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On the distribution on the roots of polynomials


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ON THE DISTRIBUTION
OF THE ROOTS OF POLYNOMIALS

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1. Introduction.

In this paper we are interested in the angular distribution of the roots of univariate polynomials. To explain our results we need to recall some definitions. If $x_1, \ldots, x_N$ is a finite sequence of points in $[0, 2\pi)$, we define the absolute discrepancy of this sequence by

$$D(x_1, \ldots, x_N) = \sup_{0 < \alpha < \beta < 2\pi} \left| \frac{\# \{j ; x_j \in [\alpha, \beta) \}}{N} - \frac{\beta - \alpha}{2\pi} \right|,$$

where $\#$ denotes the cardinality of a set. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^{n} (z - \rho_j e^{i\varphi_j}),$$

$$a_0 a_n \neq 0, \quad \rho_1, \ldots, \rho_n > 0,$$

be a polynomial of degree $n$ with complex coefficients, where $\varphi_j \in [0, 2\pi)$ for $j = 1, \ldots, n$. For $0 \leq \alpha < \beta < 2\pi$, put $N(\alpha, \beta) = \# \{j ; \varphi_j \in [\alpha, \beta) \}$. We are interested in the distribution of the points $\varphi_1, \ldots, \varphi_n$. With the previous notations,

$$D(\varphi_1, \ldots, \varphi_n) = \sup_{0 \leq \alpha < \beta < 2\pi} \left| \frac{N(\alpha, \beta)}{n} - \frac{\beta - \alpha}{2\pi} \right|.$$
to simplify the notation we put
\[ D_P = D(\varphi_1, \ldots, \varphi_n), \]
which we call absolute discrepancy of the roots of \( P \).

The first result on \( D_P \) was obtained by Erdős and Turán:

**Theorem A.** — With the above notations, for \( 0 < \alpha < \beta < 2\pi \), we have
\[
\left| N(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} n \right| \leq 16 \sqrt{n \log \frac{|P|}{\sqrt{|a_0a_n|}}},
\]
where \( |P| = \max_{|z|=1} |P(z)| \).

In other words,
\[
D_P \leq 16 \sqrt{\frac{1}{n} \log \frac{|P|}{\sqrt{|a_0a_n|}}}. \]

The proof of [ET] consists in solving several extremal problems on polynomials, using orthogonal polynomials. A few years later, Ganelius [G] proved a general theorem on conjugate functions and showed that his theorem implies a sharpening of the Erdős-Turán, namely he could replace the constant 16 by \( \sqrt{2\pi/k} = 2.5619 \ldots \), where \( k = \sum_{0}^{\infty}(-1)^{m-1}(2m+1)^{-2} = 0.915965594 \ldots \) is Catalan's constant.

The result of Ganelius is the following:

**Theorem B.** — Let \( F = f + i\tilde{f} \) be an analytic function on \( D = \{|z| < 1\} \) satisfying \( F(0) = 0 \). Suppose that \( f, \tilde{f} \) are real and \( f < H, \partial \tilde{f}/\partial \theta < K \) on \( D \) (\(*\)). Then for \( \beta > \alpha \) and \( \rho < 1 \),
\[
|\tilde{f}(\rho e^{i\beta}) - \tilde{f}(\rho e^{i\alpha})| < 2\pi \sqrt{\frac{\pi}{k}} \cdot \sqrt{HK}.
\]

For the convenience of the reader, we briefly explain how Theorem B implies Theorem A. Let us consider the polynomial
\[
Q(z) = \prod_{j=1}^{n} (1 - z \cdot e^{-i\varphi_j}).
\]

\((*)\) Here and in the sequel we often identify the complex variable \( z \) with \( \rho e^{i\theta} \).
As remarked by Schur,
\[
\rho_j \left| 1 - \frac{e^{it}}{\rho_j e^{i\varphi_j}} \right|^2 \geq \left| 1 - e^{i(t-\varphi_j)} \right|^2.
\]
Hence
\[
|Q| \leq \frac{|P|}{\sqrt{|a_0 a_n|}}.
\]
Now let
\[
f(z) = \frac{1}{\pi} \log |Q(z)|, \quad \tilde{f}(z) = \frac{1}{\pi} \sum_{j=1}^{n} \text{Arg} \left( 1 - ze^{-i\varphi_j} \right)
\]
and observe that the function \(F(z) = f + i\tilde{f}\) is analytic on \(D\) and satisfies \(F(0) = 0\). We have \(f \leq \frac{1}{\pi} \log |Q|\) and \(\partial \tilde{f} / \partial \theta < n/(2\pi)\). Moreover, it is easily seen that \(\tilde{f}\) takes the boundary value
\[
\frac{n\theta}{2\pi} - N(0, \theta) + C(Q),
\]
where
\[
C(Q) = \frac{n}{2} - \sum_{j=1}^{n} \left\{ \frac{\varphi_j}{2\pi} \right\}.
\]
Henceforth Theorem B gives
\[
D_P < \sqrt{\frac{2\pi}{k}} \cdot \sqrt{\frac{1}{n} \log |Q|} \leq \sqrt{\frac{2\pi}{k}} \cdot \sqrt{\frac{1}{n} \log \frac{|P|}{\sqrt{|a_0 a_n|}}}.
\]

