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PARTIAL INDICES OF ANALYTIC DISCS ATTACHED TO LAGRANGIAN SUBMANIFOLDS OF $\mathbb{C}^N$

by Josip GLOBEVNIK(*)

1. Introduction and the main results.

A subspace L of $\mathbb{C}^N$ is called **maximally real** if $\dim L = N$ and if $L \cap iL = \{0\}$. Denote the set of all maximally real subspaces of $\mathbb{C}^N$ by $\mathcal{T}(N)$. The set $\mathcal{T}(N)$ is an open subset of the Grassmannian of all N-dimensional (real) subspaces of $\mathbb{C}^N$. A (local) submanifold M of $\mathbb{C}^N$ is called **maximally real** if $\dim M = N$ and if $T_xM \in \mathcal{T}(N)$ for each $x \in M$.

Let Δ be the open unit disc in $\mathbb{C}$. A continuous map $f: \overline{\Delta} \to \mathbb{C}^N$, holomorphic on Δ is called an **analytic disc**; we say that f is **attached** to a maximally real submanifold M of $\mathbb{C}^N$ if $f(b\Delta) \subset M$. If this is the case then it is known that $f \in \mathcal{C}^{k-0}(\overline{\Delta})$ if M is of class $\mathcal{C}^k$ [Ch].

We denote by $GL(N,\mathbb{C})$ the group of all invertible $N \times N$ matrices with complex entries. If $P, Q \in GL(N, C)$ then the real (linear) span of the columns of P equals the real span of the columns of Q if and only if $P\overline{P^{-1}} = Q\overline{Q^{-1}}$ where $^-$ denotes complex conjugation.

Let T be a smooth loop of maximally real subspaces of $\mathbb{C}^N$, i.e. a smooth map from $b\Delta$ to $\mathcal{T}(N)$. By the preceding discussion there is a unique map $B: b\Delta \to GL(N,\mathbb{C})$ such that for each $\zeta \in b\Delta$, $B(\zeta) = A(\zeta)\overline{A(\zeta)^{-1}}$ where $A(\zeta)$ is a matrix whose columns form a basis of $T(\zeta)$. The map B is smooth. The results of Plemelj and Vekua [Pl], [Ve1], [PS] imply that

$$B(\zeta) = F(\zeta)\Lambda(\zeta)\overline{F(\zeta)^{-1}} \quad (\zeta \in b\Delta) \tag{1.1}$$

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where $F: \overline{\Delta} \to GL(N, \mathbb{C})$ is smooth and holomorphic on Δ and

$$\Lambda(\zeta) = \begin{pmatrix} \zeta^{\kappa_1} & 0 & \cdots & \cdots & 0 \\ 0 & \zeta^{\kappa_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \zeta^{\kappa_N} \end{pmatrix}$$

where $\kappa_1, \cdots, \kappa_N$ are integers [Gl1], [O]. These integers are, modulo a permutation, the same for all factorizations of B of the form (1.1). They are called **partial indices** of the loop $\zeta \mapsto T(\zeta)$. They are independent of the coordinate system on $\mathbb{C}^N$. Their sum $\kappa = \kappa_1 + \cdots + \kappa_N$ equals the Maslov index of the loop $\zeta \mapsto T(\zeta)$. Further, (1.1) implies that the loop $\zeta \mapsto T(\zeta)$ has partial indices $\kappa_1, \cdots, \kappa_N$ if and only if there is a smooth map $F: \overline{\Delta} \to GL(N, \mathbb{C})$, holomorphic on Δ, such that for each $\zeta \in b\Delta$, $T(\zeta)$ is the real span of the columns of the matrix

$$F(\zeta) \begin{pmatrix} \zeta^{\kappa_1/2} & 0 & \cdots & \cdots & 0 \\ 0 & \zeta^{\kappa_2/2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \zeta^{\kappa_N/2} \end{pmatrix}.$$

Let f be an analytic disc attached to a maximally real submanifold M of $\mathbb{C}^N$. Partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$ play a role in perturbation problems [Gl1], [O], [Gl2], [Č1], [Č2], [Č3], [Č4]. For instance, let $\kappa_1, \cdots, \kappa_N$, the partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$, satisfy $\kappa_j \geqslant -1$ for all $j, 1 \leqslant j \leqslant N$. Generalizing the results from [F], [Gl1] Y.-G. Oh [O] proved that in this case the family of all nearby analytic discs attached to M is a manifold of dimension $\kappa + N$ which varies smoothly with M in a neighbourhood of the original M. Further, if one partial index equals 2 while all other partial indices are negative then the only nearby analytic discs attached to M are the ones of the form $f \circ \omega$ where ω is an automorphism of Δ. If, in addition, at least one of the partial indices is less than -1 then there are a neighbourhood W of f and arbitrarily small perturbations $\tilde{M}$ of M such that in W there are no analytic discs attached to $\tilde{M}$ [O].

In the present paper we study the question which N-tuples of integers arise as partial indices related to analytic discs attached to maximally real submanifolds of $\mathbb{C}^N$. This question, although not stated explicitely, is contained in [O]. The author is grateful to Jean-Claude Sikorav for drawing his attention to this question.

Call a N-tuple $(\kappa_1, \cdots, \kappa_N)$ of integers **realizable** if there is a nonconstant analytic disc f attached to a maximally real submanifold M

of $\mathbb{C}^N$ such that $\kappa_1, \cdots, \kappa_N$ are the partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$. It is known that if $(\kappa_1, \cdots, \kappa_N)$ is realizable then $\kappa_j \geqslant 2$ for at least one j [Gl1], [O].

Denote by $<|>$ the Hermitian inner product on $\mathbb{C}^N$ and by $^\perp$ the orthogonal complement with respect to Re $<|>$, the real inner product on $\mathbb{C}^N$. A subspace L of $\mathbb{C}^N$ is called **Lagrangian** if $L^\perp = iL$. A Lagrangian subspace of $\mathbb{C}^N$ is necessarily maximally real. A submanifold M of $\mathbb{C}^N$ is called **Lagrangian** if $\dim M = N$ and if T_xM is Lagrangian for each $x \in M$.

THEOREM 1.1. — *A N-tuple $(\kappa_1, \cdots, \kappa_N)$ of integers is realizable if and only if $\kappa_j \geqslant 2$ for at least one j. If this is the case then there is an analytic disc f attached to a Lagrangian submanifold M of $\mathbb{C}^N$ such that $\kappa_1, \cdots, \kappa_N$ are the partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$.*

A partial result in this direction was proved by Oh who showed that if one of integers $\kappa_1, \cdots, \kappa_N$ equals 2 then there is an analytic disc f attached to a Lagrangian submanifold M of $\mathbb{C}^N$ such that $\kappa_1, \cdots, \kappa_N$ are the partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$ [O].

Following Oh we call a smooth loop $T: b\Delta \to \mathcal{T}(N)$ **realizable** if there is an analytic disc attached to a maximally real submanifold M of $\mathbb{C}^N$ such that $T(\zeta) = T_{f(\zeta)}M$ for each $\zeta \in b\Delta$. Theorem 1.1, in particular, tells that in terms of partial indices there is no difference between realizable Lagrangian loops and realizable general loops, which answers a question of Oh [O]. Another question, asked by Oh is whether a smooth loop $T: b\Delta \to \mathcal{T}(N)$ must be realizable provided that at least one of its partial indices is at least 2. By an example at the end of the paper we show that this is not the case.

