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POINTWISE MULTIPLIERS AND CORONA TYPE DECOMPOSITION IN $BMOA$

by J. M. ORTEGA and J. FÀBREGA

1. Introduction.

Let B be the unit ball of $\mathbb{C}^n$ and let S be its boundary. We will consider BMO functions on S with respect to the non-isotropic metric on S, that is, functions in $L^1(S)$ such that

$$||f||_* = \sup\left\{\frac{1}{|I_{\eta,t}|}\int_{I_{\eta,t}} |f - f_{I_{\eta,t}}|,\ \eta \in S,\ t > 0\right\} < \infty$$

where $I_{\eta,t}$ is the non-isotropic ball $I_{\eta,t} = \{z \in S;\ |1 - \bar{\eta}z| < t\}$, $|I_{\eta,t}|$ is the Lebesgue measure of $I_{\eta,t}$ and

$$f_{I_{\eta,t}} = \frac{1}{|I_{\eta,t}|}\int_{I_{\eta,t}} f d\sigma.$$

The space $BMOA$ is defined by $BMOA = H^1(B) \cap BMO(S)$. There exist several characterizations of this space. Among them we can mention the one given by the boundedness of Garsia's norm and the one given in terms of Carleson measures (see section 2 for details).

The first goal of this paper is to study the space of pointwise multipliers of $BMOA$. We denote this space by $\mathcal{M}(BMOA)$. For the one-dimensional case a characterization was obtained by D. Stegenga [S].

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This result can be generalized to the n-dimensional case. He proved that $\mathcal{M}(BMOA) = H^\infty \cap BMO_{\log}(S)$ where

$$BMO_{\log}(S) = \{f \in L^1(S);\ ||f||_{*,\log} < \infty\}$$

and

$$||f||_{*,\log} = \sup\left\{\frac{1}{|I_{\eta,t}|}\log\frac{1}{|I_{\eta,t}|}\int_{I_{\eta,t}}|f - f_{I_{\eta,t}}|d\sigma;\ \eta \in S,\ t > 0\right\}.$$

Note that this characterization is given in terms of the boundary values. Therefore, it seems interesting to obtain another characterization of this space in terms of the interior values of the function.

The first step is to give a description of the multipliers similar to the one given by Garsia's norm for $BMOA$. As a consequence of this result, we obtain a characterization of these multipliers in terms of "Carleson type" estimates for the measure $|\partial g(z)|^2(1-|z|^2)dV(z)$.

To be precise we obtain the following theorem:

THEOREM A. — *The following assertions are equivalent:*

i) *g is a pointwise multiplier of $BMOA$.*

ii) *$g \in H^\infty$ and*

$$\sup\left\{\frac{\log\frac{1}{|I_{\eta,t}|}}{|I_{\eta,t}|}\int_{I_{\eta,t}}\left|g - g_{I_{\eta,t}}\right| d\sigma;\ \eta \in S,\ t > 0\right\} < \infty.$$

iii) *$g \in H^\infty$ and for some $1 \le p < \infty$*

$$\left(\sup\left\{\log^p\frac{2}{1-|z|^2}\int_S |g(\zeta) - g(z)|^p \mathcal{P}(z,\zeta)d\sigma(\zeta);\ z \in B\right\}\right)^{\frac{1}{p}} < \infty,$$

where $\mathcal{P}(z,\zeta)$ is the Poisson-Szegö kernel given by $\mathcal{P}(z,\zeta) = \dfrac{(1-|z|^2)^n}{|1-\bar{z}\zeta|^{2n}}$.

iv) *$g \in H^\infty$ and*

$$\int_{Q_{\eta,t}}|\partial g(\zeta)|^2(1-|z|^2)dV(\zeta) \le ct^n\log^{-2}t,$$

where $Q_{\eta,t} = \{z \in B;\ |1-\bar{\eta}z| < t\}$, $\eta \in S, t > 0$.

A more complete list of characterizations of this space can be found in section 2 (see Theorems 2.7, 2.9, and 2.13).

The second problem that we study in this paper is to give a corona type decomposition for $BMOA$:

Let $g = (g_1, \dots, g_m)$ be a vector valued holomorphic function on B. We consider the map $M_g : H(B) \times \dots \times H(B) \longrightarrow H(B)$ defined by $M_g(f) = \sum_{j=1}^{m} g_j f_j$. We characterize those g's such that M_g maps $BMOA \times \dots \times BMOA$ onto $BMOA$.

Results of this type for Hardy spaces, have been studied by many authors, among them L. Carleson [C], E. Amar [A], M. Andersson and H. Carlsson [AnCa1], [AnCa2], S.Y. Li [Li] and K.C. Lin [Lin]. Corona type decompositions for Besov spaces can be found in the papers of V.A. Tolokonnikov [T], A. Nicolau [N] and ourselves [OF]. A result for the Bloch space can be found in [OF].

For $BMOA$ we obtain the following theorem:

THEOREM B. — *Let $g = (g_1, \dots, g_m)$ be a vector valued holomorphic function on B. Then the operator*

$$M_g(f)(z) = \sum_{j=1}^{m} f_j(z) g_j(z)$$

maps $BMOA \times \dots \times BMOA$ onto $BMOA$ iff the functions g_j are multipliers of $BMOA$ and satisfy the condition $\sup\{|g(z)|; z \in B\} \geq \delta > 0$.

The proof of this theorem for the unit disk of $\mathbb{C}$ can be obtained from a result of V.A. Tolokonnikov [T]. Using techniques of one complex variable, he obtained a decomposition

$$1 = \sum_{j=1}^{m} g_j h_j, \qquad h_j \in \mathcal{M}(BMOA(D)).$$

Finally, as a corollary of Theorem B we give an alternative proof of the mentioned decomposition in the Bloch space.

The paper is organized in the following way. In section 2 we prove Theorem A and we give some examples and properties of the multipliers of $BMOA$. In section 3 we prove that the conditions of Theorem B are necessary. In section 4 we recall some results about solving the $\bar{\partial}$−equation with estimates in terms of Carleson measures and $BMO(S)$ norms. Finally, in section 5 we finish the proof of Theorem B.

As usual, different constants in the inequalities will be denoted by the same c. Also, the notation $F \approx G$ means that there exist constants $c_1, c_2 > 0$ which are independent of F and G and such that $c_1 F \leq G \leq c_2 F$.

2. Pointwise multipliers of $BMOA$.

Let us start recalling several well-known definitions and results related to $BMOA$ functions.

DEFINITION 2.1. — *A positive measure μ is a Carleson measure if there exists a constant $c > 0$ such that*

$$\mu\left(Q_{\eta,t}\right) \le ct^n$$

for all $Q_{\eta,t}$.

The set of Carleson measures will be denoted by W^1.

As usual, if h is a positive function such that $h(z)dV(z)$ is a Carleson measure we will write $h \in W^1$ instead of $hdV \in W^1$.

It is well-known that a positive measure μ is Carleson if and only if

$$\sup\left\{\int_B \frac{(1-|z|^2)^n}{|1-\bar{\zeta}z|^{2n}}d\mu(\zeta);\ z \in B\right\} < \infty. \tag{2.1}$$

In order to state the main characterizations of $BMOA$ functions we need to introduce the following norms.

DEFINITION 2.2. — *For $1 \le p < \infty$ we define*

$$||f||_{*,p} = \left(\sup\left\{\frac{1}{|I_{\eta,t}|}\int_{I_{\eta,t}} |f(\zeta) - f_{I_{\eta,t}}|^p d\sigma(\zeta);\ I_{\eta,t}\right\}\right)^{\frac{1}{p}}.$$

DEFINITION 2.3 (Garsia's norms). — *For $1 \le p < \infty$ we define*

$$||f||_{G,p} = \left(\sup\left\{\int_S |f(\zeta) - f(z)|^p \mathcal{P}(z,\zeta)d\sigma(\zeta);\ z \in B\right\}\right)^{\frac{1}{p}}$$

where $\mathcal{P}(z,\zeta)$ is the Poisson-Szegö kernel i.e. $\mathcal{P}(z,\zeta) = \dfrac{(1-|z|^2)^n}{|1-\bar{z}\zeta|^{2n}}$.

The main property of this norm is its invariance by automorphisms. This follows from

$$||f||_{G,p} = \left(\sup\left\{\int_S |f(\psi(w)) - f(\psi(0))|^p d\sigma(w);\ \psi \in \operatorname{Aut}(B)\right\}\right)^{\frac{1}{p}}.$$

THEOREM 2.4. — *Let $f \in H^1(B)$. Then the following properties are equivalent:*

i) *f is in $BMOA$.*

ii) $\mu_1 = |\partial f|^2(1-|z|^2) \in W^1$.

iii) $\mu_2 = |\partial |z|^2 \wedge \partial f|^2 \in W^1$.

iv) $||f||_{*,p} < \infty, \quad 1 \leq p < \infty$.

v) $||f||_{G,p} < \infty, \quad 1 \leq p < \infty$.