Theorem B was sharpened much later in \([M]\), where, under the same hypotheses, it is proved that
\[
|\tilde{f}(\rho e^{i\beta}) - \tilde{f}(\rho e^{i\alpha})| < 2\pi \sqrt{\frac{\pi}{k} \hat{H} K},
\]
where
\[
\hat{H} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u^{+}(e^{i\theta}) d\theta \leq \max u^{+}
\]
and \(u^{+} = \max\{u, 0\}\).

Let us define \(\tilde{h}(P) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^{+} |P(e^{i\theta})| d\theta\). Since
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^{+} |Q(e^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^{+} \frac{|P(e^{i\theta})|}{\sqrt{|a_0 a_n|}} d\theta,
\]
Mignotte’s result leads to a version of Erdös-Turán’s theorem where \(\log \frac{|P|}{\sqrt{|a_0 a_n|}}\) is replaced by \(\tilde{h}(\frac{P}{\sqrt{|a_0 a_n|}})\). It is worth remarking that \(\tilde{h}(P)\)
can be much smaller than \( \log |P| \): for a discussion on the relations between these two measures, see [A].

In Section 3 we give a new (very short) proof of the result of [M], using a theorem of Kolmogorov on conjugate functions (see Section 2 for definitions and properties of conjugate and harmonic functions).

Recently, Blatt obtained a sharpening of Theorem A for square-free polynomials. He proved the following result:

**Theorem C.** — Let \( P(z) \) be a monic polynomial of degree \( n \) with all its roots \( z_j \) on the unit circle. Assume that

\[
|P| < A \quad \text{and} \quad |P'(z_j)| \geq \frac{1}{B}, \quad j = 1, \ldots, n,
\]

for some constants \( A, B > 1 \). Then,

\[
D_P \leq c \left( \frac{\log n}{n} \right) \log C_n,
\]

where \( c \) is some (non computed) absolute constant and \( C_n = \max\{A, B, n\} \).

A similar statement holds for polynomials vanishing only on \([-1, 1]\). Totik improved this last result on \([-1, 1]\] by replacing \( \log n \) with \( \log(n/\log C_n) \), provided that \( \log C_n < n/2 \).

As noticed in Blatt's paper, Theorem C is a direct consequence of the following:

**Theorem D.** — Let \( P(z) \) be a monic polynomial of degree \( n \) with all its roots on the unit circle. Then

\[
D_P \leq C \left( \frac{\log n}{n} \right) \max_{|z| \geq 1+n^{-a}} |\log |P(z)|| - n \log |z||.
\]

Since \( |z^{-n}P(z)| = |P(1/z)| \), inequality (1.2) is equivalent to

\[
D_P \leq C \left( \frac{\log n}{n} \right) |\log |P(z)||_{1/(1+n^{-a})}.
\]

In Section 4 we give a short and simple proof of the following theorem on conjugate functions:

**Theorem E.** — Let \( f \) be a real harmonic function on \( D = \{|z| < 1\} \) and let assume that its conjugate function \( \tilde{f} \) satisfies \( \partial \tilde{f}/\partial \theta \leq K \) on \( D \). Then, for any \( r \in [1/2, 1) \)

\[
|\tilde{f}| \leq \frac{6}{\pi} \left( \log \frac{2}{1-r} \right) |f|_r + 4\sqrt{3}K \frac{1-r}{r}.
\]
If we choose as before \( f = \frac{1}{\pi} \log |P| \), we obtain the following improvement of Theorem D:

**THEOREM D'.** Let \( P(z) \) be a polynomial of degree \( n \) with all its roots on the unit circle. Then

\[
D_P \leq \frac{12}{\pi^2} \left( \log \frac{2}{1 - r} \right) \left| \log |P||_{r \rightarrow} + \frac{8\sqrt{3}}{\pi} \cdot \frac{1 - r}{r} \right., \quad r \in [1/2, 1].
\]

This result implies the following improved version of Blatt’s theorem:

**THEOREM C'.** Let \( P(z) \) be a polynomial satisfying the assumptions of Theorem C. Then

\[
D_P \leq 13 \max \left\{ 1, \log \frac{2n}{\log C_n} \right\} \frac{\log C_n}{n}.
\]