2. The case when $f|b\Delta$ is an embedding.

Suppose that κ_j, $1 \leqslant j \leqslant N$, are integers and that $\kappa_1 \geqslant 2$. Assume for a moment that we want to find an analytic disc f_1 in $\mathbb{C}^2$ attached to a maximally real submanifold M_1 of $\mathbb{C}^2$ such that the loop $\zeta \mapsto T_{f_1(\zeta)}M_1$ has partial indices κ_1, κ_2, that is, there are smooth functions A, B, C, D, on $\overline{\Delta}$, holomorphic on Δ, satisfying

$$A(\zeta)D(\zeta) - B(\zeta)C(\zeta) \neq 0 \quad (\zeta \in \overline{\Delta}) \tag{2.1}$$

such that for each $\zeta \in b\Delta$, $T_{f_1(\zeta)}M_1$ is the real span of the columns of the matrix

$$\begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix} \cdot \begin{pmatrix} \zeta^{\kappa_1/2} & 0 \\ 0 & \zeta^{\kappa_2/2} \end{pmatrix}. \tag{2.2}$$

Since $f_1(b\Delta) \subset M_1$ it follows that the tangent vector to the curve $\zeta \mapsto f_1(\zeta)$ $(\zeta \in b\Delta)$ at $f_1(\zeta)$ is contained in $T_{f_1(\zeta)}M_1$, that is,

$$i\zeta f_1'(\zeta) \in T_{f_1(\zeta)}M_1 \quad (\zeta \in b\Delta). \tag{2.3}$$

Suppose now that we can find a smooth map $f_1\colon \overline{\Delta} \to \mathbb{C}^2$, holomorphic on Δ, such that $f|b\Delta$ is an embedding, and smooth functions A, B, C, D on $\overline{\Delta}$, holomorphic on Δ, satisfying (2.1), and such that for each $\zeta \in b\Delta$, $i\zeta f_1'(\zeta)$ is contained in the real span of the columns of the matrix (2.2); we denote this span by $T_1(\zeta)$. For each $\zeta \in b\Delta$, $H(\zeta) = \{ti\zeta f_1'(\zeta)\colon t \in \mathbb{R}\}^\perp$ is a real hyperplane in $\mathbb{C}^2$ and thus $H(\zeta) \cap T_1(\zeta) = L(\zeta)$ is a one dimensional subspace of $\mathbb{C}^2$. For $\varepsilon > 0$ let $I_1(\zeta, \varepsilon) = \{w \in L(\zeta) : |w| < \varepsilon\}$. Then for some sufficiently small $\varepsilon > 0$

$$M_1 = \bigcup_{\zeta \in b\Delta} [f_1(\zeta) + I_1(\zeta, \varepsilon)]$$

is a maximally real submanifold of $\mathbb{C}^2$ such that f_1 is attached to M_1 and such that $T_{f_1(\zeta)}M = T_1(\zeta)$ $(\zeta \in b\Delta)$.

We perform a similar construction in $\mathbb{C}^N = \mathbb{C}^2 \times \mathbb{C}^{N-2}$ with the analytic disc $f = (f_1, 0)$ in place of f_1. For each $\zeta \in b\Delta$ let $T_2(\zeta) \subset \mathbb{C}^{N-2}$ be the real span of the columns of the matrix

$$\begin{pmatrix} \zeta^{\kappa_3/2} & 0 & \cdots & \cdots & 0 \\ 0 & \zeta^{\kappa_4/2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \zeta^{\kappa_N/2} \end{pmatrix}.$$

For each $\zeta \in b\Delta$ let $I(\zeta, \varepsilon) = \{w \in L(\zeta) \oplus T_2(\zeta)\colon |w| < \varepsilon\}$. Then for some sufficiently small $\varepsilon > 0$

$$M = \bigcup_{\zeta \in b\Delta} [f(\zeta) + I(\zeta, \varepsilon)]$$

is a maximally real submanifold of $\mathbb{C}^N$ and f is attached to M. Further, for each $\zeta \in b\Delta$, $T_{f(\zeta)}M$ equals $T_1(\zeta) \oplus T_2(\zeta)$, that is, $T_{f(\zeta)}M$ is the real span of the columns of the matrix

$$\begin{pmatrix} A(\zeta) & B(\zeta) & 0 & \cdots & 0 \\ C(\zeta) & D(\zeta) & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \zeta^{\kappa_1/2} & 0 & \cdots & \cdots & 0 \\ 0 & \zeta^{\kappa_2/2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \zeta^{\kappa_N/2} \end{pmatrix}$$

and consequently $\kappa_1, \cdots, \kappa_N$ are partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$. This loop is a loop of Lagrangian subspaces of $\mathbb{C}^N$ provided that $\zeta \mapsto T_1(\zeta)$ is a loop of Lagrangian subspaces of $\mathbb{C}^2$. If this is the case then by a proposition of Oh [O] Prop. 6.6, one can modify M to get a Lagrangian submanifold $\tilde{M}$ of $\mathbb{C}^N$ such that f is attached to $\tilde{M}$ and such that $T_{f(\zeta)}\tilde{M} = T_{f(\zeta)}M$ $(\zeta \in b\Delta)$.

We shall see later that there are realizable N-tuples $(\kappa_1, \cdots, \kappa_N)$ such that whenever f is an analytic disc attached to a maximally real submanifold M of $\mathbb{C}^N$ such that the partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$ are $\kappa_1, \cdots, \kappa_N$, the derivative f' has at least one zero on $b\Delta$ and thus the map $f|b\Delta$ cannot be an embedding. Now a different procedure must be used to get the manifold M. Also, in these cases the proposition of Oh does not apply and we shall use an elementary construction which can also replace the proposition of Oh in the case when $f|b\Delta$ is an embedding.

3. Constructing analytic discs with tangents in Lagrangian subspaces.

We have seen in the preceding section that, on the way to proving Theorem 1.1, given $\kappa_1, \kappa_2,\ \kappa_1 \geqslant 2$, one would like to find smooth functions A, B, C, D on $\overline{\Delta}$, holomorphic on Δ, satisfying (2.1), and an analytic disc f in $\mathbb{C}^2$ such that

$$i\zeta f'(\zeta) \in T(\zeta) \quad (\zeta \in b\Delta) \tag{3.1}$$

where, for each ζ, $T(\zeta)$ is the real linear span of the columns of the matrix

$$\begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix} \cdot \begin{pmatrix} \zeta^{\kappa_1/2} & 0 \\ 0 & \zeta^{\kappa_2/2} \end{pmatrix}. \tag{3.2}$$

In our constructions we shall first construct f, A and C such that

$$\begin{pmatrix} A(\zeta) \\ C(\zeta) \end{pmatrix} \neq 0 \quad (\zeta \in \overline{\Delta}) \tag{3.3}$$

and such that for each $\zeta \in b\Delta$, $i\zeta f'(\zeta)$ is a real multiple of

$$\begin{pmatrix} A(\zeta)\zeta^{\kappa_1/2} \\ C(\zeta)\zeta^{\kappa_2/2} \end{pmatrix}.$$

We will then construct B and D such that (2.1) is satisfied.