Moreover, we have $c_p||f||_{*,p} \leq ||f||_* \leq C_p||f||_{*,p}$ *and* $c_p||f||_{G,p} \leq ||f||_* \leq C_p||f||_{G,p}$.

The equivalences between i), ii) and iii) were obtained by R. Coiffman, R. Rochberg and R. Weiss [CoRoW], [CoW]. See also [ChoaChoe] and [J]. The equivalence between i) and iv) was noted by J. Shapiro [Sh]. Finally, the equivalence between i) and v) was obtained for $p = 2$ by Garsia (see [G]) for the one-dimensional case and by Sh. Axler and J. Shapiro [AxSh] for the n-dimensional case. The case $1 \leq p < \infty$ was obtained by P.S. Chee [Chee].

The first result that we will prove is a reformulation of the Stegenga's result for $n > 1$.

We will need the following lemmas.

LEMMA 2.5. — *For $s > -1, r, t \geq 0$ and $r + t - s > n + 1$ we have*

$$\int_B \frac{(1-|\zeta|^2)^s}{|1-\bar\zeta z|^r |1-\bar\zeta w|^t} dV(\zeta)$$

$$\leq \begin{cases} \dfrac{c}{|1-\bar z w|^{r+t-s-n-1}} & \text{if } \ r-s, t-s < n+1 \\ \dfrac{c}{(1-|z|^2)^{r-s-n-1}|1-\bar z w|^t} & \text{if } \ t-s < n+1 < r-s \\ \dfrac{c}{(1-|z|^2)^{r-s-n-1}|1-\bar z w|^t} + \dfrac{c}{(1-|w|^2)^{t-s-n-1}|1-\bar z w|^r} & \\ \qquad \text{if } \ r-s, t-s > n+1. & \end{cases}$$

Proof. — The proof of this lemma is standard. See for instance [OF]. □

LEMMA 2.6. — *For $\eta \in S$ and $0 \leq t < 1$, the functions $f_{\eta,t}(z) = \log \dfrac{2}{1-(1-t)\bar\eta z}$ satisfy:*

i) $||f_{\eta,t}||_* \leq c < \infty$ *with* c *independent of* η, t.

ii) *For* $z \in Q_{\eta,t}$, *we have* $|f_{\eta,t}| \approx \log \frac{1}{t}$.

iii) $\left|(f_{\eta,t})_{Q_{\eta,t}}\right| \approx \log \frac{1}{t}$.

Proof. — To prove i), by Theorem 2.4 and (2.1) we need to show that

$$J_{\eta,t}(z) = \int_B \frac{(1-|z|^2)^n}{|1-\bar{z}\zeta|^{2n}} |\partial f_{\eta,t}(\zeta)|^2 (1-|\zeta|^2) dV(\zeta) \leq c < \infty.$$

But, this estimate follows from Lemma 2.5 and the estimate

$$J_{\eta,t}(z) \leq c \int_B \frac{(1-|z|^2)^n (1-|\zeta|^2)}{|1-\bar{z}\zeta|^{2n} |1-(1-t)\bar{\eta}\zeta|^2} dV(\zeta).$$

Parts ii) and iii) follow from

$$|1-(1-t)\bar{\eta}z| \approx t + (1-t)|1-\bar{\eta}z| \approx t, \quad z \in Q_{\eta,t}. \qquad \square$$

THEOREM 2.7. — *The following assertions are equivalent:*

i) g *is a pointwise multiplier of* $BMOA$.

ii) $g \in H^\infty$ *and*

$$\sup \left\{ \frac{1}{|I_{\eta,t}|} \log \frac{1}{|I_{\eta,t}|} \int_{I_{\eta,t}} \left| g - g_{I_{\eta,t}} \right| d\sigma; \ \eta \in S, 0 < t \right\} < \infty. \tag{2.2}$$

Proof. — First we prove that i) implies ii). Note that by closed graph theorem the map M_g is continuous.

To prove that g is bounded we recall that $|f(z)| \leq c||f||_* \log \frac{2}{1-|z|^2}$. Thus, we have

$$|g(z)| \leq c \sup_{\zeta \in B} \left\{ \left| g(\zeta) \log \frac{2}{1-\bar{z}\zeta} \log^{-1} \frac{2}{1-|\zeta|^2} \right| \right\}$$
$$\leq c \left\| g(\zeta) \log \frac{2}{1-\bar{z}\zeta} \right\|_* < \infty.$$

Let us prove that g satisfies condition (2.2). We write I instead $I_{\eta,t}$. For $f \in BMOA$, we have

$$gf - (gf)_I = g(f - f_I) + f_I(g - g_I) + f_I g_I - (fg)_I. \tag{2.3}$$

Observe that

$$\frac{1}{|I|}\int_I |g(f-f_I)|d\sigma \le ||g||_\infty ||f||_*$$

and

$$|f_I g_I - (fg)_I| = \frac{1}{|I|}\left|\int_I g(f_I - f)\right| d\sigma \le ||g||_\infty ||f||_*.$$

Thus, we have

$$\frac{1}{|I|}\int_I |f_I||g-g_I|d\sigma \le ||gf||_* + ||g||_\infty ||f||_*.$$

Finally, taking $f(\zeta) = \log\dfrac{2}{1-(1-t)\bar{\eta}z}$ and applying Lemma 2.6 we obtain (2.2).

That ii) implies i) follows from the decomposition (2.3) and the well-known fact that $|f_I| \le c||f||_* \log\dfrac{1}{|I|}$. □

Remark. — We recall (see [S], [CRW]) that $\mathcal{M}(BMO(S))$ is the subspace of bounded functions on S which satisfy condition (2.2) of Theorem 2.7. Thus, we have:

COROLLARY 2.8. — *If $g \in \mathcal{M}(BMOA)$, then*

$$g|_S, \bar{g}|_S \in \mathcal{M}(BMO(S)).$$

The next lemma gives a characterization of $\mathcal{M}(BMOA)$ in some sense similar to the one given in terms of Garsia's norm for $BMOA$.

THEOREM 2.9. — *The following assertions are equivalent:*

i) *g is a pointwise multiplier of $BMOA$.*

ii) *$g \in H^\infty$ and $|||g|||_{G,p,\log} < \infty$ for all $1 \le p < \infty$, where*

$$|||g|||_{G,p,\log} = \left(\sup\left\{\log^p \frac{2}{1-|z|^2}\int_S |g(\zeta)-g(z)|^p \mathcal{P}(z,\zeta)d\sigma(\zeta); z\in B\right\}\right)^{\frac{1}{p}}.$$

iii) *$g \in H^\infty$ and $|||g|||_{G,p,\log} < \infty$ for some $1 \le p < \infty$.*

Proof. — That ii) implies iii) is trivial. Thus, it remains to show that i) implies ii) and that iii) implies i).

First, note that in the proof of Theorem 2.7 we have shown that condition i) implies that $g \in H^\infty$. Hence, by the identity

$$g(\zeta)f(\zeta) - g(z)f(z) = g(\zeta)(f(\zeta) - f(z)) + f(z)(g(\zeta) - g(z))$$

and Theorem 2.4, we obtain

$$\begin{aligned} &\|gf\|_{G,p} \approx \|gf\|_* \leq c\|f\|_* \\ &\iff \left(\sup\left\{|f(z)|^p \int_S |g(\zeta) - g(z)|^p \mathcal{P}(z,\zeta)d\sigma(\zeta); z \in B\right\}\right)^{\frac{1}{p}} \leq c\|f\|_*. \end{aligned}$$

Hence, taking the functions $f(w) = \log \dfrac{2}{1-\bar{z}w}$ we obtain that i) implies ii). That iii) implies i) follows as before using the estimate $|f(z)| \leq c\|f\|_* \log \dfrac{2}{1-|z|^2}$. □

Finally, we will give a characterization of the multipliers of $BMOA$ in terms of an estimate of type Carleson measures. To do so, we will need some lemmas.

The first lemma was proved by J.S. Choa and B.R. Choe [ChoaChoe].