The previous assertion is trivial if \( \log C_n > \frac{n}{2} \). Assume that (1.1) holds and suppose \( \log C_n \leq \frac{n}{2} \). We apply theorem D' choosing \( r = 1 - \frac{\log C_n}{n} \). By the maximum principle \( \log^+ |P|_r \leq \log^+ A \), while by the Lagrange interpolation formula

\[
1 = \sum_{z_j, P(z_j) = 0} \frac{P(z)}{P'(z_j)(z - z_j)}
\]

we have \( \log^- |P|_r \leq \log^+ \frac{nB}{1 - r} \). Hence

\[
|\log |P||_{r \rightarrow} \leq \max \left\{ \log^+ A, \log^+ \frac{nB}{1 - r} \right\} \leq \log C_n + \log \frac{n}{1 - r} = \log \frac{n^2 C_n}{\log C_n}.
\]

Theorem D' gives

\[
D_P \leq \frac{12}{n\pi^2} \left( \log \frac{2n}{\log C_n} \right) \log \frac{n^2 C_n}{\log C_n} + \frac{16\sqrt{3}}{\pi} \log \frac{C_n}{n}
\]

\[
\leq 13 \max \left\{ 1, \log \frac{2n}{\log C_n} \right\} \frac{\log C_n}{n}.
\]

We notice that the conformal mapping \( z \mapsto \frac{1}{2} \left( z + \frac{1}{z} \right) \) which sends the unit circle onto \([-1, 1]\) can be used to get similar results on the distribution of the roots of a polynomial vanishing only on \([-1, 1]\).

In Section 5 we consider the problem of finding an upper bound for the maximum modulus of a polynomial depending on its degree and on the discrepancy. We prove the following theorem:
**THEOREM F.** — Let \( f \) be a real harmonic function on \( D = \{ |z| < 1 \} \) such that \( f(0) = 0 \) and \( \partial f / \partial \theta \leq K \) on \( D \). Let also
\[
\Delta = \max_{z, w \in D} |\hat{f}(z) - \hat{f}(w)|.
\]
Then
\[
\sup_{D} f \leq \frac{\Delta}{\pi} \left( 3 + \log \frac{2\pi K}{\Delta} \right).
\]

By applying the previous result to the function \( f = \frac{1}{\pi} \log |P| \) we obtain:

**COROLLARY.** — Let \( P(z) \) be a polynomial of degree \( n \) with all its roots on the unit circle and such that \( P(0) = 1 \). Then
\[
\log |P| \leq nD_P(3 + \log 1/D_P).
\]

Finally, in Section 6 we discuss an extremal example.

**2. Some results from harmonic analysis.**

In this section we recall some basic facts on harmonic analysis. The standard reference of all definitions and results is the book of P. Koosis ([K]).

Let \( f \) be a \( 2\pi \)-periodic real function on \( L_1(-\pi, \pi) \). Then its Hilbert transform
\[
\tilde{f}(\theta) = \int_{-\pi}^{\pi} \frac{f(\theta - t)}{2\tan t/2} \, dt
\]
exists and is finite almost a.e. (= almost everywhere). We call \( \tilde{f} \) the conjugate function of \( f \). Although \( \tilde{f} \) does not belongs to \( L_1(-\pi, \pi) \) in general, we have the following theorem of Kolmogorov (as improved by Davis) which is very important for our purposes.

**THEOREM 2.1.** — Let \( f \in L_1(-\pi, \pi) \) be a \( 2\pi \)-periodic real function and let \( \tilde{f} \) be its conjugate. Then, for any positive \( \lambda \),
\[
\mu(\theta \in [0, 2\pi); |\tilde{f}| > \lambda) < \frac{\pi^2}{8k\lambda} \int_{-\pi}^{\pi} |f(\theta)| \, d\theta
\]
where \( k \) is Catalan's constant. In this inequality, the constant \( \pi^2/8 \) is the best possible.
Kolmogorov’s proof gave no information about the best constant, which was obtained much later by Davis [D]. Also, see Baernstein [Ba] for another proof.

A real function \( f(z) \) on the open disk \( D = \{ |z| < 1 \} \) is harmonic if it is the real part of a function \( F(z) \) analytic on \( D \). We notice that \( F \) is unique to within an additive constant. The harmonic conjugate of \( f \) is the real function \( \tilde{f} \) such that \( f + i \tilde{f} \) is analytic and \( \tilde{f}(0) = 0 \). Given a function on \( D \), we often use the notation \( f(r, \theta) = f(re^{i\theta}) \). Let \( f = RF \) with \( F \) analytic on \( D \). Then the non-tangential limits

\[
\begin{align*}
  f(\theta) &:= \lim_{r \to 1^-} f(r, \varphi), & \tilde{f}(\theta) &:= \lim_{r \to 1^-} \tilde{f}(r, \varphi)
\end{align*}
\]

exist a.e. if \( F \) belongs to the Hardy space \( H_1 \), i.e. if

\[
\sup_{0 < r < 1} \int_{-\pi}^{\pi} |F(re^{i\theta})| \, d\theta < +\infty.
\]

