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ŁOJASIEWICZ INEQUALITIES FOR SETS DEFINABLE IN THE STRUCTURE $\mathbb{R}_{\mathrm{exp}}$

by TA LÊ LOI

The object to be studied in this paper is the class $\mathcal{D}$ of subsets of Euclidean spaces definable from addition, multiplication, and exponentiation. In Section 1 we recall some properties of sets in this class; we refer the reader to [3], [9], [11] for details. In the next sections we consider some variants of Łojasiewicz inequalities for this class. In Section 2 we prove the Łojasiewicz-type inequalities for $\mathcal{D}$-sets and $\mathcal{D}$-functions. Section 3 is devoted to some applications of the inequalities given in Section 2. The global Łojasiewicz inequalities with or without parameters for analytic $\mathcal{D}$-functions are presented in Section 4. The rationality of the Łojasiewicz exponents is also proved. Some of the results of this paper were announced in [10].

In the sequel, $\| \;\; \|$ and $d(\cdot,\cdot)$ denote the Euclidean norm and the Euclidean distance in $\mathbb{R}^n$ respectively, $d(x,\emptyset) \stackrel{\text{def}}{=} 1$. $B(x,r)$ denotes the open ball with center x and radius r. ${}^cX \stackrel{\text{def}}{=} \mathbb{R}^n \setminus X$ for $X \subset \mathbb{R}^n$.

1. The class of $\mathcal{D}$-sets.

1.1. DEFINITION. — *Let $\mathcal{R}_n$ denote the algebra of real-valued functions on $\mathbb{R}^n$ generated over $\mathbb{R}$ by the coordinate functions and their exponents, i.e.*

$$\mathcal{R}_n \stackrel{\text{def}}{=} [x_1,\dots,x_n,\exp(x_1),\dots,\exp(x_n)].$$

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Let $\mathcal{D}_n$ denote the class of subsets of $\mathbb{R}^n$ each of which is of the form: $\pi(f^{-1}(0))$, where $f \in \mathcal{R}_{n+m}$, $m \in \mathbb{N}$ and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the natural projection. A set A is called $\mathcal{D}$-set if $A \in \mathcal{D}_n$ for some $n \in \mathbb{N}$. A map $f : A \to B$ is called $\mathcal{D}$-map if its graph is a $\mathcal{D}$-set.

Remark. — The class $\mathcal{D}$ contains all semi-algebraic sets. A $\mathcal{D}$-set, in general, is not subanalytic (e.g. $\{(x,y) : x > 0,\ y = \exp(-\frac{1}{x})\}$). If f is a $\mathcal{D}$-function, then so is $\exp f$. If, in addition, $f > 0$, then $\log f$, f^α $(\alpha \in \mathbb{R})$ are $\mathcal{D}$-functions.

As a direct consequence of Wilkie's theorem [20] this class has the following substantial property.

1.2. THEOREM (Wilkie). — *$\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ is a Tarski system, i.e.*

If $A, B \in \mathcal{D}_n$, then $A \cup B$, $A \cap B$ and $A \setminus B \in \mathcal{D}_n$.

If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the natural projection.

Proof. — See [19], [20] (see also [11]). □

Here we shall give some geometric properties of $\mathcal{D}$-sets which will be used in the next sections.

1.3. PROPOSITION.

(i) *Every $\mathcal{D}$-set has only finitely many connected components and each component is also a $\mathcal{D}$-set.*

(ii) *The closure, the interior, and the boundary of a $\mathcal{D}$-set are $\mathcal{D}$-sets.*

(iii) *The composition of $\mathcal{D}$-maps is $\mathcal{D}$-map.*

Proof. — (i) follows from a Khovanskii result on fewnomials [6], [7] and Theorem 1.5 below (see also [3], [8]). (ii) and (iii) follow directly from Theorem 1.2. □

1.4. DEFINITION.

(i) *A $\mathcal{D}$-map $f : A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$ is called $\mathcal{D}$-analytic if there is an open neighborhood U of A in $\mathbb{R}^n$, $U \in \mathcal{D}_n$ and an analytic $\mathcal{D}$-map $F : U \to \mathbb{R}^m$ such that $F_{|A} = f$.*

(ii) *$\mathcal{D}_n$-analytic cells in $\mathbb{R}^n$ are defined by induction on n: $\mathcal{D}_1$-analytic cells are points $\{r\}$ or open intervals (a,b), $-\infty \le a < b \le +\infty$. If C is a $\mathcal{D}_n$-analytic cell and $f, g : C \to \mathbb{R}$ are $\mathcal{D}$-analytic such that $f < g$, then*

$$\begin{aligned}(f,g) &\stackrel{\text{def}}{=} \{(x,r) \in C \times \mathbb{R} : f(x) < r < g(x)\},\\ (-\infty, f) &\stackrel{\text{def}}{=} \{(x,r) \in C \times \mathbb{R} : r < f(x)\},\\ (g, +\infty) &\stackrel{\text{def}}{=} \{(x,r) \in C \times \mathbb{R} : g(x) < r\},\\ \Gamma(f) &\stackrel{\text{def}}{=} \operatorname{graph} f \quad \textit{and} \quad C \times \mathbb{R} \quad \textit{are } \mathcal{D}_{n+1}\textit{-analytic cells.}\end{aligned}$$

(iii) *A $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ is defined by induction on n: a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^1$ is a finite collection of intervals and points*

$$\{(-\infty, a_1), \ldots, (a_k, +\infty),\ \{a_1\}, \ldots, \{a_k\}\}, \quad \textit{where } a_1 < \cdots < a_k,\ k \in \mathbb{N}.$$

A $\mathcal{D}$-analytic decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into $\mathcal{D}_{n+1}$-analytic cells C such that the collection of all the projections $\pi(C)$ is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ (here $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the natural projection).

We say that a decomposition partitions A if A is a union of some cells of the decomposition.

1.5. THEOREM (van den Dries & Miller).

(I_n) *For $A_1, \ldots, A_k \in \mathcal{D}_n$ there is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning $A_1, \ldots, A_k$.*

(II_n) *For every function $f : A \to \mathbb{R}$, $A \in \mathcal{D}_n$, there is a $\mathcal{D}$-analytic decomposition of $\mathbb{R}^n$ partitioning A such that for each cell $C \subset A$ of the decomposition, the restriction $f_{|C}$ is $\mathcal{D}$-analytic.*

Proof. — For the proof see [3], [4] or [9]. □

1.6. PROPOSITION (definable selection). — *Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be a $\mathcal{D}$-set and let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be the natural projection. Then there exists a $\mathcal{D}$-map $\rho : \pi(A) \to \mathbb{R}^n \times \mathbb{R}^m$ such that $\pi(\rho(x)) = x$ for all $x \in \pi(A)$.*

Proof. — See [3], Ch. 8, Prop. 1.2. □

1.7. DEFINITION. — *The dimension of a $\mathcal{D}$-set $A \subset \mathbb{R}^n$ is defined by*

$$\dim A \stackrel{\text{def}}{=} \max\{\dim \Gamma : \Gamma \textit{ is an analytic submanifold of } \mathbb{R}^n \textit{ contained in } A\}.$$

Remark. — The class of $\mathcal{D}$-sets shares many interesting properties with those of semi-algebraic sets. For example, $\mathcal{D}$-sets admit Whitney stratification, they can be triangulated, and continuous $\mathcal{D}$-functions are piecewise trivial (see [3], [9], [11]).

2. Łojasiewicz-type inequalities.