LEMMA 2.10. — *Let $f \in H^2$. Then*

$$\int_B \tilde{\Delta}|f(z)|^2(1-|z|^2)^{-1}dV(z) \approx \int_S |f(\zeta) - f(0)|^2 d\sigma(\zeta),$$

where $\tilde{\Delta}$ is the invariant Laplacian

$$\tilde{\Delta}f(z) = 4(1-|z|^2)\sum_{i,j=1}^{n}(\delta_{i,j} - \bar{z}_i z_j)\frac{\partial^2 f(z)}{\partial z_i \partial \bar{z}_j},$$

$\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$.

For a holomorphic function f, we have

$$\begin{aligned} (2.4)\ \tilde{\Delta}|f|^2 &= 4(1-|z|^2)\left(|\partial f|^2 - |Rf|^2\right) \\ &= 4(1-|z|^2)\sum_{j=1}^{n}\left|\left(\frac{\partial}{\partial z_j} - \frac{\bar{z}_j}{|z|^2}R\right)f\right|^2 \approx (1-|z|^2)\left|\partial|z|^2 \wedge \partial f\right|^2, \end{aligned}$$

where R denotes the radial derivative.

The next lemma gives estimates between some transforms of the measures $(|f(z)|^2 + |\partial|z|^2 \wedge \partial f(z)|^2|dV(z)$ and $(|f(z)|^2 + |\partial f(z)|^2)(1 - |z|^2)dV(z)$.

LEMMA 2.11. — *Let g be a bounded holomorphic function on B. For $s > 0$ we define*

$$I_s(z) = \int_B \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}}(|g(\zeta)|^2 + |\partial|\zeta|^2 \wedge \partial g(\zeta)|^2)dV(\zeta)$$

$$J_s(z) = \int_B \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}}\left(|g(\zeta)|^2 + |\partial g(\zeta)|^2\right)(1-|\zeta|^2)dV(\zeta).$$

Then, for each s there exist constants $c_s, s' > 0$ such that

$$J_s(z) \le cI_s(z) \le c_s J_{s'}(z), \qquad z \in B.$$

Proof. — The first inequality follows from the pointwise estimate $|\partial g(\zeta)|^2(1-|\zeta|^2) = |\partial g(\zeta)|^2 - |\zeta|^2|\partial g(\zeta)|^2 \le |\partial g(\zeta)|^2 - |Rg(\zeta)|^2$.

To prove the second inequality we will use the standard representation formulas for holomorphic functions and some known integration by parts formulas (see for instance [OF]). We have

$$g(\zeta) = c_m \int_B \left(\left(I + \frac{1}{n+1+m}R\right)g(w)\right)\frac{(1-|w|^2)^{m+1}}{(1-\bar{w}\zeta)^{n+1+m}}dV(w)$$

and differentiating we obtain

$$|\partial|\zeta|^2 \wedge \partial g(\zeta)|^2 \le c\left(\int_B (|g(w)| + |\partial g(w)|)\frac{(1-|w|^2)^{m+1}}{|1-\bar{w}\zeta|^{n+\frac{3}{2}+m}}dV(w)\right)^2.$$

Hence, for $\max\left(0, \frac{1-s}{2}\right) < r < \frac{1}{2}$, Hölder inequality and the estimates of Lemma 2.5 give

$$\begin{aligned}
I_s(z) &\le c\int_B \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}}\left(\int_B (|g(w)|+|\partial g(w)|)^2\frac{(1-|w|^2)^{2m+2}}{|1-\bar{w}\zeta|^{n+2+2m-2r}}dV(w)\right)\\
&\qquad \times\left(\int_B \frac{1}{|1-\bar{w}\zeta|^{n+1+2r}}dV(w)\right)dV(\zeta)\\
&\le c_r\int_B (|g(w)| + |\partial g(w)|)^2(1-|w|^2)^{2m+2}\\
&\qquad \times\int_B \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}|1-\bar{w}\zeta|^{n+2+2m-2r}(1-|\zeta|^2)^{2r}}dV(\zeta)dV(w)\\
&\le c_r'\int_B (|g(w)| + |\partial g(w)|)^2(1-|w|^2)^{2m+2}\frac{(1-|z|^2)^{1-2r}}{|1-\bar{w}z|^{n+2+2m-2r}}dV(w)\\
&\qquad + c_r'\int_B (|g(w)| + |\partial g(w)|)^2(1-|w|^2)^{2m+2}\\
&\qquad\qquad \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}(1-|w|^2)^{2m+1}}dV(w)
\end{aligned}$$

$$\leq c_r'' \int_B \frac{(1-|z|^2)^{1-2r}}{|1-\bar{w}z|^{n+1-2r}} (|g(w)| + |\partial g(w)|)^2 (1-|w|^2) dV(w)$$
$$= c_r'' J_{1-2r}(z),$$

which proves the result. □

LEMMA 2.12. — *Let μ be a positive measure. Then the following statements are equivalent:*

i) $\mu(Q_{\eta,t}) \leq ct^n \log^{-2} t$

ii) $\sup \left\{ \log^2 \frac{2}{1-|z|^2} \int_B \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}} d\mu(\zeta);\ z \in B \right\} < \infty$, *for all* $s > 0$.

iii) $\sup \left\{ \log^2 \frac{2}{1-|z|^2} \int_B \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}} d\mu(\zeta);\ z \in B \right\} < \infty$, *for some* $s > 0$.

Proof. — The proof of this lemma is standard. To show that i) implies ii) we take

$$\Omega_0 = \emptyset, \quad \Omega_j = \{\zeta \in B;\ |1-\bar{z}\zeta| \leq 2^{j+1}(1-|z|^2) < 1\}, \quad j = 1, \ldots, N(z).$$

Thus, ii) follows from

$$\log^2 \frac{2}{1-|z|^2} \int_B \frac{(1-|z|^2)^s}{|1-\bar{z}\zeta|^{n+s}} d\mu(\zeta)$$
$$\leq c \log^2 \frac{2}{1-|z|^2} \sum_{j=1}^{N} \frac{(1-|z|^2)^s}{2^{(n+s)j}(1-|z|^2)^{n+s}} \mu(\Omega_j \setminus \Omega_{j-1}) + c$$
$$\leq c \log^2 \frac{2}{1-|z|^2} \sum_{j=1}^{N} \frac{1}{2^{sj}} \log^{-2} \frac{2}{2^j(1-|z|^2)}) + c < \infty.$$

That ii) implies iii) is trivial. Finally, that iii) implies i) follows from $|1-(1-t)\bar{\eta}z| \approx t$ for $z \in Q_{\eta,t}$ and

$$\mu(Q_{\eta,t}) \leq ct^n \log^{-2} \frac{2}{t} \log^2 \frac{2}{1-|(1-t)\bar{\eta}|^2} \int_{Q_{\eta,t}} \frac{(1-|(1-t)\eta|^2)^s}{|1-(1-t)\bar{\eta}\zeta|^{n+s}} d\mu(\zeta).$$

□

THEOREM 2.13. — *The following assertions are equivalent:*

i) *g is a pointwise multiplier of $BMOA$.*

ii) *$g \in H^\infty$ and*

$$\left(\sup \left\{ \log^2 \frac{2}{1-|z|^2} \int_B \mathcal{P}(z,\zeta) |\partial|\zeta|^2 \wedge \partial g(\zeta)|^2 dV(\zeta);\ z \in B \right\} \right)^{\frac{1}{2}} < \infty.$$

iii) $g \in H^\infty$ *and*

$$\int_{Q_{\eta,t}} |\partial|\zeta|^2 \wedge \partial g(\zeta)|^2 dV(\zeta) \le ct^n \log^{-2} t.$$

iv) $g \in H^\infty$ and

$$\left(\sup\left\{\log^2 \frac{2}{1-|z|^2} \int_B \mathcal{P}(z,\zeta)|\partial g(\zeta)|^2(1-|\zeta|^2)dV(\zeta);\ z \in B\right\}\right)^{\frac{1}{2}} < \infty.$$

v) $g \in H^\infty$ *and*

$$\int_{Q_{\eta,t}} |\partial g(\zeta)|^2(1-|\zeta|^2)dV(\zeta) \le ct^n \log^{-2} t.$$

Proof. — First, note that Lemmas 2.11 and 2.12 give the equivalences

$$\text{iii)} \Longleftrightarrow \text{ii)} \Longleftrightarrow \text{iv)} \Longleftrightarrow \text{v)}.$$

To complete the proof we will prove the equivalence between ii) and the assertion ii) of Theorem 2.9 for $p = 2$.