The space $T(\zeta)$ is Lagrangian if some basis u, w (and hence every basis) of $T(\zeta)$ satisfies $\mathrm{Im} < u|w >= 0$. Thus the loop $\zeta \mapsto T(\zeta)$ is a loop of Lagrangian subspaces if and only if

$$[A(\zeta)\overline{B(\zeta)} + C(\zeta)\overline{D(\zeta)}]\zeta^{(\kappa_1-\kappa_2)/2} \in \mathbb{R} \quad (\zeta \in b\Delta). \tag{3.4}$$

LEMMA 3.1. — *Let κ_1 and κ_2 be two integers, $\kappa_1 \geqslant \kappa_2$, and let A, C be two smooth functions on $\overline{\Delta}$, holomorphic on Δ, such that $A(\zeta) \neq 0$ ($\zeta \in \overline{\Delta}$). Given a smooth function D on $\overline{\Delta}$, holomorphic on Δ and such that $D(\zeta) \neq 0$ ($\zeta \in \overline{\Delta}$) there are a constant $\eta > 0$ and a smooth function B on $\overline{\Delta}$, holomorphic on Δ such that*

$$\det\begin{pmatrix} A(\zeta) & B(\zeta) \\ \eta C(\zeta) & D(\zeta) \end{pmatrix} \neq 0 \quad (\zeta \in \overline{\Delta}) \tag{3.5}$$

and such that for each $\zeta \in b\Delta$ the columns of the matrix

$$\begin{pmatrix} A(\zeta) & B(\zeta) \\ \eta C(\zeta) & D(\zeta) \end{pmatrix} \cdot \begin{pmatrix} \zeta^{\kappa_1/2} & 0 \\ 0 & \zeta^{\kappa_2/2} \end{pmatrix} \tag{3.6}$$

span a Lagrangian subspace of $\mathbb{C}^2$.

Remark. — Thus, if $f = (f_1, f_2)$ then $g = (f_1, \eta f_2)$ is an analytic disc such that for each $\zeta \in b\Delta$, $i\zeta g'(\zeta)$ is a real multiple of the first column of the matrix (3.6).

Proof of Lemma 3.1. — Let $\lambda = (\kappa_1 - \kappa_2)/2$. The columns of (3.6) span a Lagrangian subspace if and only if $[A(\zeta)\overline{B(\zeta)} + \eta C(\zeta)\overline{D(\zeta)}]\zeta^{\lambda} \in \mathbb{R}$ ($\zeta \in b\Delta$). We have to find B and η. Replacing D by iD and finding iB rather than B we may assume that

$$\mathrm{Re}[\overline{A(\zeta)}\zeta^{-\lambda}B(\zeta) + \eta\overline{C(\zeta)}\zeta^{-\lambda}D(\zeta)] = 0 \quad (\zeta \in b\Delta). \tag{3.7}$$

Since $A(\zeta) \neq 0$ ($\zeta \in \overline{\Delta}$) it follows that the function $h(\zeta) = A(\zeta)^{-1}$ is smooth on $\overline{\Delta}$, holomorphic on Δ and $v(\zeta) = A(\zeta)^{-1}\overline{A(\zeta)^{-1}}$ is a real function on $b\Delta$ such that

$$v(\zeta)\overline{A(\zeta)} = h(\zeta) \quad (\zeta \in b\Delta) \tag{3.8}$$

which, by (3.7) implies that

$$\operatorname{Re}[h(\zeta)\zeta^{-\lambda}B(\zeta)] = -v(\zeta)\operatorname{Re}[\eta\overline{C(\zeta)}\zeta^{-\lambda}D(\zeta)]. \tag{3.9}$$

Choose D, $D(\zeta) \neq 0$ $(\zeta \in \overline{\Delta})$. We first find a function Ω, smooth on $\overline{\Delta}$, holomorphic on Δ, such that

$$\operatorname{Re}[\zeta^{-\lambda}\Omega(\zeta)] = -v(\zeta)\operatorname{Re}[\overline{C(\zeta)}\zeta^{-\lambda}D(\zeta)] \quad (\zeta \in b\Delta). \tag{3.10}$$

Assume first that λ is a (nonnegative) integer. Since for every real smooth function γ on $b\Delta$ there is a smooth function g on $\overline{\Delta}$, holomorphic on Δ and such that $\operatorname{Re} g(\zeta) = \gamma(\zeta)$ $(\zeta \in b\Delta)$ there is an Ω satisfying (3.10). If λ is not an integer then 2λ is a positive odd integer and by (3.10) Ω has to satisfy

$$\operatorname{Re}[\zeta^{-2\lambda}\Omega(\zeta^2)] = -v(\zeta^2)\operatorname{Re}[\overline{C(\zeta^2)}\zeta^{-2\lambda}D(\zeta^2)] \quad (\zeta \in b\Delta). \tag{3.11}$$

Denote the function on the right in (3.11) by ψ. Then ψ is a smooth odd function on $b\Delta$. It is easy to see that there is a smooth odd function p on $\overline{\Delta}$, holomorphic on Δ and such that $\operatorname{Re} p(\zeta) = \psi(\zeta)$ $(\zeta \in b\Delta)$. Now p is necessarily of the form $\zeta q(\zeta^2)$ where q is smooth on $\overline{\Delta}$ and holomorphic on Δ. Then $\Omega(\zeta) = \zeta^{\lambda+1/2}q(\zeta)$ is the desired function. If $\eta > 0$ then $B(\zeta) = \eta\Omega(\zeta)/h(\zeta)$ $(\zeta \in \overline{\Delta})$ satisfies (3.9) and thus (3.7). Since A and D have no zero on $\overline{\Delta}$ it follows that one can choose $\eta > 0$ so small that (3.5) is also satisfied. This completes the proof.

4. The case when $\kappa_1 \geqslant 2$, $\kappa_2 \geqslant 2$.

PROPOSITION 4.1. — *Let $\kappa_1 \geqslant 2$, $\kappa_2 \geqslant 2$. There is a smooth map $f\colon\overline{\Delta} \to \mathbb{C}^2$, holomorphic on Δ, such that $f|b\Delta$ is an embedding and such that for each $\zeta \in b\Delta$, $i\zeta f'(\zeta)$ is contained in the real span of the columns of the matrix*

$$\begin{pmatrix} \zeta^{\kappa_1/2} & 0 \\ 0 & \zeta^{\kappa_2/2} \end{pmatrix}, \tag{4.1}$$

a Lagrangian subspace of $\mathbb{C}^2$.

PROPOSITION 4.2. — *Given $m, n \in \mathbb{N}$ there is a $\tau \in b\Delta$ such that the map $\zeta \mapsto ((\zeta+1)^m, (\zeta+\tau)^n)$ is one-to-one on $b\Delta$.*

Proof. — Let $\tau \in b\Delta$, $\tau \neq 1$, and suppose that $(\zeta_1+1)^m = (\zeta_2+1)^m$, $(\zeta_1+\tau)^n = (\zeta_2+\tau)^n$ where $\zeta_1, \zeta_2 \in b\Delta$, $\zeta_1 \neq \zeta_2$. Then $(\zeta_1+1)/(\zeta_2+1) = \alpha$,

$(\zeta_1+\tau)/(\zeta_2+\tau)=\beta$ where α is a m'th root of 1, $\alpha\neq 1$, and β is a n'th root of 1, $\beta\neq 1$. Since $\tau\neq 1$ and since $\zeta_1\neq\zeta_2$ it follows that $\alpha\neq\beta$ and that $\zeta_2=(1-\tau+\beta\tau-\alpha)/(\alpha-\beta)$. Since $|\zeta_2|=1$ it follows that

$$\left|\tau-\frac{1-\alpha}{1-\beta}\right|=\frac{|\alpha-\beta|}{1-\beta}. \tag{4.2}$$

For each $\alpha,\beta,\alpha\neq 1,\beta\neq 1,\alpha\neq\beta$, (4.2) is the equation of the circle which is not centered at the origin and which, consequently, intersects $b\Delta$ in at most two points. Let S be the set of all these intersections for all pairs α,β where α is a m'th root of 1, $\alpha\neq 1$, β is a n'th root of 1, $\beta\neq 1,\alpha\neq\beta$. Then $\tau\in S,\tau\neq 1$, has all the required properties. This completes the proof.

