k}{k'} \right) := \left( \frac{k}{k'} \right) \left\{ \frac{k}{k'} \right\}^{-1} \in \mathbb{Z}_p.
\]

Proposition 1.2.2 is still valid for the \( m \)-PD-filtration under some assumptions on the width:

1.2.5. Proposition (see [O2], 2.3.5). — The functor \( \Phi \mapsto (M, \text{Fil}) \) from the category of filtered modules transversal to \( p \) to the category of filtered modules transversal to \( (p, \{ \} ) \) is an equivalence of categories when restricted to objects of width less than \( p^{m+1} \).

1.2.6. Corollary. — The functor \( \Phi \mapsto (M, \text{Fil}) \) where \( \text{Fil}^k \) is the saturation of \( \Phi^{-1}(p^k M') \) with respect to \( (p, \{ \} ) \) is an equivalence from the category of \( p \)-isogenies of width less than \( p^{m+1} \) onto the category of filtered modules transversal to \( (p, \{ \} ) \) of width less than \( p^{m+1} \).

1.3. \( m \)-PD-structures.

We recall Berthelot’s theory of partial divided powers from [B4] which generalizes the usual divided power structures in [B1].

1.3.1. Definition. — Let \( Y \) be a formal scheme. An \( m \)-PD-structure on a coherent ideal \( I \) in \( \mathcal{O}_Y \) is the data of a PD-ideal \( J, [ \ ] \) in \( I \) such that \( I(p^m) + pI \subset J \) (where \( I(p^m) \) is the ideal locally generated by \( f p^m \) with \( f \in I \)). We say that \( I \) is an \( m \)-PD-ideal or that \( (Y, I, J) \) is a formal \( m \)-PD-scheme. We will drop \( J \), or even \( I \), from the notations when no confusion should arise. If \( f \in I \) and \( k = q p^m + r \) with \( 0 \leq r < p^m \), we write

\[
f^{(k)} := f^r (fp^m)^{[a]}.
\]
1.3.2. Definition. — Let \((S,a,b)\) be a formal \(m\)-PD-scheme. The \(m\)-PD-structure on \(a\) extends to a formal \(S\)-scheme \(X\) if the PD-structure on \(b\) extends to a PD-structure on \(X\) (compatible with \(p\)). An \(m\)-PD-structure \((I,J)\) on a formal \(S\)-scheme \(Y\) is said to be compatible with \((S,a,b)\) if the \(m\)-PD-structure on \(a\) extends to \(Y\), the PD-structure on \(J + (p)\) is compatible with the PD-structure on \(b + (p)\) and \(I \cap (b\mathcal{O}_Y + (p))\) is a sub PD-ideal of \(b\mathcal{O}_Y + (p)\). We then say that \((Y,I,J)\) is a formal \(m\)-PD-\(S\)-scheme.

1.3.3. Definition. — Let \((S,a,b)\) be a formal \(m\)-PD-scheme. A morphism of formal \(m\)-PD-\(S\)-schemes is a morphism of formal schemes \(\varphi: Y' \to Y\) such that \(\varphi^{-1}(I) \subset I'\) and \((Y',J') \to (Y,J)\) is a morphism of formal PD-schemes. If \((Y,I,J)\) is a formal \(m\)-PD-\(S\)-scheme and \(X\) is the closed formal subscheme of \(Y\) defined by \(I\), we say that \(X \hookrightarrow Y\) is an \(m\)-PD-immersion.

The following generalizes Definition 1.2.3 and agrees with Berthelot’s new definition that replaces \([B4]\) 1.3.8 and 1.3.7.

1.3.4. Proposition and Definition (see \([B5]\)). — If \((Y,I,J)\) is a formal \(m\)-PD-\(S\)-scheme, then there exists a finest ring filtration \((I,\{\}\) := \(I^\ast\) on \(\mathcal{O}_Y\) such that

(i) \(I^{(1)} = I\),

(ii) \(I^{(n)} \cap (J + b\mathcal{O}_Y + p\mathcal{O}_Y)\) is a sub PD-ideal of \(J + b\mathcal{O}_Y + p\mathcal{O}_Y\),

(iii) \(x^{(h)} \in I^{(nh)}\) whenever \(x \in I^{(n)}\).

It is called the \(m\)-PD-filtration on \(\mathcal{O}_Y\) with respect to \((I,J)\). Then \((Y,\mathcal{O}_Y,I^{(n)})\) is a filtered ringed site. Moreover, any morphism of formal \(m\)-PD-\(S\)-schemes induces a morphism of the corresponding filtered ringed sites.

Universal \(m\)-PD-immersions do exist:

1.3.5. Proposition and Definition (see \([B4]\), 2.1.1). — Let \(S\) be a formal \(m\)-PD-scheme, \(X\) a formal \(S\)-scheme to which the \(m\)-PD-structure of \(S\) extends and \(i: X \hookrightarrow Y\) an immersion into a formal \(S\)-scheme. Then \(i\) factors as an \(m\)-PD-\(S\)-immersion \(X \hookrightarrow P^n_{X/S(m)}(Y)\) followed by a morphism \(\varphi: P^n_{X/S(m)}(Y) \to Y\) having the following universal property: any morphism \(Y' \to Y\) inducing \(X' \to X\), where \(X' \hookrightarrow Y'\) is an \(m\)-PD-\(S\)-immersion whose ideal satisfies \(I^{(n+1)} = 0\), factors uniquely through \(\varphi\).
We say that $P_{X/S(m)}^n(Y)$ is the $n$-th $m$-PD-neighborhood of $X$ in $Y$ and we write $\mathcal{P}_{X/S(m)}^n(Y)$ for its structural sheaf.

1.3.6. Remark. — If $X \hookrightarrow Y$ is an immersion of schemes (locally killed by a power of $p$) then there exists an $m$-PD-$S$-immersion $X \hookrightarrow P_{X/S(m)}(Y)$ with the same universal property but without nilpotency condition on $I$. We call $P_{X/S(m)}(Y)$ the $m$-PD-neighborhood of $X$ in $Y$, and write $\mathcal{P}_{X/S(m)}^n(Y)$ for its structural sheaf.

1.3.7. Definition. — If $i$ is the diagonal immersion

$$X \hookrightarrow Y := X \times_S X,$$

then we drop $Y$ from the notations in 1.3.5 and 1.3.6 and we call $\mathcal{P}_{X/S}^n$ the sheaf of $m$-th principal parts of order at most $n$.

2. DIFFERENTIAL OPERATORS OF LEVEL $m$ AND GRIFFITHS TRANSVERSALITY

2.1. Differential operators of level $m$.

We will now recall from [B4] Berthelot’s theory of differential operators of finite level.

Let $(S, a, b)$ be a formal $m$-PD-scheme and $X$ a smooth formal $S$-scheme to which the $m$-PD-structure of $S$ extends. We consider $\mathcal{P}_{X/S(m)}^n$ as an $\mathcal{O}_X$-module using the first projection $X \times_S X \to X$ and we note $\theta: \mathcal{O}_X \to \mathcal{P}_{X/S(m)}^n$ the map induced by the second projection. We first recall the definition of differential operators of level $m$:

2.1.1. Definition. — The $\mathcal{O}_X$-dual $\mathcal{D}_{X/S}^{(m)}$ to $\mathcal{P}_{X/S(m)}^n$ is called the sheaf of differential operators of level $m$ and order at most $n$. The natural maps $\mathcal{P}_{X/S(m)}^{n'} \to \mathcal{P}_{X/S(m)}^n$ for $n \leq n'$ induce injections $\mathcal{D}_{X/S}^{(m)} \hookrightarrow \mathcal{D}_{X/S}^{(m)}$ and we set

$$\mathcal{D}_{X/S}^{(m)} = \bigcup_n \mathcal{D}_{X/S}^{(m)}.$$

Moreover, the natural maps

$$\mathcal{P}_{X/S(m)}^{n+n'} \to \mathcal{P}_{X/S(m)}^n \otimes \mathcal{P}_{X/S(m)}^{n'}$$

induce bilinear maps

$$\mathcal{D}_{X/S}^{(m)} \times \mathcal{D}_{X/S}^{(m)} \to \mathcal{D}_{X/S}^{(m)} \otimes \mathcal{D}_{X/S}^{(m)}$$

which make $\mathcal{D}_{X/S}^{(m)}$ into a ring called the ring of differential operators of level $m$. Its $p$-adic completion will be denoted by $\widehat{\mathcal{D}}_{X/S}^{(m)}$. 
2.1.2. Remark. — If \( t_1, \ldots, t_d \) are local coordinates on \( X \) and

\[ \tau_i := \theta(t_i) - t_i \quad \text{for all } i, \]

then \( \mathcal{P}_{X/S(m)}^{n} \) is a free \( \mathcal{O}_X \)-module on the \( \tau^{(k)} \) with \( |k| \leq n \).

We let \( \{ \partial^{(k)} \} \) be the dual basis to \( \{ \tau^{(k)} \} \) in \( \mathcal{D}_{X/S}^{(m)} \).

If \( k = gp^m + r < p^{m+1} \), we set

\[ \partial^{[k]} := \partial^{(k)}/q!. \]

If \( n < p^{m+1} \), then the \( \tau^k \) with \( |k| \leq n \) form a basis for \( \mathcal{P}_{X/S(m)}^{n} \) and the \( \partial^{[k]} \) form the dual basis in \( \mathcal{D}_{X/S}^{(m)} \). Note that \( \mathcal{D}_{X/S}^{(m)} \) is generated as an \( \mathcal{O}_X \)-algebra by the \( \partial^{[p]}_i = \partial^{(p)}_i \) for \( j \leq m \).

2.1.3. Remark. — If \( \varphi: Y \to X \) is a morphism of smooth formal \( S \)-schemes and \( \mathcal{F} \) is a \( \mathcal{D}_{X/S}^{(m)} \)-module then \( \varphi^{*}\mathcal{F} \) has a natural structure of \( \mathcal{D}_{Y/S}^{(m)} \)-module that can be described locally as follows. Let \( t_1, \ldots, t_d \) be local coordinates on \( X \), \( t'_1, \ldots, t'_d \) be local coordinates on \( Y \) and \( \{ \tau_i \} \) and \( \{ \tau'_i \} \) be the corresponding sections of \( \mathcal{P}_{X/S}^{n(m)} \) and \( \mathcal{P}_{Y/S}^{n(m)} \).