2.1. DEFINITION. — *For $m \in \mathbb{N}$ define:*

- $\exp_0(t) \overset{\text{def}}{=} t$, $\exp_{m+1}(t) \overset{\text{def}}{=} \exp(\exp_m(t))$, $t \in \mathbb{R}$.
- $\varphi_m(t) \overset{\text{def}}{=} \dfrac{1}{\exp_m(1/|t|)}$ *if* $t \neq 0$, $\varphi_m(0) = 0$.

Note that $\exp_m$ and φ_m are $\mathcal{D}$-functions.

2.2. PROPOSITION.

(i) $\varphi_0(t) = |t|$, *and for* $m > 0$, $\varphi_m \in C^\infty$ *and flat at* 0.

(ii) $\forall m, \ell \in \mathbb{N}$, $a \neq 0$, $\displaystyle\lim_{0 \neq t \to 0} \frac{\varphi_m(t^{\ell+1})}{\varphi_m(at^\ell)} = 0$.

Proof.

(i) Let $u(t) = \exp_m\left(\dfrac{1}{t}\right)$, $t > 0$, $m \in \mathbb{N}$, $m \neq 0$. By elementary computation we have:

$$\left(\frac{1}{u}\right)^{(k)} = \sum_{i=1}^{r(k)} \frac{1}{u^i} P_{i,k}(u', \ldots, u^{(k)}),$$

where $k, r(k) \in \mathbb{N}$, $P_{i,k} \in \mathbb{R}[x_1, \ldots, x_k]$, $\deg P_{i,k} < i$;

$$u^{(k)}(t) = Q_k\left(\frac{1}{t}, \exp\left(\frac{1}{t}\right), \ldots, \exp_{m-1}\left(\frac{1}{t}\right)\right) \cdot \exp_m\left(\frac{1}{t}\right),$$

where $Q_k \in \mathbb{R}[x_0, \ldots, x_{m-1}]$.

Therefore

$$\varphi_m^{(k)}(t) = \sum_{j=1}^{s(k)} \frac{1}{\exp_m^j(1/t)} R_j\left(\frac{1}{t}, \exp\left(\frac{1}{t}\right), \ldots, \exp_{m-1}\left(\frac{1}{t}\right)\right), \quad t > 0,$$

where $R_j \in \mathbb{R}[x_0, \ldots, x_{m-1}]$, $j = 1, \ldots, s(k)$, $s(k) \in \mathbb{N}$. So (i) follows, as $\displaystyle\lim_{t \to 0} \frac{\exp_j(1/t)}{\exp_m(1/t)} = 0$ for $0 \leq j < m$.

(ii) is proved by induction on m. □

2.3. LEMMA (van den Dries & Miller). — *Let $f : (a, +\infty) \to \mathbb{R}$ be a $\mathcal{D}$-function. Then there exist $\ell, m \in \mathbb{N}$, $C, r > 0$ such that $|f(t)| \leq C \exp_m(t^\ell)$, $\forall t \geq r$.*

Proof. — See [4], §9, Proposition 9.2 (see also [14]). □

2.4. LEMMA. — *Let $v : A \to \mathbb{R}$, $A \subset \mathbb{R}^n$, be a $\mathcal{D}$-function. Suppose that v is bounded on every bounded subset of A, i.e.*

$$(*) \qquad \forall t \geq 0 \ \text{ if } \ A_t = \{x \in A : \|x\| \leq t\} \neq \emptyset, \ \text{ then } \ \sup_{A_t} v < +\infty.$$

Then there exist $\ell, m \in \mathbb{N}$, $C > 0$: $|v(x)| \leq C(1 + \exp_m(\|x\|^\ell))$, $\forall x \in A$. In particular, the inequality holds if v is continuous and A is closed.

Proof. — Let

$$w(t) = \begin{cases} \sup\limits_{A_t} v, & \text{if } A_t \neq \emptyset, \\ 0, & \text{if } A_t = \emptyset. \end{cases}$$

Then w is well-defined, by $(*)$, and $w : \mathbb{R}^+ \to \mathbb{R}$ is a $\mathcal{D}$-function, by Theorem 1.2. By Lemma 2.3 $\exists \ell, m \in \mathbb{N}$, $r, C_1 > 0 : |w(t)| \leq C_1 \exp_m(t^\ell)$, $\forall t \geq r$. Let $C = \max(C_1, \sup\limits_{A_r}, v)$, then $C < +\infty$ by $(*)$ and

$$|w(t)| \leq C(1 + \exp_m(t^\ell)), \quad \forall t \geq 0.$$

That is

$$|v(x)| \leq C(1 + \exp_m(\|x\|^\ell), \quad \forall x \in A.$$ □

2.5. LEMMA. — *Let A be a locally closed $\mathcal{D}$-set of $\mathbb{R}^n$. Then there exists a homeomorphic $\mathcal{D}$-map from A onto a closed $\mathcal{D}$-set of $\mathbb{R}^{n+1}$.*

Proof. — By the assumption $A = \overline{A} \cap U$, where $U = {}^c\overline{A} \cup A$ is a nonempty open $\mathcal{D}$-set. $\mathbb{R}^n \ni x \mapsto d(x, {}^cU)$ is a continuous $\mathcal{D}$-function, by Theorem 1.2. So the $\mathcal{D}$-map $x \mapsto (x, d(x, {}^cU)^{-1})$ is a homeomorphism from A onto the closed $\mathcal{D}$-set $\{(x,t) \in \mathbb{R}^k \times \mathbb{R} : x \in \overline{A},\ td(x, {}^cU) = 1\}$ of $\mathbb{R}^{n+1}$. □

2.6. THEOREM. — *Let $f : A \to \mathbb{R}$. Suppose that A is locally closed in $\mathbb{R}^n$, f is a continuous $\mathcal{D}$-function and $g : \{x \in A : f(x) \neq 0\} \to \mathbb{R}$ is* *a continuous $\mathcal{D}$-function. Then there exist $M, L \in \mathbb{N}$ such that $\varphi_M(f^L) \cdot g$ can be continuously extended to A by 0 at all $x \in A$ with $f(x) = 0$.*

Proof. — By Lemma 2.5 we can suppose that A is closed. For $x \in A$, $u \geq 0$ define

$$A_{x,u} = \{y \in A : \|y - x\| \leq 1, u|f(y)| = 1\}.$$

Then $A_{x,u}$ is a compact $\mathcal{D}$-set. Let

$$v(x,u) = \begin{cases} \sup\{|g(y)| : y \in A_{x,u}\}, & \text{if } A_{x,u} \neq \emptyset \\ 0, & \text{if } A_{x,u} = \emptyset. \end{cases}$$

By Theorem 1.2 $v : A \times \mathbb{R}^+ \to \mathbb{R}$ is a $\mathcal{D}$-function.

