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NECESSARY AND SUFFICIENT CONDITIONS FOR MATRIX SUMMABILITY METHODS TO BE STRONGER THAN MULTISUMMABILITY

by W. BALSER and A. BECK

0. Introduction.

Frequently one finds a "solution" of a (non-linear) system of differential equations in the form of a power series with radius of convergence equal to zero. This raises the following natural problem which recently has attracted much attention: Find summability methods summing such *formal solutions* (on certain sectorial regions) to an analytic function which also solves the underlying differential equation.

Recently, J. Écalle has defined such a summation method which he named *multisummability*. Roughly speaking, his method is an iteration of the classical Borel summability method, and it depends upon finitely many positive parameters. Given a system of differential equations possessing a formal solution, one can explicitly compute the "correct values" for these parameters and then apply the corresponding method to sum the formal series. In addition to solving the differential equation, the sum so obtained has other natural properties; e.g. it is asymptotically equal to the formal series which we started with (when the variable tends to the origin in the corresponding region). For a self-contained presentation of the definitions and results from the theory of multisummability, we refer to [1], or the survey article of B. Malgrange [6]. Here we will use only some notation and

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basic results from this theory, which are contained in [1], 6.3, Theorem 1 and 4.3, Theorem 2.

The content of this article is as follows: Given an infinite matrix A, we give necessary and sufficient conditions under which the summation method defined by A (in series-to-sequence form) sums every formal power series which is $(k_1, \dots k_p)$-summable (in Écalle's sense) to the same sum, provided that $k_p > 1/2$. Up to now, the only example of a matrix A satisfying these conditions has been given by Jurkat [5]. In a second part we then show existence of formal power series which are summable by Jurkat's method but not $(k_1, \dots k_p)$-summable for any choice of the parameters $k_1, \dots k_p$.

This paper contains results of the second author's doctoral dissertation [2], written under the first author's direction.

1. Matrix summability methods for formal power series.

For arbitrary $d \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and $\rho \in \mathbb{R}^+ \cup \{\infty\}$, let

$$S(d, \alpha, \rho) := \left\{z = re^{i\phi} \;:\; 0 < r < \rho,\ |d - \phi| < \alpha/2\right\}$$

denote a sector (on the Riemann surface of the logarithm). A closed sector $\overline{S}$ is a set of the form

$$\overline{S}(d, \alpha, \rho) := \left\{z = re^{i\phi} \;:\; 0 < r \leq \rho,\ |d - \phi| \leq \alpha/2\right\}$$

with d and α as above, but $\rho \in \mathbb{R}^+$ (so a closed sector is automatically bounded, but does not contain the origin).

Given an infinite matrix $A = (a_{m,n})$ with $a_{m,n} \in \mathbb{C}$ for $m, n \in \mathbb{N}_0$, we call a formal power series $\hat{f}(z) = \sum f_n z^n$ (with complex coefficients f_n) *A-summable in a direction* $d \in \mathbb{R}$, if there exist $\alpha > 0$ and $\rho \in \mathbb{R}^+$, such that the following two conditions hold:

1) The series $f_m(z) := \sum\limits_{n=0}^{\infty} a_{m,n} f_n z^n$ have radius of convergence greater than or equal to ρ, for every $m \in \mathbb{N}_0$.
2) The limit $f(z) := \lim\limits_{m\to\infty} f_m(z)$ exists uniformly on every closed subsector of $S(d, \alpha, \rho)$.

The so-defined function f, which is analytic on $S(d, \alpha, \rho)$, will be called *the A-sum* of $\hat{f}$ on $S(d, \alpha, \rho)$. We sometimes say that $\hat{f}$ is A-summable on the sector $S(d,\alpha,\rho)$ if $S(d,\alpha,\rho)$ can be chosen as the sector that appears in 2). We define the *kernel functions corresponding to* A by

$$k_m(z) := \sum_{n=0}^{\infty} a_{m,n} z^n,$$

provided that the series converge. We say that $A = (a_{m,n})$ is *power series regular*, or for short: *p-regular*, if every power series $\hat{f}(z) = \sum f_n z^n$, with radius of convergence $R > 0$, is A-summable on the disc of radius R about the origin to its correct value. We call A *strongly regular*, if it satisfies the condition

$$(S) \quad \begin{cases} \text{The kernel-functions } k_m(z),\ m \in \mathbb{N}_0 \text{ are all entire functions, and} \\ \text{we have} \\ \qquad \lim_{m\to\infty} k_m(z) = \dfrac{1}{1-z} \\ \text{locally uniformly on } \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 1\}. \end{cases}$$