If \( \varphi^{*}(\tau^{(j)}_i) = \sum f_{k,l} t'_i \tau'_k \) and \( s \) is a section of \( \mathcal{F} \), we have

\[ \partial^{(j)}_i (\varphi^{*}(s)) = \sum f_{k,l} \varphi^{*}(\partial^{(l)}_k(s)). \]

As in the classical case, \( \mathcal{D}^{(m)} \)-modules have an interpretation in terms of stratifications:

2.1.4. Proposition (see [B4], 2.3.2). — If \( \mathcal{F} \) is an \( \mathcal{O}_X \)-module, it is equivalent to give it a structure of \( \mathcal{D}_{X/S}^{(m)} \)-module or an \( m \)-PD-stratification (defined in the obvious way).

2.1.5. Definition. — A \( \mathcal{D}_{X/S}^{(m)} \)-module (or \( \hat{\mathcal{D}}_{X/S}^{(m)} \)-module) is locally (topologically) quasi-nilpotent if locally, given any section \( s \), we have \( \partial^{(N)}_i(s) \to 0 \) as \( N \to \infty \) for any index \( i \).

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

2.1.6. Proposition (generalization of [B1], II. 4.1.3). — If \( X \) is a smooth \( S \)-scheme (with \( p \) locally nilpotent) and \( \mathcal{F} \) is an \( \mathcal{O}_X \)-module, it is equivalent to give it a structure of locally quasi-nilpotent \( \mathcal{D}_{X/S}^{(m)} \)-module or an \( m \)-HPD-stratification (defined in the obvious way).
We will also have to consider formal $S$-schemes that are not necessarily smooth. In order to deal with this situation we need to introduce the following terminology (see also [B4], 2.3.4 and 2.3.5):

2.1.7. DEFINITION. — Let $X$ be an $S$-scheme and $X \hookrightarrow Y$ a closed immersion into a smooth formal $S$-scheme. It follows from Proposition 4.1.5 below that $\mathcal{P}_{X/S(\mathfrak{m})}(Y)$ has a natural structure of $\mathcal{D}_{Y/S}^{(m)}$-module. A $\mathcal{P}_{X/S(\mathfrak{m})}(Y)$-$\mathcal{D}_{Y/S}^{(m)}$-module is a $\mathcal{D}_{Y/S}^{(m)}$-module $\mathcal{F}$ with a structure of $\mathcal{P}_{X/S(\mathfrak{m})}(Y)$-module such that, locally, given any sections $f$ of $\mathcal{P}_{X/S(\mathfrak{m})}(Y)$ and $s$ of $\mathcal{F}$, we have

$$\partial^{(k)}(fs) = \sum \left\{ \begin{array}{c} k \\ j \end{array} \right\} \partial^{(j)}(f) \partial^{(k-j)}(s).$$

It follows from Proposition 4.1.7 and Proposition 4.1.8 below that this definition does not depend on the choice of the local coordinate system.

2.2. Griffiths transversality for $\mathcal{D}^{(m)}$-modules.

We define Griffiths transversality for $\mathcal{D}^{(m)}$-modules and interpret it in terms of stratifications.

Let $S$ be a formal $m$-PD-scheme and $X$ a smooth formal $S$-scheme. The following generalizes the usual notion of Griffiths transversality:

2.2.1. DEFINITION. — A filtered $\mathcal{D}_{X/S}^{(m)}$-module $(\mathcal{F}, \text{Fil})$ is a $\mathcal{D}_{X/S}^{(m)}$-module $\mathcal{F}$ together with a filtration by sub $\mathcal{O}_X$-modules. We say that $(\mathcal{F}, \text{Fil})$ is Griffiths transversal if whenever $P \in \mathcal{D}_{X/S}^{(m)n}$, we have $P(\text{Fil}^k) \subset \text{Fil}^{k-n}$ and that it is horizontal if the Fil$^k$ are $\mathcal{D}_{X/S}^{(m)}$-submodules. A filtered $\mathcal{D}_{X/S}^{(m)}$-module $(\mathcal{F}, \text{Fil})$ is a complete $\mathcal{D}_{X/S}^{(m)}$-module $\mathcal{F}$ together with a filtration by complete sub $\mathcal{O}_X$-modules. We say that it is Griffiths transversal or horizontal if it is so mod $p^n$ for all $n$.

2.2.2. Remarks.

(i) What we call Griffiths transversal corresponds to what is simply called a filtration on a $\mathcal{D}$-module in the classical situation.

(ii) Assume we have local coordinates $t_1, \ldots, t_d$. In order to show that $(\mathcal{F}, \text{Fil})$ is Griffiths transversal it is sufficient to check that $\partial^{[p^j]}_i \text{Fil}^k \subset \text{Fil}^{k-p^j}$ for $j \leq m$ and all $i$.

Here is the interpretation of Griffiths transversality in terms of stratifications:
2.2.3. Definition. — Let $(\mathcal{F}, \text{Fil})$ be a filtered $\mathcal{O}_X$-module with an $m$-PD-stratification $\{\varepsilon_n : p_2^n \mathcal{F} \xrightarrow{\sim} p_1^n \mathcal{F}\}$. We call the stratification transversal if $\varepsilon_n$ induces an isomorphism between $\text{Fil}_p^n$ and $\text{Fil}_p^k$ for all $n$.

2.2.4. Proposition. — Let $\mathcal{F}$ be a $\mathcal{D}_{X/S}$-module and $\text{Fil}^k$ a filtration on $\mathcal{F}$ by sub $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is Griffiths transversal if and only if the corresponding $m$-PD-stratification is transversal.

Proof. — Let $\mathcal{I}$ be the ideal of $X$ in $P^n_{X/S(m)}$, $p_1, p_2 : P^n_{X/S(m)} \to X$ the projections, $\varepsilon : p_2^n \mathcal{F} \xrightarrow{\sim} p_1^n \mathcal{F}$ the $n$-th Taylor isomorphism of $\mathcal{F}$ and

$$\theta : \mathcal{F} \longrightarrow p_1^n \mathcal{F},$$

$$\varepsilon \longmapsto \varepsilon(1 \otimes \varepsilon)$$

the $n$-th Taylor map. Assume first the $m$-PD-stratification to be transversal. Since $\varepsilon$ induces an isomorphism between $\text{Fil}_p^n$ and $\text{Fil}_p^k$, then

$$\theta \text{Fil}^k \subset \overline{\text{Fil}}^k_{p_1} = \text{Fil}^k_{p_1} + \mathcal{I} \text{Fil}^{k-1}_{p_1} + \mathcal{I}^2 \text{Fil}^{k-2}_{p_1} + \cdots + \mathcal{I}^n \text{Fil}^{k-n}_{p_1} \subset \text{Fil}^{k-n}_{p_1}.$$

If $P : P^n_{X/S(m)} \to \mathcal{O}_X$ is a differential operator of level $m$ and order less than $n$, then $P$ acts on $\mathcal{F}$ as the composite of $\theta$ and $p_1^n(P)$ (i.e. $P(\varepsilon) = (P \otimes \text{Id})(\theta(\varepsilon))$ so that $P \text{Fil}^k \subset \text{Fil}^{k-n}$. Thus, we see that $\mathcal{F}$ is Griffiths transversal. Conversely, assume that $\mathcal{F}$ is Griffiths transversal. We want to check that $\varepsilon$ induces an isomorphism between $\overline{\text{Fil}}^k_{p_1}$ and $\text{Fil}^k_{p_1}$ and we may assume that we have local coordinates $t_1, \ldots, t_d$ on $X$. Thanks to the cocycle condition, it is sufficient to show that $\theta(\text{Fil}^k) \subset \overline{\text{Fil}}^k_{p_1}$. But if $\varepsilon \in \text{Fil}^k$ then

$$\theta(\varepsilon) = \sum \theta^{(j)}(\varepsilon) \tau^{(j)} \in \sum \mathcal{I}^{(j)} \text{Fil}^{k-j}_{p_1} = \overline{\text{Fil}}_{p_1}^{k}.$$

The same is true for hyperstratifications. Let $S$ be an $m$-PD-scheme and $X$ a smooth $S$-scheme.

2.2.5. Definition. — If $(\mathcal{F}, \text{Fil})$ is a filtered $\mathcal{O}_X$-module, we call an $m$-HPD-stratification $\varepsilon : p_2^n \mathcal{F} \xrightarrow{\sim} p_1^n \mathcal{F}$ on $\mathcal{F}$ transversal if $\varepsilon$ induces an isomorphism between $\overline{\text{Fil}}^k_{p_2}$ and $\overline{\text{Fil}}^k_{p_1}$.

2.2.6. Proposition. — An $m$-HPD-stratification $\varepsilon : p_2^n \mathcal{F} \xrightarrow{\sim} p_1^n \mathcal{F}$ on a filtered $\mathcal{O}_X$-module $(\mathcal{F}, \text{Fil})$ is transversal if and only if $(\mathcal{F}, \text{Fil})$ is Griffiths transversal.

Proof. — Same as Proposition 2.2.4.
2.3. Griffiths transversality and $p$-isogenies.

We are going to build the local version of the functor of our main theorem.

Let $S$ be a formal $m$-PD-scheme, $X$ a formal $S$-scheme, $F_0$ the relative Frobenius of $X_0$ over $S_0$ and $F : X \to X'$ a lifting of $F_0$. We assume that there are local coordinates $t_1, \ldots, t_d$ on $X$ and $X'$ such that $F^*(t_i) = t_i^p$.

We will write $X_0^{(m+1)}$ for the pull back of $X_0$ by the $m + 1$ iterate of $F_0$, and, with the usual slight abuse of notation, we will call

$$F_0^{m+1} : X_0 \to X_0^{(m+1)}$$

this $m + 1$ iterate of $F_0$ and $F^{m+1} : X \to X^{(m+1)}$ a lifting obtained by iterating the above process.