Proof of Proposition 4.1. — Denote the real span of the columns of (4.1) by $T(\zeta)$. Now, $i\zeta f'(\zeta)\in T(\zeta)$ $(\zeta\in\Delta)$ holds if and only if $(\zeta^{-\kappa_1/2}i\zeta f_1'(\zeta),\zeta^{-\kappa_1/2}i\zeta f_2'(\zeta))$ is real for each $\zeta\in b\Delta$ which happens if and only if

$$if_1'(\zeta)=\overline{if_1'(\zeta)}\zeta^{\kappa_1-2}\quad(\zeta\in b\Delta) \tag{4.3}$$

$$if_2'(\zeta)=\overline{if_2'(\zeta)}\zeta^{\kappa_2-2}\quad(\zeta\in b\Delta). \tag{4.4}$$

Define $f(\zeta)=(f_1(\zeta),f_2(\zeta))$ by

$$f_1(\zeta)=i^{-1}(\kappa_1-1)^{-1}(\zeta+1)^{\kappa_1-1}$$

$$f_2(\zeta)=i^{-1}(\kappa_2-1)^{-1}\tau^{1-\kappa_2/2}(\zeta+\tau)^{\kappa_2-1}$$

where $\tau\in b\Delta$, $\tau\neq 1$. Then (4.3), (4.4) are satisfied so $i\zeta f'(\zeta)\in T(\zeta)$ $(\zeta\in b\Delta)$. Since $\kappa_1\geqslant 2,\kappa_2\geqslant 2$ and $\tau\neq 1$ it follows that $f|b\Delta$ is an immersion. Finally, by Proposition 4.2 we can choose τ in such a way that $f|b\Delta$ is one-to-one. This completes the proof.

Remark. — By the reasoning in Section 2 this completes the proof of Theorem 1.1 in the case when at least two partial indices satisfy $\kappa_j\geqslant 2$.

5. The case when $\kappa_1\geqslant 2$ is even and $\kappa_2\leqslant 1$.

PROPOSITION 5.1. — *Let $\kappa_1\geqslant 2$ be even and let $\kappa_2\leqslant 1$. There are smooth functions A,B,C,D on $\overline{\Delta}$, holomorphic on Δ, satisfying (2.1), and a smooth map $f:\overline{\Delta}\to\mathbb{C}^2$, holomorphic on Δ, such that $f|b\Delta$ is an* *embedding and such that for each $\zeta \in b\Delta$, $i\zeta f'(\zeta)$ is contained in the real span of the columns of the matrix*

$$\begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix} \cdot \begin{pmatrix} \zeta^{\kappa_1/2} & 0 \\ 0 & \zeta^{\kappa_2/2} \end{pmatrix} \tag{5.1}$$

which is a Lagrangian subspace of $\mathbb{C}^2$.

Proof. — Denote the first factor in (5.1) by $\Theta(\zeta)$ and denote by $T(\zeta)$ the real span of the columns of (5.1). Now, $i\zeta f'(z) \in T(\zeta)$ $(\zeta \in b\Delta)$ if and only if

$$h_1(\zeta) = \overline{h_1(\zeta)}\zeta^{\kappa_1-2} \quad (\zeta \in b\Delta) \tag{5.2}$$

$$h_2(\zeta) = \overline{h_2(\zeta)}\zeta^{\kappa_2-2} \quad (\zeta \in b\Delta) \tag{5.3}$$

where

$$\begin{pmatrix} h_1(\zeta) \\ h_2(\zeta) \end{pmatrix} = \Theta(\zeta)^{-1} \begin{pmatrix} if_1'(\zeta) \\ if_2'(\zeta) \end{pmatrix}.$$

Since $\kappa_2 \leqslant 1$ and since h_2 is holomorphic, (5.3) implies that $h_2 \equiv 0$. Thus, $i\zeta f'(\zeta) \in T(\zeta)$ if and only if

$$if_1'(\zeta) = A(\zeta)h_1(\zeta), \quad if_2'(\zeta) = C(\zeta)h_1(\zeta) \quad (\zeta \in b\Delta) \tag{5.4}$$

where h_1 satisfies (5.2). Observe that $\kappa_1 - 2 = 2m \geqslant 0$ is even and put $A(\zeta) = 1$, $C(\zeta) = \zeta$ and $g(\zeta) = i^{-1}\big((m+1)^{-1}\zeta^{m+1}, (m+2)^{-1}\zeta^{m+2}\big)$ $(\zeta \in \overline{\Delta})$. Then (5.4) is satisfied with g_1, g_2 replacing f_1, f_2 and with $h_1(\zeta) = \zeta^m$ which satisfies (5.2) and consequently

$$i\zeta g_1'(\zeta) = A(\zeta)\zeta^{\kappa_1/2}, \quad i\zeta g_2'(\zeta) = C(\zeta)\zeta^{\kappa_1/2} \quad (\zeta \in b\Delta).$$

Put $D(\zeta) \equiv 1$. By Lemma 3.1 there are a smooth function B on $\overline{\Delta}$, holomorphic on Δ, and $\eta > 0$ such that (3.5) holds and such that if $f(\zeta) = \big(g_1(\zeta), \eta g_2(\zeta)\big)$ then for each $\zeta \in b\Delta$, $i\zeta f'(\zeta)$ is contained in the real span of the columns of the matrix (3.6), a Lagrangian subspace of $\mathbb{C}^2$. Clearly $f|b\Delta$ is an embedding. Replacing $C(\zeta)$ by $\eta C(\zeta)$ in (5.1) the map f and the matrix (5.1) have all the required properties. This completes the proof.

Remark. — By the reasoning in Section 2 this completes the proof of Theorem 1.1 in the case when one partial index is even and at least 2 and another partial index satifies $\kappa_j \leqslant 1$.

6. Nonimmersion in the case when $\kappa_1 \geqslant 2$ is odd and $\kappa_j \leqslant 1$ for all other j.

PROPOSITION 6.1. — *Let f be an analytic disc attached to a maximal real submanifold M of $\mathbb{C}^N$ and let $\kappa_1, \cdots, \kappa_N$ be the partial indices of the loop $\zeta \mapsto T_{f(\zeta)}M$. Suppose that there is some $\ell, 1 \leqslant \ell \leqslant N$, such that $\kappa_j \leqslant 1$ for all j, $j \neq \ell$, and such that $\kappa_\ell \geqslant 2$ is odd. Then f' has at least one zero on $b\Delta$.*

Proof. — By the assumption there is a smooth map $\Theta\colon \overline{\Delta} \to GL(N, \mathbb{C})$, holomorphic on Δ such that for each $\zeta \in b\Delta$, the space $T(\zeta) = T_{f(\zeta)}M$ is the real span of the columns of the matrix

$$\Theta(\zeta) . \begin{pmatrix} \zeta^{\kappa_1/2} & 0 & \cdots & 0 \\ 0 & \zeta^{\kappa_2/2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \zeta^{\kappa_N/2} \end{pmatrix} . \tag{6.1}$$

Since $i\zeta f'(\zeta) \in T(\zeta) \quad (\zeta \in b\Delta)$ it follows that

$$h_j(\zeta) = \overline{h_j(\zeta)} \zeta^{\kappa_j - 2} \quad (\zeta \in b\Delta, 1 \leqslant j \leqslant N) \tag{6.2}$$

where

$$\begin{pmatrix} h_1(\zeta) \\ h_2(\zeta) \\ \cdots \\ h_N(\zeta) \end{pmatrix} = \Theta(\zeta)^{-1} \begin{pmatrix} if_1'(\zeta) \\ if_2'(\zeta) \\ \cdots \\ if_N'(\zeta) \end{pmatrix} .$$