Claim. — v satisfies $(*)$ of Lemma 2.4. Indeed, for $t \geq 0$

$\sup\{v(x,u) : \|(x,u)\| \leq t\}$

$\leq \sup\{|g(y)| : \exists (x,u) \in A \times \mathbb{R}^+, \|(x,u)\| \leq t,\ \|y - x\| \leq 1,\ u|f(x)| = 1\}.$

The set $X = \{y : \exists (x,u) \in A \times \mathbb{R}^+,\ \|(x,u)\| \leq t,\ \|y - x\| \leq 1,\ u|f(y)| = 1\}$ is the image under a projection of the set

$$Y = \{(y,x,u) : x \in A,\ u \leq 0,\ \|(x,u)\| \leq t,\ \|y - x\| \leq 1,\ u|f(y)| = 1\}.$$

But Y is compact, for $A = \overline{A}$, f is continuous and $X \subset \{\|(x,u)\| \leq t, \{y - x\| \leq 1\}$. Hence X is compact. The claim follows because g is continuous. Now, by Lemma 2.4, there exist $\ell, m \in \mathbb{N}$, $C > 0$:

$$|v(x,u)| \leq C(1 + \exp_m(\|(x,u)\|^\ell)), \quad \forall (x,u) \in A \times \mathbb{R}^+.$$

Fix $x \in A$, $|v(x,u)| \leq 2C \exp_m(2^\ell \|u\|^\ell)$, $\forall u \geq \max(\|x\|, C, 1)$. This implies

$$|g(y)| \leq 2C \exp_m\Big(\frac{2^\ell}{|f(y)|^\ell}\Big),$$

when $y \in A$, $f(y) \neq 0$, $\|y - x\| \leq 1$ and $|f(y)| \leq \min\Big(\frac{1}{\|x\|}, \frac{1}{c}, 1\Big)$. That means $\varphi_m\Big(\frac{1}{2^\ell} f^\ell(y)\Big) \cdot |g(y)| \leq 2C$, when $y \in A$, $f(y) \neq 0$, $\|y - x\| \leq 1$ and $|f(y)|$ is sufficiently small. By Proposition 2.2 $\varphi_m(f^{\ell+1}) \cdot g$ can be continuously extended to 0 at all $x \in A$ with $f(x) = 0$. The theorem is proved. □

2.7. Corollary. — *Let $f, g : A \to \mathbb{R}$. Suppose that A is closed in $\mathbb{R}^n$, f, g are continuous $\mathcal{D}$-functions and $f^{-1}(0) \subset g^{-1}(0)$. Then there exist $L, M \in \mathbb{N}$, a continuous $\mathcal{D}$-function h on A such that $\varphi_M(g^L) = h \cdot f$. In particular, there exist $\ell, m \in \mathbb{N}$, $C > 0$ such that*

$$|f(x)| \geq \frac{C}{\exp_m(\|x\|^\ell)} \varphi_M(g^L(x)), \quad \forall x \in A.$$

Proof. — From the assumption $\frac{1}{f} : \{x \in A : g(x) \neq 0\} \to \mathbb{R}$ is a continuous $\mathcal{D}$-function. By Theorem 2.6 there are $L, M \in \mathbb{N}$ such that $h = \frac{\varphi_M(g^L)}{f}$ can be continuously extended to A. Moreover, from Lemma 2.4, $\exists \ell, m \in \mathbb{N}$, $C' > 0$: $|h(x)| \leq C(\exp_m(\|x\|^\ell)$, $\forall x \in A$ (we can suppose $m \neq 0$). The corollary follows. □

2.8. COROLLARY. — *Let $f : A \to \mathbb{R}$. Suppose that A is closed in $\mathbb{R}^n$, f is a continuous $\mathcal{D}$-function. Then there exist $\ell, m, L, M \in \mathbb{N}$, $C > 0$ such that*

$$|f(x)| \geq \frac{C}{\exp_m(\|x\|^\ell)} \varphi_M(d^L(x, f^{-1}(0)), \quad \forall x \in A.$$

Proof. — Apply Corollary 2.7 to $g(x) = d(x, f^{-1}(0))$. Note that g is a continuous $\mathcal{D}$-function, by Theorem 1.2. □

2.9. COROLLARY. — *Let X, Y be closed $\mathcal{D}$-sets of $\mathbb{R}^n$. Then there exist $\ell, m, L, M \in \mathbb{N}$, $C > 0$ such that*

$$d(x, X) + d(x, Y) \geq \frac{C}{\exp_m(\|x\|^\ell)} \varphi_M(l^L(x, X \cap Y)), \quad \forall x \in \mathbb{R}^n.$$

Proof. — Apply Corollary 2.7 to

$$f(x) = d(x, X) + d(x, Y) \quad \text{and} \quad g(x) = d(x, X \cap Y).$$ □

2.10. *Remark.* — In Corollaries 2.7 and 2.8, if we suppose that A is a locally closed subset of $\mathbb{R}^n$, then, from the proof of Lemmas 2.5 and 2.4, there exist $m \in \mathbb{N}$, $C > 0$ such that

$$|h(x| \leq C \exp_m(\|x\| + d(x, {}^cU)^{-1}, \quad \forall x \in A,$$

where $U = {}^c\overline{A} \cup A$. Therefore, by replacing the denominators on the right sides of the inequalities in Corollaries 2.7 and 2.8 by $\exp_m(\|x\| + d(x^cU)^{-1})$ we obtain the inequalities in the case that A is locally bounded.

Similarly, Corollary 2.9 can be somewhat generalized as follows.

2.9'. COROLLARY. — *Let X, Y be closed $\mathcal{D}$-sets in an open $\mathcal{D}$-set Ω of $\mathbb{R}^n$. Then there exist $m, L, M \in \mathbb{N}$, $C > 0$ such that*

$$d(x, X) + d(x, Y) \geq \frac{C}{\exp_m(\|x\| + d(x, {}^c\Omega)^{-1})} \varphi_M(d^L(x, X \cap Y)), \quad \forall x \in \Omega.$$

3. Applications.

As a first application of the inequalities given in Section 2 we prove the Tietze-Urysohn theorem for the class of $\mathcal{D}$-functions.

3.1. PROPOSITION. — *Let $A \subset \mathbb{R}^n$ be a locally closed $\mathcal{D}$-set and $F \subset A$ be closed in A. Suppose that $f : F \to \mathbb{R}$ is a continuous $\mathcal{D}$-function. Then there exists a continuous $\mathcal{D}$-function $\bar{f} : A \to \mathbb{R}$ such that $\bar{f}_{|F} = f$.*

Proof. — By decomposing $f = f^+ + f^-$, where $f^+ = \frac{1}{2}(|f| + f)$, $f^- = \frac{1}{2}(|f| - f)$, we can suppose that $f \geq 0$. Moreover, by Lemma 2.5, it suffices to prove the proposition for $A = \mathbb{R}^n$.

Now suppose that $f : F \to \mathbb{R}$ is a continuous $\mathcal{D}$-function, $F \subset \mathbb{R}^n$ is a closed $\mathcal{D}$-set and $f \geq 0$. Applying the inequality in Corollary 2.6 to $|f(x) - f(y)|$ and $\|x - y\|$, $(x, y) \in F \times F$, we can find $L, M \in \mathbb{N}$ ($M \geq 1$), a continuous $\mathcal{D}$-function $h : F \times F \to \mathbb{R}$ such that $\varphi_M(|f(x) - f(y)|^L) = h(x, y)\|x - y\|$, $x, y \in F$. By Lemma 2.4 there are $m \in \mathbb{N}$, $C > 0$: $h(x, y) \leq C \exp_m(\|(x, y)\|)$, $x, y \in F$ (where $\|(x, y)\| \stackrel{\text{def}}{=} (\|x\|^2 + \|y\|^2)^{1/2}$). It is easy to see that φ_M is strictly increasing on $[0, +\infty)$. So there exists the inverse function φ_M^{-1} which is also a continuous $\mathcal{D}$-function and strictly increasing on $[0, r_M)$, where $r_M \stackrel{\text{def}}{=} \lim_{t \to +\infty} \varphi_M(t) = \dfrac{1}{\exp_M(0)}$. Hence $h(x, y)\|x - y\| < r_M$, $\forall x, y \in F$. By Corollary 2.8 there are $p \in \mathbb{N}$, $C' > 0$ such that $r_M - h(x, y)\|x - y\| \geq \dfrac{C'}{\exp_p(\|(x, y)\|)}$, $\forall x, y \in F$. Define

$$k(x, y) = \min\left(C \exp_m(\|(x, y)\|)\|x - y\|,\ r_M - \frac{C'}{2\exp_p(\|(x, y)\|)}\right), \quad x, y \in \mathbb{R}^n.$$