In other words, this means that A is strongly regular if and only if the geometric series is A-summable to its natural sum on $S(\pi, 2\pi, \infty)$ and on $S(0,\varepsilon,1)$, for some $\varepsilon > 0$. It is a well-known fact (see [4], Th. 135, that a strongly regular method of summation sums all convergent power series (i.e. series with positive radius of convergence) in their whole Mittag-Leffler-star to their analytic continuation. It will turn out later that this is the right kind of regularity if we want to compare our methods with multisummability.

For $k \geq 0$ we say that $A = (a_{m,n})$ satisfies the *order condition* (O_k), if the corresponding kernel functions are all entire functions of (exponential) order $\leq k$. Furthermore, we say that $A = (a_{m,n})$ satisfies the *growth condition* (G_k) (for $k \geq 1/2$), if the following holds:

$$(G_k) \quad \begin{cases} \text{The corresponding kernel functions are all entire functions, and} \\ \text{for every } \kappa > k \text{ and arbitrary } \varepsilon \in (0, 2\pi - \pi/\kappa) \text{ there exist} \\ c_1, c_2 \in \mathbb{R}^+, \text{ such that} \\ \qquad |k_m(z)| \leq c_1 e^{c_2 |z|^\kappa}, \\ \text{for every } z \in S(\pi, 2\pi - \varepsilon - \pi/\kappa, \infty) \text{ and for every } m \in \mathbb{N}_0. \end{cases}$$

It is important to notice that the order condition (O_k) does not imply the growth condition (G_k), since the constants c_1 and c_2 in the growth condition have to be independent of m. Finally, we say (for $k > 0$) that

A satisfies the *comparison condition* (C_k), if for arbitrary $p \in \mathbb{N}$, $d \in \mathbb{R}$ and $\kappa_1 > \kappa_2 > \ldots > \kappa_p > k$, whenever $\hat{f}(z)$ is $(\kappa_1, \ldots, \kappa_p)$–summable in direction d to sum $f(z)$, then we also have A-summability of $\hat{f}(z)$ in direction d to the same sum.

2. The main result.

In what follows, we give necessary and sufficient conditions for a summation method A to satisfy the comparison condition (G_k), for $k \geq 1/2$.

LEMMA 1. — *Let $k \geq 1/2$ and A be an infinite matrix, satisfying (S), (O_k) and (G_k). Then A satisfies (C_k).*

Proof. — Let $d \in \mathbb{R}$, $p \in \mathbb{N}$ and $\kappa_1 > \kappa_2 > \ldots > \kappa_p > k$ be given and consider a series $\hat{f}(z) = \sum_0^\infty f_n z^n$, which is $(\kappa_1, \ldots, \kappa_p)$-summable in direction d. Using basic theorems from the theory of multisummability (cf. [1], Theorem 1 and 4.3, Theorem 2) we can decompose the series $\hat{f}$ into a convergent series plus finitely many so-called *moment series*; so without loss of generality we can restrict $\hat{f}$ to be one of these moment series, i.e.

$$f_n = \int_{\gamma(a)} \frac{\psi(w)}{w^{n+1}}\, dw, \tag{1}$$

where $a \in \mathbb{C} \setminus \{0\}$, $\gamma(a)$ is the path from 0 to a along $\arg w = \arg a$, and ψ is a function, L-integrable on $\gamma(a)$, satisfying

$$|\psi(w)| \leq \tilde{c}_1 e^{-\tilde{c}_2 |w|^{-\tilde{k}}} \tag{2}$$

for some $\tilde{c}_1$, $\tilde{c}_2 > 0$ and some $\tilde{k} > k$. It remains to show that this series is A-summable to the sum

$$f(z) = \int_{\gamma(a)} \frac{\psi(w)}{w - z}\, dw,$$

for $z \in S(\arg a + \pi, 2\pi - \pi/\tilde{k}, \infty)$. Using (1) and (2) one can easily see that for $m \in \mathbb{N}_0$ and $\varepsilon > 0$, condition (O_k) implies existence of $\hat{c}_1$, $\hat{c}_2 > 0$ with