2.3.1. Lemma. — If $s$ is a section of a $\mathcal{D}^{(m)}_{X^{(m+1)}/S}$-module $\mathcal{E}$, then for $k < p^{m+1}$, we have, with $a_{j,k} \in \mathbb{Z}$,

$$\varrho^{[k]}(F^{m+1*}(s)) = \sum p^j a_{j,k} t^j p^{m+1-k} F^{m+1*}(\varrho^{[j]}(s)).$$

Proof. — For $n = p^{m+1} - 1$, we have in $\mathfrak{T}^n_{X^{(m+1)}/S(m)}$

$$F^{m+1*}(\tau_i) = (t_i + \tau_i)^{p^{m+1}} - t_i^{p^{m+1}}$$

$$= \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1-k}} \tau_i^k$$

$$= \sum_{k=1}^{p^{m+1}-1} p c_{i,k} t_i^{p^{m+1-k}} \tau_i^k$$

with $c_{i,k} \in \mathbb{Z}$. Thus we can write

$$F^{m+1*}(\tau^j) = \sum p^j a_{j,k} t^j p^{m+1-k} \tau^k$$

with $a_{j,k} \in \mathbb{Z}$. Therefore, if $s$ is a section of $\mathcal{E}$, we have

$$\varrho^{[k]}(F^{m+1*}(s)) = \sum p^j a_{j,k} t^j p^{m+1-k} F^{m+1*}(\varrho^{[j]}(s)).$$

This lemma allows us to show that Frobenius pulls back transversal modules to horizontal modules:
2.3.2. Proposition. — If \( (\mathcal{E}, \text{Fil}) \) is a Griffiths transversal \( \mathcal{D}_{X^{(m+1)}/S^{-}} \) module (or \( \hat{\mathcal{D}}_{X^{(m+1)}/S^{-}} \) module) on \( X^{(m+1)} \) which is saturated with respect to \( (p, \{ \} ) \), then \( F^{m+1} (\mathcal{E}, \text{Fil}) \) is horizontal.

Proof. — We have seen that if \( s \) is a section of \( \mathcal{E} \), then for \( k < p^{m+1} \), we have

\[
\hat{\partial}^{[k]} (F^{m+1} (s)) = \sum p^{\lambda} a_{j, k} t^{j} p^{m+1-k} F^{m+1} (\hat{\partial}^{[j]} (s)).
\]

Since \( (\mathcal{E}, \text{Fil}) \) is Griffiths transversal, we know that if \( s \in \text{Fil}^{\ell} \), we have \( (\hat{\partial}^{[j]} (s)) \in \text{Fil}^{\ell-[j]} \). It follows that \( F^{m+1} (\hat{\partial}^{[j]} (s)) \in \text{Fil}^{\ell-[j]} \) so that

\[
p^{\lambda} a_{j, k} t^{j} p^{m+1-k} F^{m+1} (\hat{\partial}^{[j]} (s)) \in p^{\lambda} \text{Fil}^{\ell-[j]}.
\]

Since \( (\mathcal{E}, \text{Fil}) \) is saturated with respect to \( (p, \{ \} ) \), so is \( F^{m+1} (\mathcal{E}, \text{Fil}) \) and therefore

\[
\hat{\partial}^{[j]} (F^{m+1} (s)) = \sum p^{\lambda} a_{j, k} t^{j} p^{m+1-k} F^{m+1} (\hat{\partial}^{[j]} (s))
\]

\[
\in \sum p^{\lambda} \text{Fil}^{\ell-[j]} = \sum p^{\lambda} \text{Fil}^{\ell-[j]} \subset \text{Fil}^{\ell}.
\]

2.3.3. Theorem. — Assume \( S \) has no \( p \)-torsion. Let \( (\mathcal{E}, \text{Fil}) \) be a \( p \)-torsion free Griffiths transversal \( \hat{\mathcal{D}}_{X^{(m+1)}/S^{-}} \) module of width less than \( p^{m+1} \) which is transversal to \( (p, \{ \} ) \). Then there exists a unique \( p \)-isogeny \( \Phi : F^{m+1} \mathcal{E} \rightarrow \mathcal{F} \) of \( \hat{\mathcal{D}}_{X/S} \) modules such that \( F^{m+1} \text{Fil}^{k} \) is the saturation of \( \Phi^{-1} (p^{k} \mathcal{F}) \) with respect to \( (p, \{ \} ) \).

Proof. — Follows from Corollary 1.2.6 and Proposition 2.3.2.

2.3.4. Definition. — Given any lifting \( F : X \rightarrow X' \) of the relative Frobenius of \( X_{0} \) over \( S_{0} \), an \( F^{m+1} \)-isogeny on \( X/S \) will be a \( p \)-isogeny of the form \( \Phi : F^{m+1} \mathcal{E} \rightarrow \mathcal{F} \) where \( \mathcal{E} \) is a \( \hat{\mathcal{D}}_{X^{(m+1)}/S^{-}} \) module and \( \mathcal{F} \) is a \( \hat{\mathcal{D}}_{X/S} \) module.

2.3.5. — Theorem 2.3.3 gives a functor \( \mu \) from the category of \( p \)-torsion free Griffiths transversal \( \hat{\mathcal{D}}_{X^{(m+1)}/S^{-}} \) module of width less than \( p^{m+1} \) that are transversal to \( (p, \{ \} ) \) to the category of \( F^{m+1} \)-isogenies of width less than \( p^{m+1} \) on \( X/S \). We will show in section 3.3 that this functor is fully faithful.
3. D\((m)\)-MODULES IN CHARACTERISTIC p AND GRIFFITHS TRANSVERSALITY

3.1. p-m-curvature of a D\((m)\)-module.

We define p-m-curvature for D\((m)\)-modules in characteristic p and study the relation between it being zero and horizontal sections.

Let S be a scheme of characteristic p and X a smooth S-scheme. We let

- D\((m)\)\(_{X/S}\) be the kernel of the canonical map D\((m)\)\(_{X/S}\) \(\to\) \(\mathcal{O}_X\);
- \(\mathcal{K}\((m)\)\(_{X/S}\) be the kernel of the canonical map D\((m)\)\(_{X/S}\) \(\to\) End(\(\mathcal{O}_X\)).

3.1.1. Definitions. — Let \(\mathcal{F}\) be a D\((m)\)\(_{X/S}\)-module. The sheaf \(\mathcal{F}^\vee\) of horizontal sections of \(\mathcal{F}\) is the part of \(\mathcal{F}\) on which D\((m)\) acts as zero.

The p-m-curvature of \(\mathcal{F}\) is the restriction to \(\mathcal{K}\((m)\)\(_{X/S}\) of the canonical map D\((m)\)\(_{X/S}\) \(\to\) End(\(\mathcal{O}_X\)).

3.1.2. Remark. — Let \(\mathcal{F}\) be a D\((m)\)\(_{X/S}\)-module. Then it follows from [B4], 2.2.6, that \(\mathcal{F}\) has zero p-m-curvature if, locally on X, we have for all \(i\), \(\partial_i^{(p^{m+1})}(s) = 0\) for any \(s \in \mathcal{F}\). In particular, in case \(m = 0\), zero p-m-curvature is the same as zero p-curvature.

Let \(F : X \to X'\) be the relative Frobenius of X over S.

3.1.3. Lemma. — If \(\mathcal{E}\) is a D\((m)\)\(_{X'(m+1)/S}\)-module, then D\((m)\)\(_{X/S}\) acts as zero on sections of the form \(F^{m+1}\)\(*\)\((s)\) with \(s \in \mathcal{E}\).

Proof. — This is a local question. We have

\[
F^{m+1}\* (\tau_i) = (t_i + \tau_i)^{p^{m+1}} - t_i^{p^{m+1}} = \sum_{k=1}^{p^{m+1}} \binom{p^{m+1}}{k} t_i^{p^{m+1} - k} \tau_i^k = \tau_i^{p^{m+1}} = p! \tau_i^{\{p^{m+1}\}} = 0.
\]

It follows that, if \(0 < j < p^{m+1}\), then \(F^{m+1}\* (\tau^j) = 0\), so that, for any section \(s\) of \(\mathcal{E}\), we have \(\partial[j](F^{m+1}\* (s)) = 0\).

3.1.4. Proposition. — The trivial D\((m)\)\(_{X/S}\)-module \(\mathcal{O}_X\) has zero p-m-curvature and the canonical map \(\mathcal{O}_X^{(m+1)} \to F^{m+1}\mathcal{O}_X^\vee\) is bijective.
**Proof.** — The first assertion is an obvious consequence of the definition. The second one is local and we may therefore choose local coordinates $t_1, \ldots, t_d$. These coordinates define an étale map from $X$ to $\mathbb{A}^d_\mathbb{F}_p$. The relative Frobenius being cartesian with respect to étale morphisms and to base change, this map provides us with an isomorphism

$$F_*^{m+1} \mathcal{O}_X \cong \mathcal{O}_{X^{(m+1)}} \otimes \mathbb{F}_p[t_1, \ldots, t_d] \mathbb{F}_p[t_1, \ldots, t_d]^{(m+1)}$$

where $\mathbb{F}_p[t_1, \ldots, t_d]^{(m+1)}$ is $\mathbb{F}_p[t_1, \ldots, t_d]$ seen as a module over itself via the $(m + 1)$-st power of Frobenius. If $\mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)}$ denotes the space of polynomials of degree strictly less than $p^{m+1}$ in each variable, the canonical map

$$\mathbb{F}_p[t_1, \ldots, t_d] \otimes \mathbb{F}_p \mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)} \longrightarrow \mathbb{F}_p[t_1, \ldots, t_d]^{(m+1)}$$

is bijective and therefore

$$F_*^{m+1} \mathcal{O}_X \cong \mathcal{O}_{X^{(m+1)}} \otimes \mathbb{F}_p \mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)}.$$

Since $F_*^{m+1} \mathcal{D}_{X/S}^{(m)+}$ acts as zero on $\mathcal{O}_{X^{(m+1)}}$, we are reduced to showing that if $f \in \mathbb{F}_p[t_1, \ldots, t_d]_{<p(m+1)}$ and $\mathcal{D}_{X/S}^{(m)+}$ acts as zero on $f$, then $f \in \mathbb{F}_p$. One may first prove that if $A$ is an $\mathbb{F}_p$-algebra and $f \in A[t^{p^d}]$ is such that $\partial(p^d)(f) = 0$, then $f \in A[t^{p^{d+1}}]$ and then use induction on $d$. The details are left to the reader. \(\square\)

**3.1.5. Proposition**

(i) If $\mathcal{F}$ is a $\mathcal{D}_{X/S}^{(m)}$-module then $F_*^{m+1} \mathcal{F}^\nabla$ is a sub $\mathcal{O}_{X^{(m+1)}}$-module of $F_*^{m+1} \mathcal{F}$.