Since h_j are holomorphic and since $\kappa_j \leqslant 1$ $(2 \leqslant j \leqslant N)$, (6.2) implies that $h_j \equiv 0$ $(2 \leqslant j \leqslant N)$ and that h_1 satisfies

$$h_1(\zeta) = \overline{h_1(\zeta)} \zeta^{2m-1} \ (\zeta \in b\Delta) \tag{6.3}$$

where $m \geqslant 1$. If $f' \not\equiv 0$ then $h_1 \not\equiv 0$ and (6.3) implies that there is some $k, 0 \leqslant k \leqslant m-1$, such that $h_1(\zeta) = \zeta^k p(\zeta)$ where p is of the form

$$p(\zeta) = p_0 + p_1\zeta + \cdots + \overline{p_1}\zeta^{2n-2} + \overline{p_0}\zeta^{2n-2} + \overline{p_0}\zeta^{2n-1}$$

with $n = m - k \geqslant 1$, and $p_0 \neq 0$. If w is a zero of p then $w \neq 0$ and

$$(1/\overline{w})^{2n-1}\big(\overline{p_0} + \overline{p_1 w} + \cdots + p_1 \overline{w}^{2n-2} + p_0 \overline{w}^{2n-1}\big) = 0$$

so

$$p_0 + p_1(1/\overline{w}) + \cdots + \overline{p_1}(1/\overline{w})^{2n-2} + \overline{p_0}(1/\overline{w})^{2n-1} = 0$$

which implies that $1/\overline{w}$ is also a zero of p. Note that if $w \neq b\Delta$ then one of $w, 1/\overline{w}$ is in Δ and the other in $\mathbb{C} \setminus \overline{\Delta}$. Since the degree of p is odd it follows that at least one zero of p lies on $b\Delta$. Thus

$$\begin{pmatrix} if_1'(\zeta) \\ if_2'(\zeta) \\ \cdots \\ if_N'(\zeta) \end{pmatrix} = \Theta(\zeta) \begin{pmatrix} \zeta^k p(\zeta) \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

has at least one zero on $b\Delta$. This completes the proof.

Thus, in the situation described above, $f|b\Delta$ cannot be an embedding. We shall see that in this case we can always realize the N-tuple of partial indices by an f attached to a maximal real submanifold M in such a way that f' has only one zero $z_0 \in b\Delta$, and that $f|b\Delta$ is one-to-one. The curve $f(b\Delta)$ will have a cusp at z_0 and to put such a curve into a submanifold is more delicate than before. Let us describe briefly how to do this in $\mathbb{C}^2$. We shall see that we can get an f as above together with a smooth loop $\zeta \mapsto T(\zeta)$ of Lagrangian subspaces of $\mathbb{C}^2$ such that $i\zeta f'(\zeta) \in T(\zeta)$ $(\zeta \in b\Delta)$ and such that, in addition, there is an open arc $\lambda \subset b\Delta$ centered at z_0 such that

$$f(\zeta) \in \mathbb{R}^2 \ (\zeta \in \lambda) \quad \text{and} \quad \mathrm{T}(\zeta) = \mathbb{R}^2 \ (\zeta \in \lambda).$$

For each $\zeta \in b\Delta \setminus \{z_0\}$ let $L(\zeta) = T(\zeta) \cap \{ti\zeta f'(\zeta) : t \in \mathbb{R}\}^{\perp}$ and $I(\zeta, \varepsilon) = \{w \in L(\zeta) : |w| < \varepsilon\}$. Then for some $\varepsilon > 0$, $\bigcup_{\zeta \in b\Delta \setminus \overline{\lambda}} [f(\zeta) + I(\zeta, \varepsilon)]$ is a submanifold of $\mathbb{C}^2$. This is a strip along $f(b\Delta \setminus \overline{\lambda})$ attached to $\mathbb{R}^2$ at $I_1 = f(\zeta_1) + I(\zeta_1, \varepsilon)$, $I_2 = f(\zeta_2) + I(\zeta_2, \varepsilon)$ where ζ_1, ζ_2 are the endpoints of λ. Adding to it $I_1 \cup I_2 \cup U$ where $U \subset \mathbb{R}^2$ is an appropriate neighbourhood of $f(\lambda)$ we get the desired manifold M. We shall use a bit more complicated construction to get a Lagrangian submanifold M.

7. Construction of f and T when $\kappa_1 \geqslant 2$ is odd and $\kappa_j \leqslant 1$ for all other j.

PROPOSITION 7.1. — *Let $\kappa_1 \geqslant 2$ be odd and let $\kappa_j \leqslant 1$ $(2 \leqslant j \leqslant N)$, There are a smooth map $\Theta : \overline{\Delta} \to \mathbb{C}^N$, holomorphic on Δ, and a smooth map $f : \overline{\Delta} \to \mathbb{C}^N$, holomorphic on Δ, such that $f'(\zeta) \neq 0$ $(\zeta \in b\Delta \setminus \{-1\}$, such that $f|b\Delta$ is one-to-one and such that for each $\zeta \in b\Delta$, $i\zeta f'(\zeta)$ is* *contained in* $T(\zeta)$, *the real span of the columns of*

$$(7.1) \qquad \Theta(\zeta)\begin{pmatrix} \zeta^{\kappa_1/2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \zeta^{\kappa_N/2} \end{pmatrix}$$

which is a Lagrangian subspace of $\mathbb{C}^N$, *and such that*

$$f(\zeta) \in \mathbb{R}^N \quad (\zeta \in \lambda) \quad \text{and} \quad T(\zeta) = \mathrm{R}^N \quad (\zeta \in \lambda)$$

for an open arc $\lambda \subset b\Delta$ *centered at* -1.

Proof. — We first prove the proposition for $N = 2$. We show how to get

$$\Theta(\zeta) = \begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix}.$$

Let $\kappa_1 = 2m+1$ where $m \geqslant 1$. By the discussion in Section 6, $i\zeta f'(\zeta) \in T(\zeta)$ if and only if

$$if'(\zeta) = \begin{pmatrix} A(\zeta)h_1(\zeta) \\ C(\zeta)h_1(\zeta) \end{pmatrix}$$

where h_1 is holomorphic on Δ and satisfies

$$(7.2) \qquad h_1(\zeta) = \overline{h_1(\zeta)}\zeta^{2m-1} \quad (\zeta \in b\Delta).$$

Let D be a smoothly bounded domain, symmetric with respect to $\mathbb{R}$ such that $b\Delta$ consists of a vertical segment on the imaginary axis together with a curve ℓ of the form $\ell = \{\rho(|\theta|)e^{i\theta}: -\pi/2 \leqslant \theta \leqslant \pi/2\}$ where ρ is a strictly decreasing function on $[0,\pi/2]$, $\rho(0) = 1$, $\rho(\pi/2) = \beta > 0$. Let Φ be a the conformal map from Δ to D, $\Phi(-1) = 0$, $\Phi(1) = 1$, $\Phi(0) = 1/2$. The map Φ extends smoothly to $\overline{\Delta}$ and $\Phi'(\zeta) \neq 0$ on $\overline{\Delta}$ [Po], p. 48. Let $\lambda = \Phi^{-1}(\{it: -\beta < t < \beta\})$. Let $P(\zeta) = \Phi(\zeta)^{2m}$. If $b\Delta^+ = \{\zeta \in b\Delta: \mathrm{Im}\zeta > 0\}$ then $P|b\Delta^+$ is an embedding. We have $P(\gamma) = P(\overline{\gamma})$ whenever $\gamma \in \lambda$. Moreover, since ρ is strictly decreasing there are $m-1$ pairs of points $\gamma, \overline{\gamma}$ where $\gamma \in b\Delta \setminus \lambda$ such that $P(\gamma) = P(\overline{\gamma})$.

