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Universal Taylor series

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UNIVERSAL TAYLOR SERIES

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1. Introduction.

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with $a_n \in \mathbb{C}$; for each fixed $z \in \mathbb{C}$ denote by $L(z)$ the set of limit points in $\mathbb{C} \cup \{\infty\}$ of the sequence of partial sums

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, N = 0, 1, 2, \ldots$$

(see [12], [3]). Suppose, in addition, that for every z in a set $E \subset \{z \in \mathbb{C} : |z| = 1\}$, the series $\sum_{n=0}^{\infty} a_n z^n$ is $(C, 1)$-summable to a finite sum $\sigma(z) \in \mathbb{C}$. Then, according to a theorem due to J. Marcinkiewicz and A. Zygmund, the limit set $L(z)$ has circular structure with center $\sigma(z)$ for almost all $z \in E$ (see [12], [18], Vol. II, p. 178). For works related to the above theorem the reader is referred to [5], [6], [7], [14], [8], [9], [15].

Power series $\sum_{n=0}^{\infty} a_n z^n$ having the property that, for every z in a non-denumerable set

$$E \subset \{z \in \mathbb{C} : |z| = 1\},$$

all partial sums $S_N(z) = \sum_{n=0}^{N} a_n z^n, N = 0, 1, 2, \ldots$ are contained in the union of a finite number of circumferences $C_1(z), \ldots, C_{M(z)}(z)$, are

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investigated in [6], [7]. As it turned out, such series are $(C,1)$-summable for every $z, |z| = 1$, up to a finite set, and they are Taylor developments of rational functions of a special form. The simplest example is $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. In this example, for every z, $|z| = 1, z \neq 1$, it is true that $S_N(z) \in L(z)$ for all $N = 0, 1, 2, \ldots$. More generally, for every rational function ω of the form

$$\omega(z) = \sum_{j=1}^{M} \frac{A_j}{1-\rho_j z}$$

with $A_j \in \mathbb{C}$, $|\rho_j| = 1$, the partial sums $S_N(z), N = 0, 1, 2, \ldots$ of the Taylor development of ω belong to $L(z)$ for all z, $|z| = 1$, up to a finite set (see [9]).

The above considerations are closely related to a question which was first stated by S. K. Pichorides about the characterization of rational functions using geometric properties of the set of partial sums of their Taylor development ([16], p. 73). More precisely, we have:

Question 1. — Suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $= 1$ and that for all z in a "large" set

$$E \subset \{z \in \mathbb{C} : |z| = 1\}$$

we have $S_N(z) \in L(z)$ for all $N = 0, 1, 2, \ldots$. Is then true that the series $\sum_{n=0}^{\infty} a_n z^n$ constitutes the Taylor development of a rational function?

Question 2. — Suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ is $(C,1)$-summable on a "large" set

$$E \subset \{z \in \mathbb{C} : |z| = 1\}$$

and that $S_N(z) \in L(z)$ for all $z \in E$ and for all $N = 0, 1, 2, \ldots$. Is then true that the series $\sum_{n=0}^{\infty} a_n z^n$ constitutes the Taylor development of a rational function?

In the present paper we give a negative answer to the above Question 1. For the construction of an appropriate counterexample we use a strengthened version of a result of Chui and Parnes concerning approximation by overconvergence (cf. [2]). Question 2 remains, as far as we know, unanswered.

DEFINITION 1.1. — *A* universal Taylor series *is defined to be a Taylor series* $\sum_{n=0}^{\infty} a_n z^n$*, such that:*

(i) *For every compact set* $K \subset \{z \in \mathbb{C} : |z| \geq 1\}$ *with connected complement and for every function* $h : K \to \mathbb{C}$ *which is continuous on* K *and holomorphic in the interior of* K*, there exists a subsequence*

$$S_{\tau_m}(z) = \sum_{n=0}^{\tau_m} a_n z^n, m = 1, 2, \ldots$$

of the partial sums of the series $\sum_{n=0}^{\infty} a_n z^n$ *converging to* h *uniformly on* K*.*

(ii) *The radius of convergence of* $\sum_{n=0}^{\infty} a_n z^n$ *is exactly equal to* 1.