Let \( p \in (1, \infty) \). By a theorem of Riesz,

\[
\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |f(r, \theta)|^p \, d\theta < +\infty
\]

if and only if

\[
\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |\tilde{f}(r, \theta)|^p \, d\theta < +\infty.
\]

Therefore, if (2.1) or (2.2) holds for some \( p > 1 \), then \( f + i \tilde{f} \in H_1 \). In particular, if \( f \) or \( \tilde{f} \) are bounded, then \( f + i \tilde{f} \in H_1 \). If the non-tangential limit \( f(\theta) \) exists, it is called the boundary value of \( f \) and similarly for \( \tilde{f} \).

Let, for \( r \in [0, 1) \) and \( \theta \in \mathbb{R} \),

\[
K(\rho, \theta) = \frac{1 - \rho^2}{1 - 2\rho \cos \theta + \rho^2}, \quad \bar{K}(\rho, \theta) = \frac{2\rho \sin \theta}{1 - 2\rho \cos \theta + \rho^2}
\]

be the Poisson kernel and the conjugate Poisson kernel. Then for any real harmonic function \( f \) we have the Poisson representations

\[
\begin{align*}
  f(\rho, \varphi) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\rho/r, \theta) f(r, \varphi - \theta) \, d\theta \\
  \tilde{f}(\rho, \varphi) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \bar{K}(\rho/r, \theta) f(r, \varphi - \theta) \, d\theta
\end{align*}
\]

which hold for \( 0 \leq \rho < r < 1 \) and \( \varphi \in \mathbb{R} \). If \( f + i \tilde{f} \in H_1 \), then (2.3) and (2.4) still hold for \( r = 1 \).
Let $g \in L_1(-\pi, \pi)$ be a $2\pi$-periodic real function and let $\tilde{g}$ its conjugate function. Then
\[ f(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\rho, \theta) g(\varphi - \theta) \, d\theta \]
is harmonic and its harmonic conjugate is
\[ \tilde{f}(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{K}(\rho/\tau, \theta) g(\varphi - \theta) \, d\theta. \]

Assume further $\tilde{g} \in L_1(-\pi, \pi)$. Then the boundary values $f(\theta)$ and $\tilde{f}(\theta)$ both exist and
\[ f(\theta) = g(\theta), \quad \tilde{f}(\theta) = \tilde{g}(\theta) \quad \text{a.e.} \]

We also recall the following elementary inequalities which hold for all $\rho \in [0,1)$ and all $\theta$:
\[ 0 < \frac{1 - \rho}{1 + \rho} \leq K(\rho, \theta) \leq \frac{1 + \rho}{1 - \rho} \quad (2.6) \]
and
\[ |\tilde{K}(\rho, \theta)| \leq \frac{2\rho}{1 - \rho^2}. \quad (2.7) \]

Moreover, we notice that
\[ \int K(\rho, \theta) \, d\theta = 2 \arctg \left( \frac{1 + \rho}{1 - \rho} \cdot \tan \frac{\theta}{2} \right) + \text{constant}. \quad (2.8) \]

In particular, this implies
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\rho, \theta) \, dt = 1. \quad (2.9) \]

For the conjugate kernel, we have
\[ \int \tilde{K}(\rho, \theta) \, d\theta = \log(1 - 2\rho \cos \theta + \rho^2) + \text{constant}, \quad (2.10) \]
so that
\[ \int_{-\pi}^{\pi} |\tilde{K}(\rho, \theta)| \, d\theta = 4 \log \frac{1 + \rho}{1 - \rho}. \quad (2.11) \]

We begin this section by a very elementary lemma.

**Lemma 3.1.** — Let $g: \mathbb{R} \to \mathbb{R}$ be a $2\pi$-periodic function and suppose that there exists a constant $K$ such that

$$g(\varphi + \varepsilon) \leq g(\varphi) + \varepsilon K,$$

for any $\varphi \in \mathbb{R}$ and any $\varepsilon > 0$. Assume further that for any positive number $\lambda$ the set

$$E_\lambda = \{ \theta \in [0, 2\pi); |g(\theta)| > \lambda \}$$

satisfies

$$\mu(E_\lambda) < \frac{C}{\lambda},$$

where $\mu$ is Lebesgue measure and $C$ is some positive constant. Then

$$\max |g| \leq 2\sqrt{CK}.$$

Moreover,

$$|g^+| + |g^-| \leq 2\sqrt{2CK}.$$

**Proof.** — Put $\lambda = \sqrt{CK}$ and $A = 2\sqrt{CK}$. We first want to prove that $|g(\varphi)| \leq A$ for any $\varphi \in \mathbb{R}$.