$$\begin{aligned}\sum_{n=0}^{\infty} |a_{m,n}\, f_n| |z|^n &\leq \int_0^{|a|} \left| \frac{\psi(xe^{i \arg a})}{x} \right| \sum_{n=0}^{\infty} |a_{m,n}| \frac{|z|^n}{x^n}\, dx \\ &\leq \int_0^{|a|} \frac{\tilde{c}_1 e^{-\tilde{c}_2 |z|^{-\tilde{k}}}}{x} \hat{c}_1 e^{\hat{c}_2 |\frac{z}{x}|^{k+\varepsilon}}\, dx.\end{aligned}$$

The right hand side is finite if we take ε small enough, so we can conclude that

$$\sum_{n=0}^{\infty} a_{m,n} f_n z^n = \int_{\gamma(a)} \frac{\psi(w)}{w} k_m\left(\frac{z}{w}\right) dw$$

for every $m \in \mathbb{N}_0$ and every $z \in \mathbb{C}$.

Now let $\overline{S}_1$ be a closed subsector of $S(\arg a + \pi, 2\pi - \pi/\tilde{k}, \infty)$. Then we can choose $r > 0$, $\varepsilon > 0$ and $\kappa \in (k, \tilde{k})$ so that $\overline{S} \subset \overline{S}(\arg a + \pi, 2\pi - \varepsilon - \pi/\kappa, r)$. Doing so, we conclude from (G_k) existence of $c_1, c_2 > 0$ (independent of m) so that

$$|k_m(z/w)| \leq c_1\, e^{c_2 |r/w|^{\kappa+\varepsilon}} \qquad \forall z \in \overline{S}_1,\ \forall m \in \mathbb{N}_0.$$

From this estimate we find

$$\lim_{m \to \infty} \int_{\gamma(a)} \frac{\psi(w)}{w} k_m\left(\frac{z}{w}\right) dw = f(z),$$

uniformly on $\overline{S}_1$. □

In the following converse to Lemma 1, it is interesting to note that we only assume our matrix A to sum a very special one-parameter family of formal power series. From this we conclude that for multisummability these series play the same role as the geometric series for the question of p-regularity:

LEMMA 2. — *For $k \geq 1/2$ and a p-regular matrix A, assume that the series*

$$\hat{f}_\kappa(z) := \sum_{n=0}^{\infty} \Gamma(1 + n/\kappa) z^n$$

are A-summable in direction d, for every $\kappa > k$ and $d \in (0, 2\pi)$. Then A satisfies (S), (O_k) and (G_k).

Proof. — Under the above assumptions, the series

$$f_{m,\kappa}(z) := \sum_{n=0}^{\infty} a_{m,n} \Gamma(1 + n/\kappa) z^n$$

have positive radius of convergence (and using that $\kappa > k$ is arbitrary, this radius of convergence must even be infinite), thus represent entire functions, for every $m \in \mathbb{N}_0$ and κ as above. Termwise integration of the series can be justified to show

$$k_m(z) = \frac{1}{2\pi i} \int_{\gamma_{\kappa,\varepsilon}(\tau)} u^\kappa f_{m,\kappa}(u) e^{(z/u)^\kappa}\, d(u^{-\kappa}) \tag{3}$$

with a path of integration $\gamma_{\kappa,\varepsilon}(\tau)$ from 0 along $\arg z = \tau + (\varepsilon + 2\pi)/(4\kappa)$ to a finite point z_1 in $S(\pi, 2\pi - \pi/\kappa - \varepsilon, 1/3r(2\pi - \pi/\kappa - \varepsilon))$, then along the circle $|z| = |z_1|$ to the ray $\arg z = \tau - (\varepsilon + 2\pi)/(4\kappa)$ and from there along this ray back to the origin (with arbitrary real τ and sufficiently small $\varepsilon > 0$).