(ii) If $\mathcal{E}$ is a $\mathcal{D}_{X^{(m+1)}/S}^{(m)}$-module then $F_*^{m+1} \mathcal{E}$ has zero $p$-m-curvature.

**Proof.** — Again, these are local questions. For the first assertion, we have to show that if $s$ is a section of $\mathcal{F}^\nabla$ and $f$ is a section of $\mathcal{O}_{X^{(m+1)}}$ then

$$\partial^{(k)}((F_*^{m+1} \mathcal{F}^\nabla)(f)s) = 0 \text{ for } k \neq 0.$$ 

For the second one, we have to show that if $s$ is a section of $\mathcal{F}$ and $f$ is a section of $\mathcal{O}_X$, then

$$\partial^{(p^{m+1})}_i (f F_*^{m+1} \mathcal{E}(s)) = 0.$$ 

Using the formula

$$\partial^{(k)}(fs) = \sum \left\{ \binom{k}{j} \partial^{(j)}(f) \partial^{(k-j)}(s), \right\}$$

both statements are easy consequences of Lemma 3.1.3 and Proposition 3.1.4. \(\square\)
3.2. Cartier’s theorem for $\mathcal{D}^{(m)}$-modules.

We generalize Cartier’s theorem (see [K], 5.1) to $\mathcal{D}^{(m)}_{X/S}$-modules.

We let $S$, $X$ and $F : X \to X'$ be as in section 3.1.

3.2.1. LEMMA. — Let $t_1, \ldots, t_d$ be local coordinates on $X$ and

$$P := \sum_{k < p^{m+1}} (-t)^k \partial[k].$$

If $\mathcal{F}$ is a $\mathcal{D}^{(m)}_{X/S}$-module with zero $p$-m-curvature, then $P$ is a projector from $\mathcal{F}$ onto $\mathcal{F}^\vee$. 

Proof. — We follow the first part of the proof of Proposition 5.1 in [K]. Since $\mathcal{F}$ has zero $p$-m-curvature, we have $\partial^{(j)}(s) = 0$ for $j > p^{m+1}$. There should therefore be no confusion if we write $\partial^{(j)}(s) = 0$ for $j$ such that $\max(j_i) \geq p^{m+1}$. If $s \in \mathcal{F}$, we have

$$\partial^{(j)}(P(s)) = \partial^{(j)}\left(\sum (-t)^k \partial[k](s)\right)$$

$$= \sum \sum \partial^{(j)}((-t)^k)(\partial^{(j+k-i)} \partial[k])(s)$$

$$= \sum \sum (-1)^i \binom{k}{i} (-t)^{k-i} \binom{k + j - i}{k} \partial^{[k+j-i]}(s)$$

$$= \sum \sum (-1)^i \binom{\ell + j}{i} (-t)^\ell \binom{\ell + j}{\ell + i} \partial^{[\ell + j]}(s)$$

$$= \sum \left(\sum (-1)^i \binom{\ell + j}{i} \binom{\ell + j}{\ell + i}\right) (-t)^\ell \partial^{[\ell+j]}(s)$$

and, if $j \neq 0$, we have

$$\sum (-1)^i \binom{\ell + j}{i} \binom{\ell + j}{\ell + i} = \binom{\ell + j}{\ell} \sum (-1)^i \binom{j}{i} = 0.$$

Thus we see that $P$ maps $f$ into $\mathcal{F}^\vee$. Since $P$ restricts to the identity on $\mathcal{F}^\vee$, it is a projector from $\mathcal{F}$ onto $\mathcal{F}^\vee$. \hfill \Box

3.2.2. PROPOSITION. — Let $\mathcal{F}$ be a $\mathcal{D}^{(m)}_{X/S}$-module with zero $p$-m-curvature. Then the canonical map $F^{m+1} F^{*(m+1)} \mathcal{F}^\vee \to \mathcal{F}$ is an isomorphism.
Proof. — We follow the end of the proof of Proposition 5.1 in [K].
The question is local on $X$ and we may therefore assume that we have local
coordinates $t_1, \ldots, t_d$. We have seen in Lemma 3.2.1 that $P$ is a projector
from $\mathcal{F}$ onto $\mathcal{F}^\nabla$. It follows that the map

$$T : \mathcal{F} \rightarrow F^{m+1}_* F^{m+1}_* \mathcal{F}^\nabla,$$

$$s \mapsto \sum_{k \leq p^{m+1}} t^k \otimes P \partial^{[k]}(s)$$

is well defined. Let us show that $T$ is a right inverse to the canonical map

$$U : F^{m+1}_* F^{m+1}_* \mathcal{F}^\nabla \rightarrow \mathcal{F}.$$ If $s \in \mathcal{F}$, then

$$(U \circ T)(s) = \sum t^k P \partial^{[k]}(s)$$

$$= \sum t^k \sum (-1)^{\ell} \partial^{[\ell]} \partial^{[k]}(s)$$

$$= \sum \sum (-1)^{\ell} t^{k+\ell} \left( \frac{k + \ell}{\ell} \right) \partial^{[k+\ell]}(s)$$

$$= \sum \left( \sum (-1)^{\ell} \left( \frac{k}{\ell} \right) \right) t^k \partial^{[\ell]}(s) = s.$$

We have seen that $F^{m+1}_* \mathcal{O}_X^\nabla = \mathcal{O}_{(m+1)}$ and it follows that $U$ is a bijection
in the case $\mathcal{F} = \mathcal{O}_X$. Hence, $T$ is also a left inverse to $U$ in this case, which
implies that for any $f \in \mathcal{O}_X$, we have $T(f) = f \otimes 1$. In general, we have for

$$(\mathcal{T} \circ U)(f \otimes s) = T(f) = \sum t^k \otimes P \partial^{[k]}(fs)$$

$$= \sum t^k \otimes P \partial^{[k]}(fs)$$

$$= \left( \sum t^k \otimes P \partial^{[k]}(f) \right) (1 \otimes s)$$

$$= T(f)(1 \otimes s) = (f \otimes 1)(1 \otimes s) = f \otimes s. \qquad \Box$$

3.2.3. Proposition. — Let $\mathcal{E}$ be a $\mathcal{D}^{(m)}_{X(m+1)/S}$-module, $\mathcal{F} = F^{m+1}_* \mathcal{E}$
(as $\mathcal{D}^{(m)}_{X/S}$-module) and $\eta : \mathcal{E} \rightarrow F^{m+1}_* \mathcal{F}$ be the adjunction map. Then

(i) The map $\eta$ induces a natural isomorphism $\mathcal{E} \cong F^{m+1}_* \mathcal{F}^\nabla$ of $\mathcal{O}_{(m+1)}$-
modules.

(ii) In the situation of Lemma 3.2.1, the action of $P$ on $F^{m+1}_* \mathcal{F}$ factors
through $\eta$.

(iii) If $\mathcal{F}'$ is a sub-$\mathcal{D}^{(m)}_{X/S}$-module of $\mathcal{F}$, then the natural map

$F^{m+1}_* F^{m+1}_* \mathcal{F} \rightarrow \mathcal{F}$ induces an isomorphism $F^{m+1}_* (\eta^{-1}(F^{m+1}_* \mathcal{F}')) \cong \mathcal{F}'$.
Proof. — We know from Proposition 3.1.5 (ii) that $\mathcal{F}$ has zero p-m-curvature. It follows from Proposition 3.2.2 that

$$F^{m+1} \mathcal{E} \cong F^{m+1} F^* \mathcal{F} \mathcal{V}$$

and we use the faithful flatness of $F$ to obtain assertion (i).

In order to prove assertion (ii), we recall from Lemma 3.2.1 that the image of $P$ acting on $\mathcal{F}$ is (contained in) $\mathcal{F} \mathcal{V}$. It therefore follows from (i) that the action of $P$ on $F^{m+1} \mathcal{F}$ factors through $\eta : \mathcal{E} \cong F^{m+1} \mathcal{F} \mathcal{V} \to F^{m+1} \mathcal{F}$.

Finally, for (iii), since $\mathcal{F}$ has zero p-m-curvature, so does $\mathcal{F}'$. The map $\eta$ being functorial, it follows from (i) that it induces $\mathcal{F}' \cong F^{m+1} \mathcal{F}'$ so that

$$\mathcal{F}' \cong F^{m+1} \mathcal{F} \mathcal{V} \cong F^{m+1} (\eta^{-1}(F^{m+1} \mathcal{F}')).$$\hspace{1cm} \Box

3.2.4. Corollary (Cartier’s theorem). — The functors $\mathcal{E} \mapsto F^{m+1} \mathcal{E}$ and $\mathcal{F} \mapsto F^{m+1} \mathcal{F} \mathcal{V}$ give an equivalence between the category of $0_X^{(m+1)}$-modules and the category of $D_{X/S}^{(m)}$-modules with zero p-m-curvature. \hspace{1cm} \Box

3.3. $F^{m+1}$-p-isogenies and Griffiths transversality.

We have built in section 2.3 a functor $\mu$ that associates $F^{m+1}$-p-isogenies to some filtered $\widehat{D}^{(m)}$. We are now going to define a functor $\alpha$ from $F^{m+1}$-p-isogenies to filtered $\widehat{D}^{(m)}$-modules that will allow us to prove that $\mu$ is fully faithful.

The setting is as in section 2.3: $S$ is a p-torsion free formal scheme, $X$ is a smooth formal $S$-scheme, $F_0$ is the relative Frobenius of $X_0$ over $S_0$ and $F : X \to X'$ is a lifting of $F_0$. We also assume that there are local coordinates $t_1, \ldots, t_d$ on $X$ and $X'$ such that $F^*(t_i) = t_i^p$.

If $\Phi : F^{m+1} \mathcal{E} \to \mathcal{F}$ is an $F^{m+1}$-p-isogeny on $X/S$, we consider the filtration $M$ on $F^{m+1} \mathcal{E}$ given by

$$M^k : = \Phi^{-1}(p^k \mathcal{F})$$

and the filtration Fil on $\mathcal{E}$ given by

$$\text{Fil}^k : = \eta^{-1}(F^{m+1}_* M^k),$$

where $\eta : \mathcal{E} \to F^{m+1} F^{m+1} \mathcal{E}$ is the adjunction map. We will write $\text{Fil}$ for the saturation of Fil with respect to $(p, \{ \})$. This way, we get a functor

$$\alpha : (\Phi : F^{m+1} \mathcal{E} \to \mathcal{F}) \mapsto (\mathcal{E}, \text{Fil})$$

with values in the category of filtered $\widehat{D}^{(m)}_{X/S}$-modules transversal to $(p, \{ \})$. 