We consider the set U of universal Taylor series as a subset of the space $H(D)$ of holomorphic functions in the open unit disk D endowed with the topology of uniform convergence on compacta. Firstly, combining Mergelyan's theorem with Baire's theorem, we prove that U is a G_δ-dense subset of $H(D)$. In particular, we have $U \neq \emptyset$ and the existence of a plethora of universal Taylor series follows from this fact.

Secondly, we study some properties of universal Taylor series. We show that they are especially universal trigonometric series in the sense of D. Menchoff (cf. [13], [1], p. 439); this means that, if $\sum_{n=0}^{\infty} a_n z^n$ is a universal Taylor series, then, for every measurable function h on the unit circle T, there exists a subsequence of partial sums of $\sum_{n=0}^{\infty} a_n z^n$ converging to h almost everywhere on T.

It is also true that every Taylor series with radius of convergence greater than or equal to 1, can be expressed as the sum of two universal Taylor series.

Finally, we prove that every universal Taylor series is never the Taylor development of any rational function. Thus, every universal Taylor series serves as a counterexample to Question 1. In addition, we show that every universal Taylor series is not $(C, 1)$-summable at any point $z \in \mathbb{C}$ with $|z| = 1$, and that it cannot be continuously extended to the closed unit disk. More generally, any universal Taylor series does not belong to the Hardy space $H^1(D)$.

2. Existence of universal Taylor series.

In this section we consider the set U of universal Taylor series as a subset of the space $H(D)$ of holomorphic functions in the open unit disk D with respect to the usual topology of uniform convergence on compacta. We prove that U is a countable intersection of open dense sets. Since $H(D)$ is metrizable complete space, U is a dense G_δ-set. In particular, $U \neq \emptyset$, which automatically guarantees the existence of universal Taylor series.

LEMMA 2.1. — *There exists a sequence of infinite compact sets $K_m \subset \{z \in \mathbb{C} : |z| \geq 1\}$, $m = 1, 2, \ldots$ with connected complements, such that the following holds: every non-empty compact set $K \subset \{z \in \mathbb{C} : |z| \geq 1\}$ having connected complement is contained in some K_m.*

Proof. — Let $K \subset \{z \in \mathbb{C} : |z| \geq 1\}$ be a non-empty compact set having connected complement. If K is finite, then we can easily find an infinite compact set K' containing K with the same properties. Thus, we can assume that K is infinite. Obviously, there exists a natural number n, such that

$$K \subset \{z \in \mathbb{C} : 1 \leq |z| \leq n\}.$$

Since 0 and $n+1$ belong to the complement of K, which is connected, we can join them by a simple polygonal line Γ lying in the complement of K and having vertices with rational coordinates. The set of such polygonal lines is countable. The distance of Γ from K is strictly positive. Thus, we can find a natural number s, such that $K \subset L(n, \Gamma, s)$, where

$$L(n, \Gamma, s) := \left\{ z \in \mathbb{C} : 1 \leq |z| \leq n, \operatorname{dist}(z, \Gamma) \geq \frac{1}{s} \right\}.$$

One can check that $L(n, \Gamma, s)$ are infinite compact sets having connected complements. Since the set of $L(n, \Gamma, s)$'s is countable, an enumeration of the elements of this set gives the sequence $K_m, m = 1, 2, \ldots$ with the desired properties. □

We fix now a sequence $K_m, m = 1, 2, \ldots$ as in Lemma 2.1. Let $f_j, j = 1, 2, \ldots$ be an enumeration of all polynomials having coefficients with rational coordinates. For $f \in H(D)$ and $n \in \mathbb{Z}, n \geq 0$, we denote by $S_n(f)$ the $n-$ th partial sum of the Taylor development of f. Moreover, for any integers m, j, s, n with $m \geq 1, j \geq 1, s \geq 1$ and $n \geq 0$, we denote by $E(m, j, s, n)$ the set

$$E(m, j, s, n) := \left\{ g \in H(D) : \sup_{z \in K_m} |S_n(g)(z) - f_j(z)| < \frac{1}{s} \right\}.$$

(We remind that $U \subset H(D)$ denotes the set of universal Taylor series.)

LEMMA 2.2. — *U can be written as follows:*

$$U = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} E(m, j, s, n).$$

Proof. — The inclusion $U \subset \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} E(m, j, s, n)$ follows obviously from the definitions of U and $E(m, j, s, n)$. Let

$$f \in \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} E(m, j, s, n);$$

We shall show that $f \in U$.