Let ψ be an automorphism of B and let $z = \psi^{-1}(0)$. By Lemma 2.10 we have

$$\int_B \tilde{\Delta}|g(\psi(w)|^2(1-|w|^2)^{-1}dV(w) \approx \int_S |g(\psi(\zeta)) - g(\psi(0))|^2 d\sigma(\zeta).$$

Since $\tilde{\Delta}|f(\psi(w))|^2 = (\tilde{\Delta}|f|^2)(\psi(w))$ (see [R]), the change of variables $\psi(w) = \nu$ gives

$$\int_B \tilde{\Delta}|g(\psi(w)|^2(1-|w|^2)^{-1}dV(w) = \int_B (|\partial g(\nu)|^2 - |Rg(\nu)|^2)\mathcal{P}(z,\nu)dV(\nu).$$

Hence, by (2.4) we have

$$\sup\left\{\log^2 \frac{2}{1-|z|^2} \int_B |\partial|\nu|^2 \wedge \partial g(\nu)|^2 \mathcal{P}(z,\nu)dV(\nu), z \in B\right\}$$
$$\approx \sup\left\{\log^2 \frac{2}{1-|z|^2} \int_S |g(\zeta) - g(z)|^2 \mathcal{P}(z,\zeta)d\sigma(\zeta), z \in B\right\}$$

and thus the equivalence is proved. □

The following result gives a relation between the pointwise multipliers of $BMOA$ and the ones of the Bloch space.

PROPOSITION 2.14. — *Let g be a multiplier of the Bloch space on the unit ball B_{n-1} of $\mathbb{C}^{n-1}$. Then the extension $\tilde{g}$ defined by $\tilde{g}(z) = g(z')$, where $z' = (z_1, \ldots, z_{n-1}, 0)$, is a multiplier of $BMOA(B_n)$.*

Proof. — We recall (see [Z]) that a holomorphic function g is a multiplier of the Bloch space if and only if it satisfies

i) $g \in H^\infty$

ii) $\sup\left\{(1-|z|^2)\log\frac{2}{1-|z|^2}|\partial g(z)|;\ z\in B\right\}<\infty.$

It is clear that $\tilde{g}$ is bounded. Thus, by part ii) of Theorem 2.4, we need to show that

$$|\partial\tilde{g}(z)f(z)|^2(1-|z|^2)\in W^1.$$

By ii) it is enough to show that

$$\mu=\frac{1}{(1-|z'|^2)^2\log^2\frac{2}{1-|z'|}}(1-|z|^2)\log^2\frac{2}{1-|z|^2}\in W^1$$

i.e. $\mu\left(Q_{\eta,t}\right)\le ct^n$, which follows from elementary computations. □

As a first application of this proposition we give an example of a pointwise multiplier of $BMOA$ which is not smooth on $\bar{B}$.

Example 2.15. — The function $f(z_1,z_2,z_3)=\frac{\log\frac{2}{1-z_1^2}}{\log\frac{2}{1-z_1^2-z_2^2}}$ is a multiplier of $BMOA(B_3)$ which does not extend continuously to $\bar{B}_3$.

Proof. — By the above proposition, it is enough to show that the function $h(z_1,z_2)=f(z_1,z_2,0)$ is a multiplier of the Bloch space on B_2. This result follows trivially from the characterization of $\mathcal{M}(\mathcal{B})$ given in the proof of the above proposition. □

To finish this section we give an example which shows that there exist continuous functions on $\bar{B}$ such that are not pointwise multipliers of $BMOA$.

Example 2.16. — Let D be the unit disc of $\mathbb{C}$. Then, for $0<s<1$ the function

$$g(z)=\exp\left(\frac{z+1}{z-1}\right)\log^{-s}\frac{2}{1-z}$$

is continuous on $\bar{D}$ and it is not a multiplier of $BMOA$.

Proof. — Let $\eta=(1,0), t>0$ and $\Omega=\Omega_{\eta,t}=\{z\in Q_{\eta,t};\ 1-|z|^2\le|1-z|^2\}$.

Since $\Re e \dfrac{z+1}{z-1} = \dfrac{|z|^2-1}{|1-z|^2}$ we obtain that

$$|g'(z)|^2(1-|z|^2) \approx \frac{1-|z|^2}{|1-z|^4} \log^{-2s} \frac{1}{|1-z|}, \qquad z \in \Omega.$$

Taking polar coordinates we obtain

$$\int_\Omega |g'(z)|^2(1-|z|^2) \approx t \log^{-2s} \frac{2}{t}$$

and thus, by part v) of Theorem 2.13, g is not a multiplier of $BMOA$. □

3. Necessary conditions in Theorem B.

It is clear that if M_g maps $BMOA \times \ldots \times BMOA$ into $BMOA$ then the functions g_j are pointwise multipliers of $BMOA$.

Let us to prove that $\sup\{|g(z)|;\, z \in B\} \geq \delta > 0$. By the open map Theorem, for every function f of $BMOA$ there exist functions f_i of $BMOA$ such that

i) $f = \sum\limits_{j=1}^{m} f_i g_i$

ii) $||f_i||_* \leq c||f||_*$.

Using $|f_i(\zeta)| \leq c||f||_* \log \dfrac{2}{1-|\zeta|^2}$ for $\zeta \in B$, we obtain

$$|f(\zeta)| \leq \sum_{j=1}^{m} |f_j(\zeta)||g_j(\zeta)| \leq c||f||_* \log \frac{2}{1-|\zeta|^2} \sum_{j=1}^{m} |g_j(\zeta)|.$$

Taking the functions $f_z(\zeta) = \log \dfrac{2}{1-\bar{z}\zeta}$ we have $\|f_z\|_* \leq c$ and

$$\log \frac{2}{1-|z|^2} \leq c \log \frac{2}{1-|z|^2} \sum_{j=1}^{m} |g_j(z)|.$$

This proves the result. □

4. Estimates for the $\bar{\partial}$–equation.

We will begin recalling some results of N. Varopoulos, E. Amar, A. Bonami, A. Cumenge, M. Andersson and H. Carlsson, which permit to obtain solutions of the $\bar\partial$–equation which are functions whose boundary values are in $BMO(S)$ or $(0,q)$ forms whose coefficients have estimates of type Carleson measures.

THEOREM 4.1 [V1]. — *If ω is a $\bar\partial$–closed (0,1) form such that*

$$\text{(V)} \qquad |\omega| + (1-|z|^2)^{\frac{-1}{2}}|\bar\partial|z|^2 \wedge \omega| \in W^1$$

then there exists a function u such that $\bar\partial u = \omega$ and $u|_S \in BMO$.

THEOREM 4.2 [V2], [AnCa]. — *If ω is a $\bar\partial$–closed $(0,1)$ form such that*

$$\text{(A)} \quad (1-|z|^2)\left(|\omega|^2 + |\partial\omega|\right) + (1-|z|^2)^{\frac{1}{2}}\left(|\bar\partial|z|^2 \wedge \partial\omega| + |\partial|z|^2 \wedge \partial\omega|\right) + |\partial|z|^2 \wedge \bar\partial|z|^2 \wedge \partial\omega| \in W^1,$$

then there exists a function u such that $\bar\partial u = \omega$ and $u|_S \in BMO$.

THEOREM 4.3 [AB], [Cu]. — *Let ω be a $\bar\partial$–closed $(0,q)$ form such that*

$$\text{(C)} \qquad (1-|z|^2)^\sigma\left(|\omega| + (1-|z|^2)^{\frac{-1}{2}}|\bar\partial|z|^2 \wedge \omega|\right) \in W^1$$

for some $\sigma > 0$. Then there exists a $(0,q-1)$ form u such that $\bar\partial u = \omega$ and

i) $(1-|z|^2)^{\sigma-\frac{1}{2}}\left(|\omega| + (1-|z|^2)^{\frac{-1}{2}}|\bar\partial|z|^2 \wedge \omega|\right) \in W^1$ *if $q > 1$,*

ii) $(1-|z|^2)^{\sigma-1}|u| \in W^1$ *if $q = 1$.*

The proof of these results can be obtained using explicit integral operators. In particular, in the last result we can use Berndtsson-Andersson's kernels, which we will recall later and whose estimates will be used to prove Theorem B.

We will need kernels of the following type:

DEFINITION 4.4. — *For $r, s \geq 0, t > -1, u \geq 0$ and $0 < v + \frac{r}{2}$, we define*

$$L_{(s,t,r,u,v)}(\zeta, z) = \frac{(1-|\zeta|^2)^s(1-|z|^2)^t|\zeta - z|^r}{|1-\bar\zeta z|^u D(\zeta,z)^{n-v}},$$

where $D(\zeta, z) = |1-\bar\zeta z|^2 - (1-|\zeta|^2)(1-|z|^2)$.

Let $w(L) = n + 1 + s + t + \frac{r}{2} - u - 2(n-v)$.