If $\varphi \notin E_\lambda$ then $|g(\varphi)| \leq \lambda$ and we have nothing to prove. If $\varphi \in E_\lambda$, since $\mu(E_\lambda) < C\lambda^{-1}$, there exists $\varepsilon_1 > 0$ such that $\varepsilon_1 \leq C\lambda^{-1}$ and $\varphi - \varepsilon_1 \notin E_\lambda$, hence

$$g(\varphi) \leq g(\varphi - \varepsilon_1) + \varepsilon_1 K \leq \lambda + \frac{CK}{\lambda} = A.$$

In the same way, there exists $\varepsilon_2 > 0$ such that $\varepsilon_2 \leq C\lambda^{-1}$ and $\varphi + \varepsilon_2 \notin E_\lambda$, which implies

$$g(\varphi) \geq g(\varphi + \varepsilon_2) - \varepsilon_2 K \geq -\lambda - \frac{CK}{\lambda} = -A.$$

This proves the first assertion. To prove the second one consider the sets $E^+_\lambda = \{ \theta \in [0, 2\pi); g^+(\theta) > \lambda \}$ and $E^-_\lambda = \{ \theta \in [0, 2\pi); g^-(\theta) > \lambda \}$.

For any $\lambda > 0$ and any $\varphi, \psi \in \mathbb{R}$, the preceding argument leads to

$$g^+(\varphi) + g^-(\psi) \leq \lambda + K\mu(E^+_\lambda) + \lambda + K\mu(E^-_\lambda) \leq 2\lambda + K\mu(E_\lambda) \leq 2\lambda + \frac{KC}{\lambda},$$

and the choice $\lambda = \sqrt{CK}/2$ gives the second assertion. This concludes the proof. \qed
We denote by $|f|_r$ the sup of $|f(z)|$ on $|z| = r$ and by $|f|$ the sup of $|f(z)|$ on $|z| = 1$. When $f$ is real-valued we define the span of $f$ by the formula

$$\Delta(f) = |f^+| + |f^-|,$$

where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. Trivially, $\Delta(f) \leq 2|f|$. 

If we apply Kolmogorov's Theorem 2.1 and Lemma 3.1 (with $g = \hat{f}$), we get:

**Theorem 3.1.** — Let $f \in L_1(-\pi, \pi)$ be a $2\pi$-periodic real function and let $\hat{f}$ be its conjugate. Suppose that there exists a positive constant $K$ such that

$$\hat{f}(\varphi + \varepsilon) \leq \hat{f}(\varphi) + \varepsilon K,$$

for any $\varphi \in T$ and any $\varepsilon > 0$. Let also

$$\hat{H} = \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(\theta)| d\theta.$$

Then,

$$|\hat{f}| \leq \frac{2\pi}{k} \cdot \sqrt{\hat{H}K} \quad \text{and} \quad \Delta(\hat{f}) \leq 2\pi \cdot \sqrt{\frac{\pi}{k} \cdot \sqrt{\hat{H}K}},$$

where $k$ is Catalan's constant.

One may notice that this result is essentially the same as the refinement of Ganelius theorem published in [M]. In fact, denote by the same letter $f$ the real harmonic function on $D$ whose boundary value coincide with $f$ almost everywhere. Then, $\int_{-\pi}^{\pi} f(\theta) d\theta = f(0)$. Hence, if we further assume $f(0) = 0$, we have

$$\hat{H} = \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(\theta)| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^+(\theta) d\theta \leq \max f^+.$$

4. On Blatt's theorem.

Let $g$ be a real harmonic function on $D$ and assume that there exists a constant $K$ such that $\partial g/\partial \theta < K$ on $D$. The function $\rho \rightarrow |g|_\rho$ in general does not satisfy Lipschitz's condition. As an example, consider $g = \text{Arg}(1 - z)$. However, we have:
LEMMA 4.1 ("Turn-growth lemma"). — Let $g$ be a real harmonic function on $D$ and assume that there exists a constant $K$ such that $\frac{\partial g}{\partial \theta} < K$ on $D$. Then, for any $\rho \in [0, 1)$,

$$|g| \leq 3|g|\rho + 4\sqrt{3}K\frac{1-\rho}{1+\rho}.$$ 

Proof. — Let $\varepsilon = 2 \arctg \left( \sqrt{3} \frac{1-\rho}{1+\rho} \right) \in (0, \pi)$. Then, by (2.8)

$$(4.1) \quad \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} K(\rho, \theta) \, d\theta = \frac{2}{\pi} \arctg \left( \frac{1+\rho}{1-\rho} \cdot \frac{\varepsilon}{2} \right) = \frac{2}{3}$$

and, by (2.9)

$$(4.2) \quad \frac{1}{2\pi} \int_{\varepsilon}^{\pi} K(\rho, \theta) \, d\theta = 1 - \frac{2}{3} = \frac{1}{3}.$$ 