Let $K \subset \{z \in \mathbb{C} : |z| \geq 1\}$ be a non-empty compact set having connected complement and $h : K \to \mathbb{C}$ a function, which is continuous on K and holomorphic in the interior of K. Let $\varepsilon > 0$ and ν a natural number. We have to determine $N > \nu$, such that

$$\sup_{z \in K} |S_N(f)(z) - h(z)| < \varepsilon.$$

By Mergelyan's theorem (cf. [17]) there exists a polynomial $f_j, j = 1, 2, \ldots$ having coefficients whose coordinates are both rational, such that

$$\sup_{z \in K} |h(z) - f_j(z)| < \frac{\varepsilon}{2}.$$

We can also assume that $f_j(0) \neq f(0)$.

There exists a compact set $K_m, m = 1, 2, \ldots$ given by Lemma 2.1, such that $K \subset K_m$. For any natural number s we have $f \in \bigcup_{n=0}^{\infty} E(m, j, s, n)$. Thus, there exists an integer $n_s \geq 0$, such that

$$\sup_{z \in K_m} |S_{n_s}(f)(z) - f_j(z)| < \frac{1}{s}.$$

If the sequence $n_s, s = 1, 2, \ldots$ possesses a bounded subsequence, then there is an integer $\lambda \geq 0$ with $n_s = \lambda$ for infinitely many s. On K_m we get $S_\lambda(f) = f_j$. Since the set K_m is infinite, we obtain $S_\lambda(f) \equiv f_j$, which contradicts to the assumption $f_j(0) \neq f(0)$. Hence, the sequence $n_s, s = 1, 2, \ldots$ converges to $+\infty$ and we can determine an s, such that $\frac{1}{s} < \frac{\varepsilon}{2}$ and $n_s > \nu$.

As we have $\sup_{z\in K} |h(z) - f_j(z)| < \frac{\varepsilon}{2}$, $\sup_{z\in K_m} |S_{n_s}(f)(z) - f_j(z)| < \frac{1}{s} < \frac{\varepsilon}{2}$ and $K \subset K_m$, the triangular inequality implies

$$\sup_{z\in K} |S_{n_s}(f)(z) - h(z)| < \varepsilon$$

and $n_s > \nu$. This proves that condition (i) of Definition 1.1 is fulfilled.

Since $f \in H(D)$, the radius of convergence R of the Taylor development of f satisfies $R \geq 1$. Let $z_0 \in \mathbb{C}$ with $|z_0| = 1$. The compact set $K = \{z_0\} \subset \{z \in \mathbb{C} : |z| \geq 1\}$ has connected complement. Thus, by making use of (i), we deduce that the limit set $L(z_0)$ is equal to $\mathbb{C} \cup \{\infty\}$ and the Taylor development of f at $z = z_0$ diverges. This shows that $R = 1$ and that condition (ii) of Definition 1.1 is satisfied too. Consequently, $f \in U$, which completes the proof of our lemma. □

Lemma 2.3. — *For every integer $m \geq 1, j \geq 1, s \geq 1$, and $n \geq 0$, the set $E(m,j,s,n)$ is open in the space $H(D)$.*

Proof. — Let $f \in E(m,j,s,n)$. Then we have

$$\sup_{z\in K_m} |S_n(f)(z) - f_j(z)| < \frac{1}{s}.$$

Let $M := \sup\{|z| : z \in K_m\}$; Then $1 \leq M < +\infty$. We set now:

$$a = \frac{\frac{1}{s} - \sup_{z\in K_m} |S_n(f)(z) - f_j(z)|}{\sum_{\lambda=0}^{n} 2^\lambda M^\lambda} > 0.$$

Suppose that $g \in H(D)$ satisfies

$$\sup_{|z|\leq\frac{1}{2}} |g(z) - f(z)| < a.$$

We shall show that

$$\sup_{z\in K_m} |S_n(g)(z) - f_j(z)| < \frac{1}{s}$$

and therefore that $g \in E(m,j,s,n)$. This will prove that $E(m,j,s,n)$ is indeed open.