Then k is a continuous $\mathcal{D}$-function and $\varphi_M(|f(x) - f(y)|^L) \geq k(x, y) < r_M$, $\forall x, y \in F$. Define

$$\Delta(x, y) = \varphi_M^{-1}(k(x, y))^{1/L} + \|x - y\|, \quad x, y \in \mathbb{R}^n.$$

$$\bar{f}(x) = \inf\{f(y) + \Delta(x, y) : y \in F\}, \quad x \in \mathbb{R}^n.$$

Then $\bar{f}$ satisfies the demands of the proposition (see for example Bochnak-Coste-Roy, “Géométrie algébrique réelle”, Ch. 2, Prop. 2.6.9). □

In the differential analysis aspects, the behaviour of infinitely differentiable $\mathcal{D}$-functions, briefly C^∞ $\mathcal{D}$-functions, is quite bad.

3.2. *Examples.*

(i) (Bierstone [1], Ex. 2.18). Let F be the complement of the open subset of $\mathbb{R}^2$ defined by $\{(x,y) : 0 < y < e^{-1/x^2},\ x > 0\}$. Let f be the function on F defined by $f(x,y) = e^{-1/x^2}$ if $x > 0$, $f(x,y) = 0$ otherwise. Then f is a continuous $\mathcal{D}$-function on F, infinitely differentiable on $\operatorname{int} F$ and all partial derivatives of $f_{|\operatorname{int} F}$ extend continuously to F (in particular, to zero at the origin). But f is not the restriction of a C^∞ function on $\mathbb{R}^2$ because if $x > 0$ then the difference quotient

$$\frac{f(x,e^{-1/x^2}) - f(x,0)}{e^{-1/x^2} - 0} = 1.$$

(ii) (Malgrange [13], Ch. VI.2). Let $f(x,y) = y + e^{-1/x^2}$ if $x \neq 0$, $f(0,y) = y^2$. Then f is a C^∞ $\mathcal{D}$-function on $\mathbb{R}^2$. The ideal $fC^\infty(\mathbb{R}^2)$ is not closed in $C^\infty(\mathbb{R}^2)$ (with the topology of uniform convergence of functions and all their partial derivatives on compact sets), that is, there exists a C^∞ function g on $\mathbb{R}^2$, $g \notin fC^\infty(\mathbb{R}^2)$ but for each $(x,y) \in \mathbb{R}^2$ the Taylor series of g at (x,y) belongs to the ideal generated by the Taylor series of f at (x,y) in the ring of formal series. (In fact, f does not satisfy a Łojasiewicz inequality in any neighborhood of the origin.)

We introduce here the notion of φ_M-flatness. It measures, in a certain sense, the "degree of flatness" of the contact of the zerosets of C^∞ functions with $\mathcal{D}$-sets.

3.3. Definition. — *Let Ω be an open subset of $\mathbb{R}^n$. Let X be a closed subset in Ω, f be C^∞ function on Ω and $M \in \mathbb{N}$. f is called φ_M-flat on X iff for every $a \in \Omega$, there exists a neighborhood U_a of a such that*

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha,\ L_\alpha > 0 : |D^\alpha f(x)| \leq C_\alpha \varphi_M(d^{L_\alpha}(x,X)), \quad \forall x \in U_a \cap \Omega.$$

Let $\Phi_M(\Omega, X)$ denote the set of all C^∞ functions on Ω which are φ_M-flat on X.

3.4. Proposition. — *Let Ω be an open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be a C^∞ $\mathcal{D}$-function. Then there exists $M \in \mathbb{N}$ such that*

$$\forall g \in \Phi_M(\Omega, f^{-1}(0)),\ \exists \Psi \in C^\infty(\Omega) : g = \Psi \cdot f.$$

Proof. — By Corollary 2.8 and Remark 2.10, there are $m, M \geq 1$, $C, L > 0$ such that

$$|f(x)| \geq \frac{C\varphi_{M-1}(d^L(x, f^{-1}(0))}{\exp_M(\|x\| + d(x, {}^c\Omega)^{-1})}, \quad \forall x \in \Omega.$$

Let $a \in f^{-1}(0)$ and $g \in \Phi_M(\Omega, f^{-1}(0))$. Then there is a compact neighborhood K of a such that $\forall \alpha \in \mathbb{N}^n$, $\exists C_\alpha$, $L_\alpha > 0$:

$$|D^\alpha g(x)| \leq C_\alpha \varphi_M(d^{L_\alpha}(x, f^{-1}(0))), \quad \forall x \in K.$$

On the other hand, $\forall \beta \in \mathbb{N}^n$, $|\beta| \leq |\alpha|$, $\exists C'$, $N > 0$, $C'' > 0$:

$$\begin{aligned} |D^\beta\Big(\frac{1}{f}\Big)(x) &\leq \frac{C'}{|f^{|\beta|+1}(x)|} \\ &\leq \frac{C''}{\varphi_{M-1}(d^L(x, f^{-1}(0))^{|\alpha|+1}}, \quad \forall x \in K \setminus f^{-1}(0). \end{aligned}$$

So, from Leibnitz's formula, $\forall \alpha \in \mathbb{N}^n$, $\exists C''' > 0$:

$$|D^\alpha\Big(\frac{g}{f}\Big)(x)| \leq C''' \sum_{\beta \leq \alpha} \frac{\varphi_M(d^{L_{\alpha-\beta}}(x, f^{-1}(0))}{\varphi_{M-1}(d^L(x, f^{-1}(0))^{|\alpha|+1}}, \quad \forall x \in K \setminus f^{-1}(0).$$

Since $\lim\limits_{t \in 0^+} \dfrac{\varphi_M(t^{L'})}{\varphi_{M-1}(t^L)^{|\alpha|+1}} = 0$, $\forall L' \in \mathbb{N}$, $D^\alpha\Big(\dfrac{g}{f}\Big)$ can be continuously extended to Ω to 0 at any $x \in f^{-1}(0)$. Then, by Hesténès lemma (see, for example, [15], Ch. IV, Lemma 4.3), $\dfrac{g}{f}$ is the restriction of a C^∞ function on Ω which is flat on $f^{-1}(0)$. The proposition follows. □

3.5. PROPOSITION. — *Let X, Y be closed $\mathcal{D}$-sets in an open $\mathcal{D}$-set Ω of $\mathbb{R}^n$. Then there exists $M \in \mathbb{N}$ such that: for every $f, g \in C^\infty(\Omega)$, $f - g$ is φ_M-flat on $X \cap Y$, i.e. $f - g \in \Phi_M(\Omega, X \cap Y)$, exists $\sigma \in C^\infty(\Omega)$ such that $\sigma_{|X} = f_{|X}$, $\sigma_{|Y} = g_{|Y}$.*

Proof. — By Corollary 2.9' there are $m, L \in \mathbb{N}$, $C > 0$ ($M \geq 1$) such that

$$d(x, X) + d(x, Y) \geq \frac{C}{\exp_m(\|x\| + d(x, {}^c\Omega)^{-1})} \varphi_{M-1}(d^L(x, X \cap Y)), \quad \forall x \in \Omega.$$

