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# ON BOUNDARY SLOPES OF IMMERSED INCOMPRESSIBLE SURFACES

by Mark D. BAKER

## 1. Introduction.

Let  $M$  be a compact, orientable, irreducible 3-manifold with  $\partial M$  a torus. Hatcher [H] showed that there are only finitely many slopes on  $\partial M$  realized by boundary curves of embedded incompressible,  $\partial$ -incompressible surfaces in  $M$ .

In this paper we show that there can be infinitely many slopes on  $\partial M$  realized by the boundary curves of immersed, incompressible,  $\partial$ -incompressible surfaces in  $M$  which are embedded in a neighborhood of  $\partial M$ .

## 2. Notation and statement of results.

Let  $T_0$  be the torus with an open disk removed, pictured in Figure 1. Let  $D_x$  (resp.  $D_y$ ) denote the right handed Dehn twist about the loop  $x$  (resp. the loop  $y$ ).

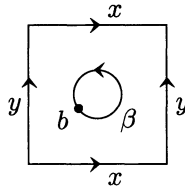


Figure 1

Consider the manifold  $M$ , a once punctured torus bundle over  $S^1$ , given by

$$M \cong T_0 \times [0, 1] / (h(s), 0) \sim (s, 1)$$

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where  $h = D_y^{-4} \circ D_x^4$ . Fix a basepoint  $b \in \partial T_0$ . The loops  $\alpha = b \times [0, 1] / \sim$ ,  $\beta = \partial T_0$  give a coordinate system for  $\partial M$ ; the loop  $\alpha^\mu \beta^\lambda$  in  $\partial M$  is represented by the pair  $(\mu, \lambda) \in \mathbb{Z}^2$  and is said to have slope  $\mu/\lambda$ .

Now, if  $S \subset M$  is an immersed surface, properly embedded in a neighborhood of  $\partial M$ , then  $\partial S$  consists of parallel simple closed curves in  $\partial M$  parametrized by a coprime pair  $(\mu, \lambda)$ .

**THEOREM.** — *For  $M$  as above, the coprime pair  $(\mu, \lambda)$  is realized by the boundary curves of an immersed, incompressible,  $\partial$ -incompressible surface provided  $\mu \geq 1$  and  $|\lambda| > \mu$ .*

*Remarks.*

1) It suffices to prove the theorem for  $(\mu, \lambda)$  with  $\mu \geq 1$  and  $\lambda > \mu$  since  $M$  admits an involution sending the curves  $\alpha, \beta$  to  $\alpha, -\beta$ . Indeed, let  $k: T_0 \rightarrow T_0$  be a reflection in the diagonal (see Figure 1) followed by  $D_y^{-4}$ . Then  $k^2$  is isotopic to  $D_y^{-4} \circ D_x^4$  and the map  $(x, t) \mapsto (k(x), t + \frac{1}{2})$  induces the desired involution on  $M$ .

2) The immersed surfaces of the theorem are virtually embedded in  $M$ : they lift to embedded surfaces in finite covers of  $M$  (see §3) and one obtains virtually Haken manifolds by Dehn filling on  $M$  with respect to these boundary slopes.

3) There exist manifolds with torus boundary ( $N$  Seifert fibered for example) for which only finitely many slopes on  $\partial N$  are realized as the boundary curves of essential immersed surfaces.

### 3. Proof of theorem.

**3.1.** — We prove our result by constructing, for each  $(\mu, \lambda)$ , a finite covering space  $\widetilde{M} \rightarrow M$  such that:

- (i) The loop  $\alpha^\mu \beta^\lambda$  lifts to loops in each of the four components of  $\partial \widetilde{M}$ .
- (ii) In  $H_1(\widetilde{M}; \mathbb{Z})$  there exists a relation of the form  $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0$  where  $\gamma_i$  is a lift of  $\alpha^\mu \beta^\lambda$  to the  $i$ -th component of  $\partial \widetilde{M}$ .

Property (ii) implies that  $\widetilde{M}$  contains an incompressible,  $\partial$ -incompressible surface  $S'$  whose boundary consists of the loops  $\gamma_1, \dots, \gamma_4$ . Indeed, if we consider a triangulation for  $\widetilde{M}$  and simplicial homology, the fact that  $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4$  is a primitive element in  $H_1(\partial \widetilde{M}; \mathbb{Z})$  that is

zero in  $H_1(\widetilde{M}; \mathbb{Z})$  implies that these loops bound an oriented 2-complex,  $K$ , in  $\widetilde{M}$  with property that an even number of triangles meet at each interior edge, with oriented sum equal to zero. Thus by cutting and pasting along the edges of  $K$ , and then pulling apart at vertices, we obtain an embedded surface which can then be compressed. Now by property (i) this surface projects to an immersed incompressible surface  $S$  in  $M$  with boundary consisting of four parallel copies of  $\alpha^\mu \beta^\lambda$ .