In fact, for $z \in K_m$, we have

$$|S_n(g)(z) - f_j(z)| \leq |S_n(g - f)(z)| + |S_n(f)(z) - f_j(z)|.$$

We write $S_n(g - f)(z) = \sum_{\lambda=0}^{n} b_\lambda z^\lambda$. Since $\sup_{|z|\leq\frac{1}{2}} |g(z) - f(z)| < a$, we get $|b_\lambda| < 2^\lambda a$. For $z \in K_m$ we obtain

$$\left|\sum_{\lambda=0}^{n} b_\lambda z^\lambda\right| < a \cdot \left[\sum_{\lambda=0}^{n} 2^\lambda M^\lambda\right].$$

Hence,

$$\sup_{z\in K_m} |S_n(g)(z) - f_j(z)| < a\cdot\left[\sum_{\lambda=0}^{n} 2^\lambda M^\lambda\right] + \sup_{z\in K_m} |S_n(f)(z) - f_j(z)| = \frac{1}{s}$$

and the proof is completed. □

LEMMA 2.4. — *For every integer $m \geq 1, j \geq 1$ and $s \geq 1$, the set $\bigcup_{n=0}^{\infty} E(m,j,s,n)$ is open and dense in the space $H(D)$.*

Proof. — By Lemma 2.3 the sets $E(m,j,s,n)$, $n = 0,1,2,\dots$ are open. Therefore the same is true for the union $\bigcup_{n=0}^{\infty} E(m,j,s,n)$. We shall prove that this set is also dense.

Let $f \in H(D), L \subset D$ be a non-empty compact disk and $\varepsilon > 0$. It suffices to find $n \geq 0$ and a $g \in E(m,j,s,n)$, such that

$$\sup_{z\in L} |f(z) - g(z)| < \varepsilon.$$

The sets K_m and L are disjoint and the compact set $K_m \cup L$ has connected complement; thus, Mergelyan's theorem can be applied to the function F with $F(z) = f_j(z)$ on K_m and $F(z) = f(z)$ on L. We find a non-zero polynomial g, such that

$$|F(z) - g(z)| < \min\left(\varepsilon, \frac{1}{s}\right) \quad \text{on} \quad K_m \cup L.$$

Moreover, we set $n := \deg(g) \geq 0$. Then

$$S_n(g) = g, \ \sup_{z\in K_m} |S_n(g)(z) - f_j(z)| < \frac{1}{s}$$

and

$$\sup_{z\in L} |f(z) - g(z)| < \varepsilon.$$

This proves that the set $\bigcup_{n=0}^{\infty} E(m,j,s,n)$ is indeed dense. □

Remark. — Let S be an arbitrary infinite subset of $\{0,1,2,\dots\}$. Then the set $\bigcup_{n\in S} E(m,j,s,n)$ is again open and dense in $H(D)$.

PROPOSITION 2.5. — *The set U of universal Taylor series is a denumerable intersection of open dense subsets of $H(D)$.*

Proof. — It suffices to combine Lemma 2.2 with Lemma 2.4. □

THEOREM 2.6. — *There exist universal Taylor series and their set U is a G_δ-dense subset of the space $H(D)$.*

Proof. — The space $H(D)$ is metrizable complete (see [11]). Thus, Baire's theorem combined with Proposition 2.5 implies that $U \neq \emptyset$ and that U is a G_δ-dense set. □

Remark. — Theorem 2.6 strengthens a result of Chui and Parnes [2]. The difference between condition (i) of Definition 1.1 and the assumption, which was used in the paper [2], is that in (i) the compact set K may meet the unit circle, whereas in [2] the hypothesis is : $K \subset \{z \in \mathbb{C} : |z| > 1\}$. The crucial point is namely that, if one wishes to give answers to questions like those being formulated in the introduction, then the necessity of getting information about the limit set $L(z)$ in the case, in which $|z| = 1$, becomes unavoidable.

Remark. — We do not know any *explicit* universal Taylor series. Nevertheless, one can use Mergelyan's theorem to provide a more constructive proof of the existence of universal Taylor series by means of a modification of the construction presented in [2]. This alternative proof avoids the use of Baire's theorem.

3. Some properties of universal Taylor series.

Universal Taylor series are *universal trigonometric series* in the sense of D. Menchoff (cf. [13], [1], Vol. II, p. 439).