The following estimates are well-known:

$$\int_B L_{(s,t,r,u,v)}(\zeta,z)dV(\zeta) \leq \begin{cases} c(1-|z|^2)^{w(L)} & w(L)<0 \\ c\log\frac{1}{(1-|z|^2)} & w(L)=0 \\ c & w(L)>0. \end{cases}$$

The next result is contained in the proof of Lemma 2.3 of [Cu].

PROPOSITION 4.5. — *Let $L = L_{(s,t,r,u,v)}$ be a kernel with $w(L) \geq 0$ and $s > 0$, and let μ be a measure in W^1. Then*

$$L(\mu) = \int_B L_{(s,t,r,u,v)}d\mu(\zeta) \in W^1.$$

To apply this result in a more general way, we introduce the following definition:

DEFINITION 4.6. — *A kernel $K(\zeta, z)$ is of type p if it satisfies*

$$|K(\zeta,z)| \leq \sum_{j=1}^{l} c_j L_{(s_j,t_j,r_j,u_j,v_j)}(\zeta,z)$$

with $w(L_j) \geq p$, $j = 1, \dots, l$.

As a consequence of Proposition 4.5 we have

COROLLARY 4.7. — *Let $K = (1-|\zeta|^2)^s K'$ with $s > 0$ and $w(K') \geq -s$. Let μ be a measure in W^1. Then*

$$\left|\int_B K d\mu(\zeta)\right| \in W^1.$$

Now, we recall the weighted kernels introduced by B. Berndtsson and M. Andersson [BAn].

Let

$$s(\zeta,z) = (1-\bar{z}\zeta)\bar{\zeta} - (1-|\zeta|^2)\bar{z}, \qquad Q(\zeta,z) = \frac{\bar{\zeta}}{1-|z|^2}$$

$$\nu(\zeta,z) = \sum_{j=1}^{n} \bar{\zeta}_j d(\zeta_j - z_j), \qquad \theta(\zeta,z) = \sum_{j=1}^{n} \bar{z}_j d(\zeta_j - z_j)$$

$$\tilde{s}(\zeta,z) = (1-\bar{z}\zeta)\nu(\zeta,z) - (1-|\zeta|^2)\theta(\zeta,z), \quad \tilde{Q}(\zeta,z) = \frac{1}{(1-|\zeta|^2)}\nu(\zeta,z).$$

For $s \geq 0$, consider the kernels

$$K^s = \sum_{k=0}^{n-1} c_{k,s} \frac{(1-|\zeta|^2)^{s+k}}{(1-\bar\zeta z)^{s+k}} \frac{\tilde s \wedge (d\tilde s)^{n-1-k} \wedge (d\tilde Q)^k}{D^{n-k}},$$

$$P^s = c_{n,s} \frac{(1-|\zeta|^2)^{s+n}}{(1-\bar\zeta z)^{s+n}} (d\tilde Q)^n$$

where $D = D(\zeta, z) = |1-\bar\zeta z|^2 - (1-|\zeta|^2)(1-|z|^2)$, and $c_{k,s} = \frac{1}{(2\pi i)^n}\binom{s+n}{k}$.

These kernels satisfy the fundamental formulas:

THEOREM 4.8 (Koppelman formulas). — *Let $K_{p,q}$ be the component of K of bidegree (p,q) in z and $(n-p, n-q-1)$ in ζ, and let $P^s_{p,q}$ be the component of P of bidegree (p,q) in z and $(n-p, n-q)$ in ζ. Then if $\omega(\zeta)$ is a (p,q) form with coefficients in $C^1(\bar B)$ one has*

$$\omega = \int_S \omega \wedge K^s_{p,q} + (-1)^{p+q+1} \int_B \bar\partial\omega \wedge K^s_{p,q} + (-1)^{p+q}\bar\partial_z \int_B \omega \wedge K^s_{p,q-1}, \quad q \geq 1$$

$$\omega = \int_S \omega \wedge K^s_{p,0} + (-1)^{p+1} \int_B \bar\partial\omega \wedge K^s_{p,0} + (-1)^p \int_B \omega \wedge P^s_{p,0}, \quad q = 0.$$

We point out that if $s > 0$ the kernel K^s vanishes on the boundary and thus the integrals on the boundary vanish, too. From now on we will assume that this condition is satisfied.

A more explicit computation of the kernels K^s and P^s gives:

LEMMA 4.9.

$$\begin{aligned} K^s = \sum_{k=0}^{n-1} c_{k,s} \frac{(1-|\zeta|^2)^{s+k}}{(1-\bar\zeta z)^{s+k} D^{n-k}} \tilde s \wedge \Big(\big((1-\bar z\zeta)d\nu - (1-|\zeta|^2)d\theta\big)^{n-1-k} \\ + (n-1-k)\big((1-\bar z\zeta)d\nu - (1-|\zeta|^2)d\theta\big)^{n-2-k} \\ \wedge \big(-d(\bar z\zeta)\wedge\nu + d|\zeta|^2 \wedge \theta\big)\Big) \wedge \big((1-|\zeta|^2)^{-k}(d\nu)^k \\ + k(1-|\zeta|^2)^{-k-1}(d\nu)^{k-1} \wedge d|\zeta|^2 \wedge \nu\big), \end{aligned}$$

$$\begin{aligned} P^s = c_{n,s} \frac{(1-|\zeta|^2)^{s+n}}{(1-\bar\zeta z)^{s+n}} \big((1-|\zeta|^2)^{-n}(d\nu)^n \\ + n(1-|\zeta|^2)^{-n-1}(d\nu)^{n-1} \wedge d|\zeta|^2 \wedge \nu\big). \end{aligned}$$

The following result, which is proved in [BrBu], gives a formula to obtain derivatives of the function $\int_B \omega \wedge K^s_{0,0}$.

LEMMA 4.10 [BrBu]. — *Let ω be a $(0,1)$ form with coefficients in $\mathcal{C}^2(\bar{B})$. Then*

$$\partial_z \int_B \omega \wedge K^s_{0,0} = \sum_{j=1}^n \int_B \frac{\partial}{\partial \zeta_i} \omega \wedge K^s_{0,0} \wedge dz_j + \int_B \bar{\partial}\omega \wedge K^s_{0,1,1} - \int_B \omega \wedge P^s_{0,1,1}$$

where $K^s_{0,1,1}$ denotes the component of K^s of bidegree $(1,0)$ in z and $(n, n-2)$ in ζ, and $P^s_{0,1,1}$ denotes the component of P^s of bidegree $(1,0)$ in z and $(n, n-1)$ in ζ.

Koppelman formulas, the above lemma and the fact that P^s is holomorphic in z give:

LEMMA 4.11. — *Let ω a $(0,1)$ form with coefficients in $\mathcal{C}^2(\bar{B})$. Then*

$$\bar{\partial}_z \partial_z \int_B \omega \wedge K^s_{0,0} = -\partial\omega + \sum_{j=1}^n \int_B \frac{\partial}{\partial \zeta_i} \bar{\partial}\omega \wedge K^s_{0,1} \wedge dz_j + \bar{\partial}_z \int_B \bar{\partial}\omega \wedge K^s_{0,1,1}.$$

The next result gives some estimates of the kernels which appears in the above lemma.

LEMMA 4.12. — *With the above notations we have:*

i) $K^s_{0,0} = K^{s,1}_{0,0} + K^{s,2}_{0,0} \wedge \bar{\partial}|\zeta|^2$ *with* $w(K^{s,1}_{0,0}) = 1$, $w(K^{s,2}_{0,0}) = \frac{1}{2}$.

ii) $K^s_{0,q} = K^{s,1}_{0,q} + K^{s,2}_{0,q} \wedge \bar{\partial}|\zeta|^2 + K^{s,3}_{0,q} \wedge \bar{\partial}|z|^2$ *with* $w(K^{s,1}_{0,q}) = 1$, $w(K^{s,2}_{0,q}) = w(K^{s,3}_{0,q}) = \frac{1}{2}$.

iii) $\bar{\partial}_z K^s_{0,1,1} = \Omega^{s,1}_{0,1,1} + \Omega^{s,2}_{0,1,1} \wedge \partial|z|^2$ *with* $w(\Omega^{s,1}_{0,1,1}) = 0$, $w(\Omega^{s,2}_{0,1,1}) = \frac{-1}{2}$.

iv) $\bar{\partial}|z|^2 \wedge \bar{\partial}_z K^s_{0,1,1} = \Lambda^{s,1}_{0,1,1} + \Lambda^{s,2}_{0,1,1} \wedge \partial|z|^2$ *with* $w(\Lambda^{s,1}_{0,1,1}) = \frac{1}{2}$, $w(\Lambda^{s,2}_{0,1,1}) = 0$.