3.3.1. Lemma. — If
\[ P := \sum_{k < p^{m+1}} (-t)^k \partial[k], \]
then there exists \( Q \), reducing to 1 mod \( p \), such that
\[ P(F^{m+1}^*(s)) = F^{m+1}^*(Q(s)) \]
for any section \( s \) of a \( \mathcal{D}^{(m)}_{X(m+1)/S} \)-module \( \mathcal{E} \).

Proof. — From Lemma 2.3.1, we deduce that
\[ \tau^k \partial[k](F^{m+1}^*(s)) = \sum p^j a_{j,k} t^j p^{m+1} F^{m+1}^*(\partial[j](s)) = F^{m+1}^*(Q_k(s)) \]
where \( Q_k := \sum p^j a_{j,k} t^j \partial[j] \) and we let
\[ Q = \sum_{k < p^{m+1}} (-1)^k Q_k. \]

The following result is of technical nature and is needed in the next proposition:

3.3.2. Lemma. — Let \( \Phi : F^{m+1}^* \mathcal{E} \to \mathcal{F} \) be an \( F^{m+1} \)-isogeny on \( X/S \) and \( M, \text{Fil} \) and \( \eta \) as above. Then \( \eta_0 : \mathcal{E}_0 \to F^{m+1}_0 F^{m+1}_0^* \mathcal{E}_0 \) is strictly compatible with the induced filtrations (i.e. we have \( \text{Fil}^k_0 = \eta_0^{-1}(F^{m+1}_0 M_0^k) \)).

Proof. — We follow the proof of Theorem 2.2 of [O1]. The map is clearly compatible with the induced filtrations and we are left with proving the strictness. Let \( s_0 \in \mathcal{E}_0 \) be such that \( \eta_0(s_0) \in F^{m+1}_0 M_0^k \). We want to prove that there exists a lifting \( s \in \mathcal{E} \) of \( s_0 \) such that \( \Phi(\eta(s)) = p^k s' \). It is clearly sufficient to show that for any \( i \) there exists a lifting \( s \in \mathcal{E} \) of \( s_0 \), and \( u \) such that \( \Phi(\eta(s) + p^i u) = p^k s' \) and then take \( i = k \). We prove this by induction on \( i \), the case \( i = 1 \) being just our assumption.

So, let us assume that \( s \in \mathcal{E} \) is a lifting of \( s_0 \) such that
\[ \Phi(\eta(s) + p^i u) = p^k s'. \]
Since \( \Phi \) is a morphism of \( \mathcal{D}^{(m)}_{X/S} \)-modules, it commutes with the operator \( P \) of the lemma. Using Lemma 3.3.1, we have
\[ p^k P(s') = P(p^k s') = P(\Phi(\eta(s) + p^i u)) \]
\[ = \Phi(P(\eta(s)) + P(p^i u)) = \Phi(\eta(Q(s)) + p^i P(u)). \]
We have seen in Proposition 3.2.3 (ii) that the action of $P$ on $F^{m+1, 1\ast}_{0} \mathcal{E}_{0}$ factors through $\eta_{0}: \mathcal{E}_{0} \to F^{m+1}_{0,\ast} F^{m+1, 1\ast}_{0} \mathcal{E}_{0}$. We can therefore write

$$P(u) = \eta(v) + pv.$$ 

It follows that

$$p^{k} P(s') = \Phi(\eta(Q(s)) + p^{i} \eta(v) + p^{i+1} w) = \Phi(\eta(Q(s) + p^{i} v) + p^{i+1} w).$$

It just remains to observe that $Q(s) + p^{i} v$ is a lifting of $s_{0}$ since $Q$ is the identity mod $p$. \hfill $\square$

3.3.3. Proposition. — Let $\Phi : F^{m+1, 1\ast} \mathcal{E} \to \mathcal{F}$ be an $F^{m+1, 1\ast}$-isogeny on $X/S$, and $M$ and Fil as above. Then we have $F^{m+1, 1\ast} Fil^{k} = M^{k}$.

Proof. — We follow the proof of Lemma 5.2.11 in [02]. The modules $\mathcal{E}$ and $\mathcal{F}$ are $p$-torsion free and the filtrations Fil$^{k}$ and $M^{k}$ are transversal to $p$. From this, we deduce that the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & F^{m+1, 1\ast} Fil^{k-1} & \overset{p}{\to} & F^{m+1, 1\ast} Fil^{k} & \to & F^{m+1, 1\ast} Fil^{0} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M^{k-1} & \overset{p}{\to} & M^{k} & \to & M^{0} & \to & 0
\end{array}
$$

has exact rows. Hence, by induction, it is sufficient to prove that $F^{m+1, 1\ast} Fil^{k} = M^{k}$. But we have seen in Proposition 3.2.3 (iii) that

$$F^{m+1, 1\ast}_{0} (\eta_{0}^{-1}(F^{m+1, 1\ast}_{0} M^{k}_{0})) = M^{k}_{0}$$

and we know from Lemma 3.3.2 that $\eta_{0}^{-1}(F^{m+1, 1\ast}_{0} M^{k}_{0}) = Fil^{k}_{0}$. \hfill $\square$

We will show in Proposition 5.2.5 that the filtration $\overline{Fil}$ in the definition of $\alpha$ is not always Griffiths transversal when $m > 0$. Nevertheless, for the functor $\mu$ of 2.3.5, we have the following:

3.3.4. Theorem. — When restricted to the essential image of $\mu$, the functor $\alpha$ is a quasi-inverse to $\mu$.

Proof. — Follows from Proposition 3.3.3. \hfill $\square$

3.3.5. Corollary. — The functor $\mu$ is fully faithful. \hfill $\square$
4. TRANSVERSAL m-CRYSTALS

4.1. m-crystals.

We recall Berthelot’s theory of m-crystals from [B3].

Let \((S, a, b)\) be a formal m-PD-scheme. If \(X\) is an \(S\)-scheme, we will always assume that the m-PD-structure of \(S\) extends to \(X\).

4.1.1. Definition. — If \(X \hookrightarrow Y\) is an m-PD-S-immersion of \(S\)-schemes, we say that \(Y\) is an m-PD-S-thickening of \(X\).

4.1.2. Definition. — Let \(X\) be an \(S\)-scheme. The \(m\)-th crystalline site of \(X/S\) is the category \(\text{Cris}^{(m)}(X/S)\) of m-PD-S-thickenings \(U \hookrightarrow Y\) with \(U\) open in \(X\), endowed with a suitable topology. As in the classical case, the site \(\text{Cris}^{(m)}(X/S)\) is functorial in \(X/S\).

4.1.3. Remark. — There exists a unique sheaf \(\mathcal{G}_X^{(n)}\) on \(\text{Cris}^{(m)}(X/S)\) whose value on \((Y, I, J)\) is \(I^{(n)}\). We will write

\[\mathcal{O}_{X/S} = \mathcal{G}_X^{(0)} \quad \text{and} \quad J_{X/S} = \mathcal{G}_X^{(1)}\]

It is clear that \((\text{Cris}^{(m)}(X/S), \mathcal{O}_{X/S}, \mathcal{G}_X^{(n)})\) is a filtered ringed site.

4.1.4. Definition. — Let \(X\) be an \(S\)-scheme. To any sheaf \(E\) on \(\text{Cris}^{(m)}(X/S)\) and any object \(Y\) of \(\text{Cris}^{(m)}(X/S)\), one associates in the obvious way a sheaf \(E_Y\) on \(Y\). If \(E\) is an \(\mathcal{O}_{X/S}\)-module, any morphism \(\varphi: Y' \to Y\) of m-PD-thickenings gives a natural morphism \(\varphi^*E_Y \to E_{Y'}\). We call \(E\) an m-crystal if these maps are all bijective.

The proofs of the following statements are straightforward generalizations of those of the analogous results from [B1]. They should appear in a forthcoming article of Berthelot as announced in [B4].

4.1.5. Proposition. — If \(X \hookrightarrow Y\) is a closed immersion of \(S\)-schemes and \(E\) is an m-crystal on \(X\), then \(i_*E\) is an m-crystal on \(Y\).

4.1.6. Corollary. — If \(\bar{S} = \text{Spec} \mathcal{O}_S/a\) and \(\bar{X} = X \times_S \bar{S}\), then the restriction functor \(\text{Cris}^{(m)}(X/S) \to \text{Cris}^{(m)}(\bar{X}/\bar{S})\) induces an equivalence between the categories of m-crystals on \(X/S\) and on \(\bar{X}/\bar{S}\).

4.1.7. Proposition. — Let \(i: X \hookrightarrow Y\) be a closed immersion of \(S\)-schemes with \(Y\) smooth. Then the functor \(E \mapsto E_Y := (i_*E)_Y\) is an equivalence of categories between m-crystals on \(X\) and locally quasi-nilpotent \(\mathcal{F}_X/S(m)\)-\(\mathcal{D}_{Y/S}^{(m)}\)-modules.
4.1.8. PROPOSITION. — Let $X$ be a smooth formal $S$-scheme and let $X_n$ denote its reduction mod $p^{n+1}$. The functor

$$E \mapsto E_X := \lim_{\longrightarrow} E_{X_n}$$

is an equivalence of categories between $m$-crystals on $X_0$ and locally topologically quasi-nilpotent complete $\hat{D}^{(m)}_{X/S}$-modules.

4.2. $T$-$m$-Crystals.

We define $T$-$m$-crystals and relate them to differential modules. Note that we call $T$-$m$-crystals what Ogus would call proto-$T$-$m$-crystals.

Let $S$ be a formal $m$-PD-scheme.

4.2.1. PROPOSITION AND DEFINITION. — Let $f : (U', Y') \to (U, Y)$ be a morphism of $m$-PD-$S$-thickenings such that $U' \to U$ is flat and $(\mathcal{F}, \Fil)$ a $T$-module on $(Y, \mathcal{O}_Y, j^{(n)})$. Then $Tf^*(\mathcal{F}, \Fil) := (f^*\mathcal{F}, \Fil^k_f)$ is a $T$-module called the $T$-inverse image of $(\mathcal{F}, \Fil)$.