PROPOSITION 3.1. — *Let $\sum\limits_{n=0}^{\infty} a_n z^n$ be a universal Taylor series. Let $h, g : T \to [-\infty, +\infty]$ be two measurable functions on the unit circle T. Then there exists a subsequence*

$$S_{k_m}\left(e^{i\vartheta}\right) = \sum_{n=0}^{k_m} a_n e^{in\vartheta}, m = 1, 2, \ldots$$

of the partial sums of the series $\sum\limits_{n=0}^{\infty} a_n z^n$, such that

$$\operatorname{Re}\left(S_{k_m}\left(e^{i\vartheta}\right)\right) \to h\left(e^{i\vartheta}\right) \quad \textit{and} \quad \operatorname{Im}\left(S_{k_m}\left(e^{i\vartheta}\right)\right) \to g\left(e^{i\vartheta}\right),$$

as $m \to +\infty$, almost everywhere on T.

Proof. — There exist two sequences of continuous functions $h_N : T \to \mathbb{R}$ and $g_N : T \to \mathbb{R}$, $N = 1, 2, \ldots$, such that

$$h_N\left(e^{i\vartheta}\right) \to h\left(e^{i\vartheta}\right) \quad \text{and} \quad g_N\left(e^{i\vartheta}\right) \to g\left(e^{i\vartheta}\right),$$

as $N \to +\infty$, almost everywhere on T.

Let E be a measurable set, $E \subset T$, with Lebesgue measure $|E| = 2\pi$, such that, for every $e^{i\vartheta} \in E$, we have $\lim_{N\to\infty} h_N\left(e^{i\vartheta}\right) = h\left(e^{i\vartheta}\right)$ and $\lim_{N\to\infty} g_N\left(e^{i\vartheta}\right) = g\left(e^{i\vartheta}\right)$.

Since h_1 and g_1 are continuous on T, using (i) of Definition 1.1, we can find an integer $k_1 \geq 0$, such that

$$\left|h_1\left(e^{i\vartheta}\right) + ig_1\left(e^{i\vartheta}\right) - S_{k_1}\left(e^{i\vartheta}\right)\right| < 1$$

for every $\vartheta \in [1, 2\pi - 1]$.

Suppose we have already defined integer numbers

$$0 \leq k_1 < k_2 < \cdots < k_{m-1}$$

for some integer $m \geq 2$. Since h_m, g_m are continuous on T, using condition (i) of Definition 1.1, we can determine an integer $k_m > k_{m-1}$, such that

$$\left|h_m\left(e^{i\vartheta}\right) + ig_m\left(e^{i\vartheta}\right) - S_{k_m}\left(e^{i\vartheta}\right)\right| < \frac{1}{m}$$

for every $\vartheta \in \left[\frac{1}{m}, 2\pi - \frac{1}{m}\right]$.

For every $z = e^{i\vartheta} \in E \setminus \{1\}$, $0 < \vartheta < 2\pi$, there exists m_0, such that $\frac{1}{m} \leq \vartheta \leq 2\pi - \frac{1}{m}$ for all $m \geq m_0$. Therefore we have

$$\left|h_m\left(e^{i\vartheta}\right) - \operatorname{Re}\left(S_{k_m}\left(e^{i\vartheta}\right)\right)\right| \leq \left|h_m\left(e^{i\vartheta}\right) + ig_m\left(e^{i\vartheta}\right) - S_{k_m}\left(e^{i\vartheta}\right)\right| < \frac{1}{m}$$

and

$$\left|g_m\left(e^{i\vartheta}\right) - \operatorname{Im}\left(S_{k_m}\left(e^{i\vartheta}\right)\right)\right| < \frac{1}{m}.$$

Since $\lim_{m\to\infty} h_m\left(e^{i\vartheta}\right) = h\left(e^{i\vartheta}\right)$ and $\lim_{m\to\infty} g_m\left(e^{i\vartheta}\right) = g\left(e^{i\vartheta}\right)$, it follows easily that

$$\lim_{m\to\infty} \operatorname{Re}\left(S_{k_m}\left(e^{i\vartheta}\right)\right) = h\left(e^{i\vartheta}\right) \quad \text{and} \quad \lim_{m\to\infty} \operatorname{Im}\left(S_{k_m}\left(e^{i\vartheta}\right)\right) = g\left(e^{i\vartheta}\right),$$

for all $e^{i\vartheta} \in E \setminus \{1\}$. As $|E \setminus \{1\}| = 2\pi$, we have the almost everywhere convergence on T. □

Since every universal Taylor series has radius of convergence 1, the sum of two universal Taylor series has radius of convergence greater than or equal to 1. In fact, the converse statement is also true.