Applying this inequality and using a similar argument as in the proof of [15], Ch. IV, Lemma 4.5, we can find $\theta \in C^\infty(\Omega \setminus (X \cap Y))$ with $\theta = 1$ on $X \setminus (X \cap Y)$, $\theta = 0$ on $Y \setminus (X \cap Y)$ and $\forall a \in \Omega$ there is a neighborhood U_a of a such that:

$$\forall \alpha \in \mathbb{N}^n, \ \exists C_\alpha, \ L_\alpha > 0 : |D^\alpha \theta(x)| \leq C_\alpha \varphi_{M-1}(d^{L_\alpha}(x, X \cap Y))^{-|\alpha|}, \quad \forall x \in U_a \setminus (X \cap Y).$$

Define $\sigma = f + \theta(g - f)$. Then $\sigma_{|X} = f_{|X}$, $\sigma_{|Y} = g_{|Y}$. Moreover, if $f - g$ is φ_M-flat on $X \cap Y$, then $\forall a \in \Omega$ there is a neighborhood $U \subset \Omega$ of a such that

$$\forall \beta \in \mathbb{N}^n, \ \exists C'_\beta, \ L'_\beta > 0 : |D^\beta(f - g)(x)| \leq C'_\beta \varphi_M(d^{L'_\beta}(x, X \cap Y)), \ \forall x \in U.$$

By Leibnitz's formula and properties of θ, shrinking U if necessary, for every $\gamma \in \mathbb{N}^n$ there is $C > 0$ such that

$$|D^\gamma \theta(g-f)(x)| \leq C \sum_{\alpha \leq \gamma} \frac{\varphi_M(d^{L'_{\gamma-\alpha}}(x, X \cap Y))}{\varphi_{M-1}(d^{L_\alpha}(x, X \cap Y))^{|\alpha|}}, \ \forall x \in U \setminus (X \cap Y).$$

Hence $\theta(g-f)$ is the restriction of a C^∞ function on Ω by Hesténès lemma, and so is σ. The proposition follows. □

4. Global Łojasiewicz inequalities for analytic $\mathcal{D}$-functions.

Throughout this section, let Ω be an open $\mathcal{D}$-set of $\mathbb{R}^n$. We are interested in the class of analytic $\mathcal{D}$-functions on Ω. First we list some properties of this class:

It is a ring containing all (restricted) polynomials.

It is a differential ring, i.e. if f is in this class, then so are $\dfrac{\partial f}{\partial x_i}$ $(i = 1, \dots, n)$.

It is closed under exponentiation, i.e. if f is in this class, then so is $\exp f$.

If f is in this class and $f > 0$, then so are $\log f$, f^α $(\alpha \in \mathbb{R})$.

It has the weakly Noetherian property, i.e. if $\{f_i,\ i \in I\}$ is a family of functions in this class, then there exist $i_1, \dots, i_k \in I$ such that

$$\bigcap_{i \in I} f_i^{-1}(0) = f_{i_1}^{-1}(0) \cap \dots \cap f_{i_k}^{-1}(0).$$

(The proofs are given in [11].)

From the rationality of Łojasiewicz's exponent of subanalytic functions on compact subanalytic sets (see [2]) and the finiteness of the number of connected components of $\mathcal{D}$-sets (Proposition 1.3), we obtain the global Łojasiewicz inequality for analytic $\mathcal{D}$-functions.

4.1. THEOREM. — *Let $f, g : \Omega \to \mathbb{R}$ be analytic $\mathcal{D}$-functions. Suppose that $f^{-1}(0) \subset g^{-1}(0)$. Then:*

(i) *There exist $L \in \mathbb{N}$, a continuous $\mathcal{D}$-function h on Ω such that*

$$h(x)f(x) = g^L(x), \quad x \in \Omega.$$

(ii) *There exist* $m \in \mathbb{N}$, α, $C > 0$ *such that*

$$|f(x)| \geq \frac{C}{\exp_m(\|x\| + d(x, {}^c\Omega)^{-1})}|g(x)|^\alpha, \quad \forall x \in \Omega.$$

(iii) *If* $F \subset \Omega$ *is a closed* $\mathcal{D}$*-set of* $\mathbb{R}^n$*, then there exist* $m_F \in \mathbb{N}$, α_F, $C_F > 0$*:*

$$|f(x)| \geq \frac{C_F}{\exp_{m_F}(\|x\|)}|g(x)|^{\alpha_F}, \quad \forall x \in F.$$

(iv) *The Łojasiewicz exponent of* g *with respect to* f *on* Ω*,*

$$\ell_\Omega(f,g) \overset{\text{def}}{=} \inf\{\alpha \in \mathbb{R}_+ : \text{ there is a continuous } \mathcal{D}\text{-function } h \text{ on } \Omega, \quad h(x)|f(x)| \geq |g(x)|^\alpha, \quad \forall x \in \Omega\},$$

is a rational number. Moreover, there exists a continuous $\mathcal{D}$*-function* h *on* Ω *such that* $h(x)|f(x)| \geq |g(x)|^{\ell_\Omega(f,g)}$, $\forall x \in \Omega$.

Proof. — Let $r(x) = \frac{1}{2}d(x, {}^c\Omega)$, $x \in \mathbb{R}^n$ ($d(x, \emptyset) = 1$, by definition). Then r is a $\mathcal{D}$-function by Theorem 1.2. Define

$$\ell(x) = \inf\{\alpha \in \mathbb{R}_+ : \exists C>0,\ C|f(x')| \geq |g(x')|^\alpha,\ \forall x' \in \overline{B}(x, r(x))\},\ x \in \Omega.$$

Note that the sets which are taken infimum are not empty by the well-known theorem on Łojasiewicz inequality for analytic functions on compact subanalytic sets. So ℓ is well-defined. ℓ is a $\mathcal{D}$-function, by Theorem 1.2.

(To assure this we can directly verify as follows: Consider

$$X = \{(x,\alpha) : x \in \Omega, \alpha \in \mathbb{R}_+, \exists C>0, \forall x' \in \overline{B}(x, r(x)), C|f(x')| \geq |g(x')|^\alpha\}.$$

First let

$$X_1 = \{(x, \alpha, C) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ : \forall x' \in \overline{B}(x, r(x)), \quad C|f(x')| \geq |g(x')|^\alpha\}.$$

Then $X = \pi_1(X_1)$, where $\pi_1(x, \alpha, C) = (x, a)$. Now define

$$X_2 = \{(x, \alpha, C, x') \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n : \|x - x'\| \leq r(x), \quad C|f(x')| < |g(x')|^\alpha\}.$$

Then ${}^cX_1 = \pi_2(X_2) \cup {}^c(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+)$, where $\pi_2(x, \alpha, C, x') = (x, \alpha, C)$. Finally, let

$$X_3 = \{(x, \alpha, C, x', v, w) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : (r(x) - \|x - x'\| - v^2)^2 + ((|g(x')|^\alpha - C|f(x')|)w^2 - 1)^2 = 0\}.$$

Then $X_2 = \pi_3(X_3)$, where $\pi_3(x, \alpha, C, x', v, w) = (x, \alpha, C, x')$. But X_3 is of the form

$$X_3 = S \cap F^{-1}(0), \quad \text{where } S \text{ is a } \mathcal{D}\text{-set and } F : S \to \mathbb{R} \text{ is a } \mathcal{D}\text{-function.}$$

So X_3 is a $\mathcal{D}$-set. This implies X is a $\mathcal{D}$-set. By a similar way one can check that

$$\operatorname{graph} \ell = \{(x, \lambda) : x \in \Omega,\ \lambda = \inf X_x\},$$

where $X_x \overset{\text{def}}{=} \{\alpha \in \mathbb{R} : (x, \alpha) \in X\}$, is a $\mathcal{D}$-set.)