Proof. — Parts i) and ii) are known (see for instance [Cu]).

To prove part iii) we obtain a decomposition

$$K^s_{0,1,1} = K^{s,1}_{0,1,1} + K^{s,2}_{0,1,1} \wedge \partial|z|^2, \text{ with } w(K^{s,1}_{0,1,1}) = 1, w(K^{s,2}_{0,1,1}) = \frac{1}{2}$$

such that $w\left(\bar\partial_z K^{s,1}_{0,1,1}\right) = 0$ and $w\left(\bar\partial_z K^{s,2}_{0,1,1}\right) = -\dfrac{1}{2}$.

We give the proof of the decomposition for the first term of the kernel K^s

$$\frac{(1-|\zeta|^2)^s}{(1-\bar\zeta z)^s D^n}\tilde s \wedge (d\tilde s)^{n-1}$$

which is more delicate, because it is the term which has components whose derivatives are not integrable on B. The decomposition of the other terms follows in the same way without the integrability problem.

First, we compute the component of $\tilde s \wedge (d\tilde s)^{n-1}$ of bidegree $(1,0)$ in z and $(n, n-2)$ in ζ.

Note that

$$\nu = \sum_{j=1}^n \bar\zeta_j d(\zeta_j - z_j) = \partial|\zeta|^2 - \partial_z(\bar\zeta z)$$

$$\theta = \sum_{j=1}^n \bar z_j d(\zeta_j - z_j) = \partial_\zeta(\bar z\zeta) - \partial|z|^2.$$

By Lemma 4.9 and an easy computation of bidegrees, we have

$$\begin{aligned}\tilde s \wedge (d\tilde s)^{n-1}\big|_{0,1,1} &= \big((1-\bar z\zeta)\nu - (1-|\zeta|^2)\theta\big) \wedge (n-1)\big(1-\bar z\zeta\big)^{n-2}\big(\bar\partial_\zeta\nu\big)^{n-2} \\ &\qquad \wedge\big(-\partial_\zeta(\bar z\zeta)\wedge\nu + \partial|\zeta|^2\wedge\theta\big)\big|_{0,1,1} \\ &= (n-1)(1-\bar\zeta z)^{n-2}(\bar\partial\partial|\zeta|^2)^{n-2} \\ &\qquad \wedge\big(-(1-\bar z\zeta)\partial_z(\bar\zeta z)\wedge\partial|\zeta|^2\wedge\partial_\zeta(\bar z\zeta) \\ &\qquad - (1-|\zeta|^2)\partial|z|^2\wedge\partial_\zeta(\bar z\zeta)\wedge\partial|\zeta|^2\big) \\ &= (n-1)(1-\bar\zeta z)^{n-2}(\bar\partial_\zeta\partial|\zeta|^2)^{n-2}\wedge\big((1-\bar z\zeta)\partial_z(\bar\zeta z) \\ &\qquad - (1-|\zeta|^2)\partial|z|^2\big)\wedge\partial_\zeta(\bar z\zeta)\wedge\partial_\zeta|\zeta|^2.\end{aligned}$$

Note that $|\partial_\zeta(\bar z\zeta)\wedge\partial_\zeta|\zeta|^2| \le c|\zeta - z|$ and that

$$(1-\bar z\zeta)\partial_z(\bar\zeta z) - (1-|\zeta|^2)\partial_z|z|^2) = (1-\bar z\zeta)\partial_z((\bar\zeta-\bar z)z) + (\zeta(\bar\zeta-\bar z))\partial_z|z|^2).$$

Thus, we have

$$\frac{(1-|\zeta|^2)^s}{(1-\bar\zeta z)^s D^n}\tilde s\wedge(d\tilde s)^{n-1}\Bigg|_{0,1,1} = K^{s,1,0}_{0,1,1} + K^{s,2,0}_{0,1,1}\wedge\partial|z|^2$$

with

$$|K^{s,1,0}_{0,1,1}| \le c\,\frac{(1-|\zeta|^2)^s|\zeta - z|^2}{|1-\bar\zeta z|^{s-n+1}|D|^n}, \qquad |K^{s,2,0}_{0,1,1}| \le c\,\frac{(1-|\zeta|^2)^s|(\bar\zeta-\bar z)\zeta||\zeta - z|}{|1-\bar\zeta z|^{s-n+2}|D|^n}$$

and

$$w\left(K^{s,1,0}_{0,1,1}\right) = 1, \qquad w\left(K^{s,2,0}_{0,1,1}\right) = \frac{1}{2}.$$

Finally, differentiating these kernels, we obtain the decompositions of iii). The same argument gives iv). □

5. Proof of sufficient conditions in Theorem B.

In the first part of this section we recall the well-known technique of Koszul's complex [H], which permits to reduce the proof of Theorem B to solve a set of $\bar{\partial}-$ equations with adequate estimates.

We will follow the notations that K.C. Lin [Lin] used to solve the analogous problem for H^p on the polydisc.

The Koszul's complex.

To simplify the notations we restrict the proof to the case $n = 3$, although the arguments hold with the obvious changes for every n.

DEFINITION 5.1. — *Let $g_1, \dots, g_m$ be holomorphic functions on B satisfying the condition $|g(z)| \geq \delta > 0$ for all $z \in B$. For $1 \leq i, j, k, l \leq m$ we define*

$$G_l = \frac{\bar{g}_l}{|g|^2}$$

$$G_{k,l} = \begin{vmatrix} G_k & G_l \\ \bar{\partial}G_k & \bar{\partial}G_l \end{vmatrix} = G_k\bar{\partial}G_l - G_l\bar{\partial}G_k$$

$$G_{j,k,l} = \begin{vmatrix} G_j & G_k & G_l \\ \bar{\partial}G_j & \bar{\partial}G_k & \bar{\partial}G_l \\ \bar{\partial}G_j & \bar{\partial}G_k & \bar{\partial}G_l \end{vmatrix} = G_j\bar{\partial}G_k \wedge \bar{\partial}G_l - G_j\bar{\partial}G_l \wedge \bar{\partial}G_k$$

$$+G_k\bar{\partial}G_l \wedge \bar{\partial}G_j - G_k\bar{\partial}G_j \wedge \bar{\partial}G_l + G_l\bar{\partial}G_j \wedge \bar{\partial}G_k - G_l\bar{\partial}G_k \wedge \bar{\partial}G_j$$

$$G_{i,j,k,l} = \begin{vmatrix} G_i & G_j & G_k & G_l \\ \bar{\partial}G_i & \bar{\partial}G_j & \bar{\partial}G_k & \bar{\partial}G_l \\ \bar{\partial}G_i & \bar{\partial}G_j & \bar{\partial}G_k & \bar{\partial}G_l \\ \bar{\partial}G_i & \bar{\partial}G_j & \bar{\partial}G_k & \bar{\partial}G_l \end{vmatrix} = G_i\bar{\partial}G_j \wedge \bar{\partial}G_k \wedge \bar{\partial}G_l - \dots$$

LEMMA 5.2 [Lin]. — *The forms $G_{k,l}, G_{j,k,l}, G_{i,j,k,l}$ are alternating and satisfy:*

$$\bar{\partial}G_{j,k,l} = \sum_{i=1}^{m} g_i G_{i,j,k,l}, \qquad \bar{\partial}G_{k,l} = \sum_{j=1}^{m} g_j G_{j,k,l}, \qquad \bar{\partial}G_l = \sum_{k=1}^{m} g_k G_{k,l}.$$

Using these properties we can obtain a solution of

$$f(z) = \sum_{j=1}^{m} g_j(z) f_j(z), \qquad f_j(z) \in H(B)$$

for $f \in H(B)$, in the following way:

Since $G_{i,j,k,l}$ are alternating $\bar\partial$−closed $(0,3)$ forms (we recall that we assume $n = 3$), there exist alternating $(0,2)$ forms $u_{i,j,k,l}$ which satisfy $\bar\partial u_{i,j,k,l} = G_{i,j,k,l} f$.

Hence, by Lemma 5.2, we have that the forms $\varphi_{j,k,l} = G_{j,k,l} f - \sum_{i=1}^{m} g_i u_{i,j,k,l}$ are alternating $\bar\partial$−closed $(0,2)$ forms. Thus, there are alternating $(0,1)$ forms $u_{j,k,l}$ such that $\bar\partial u_{j,k,l} = \varphi_{j,k,l}$.

The same argument gives that $\varphi_{k,l} = G_{k,l} f - \sum_{j=1}^{m} g_j u_{j,k,l}$ are alternating $\bar\partial$−closed $(0,1)$ forms and thus there are alternating functions $u_{k,l}$ such that $\bar\partial u_{k,l} = \varphi_{k,l}$.