PROPOSITION 3.2. — *Every Taylor series* $\sum_{n=0}^{\infty} d_n z^n$, $d_n \in \mathbb{C}$, *with radius of convergence greater than or equal to* 1, *can be expressed as the sum of two universal Taylor series.*

One can give a proof to Proposition 3.2 using Mergelyan's theorem and imitating the proof of D. Menchoff in [13]. The following short proof is due to J.-P. Kahane.

Proof. — Let $f(z) = \sum_{n=0}^{\infty} d_n z^n$; then $f \in H(D)$. We consider the homeomorphism $W : H(D) \to H(D)$, given by the translation $W(g) := g + f$, $g \in H(D)$. By Theorem 2.6 the set U of universal Taylor series is a G_δ-dense subset of $H(D)$. The same is true for its homeomorphic image $W(U) = U + f$. Since the space $H(D)$ is metrizable complete, Baire's theorem implies $U \cap (U + f) \neq \emptyset$. This shows the existence of two elements u_1, u_2 of U, such that $f = u_1 - u_2$. Since $-u_2$ belongs also to U, we are done. □

We next show that every universal Taylor series is never the Taylor development of any rational function.

The following lemma is established in [4], Ch. XIV, p. 479, I.

LEMMA 3.3. — *Let* $\sum_{n=0}^{\infty} a_n z^n$ *denote the Taylor development of a rational function* f *which is holomorphic at* 0 *and is assumed to be a non-polynomial. Denote by*

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, N = 0, 1, 2, \ldots$$

the sequence of the partial sums. Let R, $0 < R < +\infty$, *denote the radius of convergence of the series* $\sum_{n=0}^{\infty} a_n z^n$ *and let* w *be a pole of* f *satisfying* $|w| = R$ *and being of maximal multiplicity among all the poles lying on the circle of convergence. Then*

$$\lim_{N \to +\infty} S_N(w) = \infty.$$

Remark. — Consider the rational function

$$f(z) = \frac{1}{1-z^2} + \frac{1}{(1+z)^2}.$$

The partial sums $S_N(1)$ of the Taylor development of f satisfy: $S_{2k}(1) = 2k+2$ and $S_{2k+1}(1) = 0$. We see therefore that for an arbitrary pole w of f we don't have in general $\lim_{N\to+\infty} S_N(w) = \infty$. Furthermore, if w is assumed to be a pole of the highest multiplicity on the circle of convergence, then, using the formula being established in the proof of Theorem 9 of [15], one can prove the existence of a neighbourhood V of w, such that, for every $z_0 \in V$, $|z_0| > |w| = R$, we have $\lim_{N\to+\infty} S_N(z_0) = \infty$.

PROPOSITION 3.4. — *Let $\sum_{n=0}^{\infty} a_n z^n$ be a universal Taylor series. Then the series $\sum_{n=0}^{\infty} a_n z^n$ is not the Taylor development of any rational function.*

Proof. — The series $\sum_{n=0}^{\infty} a_n z^n$ is not a polynomial because its radius of convergence equals 1. According to Lemma 3.3, if $\sum_{n=0}^{\infty} a_n z^n$ were the Taylor development of a rational function, then for some $z_0 \in \mathbb{C}$ satisfying $|z_0| = 1$ we would have $\lim_{N\to\infty} S_N(z_0) = \infty$ and the limit set $L(z_0)$ would be equal to $\{\infty\}$. However, since $|z_0| = 1$, condition (i) of Definition 1.1 implies $L(z_0) = \mathbb{C} \cup \{\infty\} \neq \{\infty\}$. This leads to a contradiction. □

Remark. — Let $\sum_{n=0}^{\infty} a_n z^n$ be a universal Taylor series. Then there is no $z_0 \in \mathbb{C}$, such that $\lim_{N\to\infty} S_N(z_0) = \infty$.