By Bochnak-Risler's theorem [2], §1.3, Th. 1, $\ell(x) \in \mathbb{Q}_+$, $\forall x \in \Omega$. Therefore, by the finiteness of the number of connected components of $\mathcal{D}$-sets, $\operatorname{Im} \ell$ is a finite subset of $\mathbb{Q}_+$. This implies that there exist $a \in \Omega$, $p, q \in \mathbb{N}$ such that $\sup \operatorname{Im} \ell = \ell(a) = \dfrac{p}{q}$. So $\forall \varepsilon > 0$, $\forall x \in \Omega$, $\exists C_\varepsilon(x) > 0$: $C_\varepsilon(x)|f(x')| \geq |g(x')|^{p/q}$, $\forall x' \in \overline{B}(x, r(x))$. Let $L = \left[\dfrac{p}{q}\right] + 1$, where $[\alpha]$ denotes the integral part of $[\alpha]$. Define

$$h(x) = \begin{cases} \dfrac{g^L}{f}(x), & \text{if } x \in \Omega,\ f(x) \neq 0, \\ 0, & \text{if } x \in \Omega,\ f(x) = 0. \end{cases}$$

Then h is a continuous $\mathcal{D}$-function on Ω and $h(x)f(x) = g^L(x)$, $x \in \Omega$.

(i) is proved.

(ii) follows from (i) and Remark 2.10.

(iii) follows from (i) and Lemma 2.4.

(iv) We prove that $\ell_\Omega(f, g) = \dfrac{p}{q} = \ell(a)$. Let $\varepsilon > 0$. Define

$$h_\varepsilon(x) = \begin{cases} \dfrac{|g|^{(p/q)+\varepsilon}}{|f|}(x), & \text{if } x \in \Omega,\ f(x) \neq 0, \\ 0, & \text{if } x \in \Omega,\ f(x) = 0. \end{cases}$$

Then k_ε is a continuous $\mathcal{D}$-function on Ω. This implies $\ell_\Omega(f, g) \leq \dfrac{p}{q} + \varepsilon$.

On the other hand, by the definition of $\ell_\Omega(f, g)$, there exists a continuous $\mathcal{D}$-function h_ε on Ω such that

$$h_\varepsilon(x)|f(x)| \geq |g(x)|^{\ell_\Omega(f,g)+\varepsilon}, \quad \forall x \in \Omega.$$

Therefore,

$$\sup_{x' \in \overline{B}(a, r(a))} (h_\varepsilon(x') + 1)|f(x)| \geq |g(x)|^{\ell_\Omega(f,g)+\varepsilon}, \quad \forall x \in \overline{B}(a, r(a)).$$

So $\dfrac{p}{q} = \ell(a) \leq \ell_\Omega(f, g) + \varepsilon$. (iv) follows. □

4.2. *Remark.*

(i) In fact, in the proof of Theorem 4.1 we only require the subanalyticity of f and g, so our proof can be modified to obtain similar results in the following cases:

a) f, g, being $\mathcal{D}$-functions, are continuous and subanalytic in Ω with $f^{-1}(0) \subset g^{-1}(0)$.

b) Ω is a closed $\mathcal{D}$-set and subanalytic subset of $\mathbb{R}^n$, and f, g, being $\mathcal{D}$-functions, are continuous and subanalytic on Ω with $f^{-1}(0) \subset g^{-1}(0)$.

(Under the assumption b), instead of ℓ in the proof of the theorem we define

$$\lambda(r) = \inf\{\alpha \in \mathbb{R}_+ : \exists C > 0,$$
$$C|f(x)| \geq |g(x)|^\alpha,\ \forall x \in \overline{B}(0,r) \cap \Omega\},\ r \geq d(0,\Omega).$$

The remaining argument can be modified without difficulty.)

(ii) By the above remark, the rationality of Łojasiewicz exponent for continuous semi-algebraic functions on closed or open semi-algebraic sets follows. In [5] Fekak proved this, but his method relied upon the theory of the real spectrum.

(iii) Similar remarks can be made for Propositions 4.3 and 4.4 below.

4.3. PROPOSITION. — *Let $f : \Omega \to \mathbb{R}$ be an analytic $\mathcal{D}$-function. Then:*

(i) *There exist $m \in \mathbb{N}$, α, $C > 0$ such that*

$$|f(x) \geq \frac{C}{\exp_m(\|x\| + d(x, {}^c\Omega)^{-1})} d^\alpha(x, f^{-1}(0)),\ \forall x \in \Omega.$$

(ii) *There exist $m_F \in \mathbb{N}$, α_F, $C_F > 0$ such that*

$$|f(x) \geq \frac{C_F}{\exp_{m_F}(\|x\|)} d^{\alpha_F}(x, f^{-1}(0)),\ \forall x \in F,$$

where F is as in (iii) *of Theorem 4.1.*

(iii) $\ell_\Omega(f) \overset{\text{def}}{=} \inf\{\alpha \in \mathbb{R}_+$: *there is a continuous $\mathcal{D}$-function h on Ω, $h(x)|f(x)| \geq d^\alpha(x, f^{-1}(0))$, $\forall x \in \Omega\}$ is a rational number. Moreover, there exists a continuous $\mathcal{D}$-function h on Ω such that $h(x)|f(x)| \geq d^{\ell_\Omega(f)}(x, f^{-1}(0))$, $\forall x \in \Omega$.*

Proof. — Using the notation of the proof of Theorem 4.1, we now define

$$\ell_1(x) = \inf\{\alpha \in \mathbb{R}_+ : \exists C > 0,$$
$$C|f(x')| \geq d^\alpha(x', f^{-1}(0),\ x' \in \overline{B}(x, r(x))\},\ x \in \Omega.$$

Arguing as in the proof of the theorem, now replacing ℓ by ℓ_1, we obtain the desired result. □

4.4. PROPOSITION. — *Let*

$$X = \bigcup_{i=1}^{p} \{x \in \Omega : f_i(x) = 0,\ g_{i1}(x) > 0, \dots, g_{ir}(x) > 0\}$$

and

$$Y = \bigcup_{j=1}^{q} \{x \in \Omega : h_j(x) = 0,\ k_{j1}(x) > 0, \dots, k_{js}(x) > 0\},$$

where the $f_i, h_j, g_{i1}, \dots, g_{ir}, k_{j1}, \dots, k_{js}$ *are analytic* $\mathcal{D}$*-functions on* Ω. *Suppose that* X, Y *are closed in* Ω. *Then:*

(i) *There exist* $m \in \mathbb{N}$, α, $C > 0$ *such that*

$$d(x, X) + d(x, Y) \geq \frac{C}{\exp_m(\|x\| + d(x, {}^c\Omega)^{-1})} d^\alpha(x, X \cap Y),\ \forall x \in \Omega.$$

(ii) *There exist* $m_F \in \mathbb{N}$, α_F, $C_F > 0$ *such that*

$$d(x, X) + d(x, Y) \geq \frac{C_F}{\exp_{m_F}(\|x\|)} d^{\alpha_F}(x, X \cap Y),\ \forall x \in F,$$

where F *is as in* (iii) *of Theorem 4.1.*

(iii) $\ell_\Omega(X, Y) \overset{\text{def}}{=} \inf\{\alpha \in \mathbb{R}_+$: *there exist a continuous* $\mathcal{D}$*-function* h *on* Ω, $h(x)(d(x, X) + d(x, Y)) \geq d^\alpha(x, X \cap Y),\ \forall x \in \Omega\}$ *is a rational number. Moreover, there exists a continuous* $\mathcal{D}$*-function* h *on* Ω *such that* $h(x)(d(x, X) + d(x, Y)) \geq d^{\ell_\Omega(X,Y)}(x, X \cap Y)$, $\forall x \in \Omega$.