Proof. — This follows from Proposition 1.1.7 (ii) and Proposition 1.1.8. \qed

4.2.2. DEFINITION. — Let $X$ be an $S$-scheme. If $E$ is any $T$-module on $\text{Cris}(X/S)^{(m)}$ and $Y$ any object of $\text{Cris}(X/S)^{(m)}$, then $E_Y$ is in a natural way a $T$-module. If $f : Y' \to Y$ is a morphism in $\text{Cris}(X/S)^{(m)}$, then there is a natural morphism of filtered modules $Tf^*E_Y \to E_{Y'}$. We call $E$ a $T$-$m$-crystal if these maps are all isomorphisms of filtered modules (i.e. such that $\Fil^k_f = \Fil^k$).

The category of $T$-$m$-crystals is functorial with respect to flat morphisms: if $\varphi : X' \to X$ is a flat morphism and $E$ a $T$-$m$-crystal on $X/S$, then

$$T\varphi^*(E, \Fil) := (\varphi^*E, \Fil^k_\varphi)$$

is a $T$-$m$-crystal.

4.2.3. Example. — The trivial $T$-$m$-crystal is $(\mathcal{O}_{X/S}, j^{(k)}_{X/S})$ whose value at $X$ is the trivial filtered module $\mathcal{O}_X = \Fil^0 \supset \Fil^1 = 0$.

The following generalize Proposition 3.2.2 and Theorem 3.2.3 of [O2]:

---

4.1.8. PROPOSITION. — Let $X$ be a smooth formal $S$-scheme and let $X_n$ denote its reduction mod $p^{n+1}$. The functor

$$E \mapsto E_X := \lim_{\longrightarrow} E_{X_n}$$

is an equivalence of categories between $m$-crystals on $X_0$ and locally topologically quasi-nilpotent complete $\hat{D}^{(m)}_{X/S}$-modules.

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4.2.1. PROPOSITION AND DEFINITION. — Let $f : (U', Y') \to (U, Y)$ be a morphism of $m$-PD-$S$-thickenings such that $U' \to U$ is flat and $(\mathcal{F}, \Fil)$ a $T$-module on $(Y, \mathcal{O}_Y, j^{(n)})$. Then $Tf^*(\mathcal{F}, \Fil) := (f^*\mathcal{F}, \Fil^k_f)$ is a $T$-module called the $T$-inverse image of $(\mathcal{F}, \Fil)$.

Proof. — This follows from Proposition 1.1.7 (ii) and Proposition 1.1.8. \qed

4.2.2. DEFINITION. — Let $X$ be an $S$-scheme. If $E$ is any $T$-module on $\text{Cris}(X/S)^{(m)}$ and $Y$ any object of $\text{Cris}(X/S)^{(m)}$, then $E_Y$ is in a natural way a $T$-module. If $f : Y' \to Y$ is a morphism in $\text{Cris}(X/S)^{(m)}$, then there is a natural morphism of filtered modules $Tf^*E_Y \to E_{Y'}$. We call $E$ a $T$-$m$-crystal if these maps are all isomorphisms of filtered modules (i.e. such that $\Fil^k_f = \Fil^k$).

The category of $T$-$m$-crystals is functorial with respect to flat morphisms: if $\varphi : X' \to X$ is a flat morphism and $E$ a $T$-$m$-crystal on $X/S$, then

$$T\varphi^*(E, \Fil) := (\varphi^*E, \Fil^k_\varphi)$$

is a $T$-$m$-crystal.

4.2.3. Example. — The trivial $T$-$m$-crystal is $(\mathcal{O}_{X/S}, j^{(k)}_{X/S})$ whose value at $X$ is the trivial filtered module $\mathcal{O}_X = \Fil^0 \supset \Fil^1 = 0$.

The following generalize Proposition 3.2.2 and Theorem 3.2.3 of [O2]:
4.2.4. **Proposition.** — If \( i : X \hookrightarrow Y \) is a closed immersion into a smooth \( S \)-scheme and \( E \) a \( T \)-\( m \)-crystal on \( X/S \), then

\[
i_*(E, \text{Fil}) := (i_*E, i_*\text{Fil})
\]

is a \( T \)-\( m \)-crystal which is transversal to \((i_*j_{X/S}, \{ \})\).

**Proof.** — Same proof as [O2], 3.2.2. \( \square \)

4.2.5. **Proposition.** — Let \( i : X \hookrightarrow Y \) be a closed \( S \)-immersion into a smooth \( S \)-scheme. Then the functor \( E \mapsto E_Y \) is an equivalence of categories between \( T \)-\( m \)-crystals on \( X \) and Griffiths transversal locally quasi-nilpotent \( \mathcal{P}_{X/S(m)}(Y) \)-\( \mathcal{D}^{(m)}_{Y/S} \)-modules which are transversal to the \( m \)-PD-filtration of \( \mathcal{P}_{X/S(m)}(Y) \).

**Proof.** — Let \( p_1, p_2 : P_X(Y^2) \to P_X(Y) \) be the projections. If \( E \) is a \( T \)-\( m \)-crystal, we have an isomorphism of filtered modules

\[
\varepsilon : Tp^*_2E_Y \cong E_Y \leftrightarrow Tp^*_1E_Y,
\]

which means that the HPD-stratification \( \varepsilon : p^*_2\mathcal{F} \cong p^*_1\mathcal{F} \) is transversal and therefore, by Proposition 2.2.6, that \( E_Y \) is Griffiths transversal.

Conversely, let \( \mathcal{F} \) be a Griffiths transversal locally quasi-nilpotent \( \mathcal{P}_{X/S(m)}(Y) \)-\( \mathcal{D}^{(m)}_{Y/S} \)-module which is transversal to the \( m \)-PD-filtration of \( \mathcal{P}_{X/S(m)}(Y) \). There exists, by Proposition 4.1.7, a unique \( m \)-crystal \( E \) such that \( E_Y = \mathcal{F} \). Let \( X \hookrightarrow T \) be an \( m \)-PD-thickening. Since \( Y \) is smooth, \( i \) extends locally on \( T \) to a map \( f : T \to Y \) which in turn extends to an \( m \)-PD-morphism \( g : T \to P_X(Y) \). We then set

\[
\text{Fil}^k E_T = \overline{\text{Fil}}^k_g,
\]

so that \((E_T, \text{Fil}) = Tg^*(\mathcal{F}, \text{Fil})\). If this is well defined, it is clear that we obtain a quasi-inverse to our functor. It is actually sufficient to check that the HPD-stratification \( \varepsilon : p^*_2\mathcal{F} \cong p^*_1\mathcal{F} \) is transversal. But this follows again from Proposition 2.2.6. \( \square \)

4.2.6. **Corollary.** — Let \( X \) be a smooth formal \( S \)-scheme. Then the functor \( E \mapsto E_X \) is an equivalence of categories between \( T \)-\( m \)-crystals on \( X_0/S \) and locally topologically quasi-nilpotent Griffiths transversal complete \( \widehat{\mathcal{D}}^{(m)}_{X/S} \)-modules transversal to \((p, \{ \})\). \( \square \)
4.3. $T$-m-crystals and $F$-m-spans.

We define $F$-m-spans and use them to describe $T$-m-crystals.

Let $S$ be a formal $m$-PD-scheme, $X$ a smooth $S_0$-scheme, and $F : X \to X'$ the relative Frobenius of $X$ over $S_0$.

4.3.1. Definition. — If $(E, \text{Fil})$ is a filtered $m$-crystal where the $\text{Fil}^k$ are not merely sub modules but sub $m$-crystals, then we say that $(E, \text{Fil})$ is horizontal.

Note that a horizontal filtered $m$-crystal is not a $T$-m-crystal. Let us describe the saturation process:

4.3.2. Proposition

(i) Any horizontal filtered $m$-crystal $(E, \text{Fil})$ on $X/S$ that is almost transversal to $(p, \{ \})$ is almost transversal to $(I_{X/S}, \{ \})$. In particular, $(E, \text{Fil})$ is a $T$-m-crystal.

(ii) The functor $(E, \text{Fil}) \to (E, \text{Fil})$ from the category of horizontal filtered $m$-crystals on $X/S$ that are transversal to $(p, \{ \})$, to the category of $T$-m-crystals is fully faithful.

Proof.

(i) Let $X \hookrightarrow T$ be an $m$-PD-immersion and $I$ the ideal of $X$ in $T$. We have to show that $(E_T, \text{Fil})$ is almost transversal to $(I, \{ \})$. This question is local on $T$. The scheme $X$ being smooth over $S_0$, it locally lifts to a smooth formal scheme $Y$ over $S$. Since $Y$ is smooth and $X \hookrightarrow T$ is nilpotent, there exists, locally on $T$, a map $\varphi : T \to Y$ that induces the identity on $X$. The $m$-PD-structure on $T$ is compatible with $(p, \{ \})$, so that the map $\varphi$ is an $m$-PD-morphism. Since $(E_Y, \text{Fil})$ is almost transversal to $(p, \{ \})$, it follows from Proposition 1.1.8 that $(E_T, \text{Fil})$ is almost transversal to $(I, \{ \})$. Applying Proposition 1.1.7 (ii), we get the last assertion.

(ii) We have to show that $\text{Fil}^k$ determines $\text{Fil}^k$. This is a local question on $X$. The scheme $X$ being smooth over $S_0$, it locally lifts to a smooth formal scheme $Y$ over $S$. Since $(E_Y, \text{Fil})$ is saturated with respect to $(p, \{ \})$, we have $\text{Fil}^k E_Y = \text{Fil}^k E_Y$. It follows from Corollary 4.2.6 that $\text{Fil}^k E$ is determined by $\text{Fil}^k E_Y$ and hence by $\text{Fil}^k E$.

4.3.3. Definition. — If $(E, \text{Fil})$ is in the image of this last functor, we call it a horizontal $T$-m-crystal.

We are now able to globalize the local results of parts 2 and 3:
4.3.4. **Proposition.** — If \((E, \text{Fil})\) is a \(T\)-\(m\)-crystal on \(X^{(m+1)}/S\), then \(TF^{m+1}(E, \text{Fil})\) is a horizontal \(T\)-\(m\)-crystal.

**Proof.** — This follows from Proposition 2.3.2 and Proposition 4.3.2 (i).

4.3.5. **Definition.** — An \(F\)-\(m\)-span is a \(p\)-isogeny \(\Phi: F^{m+1} E \to E'\) of \(m\)-crystals.