PROPOSITION 3.5. — *Let $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be a universal Taylor series. Then $\sum_{n=0}^{\infty} a_n z^n$ does not belong to the Hardy space $H^2(D)$. In particular, $\sum_{n=0}^{\infty} a_n z^n$ does not extend continuously to the entire closed unit disk $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.*

Proof. — According to condition (i) of Definition 1.1 we can determine two strictly increasing sequences ℓ_n, m_n, $n = 1, 2, \ldots$ of natural numbers satisfying $\ell_n < m_n < \ell_{n+1}$ for all $n = 1, 2, \ldots$ and such that $S_{\ell_n}\left(e^{i\theta}\right)$,

$n = 1, 2, \ldots$ converges to 1 uniformly on $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ and $S_{m_n}(e^{i\theta})$, $n = 1, 2, \ldots$ converges to 0 uniformly on the same interval $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, where $S_N(z) = \sum\limits_{n=0}^{N} a_n z^n$.

We obtain

$$\sum_{k=\ell_n+1}^{m_n} |a_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| S_{m_n}(e^{i\theta}) - S_{\ell_n}(e^{i\theta}) \right|^2 d\theta$$
$$\geq \frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left| S_{m_n}(e^{i\theta}) - S_{\ell_n}(e^{i\theta}) \right|^2 d\theta.$$

This last expression converges to

$$\frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1 d\theta = \frac{1}{2}, \quad \text{as } n \to +\infty.$$

Thus, the sequence a_n, $n = 0, 1, 2, \ldots$ does not belong to l^2 and therefore $\sum\limits_{n=0}^{\infty} a_n z^n$ does not belong to $H^2(D)$. □

Proposition 3.5 implies obviously the following:

COROLLARY 3.6. — *Let $\sum\limits_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{C}$, be a universal Taylor series. Then there is no subsequence of the partial sums of $\sum\limits_{n=0}^{\infty} a_n z^n$ converging uniformly on the entire unit circle $T = \{z \in \mathbb{C} : |z| = 1\}$.*

Using a formula of Rogosinski (cf. [18], Vol. I, Ch. III, 12.16, p. 114), E.S. Katsoprinakis proved the following:

PROPOSITION 3.7. — *Let $\sum\limits_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{C}$, be a universal Taylor series. Then, for every $z_0 \in \mathbb{C}, |z_0| = 1$, the series $\sum\limits_{n=0}^{\infty} a_n z_0^n$ is not $(C, 1)$-summable to a finite sum.*

Since every H^1-Taylor series is almost everywhere $(C, 1)$-summable on the unit circle (see e.g. [10]), any universal Taylor series does not belong to the Hardy space $H^1(D)$.

4. The counterexample.

In this section we prove that every universal Taylor series serves as a counterexample to Question 1 stated in the introduction.

PROPOSITION 4.1. — *Let $\sum_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{C}$, be a universal Taylor series. Then we have:*

(a) *The radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is exactly equal to 1.*

(b) *For every $N = 0, 1, 2, \ldots$ and $z \in \mathbb{C}$, with $|z| = 1$, we have $S_N(z) \in L(z)$, where $S_N(z) = \sum_{n=0}^{N} a_n z^n$ and $L(z)$ is the limit set of the sequence $S_\lambda(z), \lambda = 0, 1, 2, \ldots$*

(c) *The series $\sum_{n=0}^{\infty} a_n z^n$ is not the Taylor development of any rational function.*

Proof. — Statement (a) coincides with condition (ii) of Definition 1.1. We have already seen that condition (i) of Definition 1.1 implies that for every $z, |z| \geq 1$, the limit set $L(z)$ equals to $\mathbb{C} \cup \{\infty\}$; consequently, $S_N(z) \in L(z)$ for all $N = 0, 1, 2, \ldots$. This proves (b). Finally, Proposition 3.4 proves (c). □

Remark. — To give a counterexample to Question 1. stated in the introduction, it is sufficient to find a universal Taylor series, which is not the Taylor development of any rational function. This can be done without making use of Proposition 3.4. J.-P. Kahane suggested the following argument.

By Theorem 2.6 the set U of universal Taylor series is a G_δ-dense subset of the complete metrizable space $H(D)$. Let G denote the set of elements of $H(D)$ which are not holomorphically extendable to any domain strictly containing the open unit disc D. It is known (cf. [11]) that G contains a G_δ-dense subset of $H(D)$. By Baire's theorem we have $U \cap G \neq \emptyset$. Let $f \in U \cap G$; then f is a universal Taylor series and it is not the Taylor development of any rational function because it is not extendable.

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