Proof. — Using the notation in the proof of Theorem 4.1, instead of ℓ, we define

$$\ell_2(x) = \inf\{\alpha \in \mathbb{R}_+ : \exists C > 0,$$
$$C(d(x', X) + d(x', Y)) \geq d^\alpha(x', X \cap Y),\ x' \in \overline{B}(x, r(x))\},\ x \in \Omega.$$

Once again, carrying out the same argument as in the proof of the theorem, but now replacing ℓ by ℓ_2, we obtain the proposition. □

4.5. *Remark.* — Tougeron in [18] proved global Łojasiewicz inequalities for functions in the exponential extensions of rings contained in a certain class of weakly Noetherian rings of analytic functions (see [16], [17] for the precise definition). Here we present a simple proof of these inequalities for analytic $\mathcal{D}$-functions, this class is much larger than the exponential extension of the ring of polynomials.

As an application of the global Łojasiewicz inequality in Theorem 4.1, we prove the next proposition which is analogous to a theorem of Łojasiewicz [12], Th. 3, p. 98.

Let $\mathcal{A}_n$ denote the smallest ring of real-valued functions on $\mathbb{R}^n$ containing all polynomials and closed under exponentiation. A subset X of $\mathbb{R}^n$ is called *$\mathcal{A}_n$-semianalytic*, iff it can be globally described by functions in $\mathcal{A}_n$, i.e.

$$X = \bigcup_{i=1}^{p} \{x \in \mathbb{R}^k : f_i(x) = 0,\ g_{ij}(x) > 0,\ j = 1, \ldots, q\},$$

where the f_i, g_{ij} are in $\mathcal{A}_n$, $p, q \in \mathbb{N}$.

Note that functions in $\mathcal{A}_n$ (resp. $\mathcal{A}_n$-semianalytic sets) are $\mathcal{D}$-functions (resp. $\mathcal{D}$-sets).

4.6. LEMMA. — *Let X be an open $\mathcal{A}_n$-semianalytic subset of $\mathbb{R}^n$. Let*

$$U = \{x \in \mathbb{R}^n : g_1(x) > 0, \ldots, g_q(x) > 0\}$$

and

$$Z = \{x \in U : f(x) = 0\},$$

where $f, g_1, \ldots, g_q \in \mathcal{A}_n$. If $Z \subset X$, then there exist α, $m \in \mathbb{N}$, $C > 0$ such that

$$Z \subset \{x \in U : |f(x)| \exp_m(\|x\|^2) < C|g(x)|^\alpha\} \subset X, \text{ where } g \overset{\text{def}}{=} \prod_{j=1}^{q} g_j.$$

Proof. — Apply Theorem 4.1 and Remark 4.2 (i), b) to the restrictions of f and g to $\overline{U} \setminus X$. There are α, $m \in \mathbb{N}$, $C > 0$ such that

$$|f(x)| \geq \frac{C}{\exp_m(\|x\|^2)} |g(x)|^\alpha, \quad \forall x \in \overline{U} \setminus X.$$

Therefore $Z \subset \{x \in U : |f(x)| \exp_m(\|x\|^2) < C|g(x)|^\alpha\} \subset X$ as desired. □

4.7. PROPOSITION. — *Let X be an $\mathcal{A}_n$-semianalytic subset of $\mathbb{R}^n$. If X is open (resp. closed), then X can be represented in the form*

$$X = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s} \{x \in \mathbb{R}^n : h_{ij}(x) > 0\} \quad (\text{resp. } X = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s} \{x \in \mathbb{R}^n : h_{ij}(x) \geq 0\})$$

where the h_{ij} are in $\mathcal{A}_n$.

Proof. — It is sufficient to prove the proposition for the open case. Suppose that X is an open $\mathcal{A}_n$-semianalytic subset of $\mathbb{R}^n$. Then $X = \bigcup_{i=1}^{p} U_i \cap Z_i$, where $U_i = \{x \in \mathbb{R}^n : g_{ij}(x) > 0,\ j = 1, \dots, q\}$, $Z_i = \{x \in U_i : f_i(x) = 0\}$ with $f_i, g_{ij} \in \mathcal{A}_n$, $i = 1, \dots, p$, $j = 1, \dots, q$. Fix i and apply the lemma with $U = U_i$, $Z = Z_i$. There exist α_i, $m_i \in \mathbb{N}$, $C_i > 0$ such that

$$Z_i \subset U_i \cap \{x \in \mathbb{R}^n : \psi_i(x) > 0\} \subset X,$$

where $\psi_i = C_i^2 \prod_{i=1}^{q} g_{ij}^{2\alpha_i} - f_i^2 \exp_{m_i}(\|x\|^2)$. (Note that $\psi_i \in \mathcal{A}_n$.) Therefore

$$X \subset \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{x \in \mathbb{R}^n : g_{ij}(x) > 0,\ \psi_i(x) > 0\} \subset X.$$

This implies that X can be represented in the desired form. □

The remain part of this section is devoted to Łojasiewicz inequalities with parameters. Let

$X \subset \mathbb{R}^n \times \mathbb{R}^m$ and (x, t) be the coordinate of $\mathbb{R}^n \times \mathbb{R}^m$,

$X_t \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : (x, t) \in X\}$ be the fibre of X over $t \in \mathbb{R}^m$,

$X|T \stackrel{\text{def}}{=} X \cap \mathbb{R}^n \times T$ be the restriction of X to $T \subset \mathbb{R}^m$.

If $f : X \to \mathbb{R}$, then define $f_t(x) = f(x, t)$.

4.8. THEOREM. — *Let $X \subset \mathbb{R}^n \times \mathbb{R}^m$ be a locally closed $\mathcal{D}$-set and let $f, g : X \to \mathbb{R}$ be $\mathcal{D}$-functions. Let $T = \pi(X)$, where $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection on the last m coordinates. Suppose that for all $t \in T$, X_t is open and f_t, g_t are analytic on X_t with $f_t^{-1}(0) \subset g_t^{-1}(0)$. Then there exist a $\mathcal{D}$-analytic decomposition of T into cells $C_1, \dots, C_k$ and $r_1, \dots, r_k \in \mathbb{Q}_+$ such that:*

(i) *For each $t \in C_i$, $\ell_{X_t}(f_t, g_t) = r_i$ $(i = 1, \dots, k)$.*

(ii) *For each $i \in \{1, \dots, k\}$ there exists a continuous $\mathcal{D}$-function h_i on $X|C_i$:*

$$h_i(x, t)|f(x, t)| \geq |g(x)|^{r_i}, \quad \forall (x, t) \in X|C_i.$$

Proof. — Define

$$r(x,t) = \frac{1}{2} d(x, {}^{c}X_t), \ x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}^m,$$

$$\ell(x,t) = \inf\{\alpha \in \mathbb{R}_+ : \exists C > 0, \quad C|f(x',t)| \geq |g(x',t)|^\alpha, \ \forall x' \in \overline{B}(x, r(x,t))\}, \ (x,t) \in X,$$

$$L(t) = \sup_{x \in X_t} \ell(x,t), \quad t \in T.$$

Then r, ℓ are $\mathcal{D}$-functions. From the proof of Theorem 4.1, for each $t \in T$ the set $\{\ell(x,t) : x \in X_t\}$ is a finite subset of $\mathbb{Q}_+$. So $L(t) \in \mathbb{Q}_+$, $\forall t \in T$, It can be verified that L is a $\mathcal{D}$-function (as in the proof of Theorem 4.1). By the finiteness of the number of the connected components of $\mathcal{D}$-sets, $\operatorname{Im} L$ has finitely many components. So $\operatorname{Im} L = \{r_1, \ldots, r_s\}$, where $r_i \in \mathbb{Q}_+$, $i = 1, \ldots, s$.