Now assume $|g| = -g(\varphi)$ for some $\varphi \in \mathbb{R}$ (otherwise $|g| = |g^+|$ and a similar argument applies). Since $g$ is bounded on $|z| < 1$, Poisson’s Formula (2.3) applies and we have, by (4.2),

$$-|g|\rho \leq g(\rho, \varphi + \varepsilon) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\rho, \theta) g(\varphi + \varepsilon - \theta) \, d\theta$$

$$(4.3) \quad \leq \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} K(\rho, \theta) g(\varphi + \varepsilon - \theta) \, d\theta + \frac{1}{3}|g|.$$ 

By our assumption we have $g(\varphi + \varepsilon - \theta) \leq g(\varphi) + K(\varepsilon - \theta)$ for $\theta \leq \varepsilon$. Moreover $K(\rho, \theta) > 0$, whence, by (4.1),

$$\frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} K(\rho, \theta) g(\varphi + \varepsilon - \theta) \, d\theta \leq \frac{2}{3} (g(\varphi) + K\varepsilon) - \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} K(\rho, \theta) \theta \, d\theta.$$ 

Since $\theta \mapsto \theta K(\rho, \theta)$ is odd we have $\int_{-\varepsilon}^{\varepsilon} K(\rho, \theta) \theta \, d\theta = 0$ and we obtain

$$\frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} K(\rho, \theta) g(\varphi + \varepsilon - \theta) \, d\theta \leq \frac{2}{3} (-|g| + K\varepsilon).$$

Now (4.3) gives

$$-|g|\rho \leq \frac{2}{3} (-|g| + K\varepsilon) + \frac{1}{3}|g| = \frac{2}{3} K\varepsilon - \frac{1}{3}|g|$$

and, since $\varepsilon \leq 2\sqrt{3} \frac{1-\rho}{1+\rho}$,

$$|g| \leq 3|g|\rho + 4\sqrt{3}K\frac{1-\rho}{1+\rho}.$$ 

The next lemma is an easy consequence of Poisson’s formula.
Lemma 4.2. — Let \( f \) be a real harmonic function on \(|z| < 1\) and let \( 0 < \rho < r < 1 \). Then,
\[
|f|_\rho \leq \frac{2}{\pi} \left( \log \frac{r + \rho}{r - \rho} \right) |f|_r.
\]
Moreover, if \( f + i\tilde{f} \in H_1 \), we also have
\[
|\tilde{f}|_\rho \leq \frac{2\rho}{1 - \rho^2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)| d\theta.
\]

Proof. — By Poisson’s formula (2.4)
\[
\tilde{f}(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{K}(\rho / r, \theta) f(r, \varphi - \theta) d\theta.
\]
It follows by (2.11) that
\[
|\tilde{f}|_\rho \leq \frac{2}{\pi} \left( \log \frac{r + \rho}{r - \rho} \right) |f|_r.
\]
Assume now \( f + i\tilde{f} \in H_1 \). Then Poisson’s formula (2.4) still holds for \( r = 1 \) and we find
\[
\tilde{f}(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{K}(\rho, \theta) f(\varphi - \theta) d\theta.
\]
Therefore, by (2.7),
\[
|\tilde{f}(\rho, \varphi)| \leq \frac{2\rho}{1 - \rho^2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)| d\theta.
\]

Lemma 4.1 and the first part of Lemma 4.2 lead to Theorem E announced in the introduction:

Theorem 4.1. — Let \( f \) be a real harmonic function on \( D \) and let assume that its conjugate function \( \tilde{f} \) satisfies \( \partial \tilde{f} / \theta < K \) on \( D \). Then, for any \( r \in [1/2, 1) \)
\[
|\tilde{f}| \leq \frac{6}{\pi} \left( \log \frac{2}{1 - r} \right) |f|_r + 4\sqrt{3} K \frac{1 - r}{r}.
\]

Proof. — Let \( \rho, r \) such that \( 0 \leq \rho < r < 1 \). From Lemmas 2 and 3 we obtain
\[
|\tilde{f}| \leq \frac{6}{\pi} \left( \log \frac{r + \rho}{r - \rho} \right) |f|_\rho + 4\sqrt{3} K \frac{1 - \rho}{1 + \rho}.
\]
Now choose \( \rho = 2r - 1 \). \( \square \)
The second part of Lemma 4.2 leads to an elementary proof (with a worse constant) of Ganelius-Mignotte’s theorem:

**Theorem 4.2.** — Let \( f \) be a real harmonic function on \( D \) such that \( \partial \tilde{f} / \partial \theta < K \) on \( D \). Then

\[
|\tilde{f}| \leq 4\sqrt{3\sqrt{3} \cdot \sqrt{H} K}
\]

where

\[
H = \frac{1}{4\pi} \int_{-\pi}^{\pi} |f(\theta)| d\theta.
\]