Let M be an automorphism of Δ mapping -1 to -1, 1 to 1 and 0 to $\tau < 0$. Then $\Psi = \Phi \circ M$ has properties analogous to Φ. Define $Q = i\Psi^{2m+1}$. Then $Q|\lambda$ is one-to-one. Also, there are m pairs of points $\gamma, \overline{\gamma} \in b\Delta \setminus \lambda$ such that $Q(\gamma) = Q(\overline{\gamma})$. By choosing τ in a right way we achieve that for each such pair, $P(\gamma) \neq P(\overline{\gamma})$. Define

$$f(\zeta) = \left(\Phi(\zeta)^{2m}, i\Psi(\zeta)^{2m+1}\right) \quad (\zeta \in \overline{\Delta}).$$

Then $f|b\Delta$ is one-to-one. Recall that Φ, Ψ are smooth on $\overline{\Delta}$ and that Φ', Ψ' have no zero on $\overline{\Delta}$. Thus, f is smooth on $\overline{\Delta}$ and the only zero of f' on $b\Delta$ is -1.

By reflection, Φ and Ψ extend holomorphically to a neighbourhood of -1. We have $\Phi(\zeta) = (\zeta+1)G(\zeta)$, $\Psi(\zeta) = (\zeta+1)H(\zeta)$ where G, H are smooth on $\overline{\Delta}$, holomorphic on Δ, and have no zero on $\overline{\Delta}$. We have $f_1'(\zeta) = 2m\Phi(\zeta)^{2m-1}\Phi'(\zeta) = 2m(\zeta+1)^{2m-1}G(\zeta)^{2m-1}\Phi'(\zeta)$. Define $A(\zeta) = 2miG(\zeta)^{2m-1}\Phi'(\zeta)$. Then $A(\zeta) \neq 0$ $(\zeta \in \overline{\Delta})$ and

$$if_1'(\zeta) = A(\zeta)(\zeta+1)^{2m-1} \quad (\zeta \in b\Delta). \tag{7.3}$$

Similarly

$$if_2'(\zeta) = C(\zeta)(\zeta+1)^{2m-1} \quad (\zeta \in b\Delta) \tag{7.4}$$

where $C(\zeta) = -(2m+1)(\zeta+1)H(\zeta)^{2m}\Psi'(\zeta)$. Since $h_1(\zeta) = (\zeta+1)^{2m-1}$ satisfies (7.2) it follows that for $\zeta \in \lambda$, $i\zeta f'(\zeta)$ is a real multiple of $\left(A(\zeta)\zeta^{\kappa_1/2}, C(\zeta)\zeta^{\kappa_1/2}\right)$; in particular, $\left(A(\zeta)\zeta^{\kappa_1/2}, C(\zeta)\zeta^{\kappa_2/2}\right) \in \mathbb{R}^2$ $(\zeta \in \lambda)$ and $i\zeta f'(\zeta) \in T(\zeta)$ $(z \in \lambda)$ for any B, D. Choose a smooth function $\varphi: b\Delta \to b\Delta$ such that $\varphi(\zeta) = \zeta^{-\kappa_2/2}$ $(\zeta \in \lambda)$ and such that the winding number of φ around 0 is 0. By the one dimensional version of (1.1) there is a smooth function D on $\overline{\Delta}$, holomorphic on Δ, such that $D(\zeta) \neq 0$ $(\zeta \in \overline{\Delta})$ and such that for each $\zeta \in b\Delta$, $D(\zeta)$ is a real multiple of $\varphi(\zeta)$. By Lemma 3.1 there are a smooth function B on $\overline{\Delta}$, holomorphic on Δ, and an $\eta > 0$ such that if we change f to $f(\zeta) = \left(\Phi(\zeta)^{2m}, \eta i\Psi(\zeta)^{2m+1}\right)$ and replace C in (7.1) by ηC, then $f|b\Delta$ is one-to- one, $f'(\zeta) \neq 0$ $(\zeta \in b\Delta, \zeta \neq -1)$, $f(\zeta) \in \mathbb{R}^2$ $(\zeta \in \lambda)$, and $\zeta \mapsto T(\zeta)$ is a smooth loop of Lagrangian subspaces of $\mathbb{C}^2$ such that $i\zeta f'(\zeta) \in T(\zeta)$ $(\zeta \in b\Delta)$ and such that $T(\zeta) \subset \mathbb{C} \times \mathbb{R}$ $(\zeta \in \lambda)$. It remains to show that $T(\zeta) = \mathbb{R}^2$ $(\zeta \in \lambda)$. It is easy to see that a two dimensional subspace of $\mathbb{C} \times \mathbb{R}$ is Lagrangian if and only if it is of the form $L \times \mathbb{R}$ where L is a (real) one dimensional subspace of $\mathbb{C}$. Thus, if a two dimensional subspace E of $\mathbb{C}\times\mathbb{R}$ is Lagrangian and contains a vector in $\mathbb{R}^2$ that is not parallel to $\{0\} \times \mathbb{R}$ then $E = \mathbb{R}^2$. Note that there are an open segment I in $\mathbb{R}$ and a smooth real function ψ on I such that $f(\lambda \setminus \{-1\}) = \{(t, \psi(t)): t \in I\} \cup \{(t, -\psi(t)): t \in I\}$ which implies that for $\zeta \in \lambda \setminus \{-1\}$, the vector $i\zeta f'(\zeta) \in T(\zeta) \cap \mathbb{R}^2$ is not parallel to $\{0\} \times \mathbb{R}$. Since $T(\zeta) \subset \mathbb{C} \times \mathbb{R}$ $(\zeta \in \lambda)$ the preceding discussion implies that $T(\zeta) = \mathbb{R}^2$ $(\zeta \in \lambda)$. This completes the proof in the case $N = 2$.

As in the case of D above, for each j, $3 \leqslant j \leqslant N$, we get a smooth function D_j on $\overline{\Delta}$, holomorphic on Δ, such that $D_j(\zeta) \neq 0$ $(\zeta \in \overline{\Delta})$ and such that for each $\zeta \in \lambda$, $D_j(\zeta)$ is a real multiple of $\zeta^{-\kappa_j/2}$. Denote by f_1, f_2 the components of f obtained above and define f in $\mathbb{C}^N$ by $f = (f_1, f_2, 0, \cdots, 0)$ $(\zeta \in \overline{\Delta})$. Further, define

$$\Theta(\zeta) = \begin{pmatrix} A(\zeta) & B(\zeta) & 0 & 0 & \cdots & 0 \\ C(\zeta) & D(\zeta) & 0 & 0 & \cdots & 0 \\ 0 & 0 & D_3(\zeta) & 0\cdots & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & D_N(\zeta) \end{pmatrix}.$$

It is easy to see that f and Θ have all the required properties. This completes the proof.