Finally, defining $f_l = G_l f - \sum_{k=1}^{m} g_k u_{k,l}$ we obtain the solution.

The next lemmas are devoted to obtain the adequate estimates of the forms which appears in the above scheme, to conclude finally that the functions f_l are in $BMOA$.

LEMMA 5.3. — *If* $g_1, \dots, g_m \in \mathcal{M}(BMOA)$, *then* $G_l \in \mathcal{M}(BMO(S))$.

Proof. — We have to prove that G_l are bounded functions and that satisfy condition (2.2) of Theorem 2.7.

Corollary 2.8 gives that $\bar g_j, j = 1, \dots, m$ and $|g|^2$ are multipliers of $BMO(S)$.

Thus, to obtain the result, we need to show that $\frac{1}{|g|^2}$ is a multiplier of $BMO(S)$. But clearly $\frac{1}{|g|^2}$ is bounded and satisfy

$$\begin{aligned}
&\frac{1}{|I|} \log \frac{1}{|I|} \int_I \left| \frac{1}{|g|^2} - \left(\frac{1}{|g|^2}\right)_I \right| d\sigma \\
&\le \frac{1}{|I|} \log \frac{1}{|I|} \int_I \left| \frac{1}{|g|^2} - \frac{1}{(|g|^2)_I} \right| d\sigma + \frac{1}{|I|} \log \frac{1}{|I|} \left| \int_I \left(\frac{1}{(|g|^2)_I} - \frac{1}{|g|^2} \right) d\sigma \right| \\
&\le \frac{c}{|I|} \log \frac{1}{|I|} \int_I \left| |g|^2 - (|g|^2)_I \right| d\sigma < \infty
\end{aligned}$$

which proves the result. □

LEMMA 5.4. — *If $g_1, \dots, g_m \in \mathcal{M}(BMOA)$, and $f \in BMOA$, then*

i) $(1-|z|^2)|G_{i,j,k,l}f| + (1-|z|^2)^{\frac{1}{2}}|\bar\partial|z|^2 \wedge G_{i,j,k,l}f| \in W^1$

ii) $(1-|z|^2)^{\frac{1}{2}}|G_{j,k,l}f| + |\bar\partial|z|^2 \wedge G_{j,k,l}f| \in W^1$

iii) $(1-|z|^2)^{\frac{1}{2}}|\bar\partial G_{k,l}f| + |\bar\partial|z|^2 \wedge \bar\partial G_{k,l}f| \in W^1$

iv) $(1-|z|^2)\big(|G_{k,l}|^2 + |\partial(G_{k,l}f)|\big) + (1-|z|^2)^{\frac{1}{2}}\big(|\partial|z|^2 \wedge \partial G_{k,l}f| + |\bar\partial|z|^2\partial(G_{k,l}f)|\big) + |\bar\partial|z|^2 \wedge \partial|z|^2 \wedge \bar\partial G_{k,l}f| \in W^1$

v) $(1-|z|^2)^{\frac{3}{2}}|\bar\partial\partial(G_{k,l}f)| + (1-|z|^2)\big(|\partial|z|^2 \wedge \bar\partial\partial G_{k,l}f| + |\bar\partial|z|^2 \wedge \bar\partial\partial(G_{k,l}f)|\big) + (1-|z|^2)^{\frac{1}{2}}|\bar\partial|z|^2 \wedge \partial|z|^2 \wedge \bar\partial\partial G_{k,l}f| \in W^1$.

Proof. — Let R be the radial derivative and let T_j be the complex vector fields defined by

$$T_j = \frac{\partial}{\partial z_j} - \frac{\bar z_j}{|z|^2}R, \quad j = 1, \dots, n.$$

Thus, we have

$$\partial h(z) = \sum_{j=1}^{n} T_j h(z) dz_j + \frac{1}{|z|^2} Rh(z)\partial|z|^2. \tag{5.1}$$

We define $|Tg| = \sum_{i,j} |T_j g_i|$ and $|Tf| = \sum_j |T_j f|$. Note that $(1-|z|^2)|Tg|^2 \in L^\infty(B)$.

Let us prove i). By decomposition (5.1) we have

$$\begin{aligned}
&(1-|z|^2)|G_{i,j,k,l}f| + (1-|z|^2)^{\frac{1}{2}}|\bar\partial|z|^2 \wedge G_{i,j,k,l}f| \\
&\quad\le c(1-|z|^2)|Tg|^2|\partial g||f| + c(1-|z|^2)^{\frac{1}{2}}|Tg|^3|f| \\
&\quad\le c(1-|z|^2)|\partial g|^2|f|^2 + c|Tg|^2 + c|Tg|^2|f| \\
&\quad\le c(1-|z|^2)|\partial(gf)|^2 + c(1-|z|^2)|\partial f|^2 + c|Tg|^2 + c|T(gf)|^2 + c|Tf|^2.
\end{aligned}$$

Hence, by Theorem 2.4 and the fact that the functions $g_j \in \mathcal{M}(BMOA)$, we obtain i). The proof of the other parts follows in the same way. □

Proof of Theorem B. — We will follow the same notations that in the section of the Koszul's complex. Moreover, we take the kernels K^s with $s > \frac{3}{2}$.

By part i) of Lemma 5.4 the forms $G_{i,j,k,l}$ satisfy condition (C) of Theorem 4.3 with $\sigma = 1$. Therefore, the forms $u_{i,j,k,l}$, which solves the equation $\bar{\partial} u_{i,j,k,l} = G_{i,j,k,l} f$, can be taken satisfying condition (C) with $\sigma = \frac{1}{2}$. Thus, by part ii) of Lemma 5.4, the $(0,2)$ forms $\varphi_{j,k,l} = G_{j,k,l} f - \sum_{i=1}^{m} g_i u_{i,j,k,l}$ satisfy condition (C) with $\sigma = \frac{1}{2}$. Therefore, the $(0,1)$ forms $u_{j,k,l}$, which solve the equations $\bar{\partial} u_{j,k,l} = \varphi_{j,k,l}$, can be taken satisfying condition (C) with $\sigma = 0$ i.e. condition (V) of Theorem 4.1.

The next step is to prove that the functions $u_{k,l}$, which solve the equations $\bar{\partial} u_{k,l} = \varphi_{k,l} = G_{k,l} f - \sum_{j=1}^{m} g_j u_{j,k,l}$ have the boundary values in BMO. Observe that $\varphi_{k,l}$ does not satisfy the conditions of Theorem 4.1 or 4.2. However, $G_{k,l} f$ satisfies condition (A) of Theorem 4.2 (see part iv) of Lemma 5.4) and $\sum_{j=1}^{m} g_j u_{j,k,l}$ satisfies condition (V) of Theorem 4.1, but these two $(0,1)$-forms are not $\bar{\partial}$−closed. Thus, we need to find a decomposition $\varphi_{k,l} = \varphi_{k,l,1} + \varphi_{k,l,2}$ such that $\varphi_{k,l,1}$ is a $\bar{\partial}$−closed $(0,1)$ form which satisfies condition (V) and $\varphi_{k,l,2}$ is a $\bar{\partial}$−closed $(0,1)$ form which satisfies condition (A).

Assuming that we have this result, by Theorems 4.1 and 4.2, we can take the functions $u_{k,l}$ such that the boundary values are in $BMO(S)$. Finally, by Lemma 5.3, we conclude that the functions $f_l = G_l f - \sum_{k=1}^{m} g_k u_{k,l}$ are in $BMOA$ and hence Theorem B is proved.

To finish the proof we obtain the decomposition of $\varphi_{k,l}$.

Define

$$\varphi_{k,l,1} = \int_B \bar{\partial} G_{k,l} f \wedge K^s_{0,1} - \sum_{j=1}^{m} g_j u_{j,k,l}$$

$$\varphi_{k,l,2} = G_{k,l} f - \int_B \bar{\partial} G_{k,l} f \wedge K^s_{0,1} = -\bar{\partial}_z \int_B G_{k,l} f \wedge K_{0,0}.$$

The last inequality follows from Koppelman formulas. It is clear that the forms $\varphi_{k,l,1}, \varphi_{k,l,2}$ are $\bar{\partial}$−closed and that $\varphi_{k,l} = \varphi_{k,l,1} + \varphi_{k,l,2}$.

Let us show that $\varphi_{k,l,1}$ satisfies condition (V). By part ii) of Lemma 4.12, we have $w((1-|\zeta|^2)^{\frac{-1}{2}} K^s_{0,1}) = 0$ and by part iii) of Lemma 5.4, we have that $(1-|\zeta|^2)^{\frac{1}{2}} |\bar{\partial} G_{k,l} f| \in W^1$. Thus, the result follows from Corollary 4.7.