**Proof.** — From Lemma 4.1 (with \( g = \tilde{f} \)) and Lemma 4.2 (since \( \tilde{f} \) is bounded, \( \tilde{f} + i\tilde{f} \in H_1 \)) we obtain

\[
|\tilde{f}| \leq \frac{12\rho H}{1 - \rho^2} + 4\sqrt{3}K \frac{1 - \rho}{1 + \rho}.
\]

Let \( \alpha, \beta > 0 \) and \( u(\rho) = \frac{\alpha \rho}{1 - \rho^2} + \frac{\beta(1 - \rho)}{1 + \rho} \). Then \( \inf_{0 < \rho < 1} u(\rho) \leq \sqrt{\alpha \beta} \). In fact, if \( \beta \leq \alpha \) we have \( u(0) = \beta \leq \sqrt{\alpha \beta} \); otherwise \( \rho_0 = 1 - \frac{\sqrt{\alpha}}{\beta} \in (0,1) \) and \( u(\rho_0) = \sqrt{\alpha \beta} \). Using this remark with \( \alpha = 12H \) and \( \beta = 4\sqrt{3}K \) we obtain

\[
|\tilde{f}| \leq \sqrt{6H} \cdot 4\sqrt{3}K = 4\sqrt{3\sqrt{3}HK} < 9.119 \sqrt{HK}.
\]

**5. Upper bounds for \( \max f \).**

The aim of this section is to give an upper bound for the maximum of an harmonic function \( f \) such that \( \partial \tilde{f} / \partial \theta \) is bounded on \( D \).

**Theorem 5.1.** — Let \( f \) be a real harmonic function on \( D \) such that \( f(0) = 0 \) and \( \partial \tilde{f} / \partial \theta < K \) on \( D \) for some \( K > 0 \). Then,

\[
\sup_{D} f \leq \frac{\Delta(\tilde{f})}{\pi} \left( 3 + 2 \log \frac{2\sqrt{3} \pi K}{\Delta(\tilde{f})} \right).
\]

**Proof.** — Let \( \varphi \in \mathbb{R} \) and let \( \rho \in (0,1) \). We apply Poisson’s formula (2.4) to the harmonic function \( \tilde{f} \). Since \( f(0) = 0 \) we have \( \tilde{f} = -f \).
Moreover, since \( \hat{f} \) is bounded, \( f + i \hat{f} \in H_1 \) and (2.4) still holds with \( r = 1 \).
By using (2.11) we get

\[
(5.1) \quad f(\rho, \varphi) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{K}(\rho, \theta) \hat{f}(\varphi - \theta) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} \tilde{K}(\rho, \theta) (\hat{f}(\varphi + \theta) - \hat{f}(\varphi - \theta)) \, d\theta \leq \frac{2\Delta(\hat{f})}{\pi} \log \frac{1+\rho}{1-\rho}.
\]

Since \( \partial \hat{f}/\partial \theta = \rho(\partial f/\partial \rho) \), we have \( f(\varphi) \leq f(\rho, \varphi) + K \log 1/\rho \). Therefore (5.1) gives

\[
\max f \leq \frac{2\Delta(\hat{f})}{\pi} \log \frac{2}{1-\rho} + K \log 1/\rho.
\]

Now choose \( \rho = K\pi/(\Delta(\hat{f}) + K\pi) \). Since \( \Delta(\hat{f}) \leq 2\pi K \), we obtain

\[
\max f \leq \frac{2\Delta(\hat{f})}{\pi} \log \frac{2\pi K}{\Delta(\hat{f})} + \left( K + \frac{\Delta(\hat{f})}{\pi} \right) \log \left( 1 + \frac{\Delta(\hat{f})}{\pi K} \right)
\]

\[
\leq \frac{\Delta(\hat{f})}{\pi} \left( 3 + 2 \log \frac{2\sqrt{3}\pi K}{\Delta(\hat{f})} \right). \tag*{\square}
\]

We end this section with a further remark concerning harmonic functions.

**Proposition 5.1.** — Let \( f \) be an harmonic function on \( \mathbb{D} \) and assume that \( f + i \hat{f} \in H_1 \). Then, for \( 0 \leq \rho < 1 \) and \( \varphi \in \mathbb{R} \),

(i) \[ -\frac{1+\rho}{1-\rho} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} f^- (\theta) \, d\theta \leq f(\rho, \varphi) \leq \frac{1+\rho}{1-\rho} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} f^+ (\theta) \, d\theta. \]

Moreover, if \( f(\theta) \leq 0 \), then

(ii) \[ f(\rho, \varphi) \leq \frac{1-\rho}{1+\rho} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \, d\theta. \]