The cover  $\widetilde{M}$  is obtained by constructing a cover  $F \rightarrow T_0$  to which  $h = D_y^{-4} \circ D_x^4$  lifts to a homeomorphism  $\tilde{h}: F \rightarrow F$ . Then  $\widetilde{M}$  is the mapping torus of the pair  $(F, \tilde{h})$ .

**3.2.** — We construct  $F \rightarrow T_0$  by cutting and pasting together copies of the following two covers of  $T_0$ :

a) The four-fold cover  $X' \rightarrow T_0$  corresponding to the kernel of the map  $\theta: \pi_1(T_0) \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$  defined by  $\theta([x]) = (1, 0)$  and  $\theta([y]) = (0, 1)$ .

b) The two-fold cover  $Y' \rightarrow T_0$  corresponding to the kernel of the map  $\theta: \pi_1(T_0) \rightarrow \mathbb{Z}/2$  defined by  $\theta([x]) = 0$  and  $\theta([y]) = 1$ .

Note that  $X'$  (resp.  $Y'$ ) is a torus with four (resp. two) boundary circles.

Now alter  $X'$  (resp.  $Y'$ ) by making two (resp. one) vertical cuts  $\tau_1, \tau_2$  (resp.  $\tau$ ) between boundary circles as shown in Figure 2a (resp. Figure 2b). Denote the cut surfaces by  $X$  and  $Y$ .

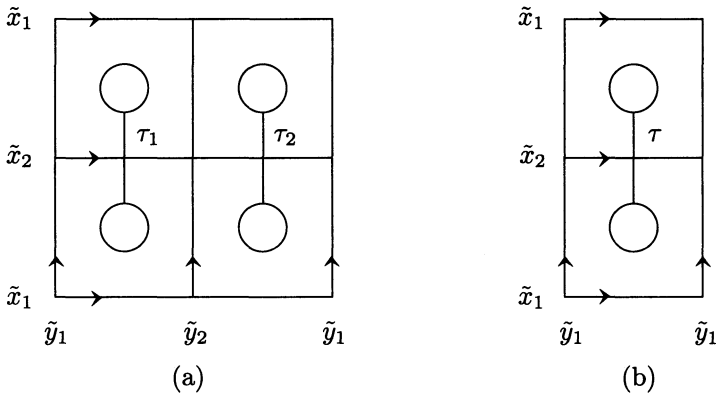


Figure 2

**3.3.** — Since  $\lambda > \mu$ , we have that  $\lambda - \mu = 2r + s$  for  $0 \leq s \leq 1$ . Then  $F \rightarrow T_0$  is obtained by gluing together  $\mu + 2r$  copies of  $X$  and  $2s$  copies

of  $Y$  as indicated in steps i) – v) below. The cover  $F$  for  $(\mu, \lambda) = (2, 7)$ , with edges glued as numbered, is illustrated in Figure 3.

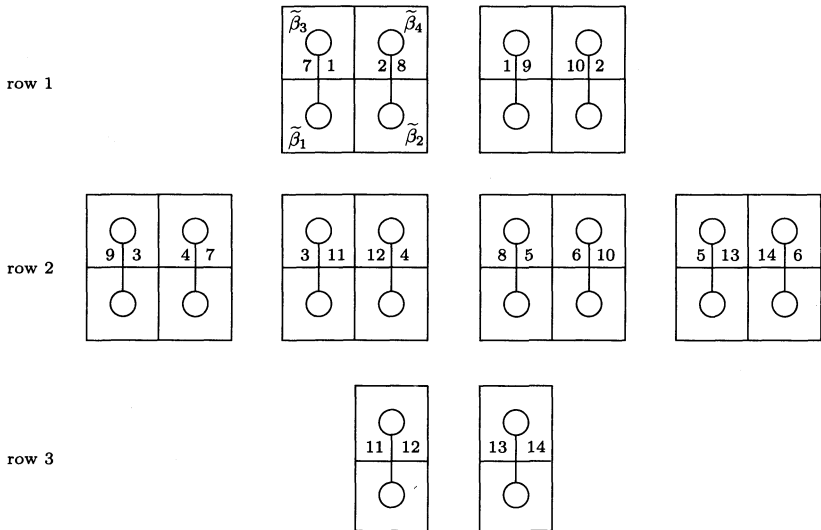


Figure 3

i) First arrange  $\mu$  copies of  $X$ , indexed  $X_1, \dots, X_\mu$ , in a row, followed by the remaining  $2r$  copies of  $X$ , indexed  $X_1^2, \dots, X_{2r}^2$  in a second row. The  $2s$  copies of  $Y$ , indexed  $Y_1, \dots, Y_{2s}$  form the third row.