4.3.6. **Theorem.** — Assume \(S\) has no \(p\)-torsion. Let \((E, \text{Fil})\) be a \(p\)-torsion free \(T\)-\(m\)-crystal on \(X^{(m+1)}/S\) of width less than \(p^{m+1}\). Then there exists a unique \(F\)-\(m\)-span \(\Phi: F^{m+1} E \to E'\) of width less than \(p^{m+1}\) such that the saturations of \(F^{m+1}\text{Fil}^k\) and \(\Phi^{-1}(p^k E')\) with respect to \((\mathcal{O}_X/S, \{\})\) coincide. This construction is functorial in \((E, \text{Fil})\) and the functor is fully faithful.

**Proof.** — Follows from Theorem 2.3.3, Proposition 4.3.2 (ii) and Corollary 3.3.5.

5. **Comparison of Transversality Properties for Various Levels**

From now on, \(m'\) will be an integer larger than \(m\) and \(\{\}\)' will denote divided powers of level \(m'\). We will also write \(d := m' - m\).

5.1. **Changing level and Griffiths transversality.**

After recalling how to obtain a \(\mathcal{D}^{(m)}\)-module from a \(\mathcal{D}^{(m')}\)-module, we show that, for filtered \(\mathcal{D}^{(m')}\)-modules transversal to \(p\) of width at most \(p^{m+1}\), Griffiths transversality can be checked on the corresponding filtered \(\mathcal{D}^{(m)}\)-module. We give a counterexample for higher width.

5.1.1. — We recall some results from [B4].

(i) If \(Y\) is a formal scheme and \(I\) is a coherent ideal in \(\mathcal{O}_Y\), then any \(m\)-PD-structure \((J, [])\) on \(I\) is also an \(m'\)-PD-structure on \(I\). If \((S, a, b)\) is a formal \(m\)-PD-scheme and \((Y, I, J)\) is a formal \(m\)-PD-\(S\)-scheme, then it is also a formal \(m'\)-PD-\(S\)-scheme. We should also remark that the \(m'\)-PD-filtration is finer than the \(m\)-PD-filtration.

(ii) Let \(S\) be a formal \(m\)-PD-scheme, \(X\) a formal \(S\)-scheme to which the \(m\)-PD-structure of \(S\) extends and \(i: X \hookrightarrow Y\) an immersion into a formal
$S$-scheme, then there are canonical maps $P^n_{X/S(m')} (Y) \to P^n_{X/S(m)} (Y)$. They are bijective for $n < p^{m+1}$.

(iii) Assume now that $X$ is smooth over $S$. Then we get canonical maps

$$\mathcal{D}_{X/S} (m) \to \mathcal{D}_{X/S} (m')$$

that are bijective for $n < p^{m+1}$. They glue to give canonical maps $\mathcal{D}_{X/S} (m) \to \mathcal{D}_{X/S} (m')$ and, after completion, $\widehat{\mathcal{D}}_{X/S} (m) \to \widehat{\mathcal{D}}_{X/S} (m')$. We can therefore consider any $\mathcal{D}_{X/S} (m')$-module (resp. $\widehat{\mathcal{D}}_{X/S} (m')$-module) as a $\mathcal{D}_{X/S} (m)$-module (resp. $\widehat{\mathcal{D}}_{X/S} (m)$-module).

(iv) Assume moreover that $S$ has no $p$-torsion. Then one easily checks that the obvious functor from $\widehat{\mathcal{D}}_{X/S} (m')$-modules to $\widehat{\mathcal{D}}_{X/S} (m)$-modules is faithful. It is even fully faithful when restricted to $p$-torsion free objects.

Let $S$ be a formal $m$-PD-scheme and $X$ a smooth formal $S$-scheme to which the $m$-PD-structure of $S$ extends. If $(\mathcal{F}, \text{Fil})$ is a Griffiths transversal $\mathcal{D}_{X/S} (m')$-module (resp. $\widehat{\mathcal{D}}_{X/S} (m')$-module), then it is also Griffiths transversal as a $\mathcal{D}_{X/S} (m)$-module (resp. $\widehat{\mathcal{D}}_{X/S} (m)$-module).

The converse is true under some additional hypothesis:

5.1.2. Proposition. — Let $(\mathcal{F}, \text{Fil})$ be a filtered $\mathcal{D}_{X/S} (m')$-module of width at most $p^{m+1}$ that is Griffiths transversal as a $\mathcal{D}_{X/S} (m)$-module and transversal to $p$. Then it is also Griffiths transversal as a $\mathcal{D}_{X/S} (m')$-module.

Proof. — We have to show that, if $P \in \mathcal{D}_{X/S} (m')$ is an $m'$-PD-differential operator of order at most $n$, then $P(\text{Fil}^k) \subset \text{Fil}^{k-n}$. Thanks to 5.1.1 (iii), we may assume that $n \geq p^{m+1}$. We proceed by induction on $k$.

- If $k \leq p^{m+1}$, then $\text{Fil}^{k-n} = \mathcal{F}$ and our assertion is trivial.
- If $k > p^{m+1}$, transversality to $p$ and the condition on the width give us that $\text{Fil}^k = p \text{Fil}^{k-1}$. It follows that

$$P(\text{Fil}^k) = pP(\text{Fil}^{k-1}) \subset p\text{Fil}^{k-1-n} \subset \text{Fil}^{k-n}. \quad \Box$$

The bound on the width is sharp as the following shows:
5.1.3. Example. — We take $X$ to be the affine line over $S$ and we consider $(\mathcal{F}, \text{Fil})$ where $\mathcal{F} = \mathcal{O}_X$ and $\text{Fil}$ is defined as follows:

- for $0 \leq k \leq p^{m+1}$, $\text{Fil}^k$ is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k$;
- for $k > p^{m+1}$, $\text{Fil}^k$ is the ideal generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $p^{k-p^{m+1}+1}t^{p^{m+1}}$.

It is clear that $(\mathcal{F}, \text{Fil})$ is a filtered $\mathcal{D}^{(m')}_{X/S}$-module of width $p^{m+1} + 1$. It is transversal to $p$ because, for $k \leq p^{m+1}$, both $(p) \cap \text{Fil}^k$ and $p \text{Fil}^{k-1}$ are generated by the elements $p^{k-i}t^i$ for $0 \leq i \leq k - 1$, together with $pt^k$, while $(p) \cap \text{Fil}^{p^{m+1}+1}$ and $p \text{Fil}^{p^{m+1}}$ are generated by the elements $p^{p^{m+1}+1-i}t^i$ for $0 \leq i \leq p^{m+1} - 1$, together with $pt^{p^{m+1}}$.

To show that $(\mathcal{F}, \text{Fil})$ is Griffiths transversal as a $\mathcal{D}^{(m')}_{X/S}$-module, let us remark that

$$\vartheta^r(p^{k-i}t^i) = \begin{cases} \binom{i}{r} p^{k-i}t^{i-r} & \text{if } r \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\vartheta^r(\text{Fil}^k) \subset \text{Fil}^{k-r}$ when $0 \leq k \leq p^{m+1}$. Moreover, when $r \leq p^m$, we have $(p^{m+1}) \subset (p)$ and therefore

$$\vartheta^r(\text{Fil}p^{m+1}) \subset p\text{Fil}^{m+1} \subset \text{Fil}^{p^{m+1}+1-r}.$$ Nevertheless, $(\mathcal{F}, \text{Fil})$ is not Griffiths transversal as a $\mathcal{D}^{(m')}_{X/S}$-module because $t^{p^{m+1}} \in \text{Fil}^{p^{m+1}+1}$ but $\vartheta^{p^{m+1}}(t^{p^{m+1}}) = 1 \notin \text{Fil}^1$.

5.2. Frobenius descent and $F^{m+1}$-isogenies.

We are going to apply Berthelot’s theory of Frobenius descent to $F^{m+1}$-isogenies and use it to study the question of the surjectivity of the functor $\mu$ of 2.3.5.

Let $S$ be a formal $m$-PD-scheme and $X$ a smooth formal $S$-scheme to which the $m$-PD-structure of $S$ extends. Let $F_0$ be the relative Frobenius of $X_0$ over $S_0$ and $F : X \rightarrow X'$ a lifting of $F_0$. We briefly recall Berthelot’s unpublished theory of Frobenius descent.

5.2.1. Proposition (see [B5]). — The morphism

$$F^d \times_S F^d : X \times_S X \rightarrow X^{(d)} \times_S X^{(d)}$$

induces for all $n$, a unique morphism

$$F^d : P^n_{X/S(m')} \rightarrow P^n_{X^{(d)}/S(m)}$$

compatible with the PD-structures (taking into account the PD-ideal of $S$). It is also compatible with the partial divided power filtrations.
It follows that, if $E$ is a $\mathcal{D}^{(m)}_{X(d)/S}$-module, then $F^d(E)$ has a natural structure of $\mathcal{D}^{(m)}_{X/S}$-module.

**5.2.2. Theorem (see [B5]).** — If $S$ is a scheme, the functor $E \mapsto F^d(E)$ induces an equivalence between the categories of $\mathcal{D}^{(m)}_{X(d)/S}$-modules and $\mathcal{D}^{(m)}_{X/S}$-modules.

It follows that the functor $E \mapsto F^d(E)$ induces an equivalence between the category of complete $\mathcal{D}^{(m)}_{X(d)/S}$-modules and the category of complete $\mathcal{D}^{(m)}_{X/S}$-modules. From Proposition 1.2.2, we get an equivalence between the category of $p$-isogenies of complete $\mathcal{D}^{(m)}_{X(d)/S}$-modules and the category of $p$-isogenies of complete $\mathcal{D}^{(m)}_{X/S}$-modules. Thus, we get:

**5.2.3. Corollary.** — The functor $F^d$ makes the full subcategory of $F^{m+1}$-$p$-isogenies on $X(d)/S$ consisting of those $\Phi : F^{m+1}E \to \mathcal{F}$ where $E$ is a $\mathcal{D}^{(m')}_{X^{(m'+1)}/S}$-module equivalent to the category of $F^{m'+1}$-$p$-isogenies on $X/S$.

**5.2.4. Lemma.** — Let $(\mathcal{F}, \text{Fil})$ be a filtered $\mathcal{D}^{(m)}_{X/S}$-module of width less than $p^{m+1}$ that is transversal to $p$ and $\overline{\text{Fil}}$ the saturation of Fil with respect to $(p, \{ \})$. Then $(\mathcal{F}, \text{Fil})$ is Griffiths transversal if and only if $(\mathcal{F}, \overline{\text{Fil}})$ is Griffiths transversal.