**Proof.** — By Poisson’s formula (2.3), for any \( \rho \in (0,1) \) and for any \( \varphi \in \mathbb{R} \) we have

\[
f(\rho, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\rho, \theta - \varphi) f(\theta) \, d\theta.
\]

Thus, by (2.6)

\[
-\frac{1+\rho}{1-\rho} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} f^- (\theta) \, d\theta \leq f(\rho, \varphi) \leq \frac{1+\rho}{1-\rho} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} f^+ (\theta) \, d\theta,
\]

which proves (i).
Assume now $f(\theta) \leq 0$. Then
\[-f(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(\rho, \theta - \varphi) (-f(\theta)) \, d\theta \geq \frac{1 - \rho}{1 + \rho} \times \frac{1}{2\pi} \int_{-\pi}^{\pi} (-f(\theta)) \, d\theta,\]
which leads to (ii).

\begin{corollary}
Let $P$ be a polynomial with no zeros for $|z| < 1$. Then, for $0 < p < 1$ and $\varphi \in \mathbb{R}$,
\[-\frac{1 + \rho}{1 - \rho} \left( \bar{h}(P) + \log M(P) \right) \leq \log |P(\rho e^{i\varphi})| \leq \frac{1 + \rho}{1 - \rho} \times \bar{h}(P).\]

\textbf{Proof.} — Use (i) and the relation
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |P| = \log M(P) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |P| - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |P|. \quad \Box
\]

\begin{corollary}
Let $P$ be a polynomial with no zeros for $|z| < 1$. Then, for $0 < r < 1$,
\[|P|^r \leq |P|^{2r} M(P)^{1-2r}.\]

\textbf{Proof.} — This is an easy consequence of (ii). \quad \Box
\end{corollary}

6. An extremal example.

Let $x$ be a positive real number and consider the set $\Lambda_x$ of polynomials $P(z) = a_n z^n + \cdots + a_1 z + a_0$ such that $a_0 a_n \neq 0$ and
\[
\log \frac{|P|}{\sqrt{|a_0 a_n|}} \leq x \cdot n.
\]
Let
\[f(x) = \sup_{P \in \Lambda_x} D_P.\]
Then, $f$ is a non-decreasing function and Erdös-Turan’s theorem implies the inequality
\[f(x) \leq c \sqrt{x}, \quad c = \sqrt{2\pi/k}.\]
The aim of this section is to prove that this inequality is essentially sharp.

\begin{theorem}
For any $x \in (0, 1/2)$ we have $f(x) \geq \sqrt{2x}$.
\end{theorem}
Let $n, r$ two positive integers with $r < n$. By the results of [ET], §14,

\begin{equation}
(6.1) \quad P(z) = \frac{r^{n+r}}{(1+z)^r} \int_{-1}^{z} (z-t)^{r-1}(1+t)^{r} t^{n-r} \, dt
\end{equation}

is a monic polynomial of degree $n$ vanishing at $-1$ with multiplicity $r$ such that

\begin{equation}
(6.2) \quad \log \|P\| = \frac{1}{2} \sum_{v=n-r+1}^{n} \log \left(1 + \frac{r}{v}\right) \leq \frac{r^2}{2(n-r)},
\end{equation}

where $\|P\|$ is the euclidean norm of the polynomial $P$, i.e. the quadratic mean of the moduli of the coefficients of $P$. Moreover, by (6.1)

\begin{equation}
(6.3) \quad a_0 = P(0) = (-1)^{n-r} r^{n+r} \int_{0}^{1} (1-s)^r s^{n-1} \, ds = (-1)^{n-r} \frac{r}{n}.
\end{equation}

Since $P$ has a root at $-1$ of multiplicity $\geq r$ we have $D_P \geq \frac{r}{n}$. On the other hand, by (6.2) and (6.3) we obtain

\begin{equation*}
\log \frac{|P|}{\sqrt{|a_0 a_n|}} \leq \log \frac{\sqrt{n} \|P\|}{\sqrt{|a_0|}} \leq \frac{r^2}{2(n-r)} + \frac{1}{2} \log \frac{n^2}{r} \leq \frac{r^2}{2(n-r)} + \log n.
\end{equation*}

Hence

\begin{equation*}
\frac{r}{n} \leq D_P \leq f \left( \frac{r^2}{2n(n-r)} + \frac{\log n}{n} \right).
\end{equation*}

Let now $x \in (0, 1/2)$ and choose a sequence $(n_k, r_k)$ such that $n_k \to +\infty$ and

\begin{equation*}
\frac{r_k^2}{2n_k(n_k - r_k)} + \frac{\log n_k}{n_k}
\end{equation*}

increases to $x$ as $k \to +\infty$. Then we have $r_k/n_k \leq f(x)$ and, when $k \to +\infty$,

\begin{equation*}
\sqrt{2x} \leq f(x).
\end{equation*}

\[\square\]

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