Note that row 1 is never empty whereas either row 2 or row 3 (but not both) can be.

ii) In the first row, glue the right edge of  $\tau_1$  (resp. left edge of  $\tau_2$ ) in  $X_i$  to the left edge of  $\tau_1$  (resp. right edge of  $\tau_2$ ) in  $X_{i+1}$  for  $i = 1, \dots, \mu - 1$ . In the second row, perform the same gluing as in the first row, first on the  $X_1^2, \dots, X_r^2$  for  $i = 1, \dots, r - 1$  and then on  $X_{r+1}^2, \dots, X_{2r}^2$  for  $i = r + 1, \dots, 2r - 1$ .

The gluing pattern for the remaining edges depends on the values of  $r$  and  $s$ . The three distinct cases are treated in iii) – v) below.

iii) *The case  $r > 0$  and  $s = 1$ .* There are sixteen remaining edges to glue in this case. Glue the left edge of  $\tau_1$  in  $X_1$  to the right edge of  $\tau_2$  in  $X_1^2$ ; the right edge of  $\tau_2$  in  $X_1$  to the left edge of  $\tau_1$  in  $X_{r+1}^2$ ; the right edge of  $\tau_1$

in  $X_\mu$  to the left edge of  $\tau_1$  in  $X_1^2$ ; the left edge of  $\tau_2$  in  $X_\mu$  to the right edge of  $\tau_2$  in  $X_{r+1}^2$ .

Finally, attach  $Y_1$  to  $X_r$  and  $Y_2$  to  $X_{2r}$  by, in each case, gluing the left edge (resp. right edge) of  $\tau$  to the right edge of  $\tau_1$  (resp. left edge of  $\tau_2$ ).

iv) *The case  $r > 0$  and  $s = 0$ .* Glue the copies of  $X$  as in iii). Finish by gluing the right edge of  $\tau_1$  to the left edge of  $\tau_2$ , first in  $X_1^2$  and then in  $X_{2r}^2$ .

v) *The case  $r = 0$  and  $s = 1$ .* There are eight edges to glue. Glue the left edge of  $\tau_1$  (resp. right edge of  $\tau_2$ ) in  $X_1$  to the right edge of  $\tau$  in  $Y_1$  (resp. left edge of  $\tau$  in  $Y_2$ ).

Finally, pair the right edge of  $\tau_1$  (resp. left edge of  $\tau_2$ ) in  $X_\mu$  with the left edge of  $\tau$  in  $Y_1$  (resp. right edge of  $\tau$  in  $Y_2$ ).

Note that the surface  $F$  is indeed a cover of  $T_0$ ; some of its properties are given in:

LEMMA 1. — *The surface  $F$  is a  $4\lambda$ -fold cover of  $T_0$ . It is of genus  $(2\lambda - 1)$  with four boundary components, each of which projects  $\lambda$  to 1 onto  $\beta = \partial T_0$ .*

3.4. — Now the loop  $x$  (resp.  $y$ ) in  $T_0$  is covered by loops in  $F$  which project  $d$  to 1 onto  $x$  for  $d \in 1, 2, 4$  (resp. 2 to 1 onto  $y$ ). Thus  $D_x^4$  and  $D_y^{-4}$  lift to appropriate powers of Dehn twist homeomorphisms in  $F$ , whence  $h = D_y^{-4} \circ D_x^4$  lifts to a homeomorphism  $\tilde{h}: F \rightarrow F$ , which fixes  $\partial F$  pointwise. Denote by  $\tilde{M}$  the mapping torus of  $(F, \tilde{h})$ :

$$\tilde{M} = F \times [0, 1] / (\tilde{h}(s), 0) \sim (s, 1).$$

From the construction of  $\tilde{M}$ , it is clear that we can choose on each boundary torus,  $T_i$ , of  $\tilde{M}$  a pair of loops  $\tilde{\alpha}_i, \tilde{\beta}_i$  which cover  $\alpha, \beta$  in  $\partial M$ . The loop  $\tilde{\alpha}_i$  (resp.  $\tilde{\beta}_i$ ) projects 1 to 1 onto  $\alpha$  (resp.  $\lambda$  to 1 onto  $\beta$ ). We will index the four tori in all cases so that the labelling matches that of  $X_1$  in Figure 3.

3.5. — It is clear from the preceding paragraph that the loops  $\tilde{\alpha}_i^\mu \tilde{\beta}_i^\lambda$  in  $\partial \tilde{M}$  project homeomorphically to  $\alpha^\mu \beta^\lambda$  in  $\partial M$ . Thus property (i) of §3.1 is verified and all that remains to show is that property (ii) holds.