By assumption, for every d as above there exist $\varepsilon(d)$, $r(d) > 0$ so that the functions $f_{m,\kappa}(z)$, as $m \to \infty$, converge uniformly on $\overline{S}(d, \varepsilon(d), r(d))$ to a function $f(z)$ which is independent of d. For every α (small), we conclude (using the compactness of the interval $[\alpha, 2\pi-\alpha]$) existence of $r(\alpha)$ such that the convergence is uniform (hence the sequence is uniformly bounded) on $\overline{S}(\pi, 2(\pi-\alpha), r(\alpha))$. Choosing τ such that $\gamma_{\kappa,\varepsilon}(\tau)$ lies in $\overline{S}(\pi, 2(\pi-\alpha), r(\alpha))$, we can estimate (3) in a straightforward manner, and varying τ, we so obtain (G_k). Moreover, letting $m \to \infty$ in (3), and observing that uniform convergence allows interchanging limit and integration, we find that the kernel functions tend to $1/(1-z)$ in $S(\tau, \varepsilon/(2\kappa), \infty)$. Varying τ and using that κ can be taken arbitrarily large, one can easily prove (S) (note that p-regularity implies convergence of $k_m(z)$ in the unit disc). □

Lemma 1 and Lemma 2 together now imply our main result:

THEOREM 1. — *Let $k \geq 1/2$. A p-regular matrix A satisfies the comparison-condition (C_k) if and only if it satisfies the conditions (S), (O_k) and (G_k).*

The following result shows that from matrices A satisfying (C_k) we can construct other matrices satisfying $(C_{\tilde{k}})$, for some $0 < \tilde{k} < k$, by deleting certain columns of A.

THEOREM 2. — *Let $k > 0$ be given and $A = (a_{m,n})$, $m, n \in \mathbb{N}_0$ satisfy (C_k). If we define $B = (b_{m,n})$ by $b_{m,n} := a_{m,2n}$, $m, n \in \mathbb{N}_0$, then B satisfies $(C_{\frac{k}{2}})$.*

Proof. — Let $d \in \mathbb{R}$, $(\kappa_1, \ldots, \kappa_p) \in \mathbb{R}^p$ with $\kappa_1 > \ldots > \kappa_p > k/2$ be given. If $\hat{f}(z) = \sum_{n=0}^{\infty} f_n z^n$ is $(\kappa_1, \ldots, \kappa_p)$-multisummable in direction d to sum $f(z)$, then it is well-known (see [1]) that the series $\hat{f}(z^2)$ is $(2\kappa_1, \ldots, 2\kappa_p)$-multisummable in direction $d/2$ to sum $f(z^2)$. Since $2\kappa_p > k$ and A satisfies (C_k), there exist $\alpha, r > 0$, such that

$$\sum_{n=0}^{\infty} a_{m,2n} f_n z^{2n} \to f(z^2) \qquad (m \to \infty)$$

uniformly on closed subsectors of $S(d/2, \alpha, r)$. Hence we get that

$$\sum_{n=0}^{\infty} b_{m,n} f_n z^n = \sum_{n=0}^{\infty} a_{m,2n} f_n z^n$$

converges to $f(z)$ uniformly on closed subsectors of $S(d, 2\alpha; r^2)$. □

As an application of Theorem 2, consider a matrix A_α depending upon a parameter α, and assume that (C_k) holds for every α. If for every α there exists an $\tilde{\alpha}$ such that for every m, n

$$a_{m,n}(\alpha) = a_{m,2n}(\tilde{\alpha}),$$

then Theorem 2 implies that each A_α satisfies $(C_{k/2})$. Repeating this argument we eventually obtain (C_0). For Jurkat's method (see the following section) one can use this technique to prove (C_0).

3. Jurkat's method.

In 1993, W.B. Jurkat [5] studied the matrices $J_\alpha = (j_{m,n}(\alpha))$ with

$$(j_{m,n}(\alpha)) = \exp(-\delta_m \, \lambda(\alpha n)),$$

where δ_m may be any positive sequence tending to zero as $m \to \infty$, α is a positive real parameter, and

$$\lambda(u) = u \log(u+3) \log\log(u+3).$$

This method had already been introduced by G.H. Hardy in [3] in connection with summing special power series with rapidly growing coefficients. Jurkat showed that his method satisfies (C_0), but it was open whether his method was strictly stronger than, or even equivalent to multisummability. The following Theorem 3 settles this question.