8. Putting curves into Lagrangian submanifolds.

LEMMA 8.1. — *Let $\lambda \subset \mathbb{R}$ be a neighbourhood of 0, let g_j, $2 \leqslant j \leqslant N$, h_j, $1 \leqslant j \leqslant N$, be smooth real functions on λ such that $g_j(0) = g_j'(0) = 0$ $(2 \leqslant j \leqslant N)$ and $h_j(0) = h_j'(0) = 0$ $(1 \leqslant j \leqslant N)$. Write*

$$\nu(t) = \big(t + ih_1(t), g_2(t) + ih_2(t), \cdots, g_N(t) + ih_N(t)\big) \quad (t \in \lambda)$$

and let $L: \lambda \to \mathcal{T}(N)$ be a smooth map such that $L(0) = \mathbb{R}^N$, such that $L(t)$ is Lagrangian for each $t \in \lambda$, and such that $\nu'(t) \in L(t)$ $(t \in \lambda)$.

There are $\delta > 0$ and smooth functions $\varphi_1, \cdots, \varphi_N$ on $U = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : |x_1| < \delta\}$ such that

$$M = \{(x_1 + i\varphi_1(x), \cdots, x_N + i\varphi_N(x)) : x \in U\} \tag{8.1}$$

is a Lagrangian submanifold of $\mathbb{C}^N$ such that

$$\{\nu(t) : |t| < \delta\} \subset M \tag{8.2}$$

and

$$T_{\nu(t)}M = L(t) \quad (|t| < \delta). \tag{8.3}$$

Proof. — Note that M of the form (8.1) is Lagrangian if and only if the form $\sum_{j=1}^{N} \varphi_j(x)dx_j$ is closed on U; U being simply connected this is equivalent to the existence of a smooth function u on U such that $\dfrac{\partial u}{\partial x_j} \equiv \varphi_j$ $(1 \leqslant j \leqslant N)$. We shall say that M is defined by u.

It is easy to see that there is a neighbourhood $W \subset \mathcal{T}(N)$ of $\mathbb{R}^N$ such that each $E \in W$ contains a unique N-tuple of vectors of the form

$$\begin{aligned}&(1+i\beta_{11}, i\beta_{12}, \cdots, i\beta_{1N})\\&(i\beta_{21}, 1+i\beta_{22}, \cdots, i\beta_{2N})\\&\cdots\\&(i\beta_{N1}, i\beta_{N2}, \cdots, 1+i\beta_{NN})\end{aligned}$$

with β_{jk} real. These vectors depend smoothly on E. Obviously they are linearly independent over $\mathbb{R}$ and so they form a basis of E.

Thus there are $\delta > 0$ and smooth real functions β_{jk} on $(-\delta, \delta)$, $1 \leqslant j, k \leqslant N$, such that for each $x_1, |x_1| < \delta$,

$$\begin{aligned}&(1+i\beta_{11}(x_1), i\beta_{12}(x_1, \cdots, i\beta_{1N}(x_1))\\&\cdots\\&(i\beta_{N1}(x_1), \cdots, i\beta_{N,N-1}(x_1), 1+i\beta_{NN}(x_1))\end{aligned}$$

is a basis of $L(x_1)$. Since $L(x_1)$ is Lagrangian for each $x_1, |x_1| < \delta$, we have

$$\beta_{jk}(x_1) = \beta_{kj}(x_1) \quad (|x_1| < \delta,\ 1 \leqslant j, k \leqslant N). \tag{8.4}$$

Write $\rho(t) = (t, g_2(t), \cdots, g_N(t))$. We want M of the form (8.1). Now (8.2) implies that

$$\varphi_j(\rho(x_1)) = h_j(x_1) \quad (|x_1| < \delta, 1 \leqslant j \leqslant N). \tag{8.5}$$

Moreover, if $|x_1| < \delta$ then $T_{\nu(x_1)}M$ is spanned by the vectors

$$\left\{\begin{aligned}&\Big(1+i\frac{\partial\varphi_1}{\partial x_1}(\rho(x_1), i\frac{\partial\varphi_2}{\partial x_1}(\rho(x_1)), \cdots, i\frac{\partial\varphi_N}{\partial x_1}(\rho(x_1))\Big)\\&\Big(i\frac{\partial\varphi_1}{\partial x_2}(\rho(x_1)), 1+i\frac{\partial\varphi_2}{\partial x_2}(\rho(x_1)), \cdots, i\frac{\partial\varphi_N}{\partial x_2}(\rho(x_1))\Big)\\&\cdots\\&\Big(i\frac{\partial\varphi_1}{\partial x_N}(\rho(x_1)), i\frac{\partial\varphi_2}{\partial x_N}(\rho(x_1)), \cdots, 1+i\frac{\partial\varphi_N}{\partial x_N}(\rho(x_1))\Big)\end{aligned}\right\} \tag{8.6}$$

so by (8.3) this must be a basis of $L(x_1)$. Expressing these vectors as real linear combinations of the basis vectors above we get

$$\frac{\partial\varphi_j}{\partial x_k}(\rho(x_1)) = \beta_{kj}(x_1) \quad (|x_1| < \delta,\ 1 \leqslant k, j \leqslant N). \tag{8.7}$$

Recall that by our assumption $\nu'(t) \in L(t)$ $(t \in \lambda)$. By (8.5), $\nu'(x_1) \in T_{\nu(x_1)}M$ $(|x_1| < \delta)$ and the form of $\nu'(x_1)$ shows that $\nu'(x_1)$, and the last

$N-1$ vectors in (8.6), form a basis of $T_{\nu(x_1)}M$. Thus, to get (8.3) it is enough to assume only that the last $N-1$ vectors in (8.6) are contained in $L(x_1)$, that is, that (8.7) holds only for $k \geqslant 2$. The first equation in (8.7) will then be automatically satisfied. Note that (8.5) and (8.7) imply that

$$(8.8)\quad h_j'(x_1) = \beta_{1j}(x_1) + g_2'(x_1)\beta_{2j}(x_1) + \cdots \\ + g_N'(x_1)\beta_{Nj}(x_1) \quad (|x_1| < \delta,\ 2 \leqslant j \leqslant N).$$

For $x \in U$ let

$$\varphi_1(x) = h_1(x_1) + \sum_{k=2}^{N} \beta_{k1}(x_1)\big(x_k - g_k(x_1)\big) \\ + \frac{1}{2}\sum_{j=2}^{N}\left[\sum_{k=2}^{N} \beta_{kj}'(x_1)\big(x_k - g_k(x_1)\big)\right]\big(x_j - g_j(x_1)\big)$$

$$\varphi_j(x) = h_j(x_1) + \sum_{k=2}^{N} \beta_{kj}(x_1)(x_k - g_k(x_1)) \quad (2 \leqslant j \leqslant N).$$

Using (8.4) and (8.8) it is easy to see that φ_k, $1 \leqslant k \leqslant N$, satisfy (8.5) and (8.7) for $2 \leqslant k \leqslant N$, $1 \leqslant j \leqslant N$, and that the form $\sum_{k=1}^{N} \varphi_j(x)dx_j$ is closed on U. This completes the proof.