**3.6. LEMMA 2.** — *Let  $\gamma_i$  denote the loop  $\tilde{\alpha}_i^\mu \tilde{\beta}_i$ . Then, in  $H_1(\widetilde{M}; \mathbb{Z})$ ,*

$$\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0.$$

Hence there is a properly embedded incompressible surface  $S' \subset \widetilde{M}$  with  $\partial S'$  the collection of boundary curves  $\tilde{\alpha}_i^\mu \tilde{\beta}_i$  which projects to an immersed incompressible surface  $S$  in  $M$  with boundary consisting of four parallel copies of  $\alpha^\mu \beta^\lambda$ .

*Proof.* — It suffices to show that in  $H_1(\widetilde{M}; \mathbb{Z})$  the following relation holds:

$$(*) \quad (\mu \tilde{\alpha}_1 + \tilde{\beta}_1) - (\mu \tilde{\alpha}_2 + \tilde{\beta}_2) + (\mu \tilde{\alpha}_3 + \tilde{\beta}_3) - (\mu \tilde{\alpha}_4 + \tilde{\beta}_4) = 0.$$

Note that

(a)  $\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3 + \tilde{\beta}_4 = 0$  the relation being given by the fiber surface  $F$ , and

(b)  $\mu(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \tilde{\alpha}_3 - \tilde{\alpha}_4) + 2(\tilde{\beta}_1 + \tilde{\beta}_3) = 0$ .

Relations (a) and (b) imply (\*).

One sees (b) as follows: Consider the loops  $\tilde{y}_1, \tilde{y}_2$  on  $X$ , pictured in Figure 2, and let  $\tilde{y}_{1,i}$  and  $\tilde{y}_{2,i}$  denote the corresponding loops in  $X_i \subset F$ ,  $i = 1, \dots, \mu$  (see §3.3). Now on each of these  $X_i$ , let  $\gamma_i \subset X_i$  (resp.  $\delta_i \subset X_i$ ) denote a horizontal, properly embedded arc between  $\tilde{\beta}_1$  and  $\tilde{\beta}_2$  crossing  $\tilde{y}_{2,i}$  (resp. between  $\tilde{\beta}_3$  and  $\tilde{\beta}_4$  crossing  $\tilde{y}_{1,i}$ ). Then the disks  $\gamma_i \times I \subset F \times I$  and  $\delta_i \times I \subset F \times I$  provide the relations

$$\begin{aligned} \tilde{\alpha}_1 - \tilde{\alpha}_2 + (\tilde{h}(\gamma_i) * \gamma_i^{-1}) &= 0, \\ \tilde{\alpha}_3 - \tilde{\alpha}_4 + (\tilde{h}(\delta_i) * \delta_i^{-1}) &= 0, \end{aligned}$$

where  $\tilde{h}(\gamma_i) * \gamma_i^{-1}$  (resp.  $\tilde{h}(\delta_i) * \delta_i^{-1}$ ) denotes path composition and is equal to  $2\tilde{y}_{2,i}$  (resp.  $-2\tilde{y}_{2,i}$ ).

Combining the  $2\mu$  relations thus obtained gives:

$$\mu(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \tilde{\alpha}_3 - \tilde{\alpha}_4) + 2(\tilde{y}_2, 1 + \dots + \tilde{y}_2, \mu) - 2(\tilde{y}_{1,1} + \dots + \tilde{y}_{1,\mu}) = 0$$

and hence

$$\mu(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \tilde{\alpha}_3 - \tilde{\alpha}_4) + 2(\tilde{\beta}_1 + \tilde{\beta}_3) = 0$$

since the sum of loops about the boundary circles contributing to  $\tilde{\beta}_1$  and  $\tilde{\beta}_3$  in the  $X_i^2$  and  $Y_i$  are all homologous to zero. □

#### 4. Concluding remarks.

1) A once-punctured torus bundle will have infinitely many slopes realized by immersed incompressible surfaces if its characteristic homeomorphism is of the form  $D_x^{r_1} \circ D_y^{s_1} \circ \dots \circ D_x^{r_k} \circ D_y^{s_k}$  where  $2 \mid s_i$  and  $4 \mid r_i$ ,  $i = 1, \dots, k$ , provided that  $s_1 + \dots + s_k \neq 0$ . Hence the same will be true for any once-punctured torus bundle whose monodromy (in  $\mathrm{SL}_2(\mathbb{Z})$ ) has a power that is conjugate to the monodromy of one of the above bundles.

2) The cut and paste techniques in §3 can also be used to produce families of once-punctured surface bundles over  $S^1$  of any given genus  $g > 1$  having infinitely many slopes realized by immersed incompressible surfaces.

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