Let S be a fixed but arbitrary sector (of small opening). By $C(J_\alpha, S)$ we denote the set of all formal power series that are J_α-summable on $S(d, \alpha, r)$. Given a formal power series $\hat{f}(z) = \sum f_n z^n \in C(J_\alpha, S)$ we define

$$f_m(z) := \sum_{n=0}^{\infty} j_{m,n}(\alpha) f_n z^n.$$

Hence $\hat{f} \in C(J_\alpha, S)$ implies that

$$f(z) := \lim_{m \to \infty} f_m(z)$$

exists uniformly on closed subsectors of $S(d,\alpha,r)$. Now let $\overline{S}_k$, $k \in \mathbb{N}_0$ denote a sequence of closed subsectors of $S(d,\alpha,r)$, that satisfies $\overline{S}_0 := \overline{S}(d,\alpha/2,r/2)$, $\overline{S}_k \subseteq \overline{S}_{k+1}$ $\forall k \in \mathbb{N}_0$ and $\forall z \in S(d,\alpha,r)$ $\exists k_0 \in \mathbb{N}_0$ such that $z \in \overline{S}_{k_0}$. Furthermore, let ρ_k be a sequence in $(0,r)$ with $\rho_k \to r$. It is easy to see that for arbitrary $k \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$

$$\|\hat{f}\|_{\overline{S}_k} := \sup_{m\in\mathbb{N}_0} \sup_{z\in\overline{S}_k} |f_m(z)|,$$

and

$$\|\hat{f}\|_{\rho_k,m} := \sup_{|z|\le\rho_k} |f_m(z)|$$

are norms on $C(J_\alpha, S)$. The linear space $C(J_\alpha, S)$ can be seen to be a Fréchet-space with respect to the above norms. To show the existence of $\hat{f} \in C(J_\alpha, S)$ that is not multisummable in any direction, we proceed as follows: For $l_m := \sqrt{\log\log(m+3)}$, define

$$c_m := \min\left\{\frac{e^{-2m\log(m+3)l_m}}{m!}, \frac{1}{h_m m!}\right\}, \qquad m \in \mathbb{N}_0,$$

where

$$h_m := \sup_{p\in\mathbb{N}_0} \sup_{z\in\overline{S}(d,\alpha,r)} \left|\sum_{n=0}^{\infty} j_{p,n}(\alpha)\Gamma(1+l_m n)z^n\right|$$

(note that h_m is finite since $\sum \Gamma(1+l_m n)z^n \in C(J_\alpha, S)$). Furthermore, the series

$$f_n := \sum_{m=0}^{\infty} c_m \Gamma(1+l_m n)$$

converges for all $n \in \mathbb{N}_0$ and therefore

$$\hat{f}(z) := \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} c_m\Gamma(1+l_m n)\right) z^n$$

is a well-defined formal power series. By checking that there exist no positive constants C, K, k such that $|f_n| \le CK^n\Gamma(1+n/k)$ for all $n \in \mathbb{N}_0$, we conclude that $\hat{f}$ cannot be multisummable in any direction. Consider now the sequence $\hat{f}_j$ of formal power series

$$\hat{f}_j(z) := \sum_{m=0}^{j}\left(\sum_{n=0}^{\infty} c_m\Gamma(1+l_m n)z^n\right)$$

which is a linear combination of series being in $C(J_\alpha, S)$, hence itself is in $C(J_\alpha, S)$, for every $j \in \mathbb{N}_0$. Because of our definition of the c_m the series $\hat{f}_j$ may be shown to be a Cauchy sequence in $C(J_\alpha.S)$ (with respect to the above norms). Since $C(J_\alpha, S)$ is a Fréchet-space, this shows that $\hat{f} \in C(J_\alpha, S)$. Altogether we get the following

THEOREM 3. — *There exist formal power series, that are not multisummable, but J_α-summable by Jurkat's method.*

BIBLIOGRAPHY

[1] W. BALSER, From Divergent Power Series to Analytic Functions, Lecture Notes in Mathematics 1582, Springer-Verlag, 1994.

[2] A. BECK, Matrix-Summationsverfahren und Multisummierbarkeit, Dissertation, Ulm, 1995.

[3] G.H. HARDY, Note on a divergent series, Proc. Cambridge Phil. Soc., (1941), 1–8.

[4] G.H. HARDY, Divergent Series, Oxford, 1956.

[5] W.B. JURKAT, Summability of asymptotic power series, Asympt. Anal., 7 (1993), 239–250.

[6] B. MALGRANGE, Sommation des séries divergentes, Expo. Math., 13 (1995), 163–222.

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