**Proof.** — The filtrations are identical up to order $(p^{m+1} - 1)$ and, for any $k \geq 0$, we have

$$\text{Fil}^{p^{m+1} - 1 + k} = p^k \text{Fil}^{p^{m+1} - 1} \quad \text{and} \quad \overline{\text{Fil}}^{p^{m+1} - 1 + k} = (p)^{(k)} \overline{\text{Fil}}^{p^{m+1} - 1}. \quad \square$$

Assume now that $S$ is a $p$-torsion free formal PD-scheme and that there are local coordinates $t_1, \ldots, t_d$ on $X$ and $X'$ such that $F^*(t_i) = t_i^p$.

**5.2.5. Proposition.** — The functor $\mu$ of 2.3.5 is not in general an equivalence of categories for $m > 0$. However, it becomes an equivalence when restricted to objects of width at most $p$.

**Proof.** — Let $\Phi : F^{m+1}E \to \mathcal{F}$ be an $F^{m+1}$-$p$-isogeny on $X/S$ of width less than $p^{m+1}$. By Corollary 5.2.3, it corresponds to a unique $F$-$p$-isogeny $\Phi^0 : F^*E \to \mathcal{F}'$ on $X^{(m)}/S$. We have shown in section 3.3 how to associate to $\Phi^0$ a filtration Fil on $E$ that is transversal to $p$. Thanks to
Proposition 3.3.3 and [O2], 5.2.12, the filtered module \((E, \text{Fil})\) is Griffiths transversal as a \(\hat{\mathcal{D}}_{X(m+1)/S}^{(0)}\)-module. It follows from Lemma 5.2.4 and Proposition 3.3.3 that \(\Phi\) will be in the essential image of \(\mu\) if and only if \((E, \text{Fil})\) is Griffiths transversal as a \(\hat{\mathcal{D}}_{X(m+1)/S}^{(m)}\)-module. If the width is at most \(p\) this is always the case by Proposition 5.1.2, while Example 5.1.3 shows that this needs not happen for higher width.

5.2.6. Example. — For \(m > 0\), we can give an explicit \(F^{m+1}-p\)-isogeny of width less than \(p^{m+1}\) on the formal affine line \(X\) which is not in the essential image of \(\mu\). We take \(E = \mathcal{O}_{X(m+1)}\) and we let \(\mathcal{F}\) be the ideal of \(\mathcal{O}_X\) generated by the elements \(p^{p+1-i}t^ip^{m+1}\) for \(0 \leq i \leq p-1\), together with \(t^{2(p+1)}\). It is a sub \(\hat{\mathcal{D}}_{X}^{(m)}\)-module of \(\mathcal{O}_X\) and we let the \(p\)-isogeny \(\Phi : F^{m+1}E \to \mathcal{F}\) be multiplication by \(p^{p+1}\). If we apply the functor \(\alpha\) to this \(F^{m+1}-p\)-isogeny, we get the saturation of the following filtration:

- for \(0 \leq k \leq p\), \(\text{Fil}^k\) is the ideal generated by the elements \(p^{k-i}t^i\) for \(0 \leq i \leq k\);
- for \(k > p\), \(\text{Fil}^k\) is the ideal generated by the elements \(p^{k-i}t^i\) for \(0 \leq i \leq p-1\), together with \(p^{k-p-1}t^p\).

It is not Griffiths transversal because \(t^p \in \text{Fil}^{p+1}\) but \(\partial[p](t^p) = 1\) is not in \(\text{Fil}^1\) and we can use Lemma 5.2.4.

5.2.7. Remark. — Let \(\Phi : F^*E \to \mathcal{F}\) and \(\Phi' : F^*\mathcal{F} \to \mathcal{G}\) be two \(F\)-\(p\)-isogenies of width less than \(p\). From [O2], 5.2.13, or Proposition 5.2.5, they are in the essential image of the functor \(\mu\) for level 0. Assume that \(E\) and \(\mathcal{G}\) are \(\hat{\mathcal{D}}^{(1)}\)-modules and that \(\Phi' \circ F^*(\Phi) : F^2E \to \mathcal{G}\) is a morphism of \(\hat{\mathcal{D}}^{(1)}\)-modules. Then it is an \(F^2-p\)-isogeny of width less than \((2p-1)\), and one may wonder if it is in the essential image of \(\mu\). One can show that this is true if \(p = 2\), but if \(p > 2\) the answer is no in general as the following example on the formal affine line shows:

We take \(E = \mathcal{O}\), we let \(\mathcal{F}\) be the ideal of \(\mathcal{O}\) generated by \(p^2, pt^p\) and \(t^{2p}\), and \(\mathcal{G}\) be the ideal of \(\mathcal{O}\) generated by the elements \(p^{p+1-i}t^ip^{2p}\) with \(0 \leq i \leq p-1\), together with \(t^{3p}\). The \(p\)-isogenies \(\Phi\) and \(\Phi'\) are multiplication by \(p^2\) and \(p^{p-1}\), respectively. The composition of \(F^*(\Phi)\) and \(\Phi\) is Example 5.2.6 in the case \(m = 1\).

5.3. Changing level for \(T-m\)-crystals and \(F-m\)-spans.

We study the behavior of the functors relating \(T-m\)-crystals and \(F-m\)-spans when the level changes and derive some consequences.
5.3.1. Lemma. — The functor «saturation with respect to \((p, \{ \} )\)» from the category of filtered modules transversal to \((p, \{ \} )'\) to the category of filtered modules transversal to \((p, \{ \} )\) gives an equivalence of categories when restricted to objects of width less than \(p^{m+1}\).

Proof. — This is an immediate consequence of Proposition 1.2.5. □

Let \((S, a, b)\) be a formal \(m\)-PD-scheme. If \(X\) is an \(S\)-scheme, it follows from 5.1.1 (i) that \(\text{Cris}^{(m)}(X/S)\) is a subsite of \(\text{Cris}^{(m')}(X/S)\). By restriction, any sheaf on \(\text{Cris}^{(m')}(X/S)\) defines a sheaf on \(\text{Cris}^{(m)}(X/S)\). The \(m'\)-PD-filtration restricts to a filtration on the structural sheaf \(\mathcal{O}^{(m)}_{X/S}\) of \(\text{Cris}^{(m)}(X/S)\) that is finer than the \(m\)-PD-filtration.

Using restriction and then saturation with respect to the \(m\)-PD-filtration, any \(T\)-module \(E\) on \(\text{Cris}^{(m')}(X/S)^{(m')}\) defines a \(T\)-module on \(\text{Cris}^{(m)}(X/S)^{(m)}\). It is clear that this process is functorial and that, when applied to \(T\)-\(m'\)-crystals, it produces \(T\)-\(m\)-crystals.

Assume from now on that \(S\) has no \(p\)-torsion and that \(X\) is a smooth \(S_0\)-scheme.

5.3.2. Proposition. — Consider the functor that associates a \(T\)-\(m\)-crystal to a \(T\)-\(m'\)-crystal. Restricted to \(p\)-torsion free \(T\)-\(m'\)-crystals of width less than \(p^{m+1}\), it is fully faithful and its essential image is the full subcategory of \(p\)-torsion free \(T\)-\(m\)-crystals of width less than \(p^{m+1}\) whose underlying crystal is the restriction of an \(m'\)-crystal.

Proof. — This is a local question and all our constructions are functorial. Using Corollary 4.2.6 and Lemma 5.3.1, the first assertion is a consequence of 5.1.1 (iv) and the second follows from Proposition 5.1.2. □

Let \(F : X \to X'\) be the relative Frobenius of \(X\) over \(S_0\). We will write \((X/S)^{(m)}_{\text{cris}}\) for the crystalline topos of level \(m\). In [B3] Berthelot shows that the morphism of crystalline topos of level \(m\) induced by \(F^d\) factors canonically through the restriction map \((X/S)^{(m)}_{\text{cris}} \to (X/S)^{(m')}_{\text{cris}}\) to give a morphism

\[F^d : (X/S)^{(m')}_{\text{cris}} \to (X^{(d)}/S)^{(m)}_{\text{cris}}.\]

Under the equivalence of Corollary 4.1.8, this construction is compatible with that of Proposition 5.2.1.
5.3.3. Proposition. — The functor $F^a$ makes the full subcategory of $F$-$m$-spans on $X^{(d)}/S$ consisting of those $\Phi : F^{m+1}E \to E'$ where $E$ is an $m'$-crystal on $X^{(m'+1)}/S$ equivalent to the category of $F$-$m'$-spans on $X/S$.

Proof. — This is again a local question. Using Corollary 4.2.6, the assertion reduces to Proposition 5.2.3. \qed

5.3.4. Remark. — When restricted to objects of width less than $p^{m+1}$, we have commutative diagrams:

\[
\begin{array}{ccc}
p\text{-torsion free} & \longrightarrow & F\text{-}m\text{'-spans on } X/S \\
T\text{-}m\text{'-crystals on } X^{(m'+1)}/S & \downarrow & \\
p\text{-torsion free} & \longrightarrow & F\text{-}m\text{-spans on } X^{(d)}/S \\
T\text{-}m\text{-crystals on } X^{(m+1)}/S & \downarrow & \\
\end{array}
\]

where the horizontal arrows come from Theorem 4.3.6 and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3; and, when $S$ is a PD-scheme:

\[
\begin{array}{ccc}
p\text{-torsion free} & \longrightarrow & F\text{-}m\text{-spans on } X/S \\
T\text{-}m\text{-crystals on } X^{(m+1)}/S & \downarrow & \\
p\text{-torsion free} & \longrightarrow & F\text{-}0\text{-spans on } X^{(m)}/S \\
T\text{-}0\text{-crystals on } X^{(m+1)}/S & \downarrow & \\
\end{array}
\]

where the top arrow comes from Theorem 4.3.6, the bottom one from Theorem 5.2.13 of [O2] and the vertical ones from Proposition 5.3.2 and Proposition 5.3.3.

5.3.5. Proposition. — The construction of 4.3.6 does not give an equivalence of categories in general. However, if $S$ is a PD-scheme, it becomes an equivalence when restricted to objects of width at most $p$.

Proof. — Follows from Corollary 4.2.6 and Proposition 5.2.5. \qed
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