Let $T_i = T \cap L^{-1}(r_i)$, $i = 1, \ldots, s$. Then, from Theorem 4.1,

$$\ell_{X_t}(f_t, g_t) = r_i, \quad \forall t \in T_i, \ i = 1, \ldots, s.$$

Now fix $i \in \{1, \ldots, s\}$. Define

$$k_i(x,t) = \begin{cases} \dfrac{|g_t(x)|^{r_i}}{|f_t(x)|}, & \text{if } x \in X_t, \ f_t(x) \neq 0 \\ 0, & \text{if } x \in X_t, \ f_t(x) = 0 \end{cases}$$

and

$$C_i(x,t) = \sup\{k_i(x',t) : x' \in \overline{B}(x, r(x,t))\}, \ (x,t) \in X|T_i.$$

Then k_i, C_i are $\mathcal{D}$-function. Note that r is continuous in x for each fixed $t \in T$. By the cellwise triviality theorem [3], Ch. 10 (2.6) and Theorem 1.5, there exists a $\mathcal{D}$-analytic decomposition of T_i into finite number of cells T_{ij} such that $r|(X|T_{ij})$ are continuous.

The theorem follows from the following

4.9. LEMMA. — *Let $Y_{ij} = \{(x,t) \in X|T_{ij} : k_i|(X|T_{ij})$ is not bounded in every neighborhood of $(x,t)\}$. The Y_{ij} is a $\mathcal{D}$-set and $\dim \pi(Y_{ij} < \dim T_{ij}$.*

Let us suppose that the lemma is true. Then, by Theorem 1.5, T_{ij} can be partitioned into finite cells $T_{ij\beta}$ such that for each $\beta k_i|(X|T_{ij\beta})$ is locally bounded. Since the $T_{ij\beta}$ are cells, they are locally closed in $\mathbb{R}^m$. Then $X|T_{ij\beta}$ are locally closed in $\mathbb{R}^n \times \mathbb{R}^m$. By Lemmas 2.4 and 2.5, for each (i,j,β) there is a continuous $\mathcal{D}$-function $h_{ij\beta} : X|T_{ij\beta} \to \mathbb{R}$ such that $k_i|(X|T_{ij\beta}) \leq h_{ij\beta}$. The cells $T_{ij\beta}$ and the functions $h_{ij\beta}$ satisfy (i), (ii) (with the exponents $r_{ij\beta} := r_i$).

It remains to prove Lemma 4.9. By the definition of Y_{ij},

$$Y_{ij} = \Big\{(x,t) \in X|T_{ij} : \forall\varepsilon > 0,\ \forall\delta > 0,\ \exists(x',t') \in X|T_{ij},\\ \|(x',t') - (x,t)\| < \varepsilon,\ k_i(x',t') > \frac{1}{\varepsilon}\Big\}.$$

So, by Theorem 1.2, Y_{ij} and $\pi(Y_{ij})$ are $\mathcal{D}$-sets. Suppose the assertion of the lemma is false, i.e. $\dim\pi(Y_{ij}) = \dim T_{ij}$. Then, by Theorem 1.5, there is a cell $U \subset \pi(Y_{ij})$ with $\dim U = \dim T_{ij}$. By Proposition 1.6 there is a $\mathcal{D}$-map $\rho : U \to Y_{ij}|U$ such that $\rho(t) = (x(t),t)$, $t \in U$. By theorem 1.5, shrinking U, we may suppose that ρ is analytic. So $\rho(U)$ is a cell contained in $Y_{ij}|U$. Since $C|\rho(U)$ is a $\mathcal{D}$-function, by Theorem 1.5, we may also suppose that $C|\rho(U)$ is analytic. Let $(x_0,t_0) \in \rho(U)$ and K be a compact neighborhood of (x_0,t_0) in $\rho(U)$. Then there are $\delta, M > 0$ such that

$$r(x,t) \geq \delta \ \text{ and } \ C_i(x,t) \leq M, \quad \forall(x,t) \in K.$$

Since $(x_0,t_0) \in Y_{ij}|U$ and K is its neighborhood, there exists $(x,t) \in X|T_{ij}$ such that

$$\|(x,t) - (x_0,t_0)\| < \frac{\delta}{2},\ k_i(x,t) > M,\ t \in \pi(K),\ \|\rho(t) - \rho(t_0)\| < \frac{\delta}{2}.$$

From the last inequality $x \in \overline{B}(x(t), r(t), t))$. This implies

$$k_i(x,t) \leq \sup\{k_i(x',t) : x' \in \overline{B}(x(t), r(x(t),t))\} = C_i(x,t) \leq M.$$

It is a contradiction. The proof of the theorem is complete. □

The following propositions have a similar argument.

4.10. PROPOSITION. — *Under the assumption of Theorem 4.8, there exist a $\mathcal{D}$-analytic decomposition of T into cells $C_1,\dots,C_k$ and $r_1,\dots,r_k \in \mathbb{Q}_+$ such that:*

(i) *For each $t \in C_i$, $\ell_{X_t}(f_t) = r_i$, $i = 1,\dots,k$.*

(ii) *For each $i \in \{1,\dots,k\}$ there exists a continuous $\mathcal{D}$-function h_i on $X|C_i$:*

$$h_i(x,t)|f(x,t)| \geq d^{r_i}(x, f_t^{-1}(0)), \quad \forall(x,t) \in X|C_i.$$

4.11. PROPOSITION. — *Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be a locally closed $\mathcal{D}$-set. Suppose that A_t is open for each $t \in \mathbb{R}^m$. Let*

$$X = \bigcup_{i=1}^{p}\{(x,t) \in A : f_i(x,t) = 0,\ g_{i1}(x,t) > 0,\dots,g_{ir}(x,t) > 0\},$$

$$Y = \bigcup_{j=1}^{q}\{(x,t) \in A : h_j(x,t) = 0,\ k_{j1}(x,t) > 0,\dots,k_{js}(x,t) > 0\},$$

where the $f_i, h_j, g_{i1}, \ldots, g_{ir}, k_{j1}, \ldots, k_{js}$ are $\mathcal{D}$-functions.

Suppose that $(f_i)_t, (h_j)_t, \ldots, (g_{i1})_t, \ldots, (g_{ir})_t, (k_{j1})_t, \ldots, (k_{js})_t$ are analytic on A_t and X_t, Y_t are closed in A_t, for all $t \in T$. Then there exist a $\mathcal{D}$-analytic decomposition of T into cells $C_1, \ldots, C_k$ and $r_1, \ldots, r_k \in \mathbb{Q}_+$ such that:

(i) *For each $t \in C_i$, $\ell_{A_t}(X_t, Y_t) = r_i$, $i = 1, \ldots, k$.*

(ii) *For each $i \in \{1, \ldots, k\}$ there exists a continuous $\mathcal{D}$-function h_i on $A|C_i$:*

$$h_i(x,t)(d(x, X_t) + d(x, Y_t)) \geq d^{r_i}(x, X_t \cap Y_t), \quad \forall (x,t) \in A|C_i.$$

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