9. Completion of the proof of Theorem 1.1.

We keep the notation from Section 8. For $\tau > 0$ let $U_\tau = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : |x_j - g_j(x_1)| < \tau \quad (2 \leqslant j \leqslant N), |x_1| < \delta\}$ and assume that M_1, M_2 are two Lagrangian submanifolds, both graphs of the form (8.1) with U replaced by U_τ, given by functions u_1 and u_2 respectively, both satisfying (8.2) and (8.3). The discussion in Section 8 shows that $\Phi = u_1 - u_2$ must satisfy

$$(9.1)\qquad \frac{\partial \Phi}{\partial x_j}(\rho(x_1)) = 0 \quad (1 \leqslant j \leqslant N, |x_1| < \delta)$$

$$(9.2)\qquad \frac{\partial^2 \Phi}{\partial x_j \partial x_k}(\rho(x_1)) = 0 \quad (1 \leqslant j, k \leqslant N, |x_1| < \delta).$$

By (9.1) Φ is constant on $\{\rho(x_1) \colon |x_1| < \delta\}$. Since by M_1, M_2 the functions u_1, u_2 are determined only up to additive constants we may with no loss of generality assume that

$$\Phi(\rho(x_1)) = 0 \quad (|x_1| < \delta). \tag{9.3}$$

Converse is obviously true so we get

PROPOSITION 9.1. — *Let M_1 be given by u_1 on U_τ and assume that M_1 satisfies (8.2) and (8.3). Then a Lagrangian submanifold M_2, a graph of the form (8.1) over U_τ, satisfies (8.2) and (8.3) if and only if it is given by $u_2 = u_1 + \Phi$ where Φ satisfies (9.1), (9.2) and (9.3).*

Suppose that Φ satisfies (9.1), (9.2) and (9.3). If Ψ is any real smooth function on U_τ then $\Psi\Phi$ also satisfies (9.1), (9.2) and (9.3). This implies the following patching lemma.

LEMMA 9.2. — *Let M_1 and M_2, given by u_1 and u_2 on U_τ, satisfy (8.2) and (8.3). Let $0 < \delta' < \delta$. Then there is a Lagrangian submanifold M, a graph over U_τ, satisfying (8.2) and (8.3) such that M is given by u_1 on $U_\tau \cap \{-\delta < x_1 < -\delta'\}$ and by u_2 on $U_\tau \cap \{\delta' < x_1 < \delta\}$.*

Proof. — Let ψ be a real smooth function on $(-\delta, \delta)$, $\psi \equiv 0$ on $(-\delta, -\delta')$, $\psi \equiv 1$ on (δ', δ). Define $u(x) = u_1(x) + \psi(x_1)\big(u_2(x) - u_1(x)\big)$ $(x \in U_\tau)$. By the preceding discussion u has all the required properties. This completes the proof.

It is now easy to get a Lagrangian "strip" M along $f(b\Delta)$, similar to the one described at the end of Section 6. We use Proposition 7.1. Observe that $L \in \mathcal{T}(N)$ is Lagrangian if and only if $L = U(\mathbb{R}^N)$ for some unitary map on $\mathbb{C}^N$. Lemma 8.1 gives a candidate for a piece of M locally, near every point of $f(b\Delta \setminus \{-1\})$ since $f|(b\Delta \setminus \{-1\})$ is an embedding. By Proposition 7.1 we may assume that near each point of $f(\lambda)$ the candidate for M is $\mathbb{R}^N$. Now we use Lemma 9.1 to patch these pieces together into a Lagrangian submanifold in such a way that near the cusp $f(-1)$, M remains a piece of $\mathbb{R}^N$. This completes the proof of Theorem 1.1.

It is clear that one can apply the above patching procedure in the place of the proposition of Oh in Sections 4 and 5.

10. Realizability of loops of maximally real subspaces.

PROPOSITION 10.1. — *Let $(\kappa_1, \cdots, \kappa_N)$ be a realizable N-tuple such that $\kappa_j \leqslant 2$ $(1 \leqslant j \leqslant N)$. There is a smooth loop $\zeta \mapsto T(\zeta)$ of Lagrangian subspaces with partial indices $\kappa_1, \cdots, \kappa_N$, which is not realizable.*

Proof. — Let $2^{-1/2} < \rho < 1$ and let $\omega(\zeta) = 4i\rho(1+\rho\zeta)^3$ ($\zeta \in \overline{\Delta}$). For each $\zeta \in b\Delta$ let $T(\zeta)$ be the real span of the columns of the matrix

$$\omega(\zeta)\begin{pmatrix} \zeta^{\kappa_1/2} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \zeta^{\kappa_N/2} \end{pmatrix}.$$

Since $(\kappa_1, \cdots, \kappa_N)$ is realizable at least one of κ_j satisfies $\kappa_j \geqslant 2$. Thus, by our assumption we may, with no loss of generality, assume that for some ℓ, $1 \leqslant \ell \leqslant N$ we have $\kappa_j = 2$ ($1 \leqslant j \leqslant \ell$) and $\kappa_j < 1$ ($\ell + 1 \leqslant j \leqslant N$).

Assume now that there is an analytic disc f attached to a maximally real submanifold M of $\mathbb{C}^N$ such that $T(\zeta) = T_{f(\zeta)}M$ for each $\zeta \in b\Delta$. Then $i\zeta f'(\zeta) \in T(\zeta)$ ($\zeta \in b\Delta$) which implies that for each $\zeta \in b\Delta$, all entries of the column

$$\omega(\zeta)^{-1}\begin{pmatrix} \zeta^{-\kappa_1/2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \zeta^{-\kappa_N/2} \end{pmatrix}\begin{pmatrix} i\zeta f_1'(\zeta) \\ \cdots \\ i\zeta f_N'(\zeta) \end{pmatrix}$$

are real which, as in the proof of Proposition 6.1, implies that

$$if_j'(\zeta) = \omega(\zeta)h_j(\zeta) \quad (1 \leqslant j \leqslant N)$$

where

$$h_j(\zeta) = \zeta^{\kappa_j - 2}\overline{h_j(\zeta)} \quad (\zeta \in b\Delta,\ 1 \leqslant j \leqslant N).$$

This implies that $h_j \equiv 0$ ($1 \leqslant j \leqslant N$) and that h_j is a real constant γ_j if $1 \leqslant j \leqslant \ell$. Since f is not a constant at least one of γ_j is different from 0. It follows that there are $\alpha_j \in \mathbb{C}$ such that

$$f_j(\zeta) = \gamma_j(1+\rho\zeta)^4 + \alpha_j \quad (\zeta \in b\Delta,\ 1 \leqslant j \leqslant \ell)$$
$$f_j(\zeta) = \alpha_j \quad (\zeta \in b\Delta,\ \ell + 1 \leqslant j \leqslant N).$$

Observe that the circle $\{1 + \rho e^{i\theta} : 0 \leqslant \theta < 2\pi\}$ intersects each of the rays $\{te^{i\theta} : t > 0\}$ in two points. Thus there are points $p, q \in b\Delta$, $p \neq q$, $p = \overline{q}$, such that the curve $\{(1+\rho\zeta)^4 : \zeta \in b\Delta\}$ intersects itself at the point $w = (1+\rho p)^4 = (1+\rho q)^4$ transversely, that is, there is a number $\eta \in \mathbb{C}$, $\eta \notin \mathbb{R}$, such that $ip4\rho(1+\rho p)^3 = \eta iq4\rho(1+\rho q)^3$. It follows that the immersed curve $\{f(\zeta) : \zeta \in b\Delta\}$ intersects itself at $f(p) = f(q) = (\gamma_1 w + \alpha_1, \cdots, \gamma_\ell w + \alpha_\ell, \alpha_{\ell+1}, \cdots, \alpha_N) = u$ and that $ipf'(p) = \eta iqf'(q)$ which implies that the real span of $ipf'(p)$, $iqf'(q)$ is a complex line that we denote by L. Since f is attached to M it follows that $ipf'(p)$ and $iqf'(q)$ belong to T_uM which implies that T_uM contains L. This contradicts the fact that M is totally real. The proof is complete.

It remains an open problem to show that for each realizable N- tuple $(\kappa_1, \cdots, \kappa_N)$ there is a nonrealizable smooth loop $T: b\Delta \to \mathcal{T}(N)$ with partial indices $\kappa_1, \cdots, \kappa_N$.

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