To prove that $\varphi_{k,l,2}$ satisfies condition (A) we need to show that:

i) $(1-|z|^2)|\varphi_{k,l,2}|^2 \in W^1$

ii) $(1-|z|^2)|\partial\varphi_{k,l,2}| \in W^1$

iii) $(1-|z|^2)^{\frac{1}{2}}\left(|\partial|z|^2 \wedge \partial\varphi_{k,l,2}| + |\bar\partial|z|^2 \wedge \partial\varphi_{k,l,2}|\right) \in W^1$

iv) $|\bar\partial|z|^2 \wedge \partial|z|^2 \wedge \partial\varphi_{k,l,2}| \in W^1$.

To prove i) note that $(1-|z|^2)|G_{k,l}f|^2 \leq c(1-|z|^2)|\partial g|^2|f|^2 \in W^1$. Thus, to obtain i), we need to show that

$$\mu = (1-|z|^2)\left|\int_B \bar\partial G_{k,l}f \wedge K_{0,1}\right|^2 \in W^1.$$

Following the notations used in the above lemma, we have $|\bar\partial G_{k,l}| \leq c|Tg||\partial g|$ and thus Hölder inequality gives

$$\mu \leq c(1-|z|^2)\left(\int_B |Tg|^2(1-|\zeta|^2)^{-\frac{1}{2}}|K^s_{0,1}|dV\right)$$
$$\left(\int_B |\partial g|^2|f|^2(1-|\zeta|^2)^{\frac{1}{2}}|K^s_{0,1}|dV\right).$$

But

$$\begin{aligned} v(z) &= (1-|z|^2)\int_B |Tg|^2(1-|\zeta|^2)^{-\frac{1}{2}}|K^s_{0,1}|dV \\ &\leq c(1-|z|^2)\int_B (1-|\zeta|^2)^{-\frac{3}{2}}|K^s_{0,1}|dV \end{aligned}$$

and thus $v(z)$ is a bounded function on B. Since $w((1-|\zeta|^2)^{-\frac{1}{2}}|K^s_{0,1}|) = 0$ and $(1-|\zeta|^2)|\partial g|^2|f|^2 \in W^1$, Corollary 4.7 gives

$$\int_B |\partial g|^2|f|^2(1-|\zeta|^2)^{\frac{1}{2}}|K^s_{0,1}|dV \in W^1.$$

Hence, $\mu \in W^1$.

To prove part ii), iii) and iv) we will use Lemma 4.11. By this lemma we have

$$\begin{aligned} \partial_z\varphi_{k,l,2} &= \bar\partial_z\partial_z\int_B G_{k,l}f \wedge K^s_{0,0} \\ &= -\partial(G_{k,l}f) + \sum_{j=1}^n \int_B \frac{\partial}{\partial\zeta_j}\bar\partial(G_{k,l}f) \wedge K^s_{0,1} \wedge dz_j \\ &\qquad + \bar\partial_z\int_B \bar\partial(G_{k,l}f) \wedge K^s_{0,1,1}. \end{aligned}$$

Hence, we need to show that conditions ii), iii) and iv) are true for the three terms which appears in the right part of the last formula. But, by part iv) of Lemma 5.4 it is clear that $\partial(G_{k,l}f)$ satisfies the conditions. Thus we need to prove that the two last terms satisfy the conditions. Let us prove ii).

That

$$(1-|z|^2)\left|\sum_{j=1}^n \int_B \frac{\partial}{\partial \zeta_j}\bar{\partial}(G_{k,l}f)\wedge K^s_{0,1}\wedge dz_j\right| \in W^1$$

follows from Corollary 4.7, $w\left((1-|\zeta|^2)^{\frac{-3}{2}}(1-|z|^2)|K^s_{0,1}|\right)=0$ and $(1-|\zeta|^2)^{\frac{3}{2}}\left|\bar{\partial}\partial(G_{k,l}f)\right| \in W^1$ (see Lemmas 4.12 and 5.4).

Finally, by Lemmas 4.12 and 5.4, we have

$$w\left((1-|\zeta|^2)^{\frac{-1}{2}}(1-|z|^2)|\bar{\partial}_z K^s_{0,1,1}|\right)=0 \quad\text{and}\quad (1-|\zeta|^2)^{\frac{1}{2}}\left|\bar{\partial}G_{k,l}f\right| \in W^1.$$

Hence, by Corollary 4.7, we obtain

$$(1-|z|^2)\left|\bar{\partial}_z \int_B \bar{\partial}(G_{k,l}f)\wedge K^s_{0,1,1}\right| \in W^1,$$

which ends the proof of ii).

To prove that

$$h(z)=(1-|z|^2)^{\frac{1}{2}}\left|\partial|z|^2\wedge\left(\sum_{j=1}^n \int_B \frac{\partial}{\partial\zeta_j}\bar{\partial}(G_{k,l}f)\wedge K^s_{0,1}\wedge dz_j\right)\right| \in W^1$$

note that

$$\begin{aligned}|h(z)| &\le (1-|z|^2)^{\frac{1}{2}}\sum_{i,j=1}^n\left|\int_B\left(\bar{z}_j\frac{\partial}{\partial\zeta_i}-\bar{z}_i\frac{\partial}{\partial\zeta_j}\right)\bar{\partial}(G_{k,l}f)\wedge K^s_{0,1}\right|\\ &\le (1-|z|^2)^{\frac{1}{2}}\sum_{i,j=1}^n\int_B\left|\left(\bar{\zeta}_j\frac{\partial}{\partial\zeta_i}-\bar{\zeta}_i\frac{\partial}{\partial\zeta_j}\right)\bar{\partial}(G_{k,l}f)\right||K^s_{0,1}|dV\\ &\quad+(1-|z|^2)^{\frac{1}{2}}\sum_{i,j=1}^n\int_B\left|\partial\bar{\partial}(G_{k,l}f)\right||\zeta-z||K^s_{0,1}|dV.\end{aligned}$$

Thus the result follows from

$$w\left((1-|z|^2)^{\frac{1}{2}}(1-|\zeta|^2)|^{-1}|K^s_{0,1}|\right)=0$$

$$(1-|\zeta|^2)\left|\left(\bar{\zeta}_j\frac{\partial}{\partial\zeta_i}-\bar{\zeta}_i\frac{\partial}{\partial\zeta_j}\right)\bar{\partial}(G_{k,l}f)\right| \in W^1,$$

$$w\left((1-|z|^2)^{\frac{1}{2}}(1-|\zeta|^2)^{\frac{-3}{2}}|z-\zeta||K^s_{0,1}|\right)=0 \quad\text{and}\quad (1-|\zeta|^2)^{\frac{3}{2}}\left|\partial\bar{\partial}G_{k,l}f\right|\in W^1.$$

The estimates of the other terms follows in the same way. □

Finally, as a consequence of this theorem and Proposition 2.14, we obtain a corona type decomposition for the Bloch space $\mathcal{B}$. We point out that this result was obtained by ourselves [OF] by other method, using some explicit division formulas.

We recall that

$$\mathcal{B}(B) = \left\{f \in H(B); ||f||_{\mathcal{B}} = \sup\left\{(1-|z|^2)\,|(I+R)f(z)|\,, z \in B\right\} < \infty\right\}.$$

COROLLARY 5.5. — *M_g maps $\mathcal{B} \times \dots \mathcal{B}$ onto $\mathcal{B}$ iff the functions g_j are pointwise multipliers of $\mathcal{B}$ and satisfy* $\sup\{|g(z)|; z \in B\} \geq \delta > 0$.

Proof. — The necessity of the conditions follows in the same way that for the $BMOA$ space (see section 3 or [OF]).

Denote by B_k the unit ball in $\mathbb{C}^k$ and by $\tilde{h}(z_1,\dots,z_{n+1}) = h(z_1,\dots,z_n)$. An elementary computation gives that $\tilde{h} \in BMOA(B_{n+1})$, if $h \in \mathcal{B}(B_n)$. Moreover, by Proposition 2.14, we have that $\tilde{g}_j \in \mathcal{M}(BMOA(B_{n+1}))$. Thus, by Theorem B, if $f \in \mathcal{B}(B_n)$ there exist functions $f_j \in BMOA(B_{n+1})$ such that $\tilde{f} = \sum_{j=1}^{m} \tilde{g}_j f_j$. Therefore, the result follows from

$$BMOA(B_{n+1})|_{B_n} \subset \mathcal{B}(B_{n+1})|_{B_n} = \mathcal{B}(B_n). \qquad \square$$

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