EDOARDO BALLICO

A splitting theorem for the Kupka component of a foliation of $\mathbb{C}P^n$, $n \geq 6$. Addendum to a paper by O. Calvo-Andrade and N. Soares


<http://www.numdam.org/item?id=AIF_1995__45_4_1119_0>
A codimension one singular holomorphic foliation \( F \) of \( \mathbb{CP}^n \) is given by \( \omega \in H^0(\mathbb{CP}^n, \Omega(k)) \) (for some \( k \)) with \( \omega \neq 0 \), \( \omega \) not vanishing on a hypersurface. The Kupka subset \( K(F) := \{ P \in \mathbb{CP}^n : \omega(P) = 0, d\omega(P) \neq 0 \} \) of the singular set \( S(F) := \{ P \in \mathbb{CP}^n : \omega(P) = 0 \} \) of \( F \) has remarkable properties (e.g. if not empty it is a smooth submanifold of pure codimension 2 with strong stability properties with respect to deformations of \( F \)). For much more on this topic, see [GLM] and [CS]. Let \( K \neq \emptyset \) be a Kupka component of \( F \), i.e. ([CS]) a connected component of \( K(F) \). It was proved in [CL] that if \( K \) is a complete intersection, then \( F \) has a meromorphic first integral. Motivated by this result in [CS] it was conjectured and proved in some cases that every Kupka component is a complete intersection. Here we prove the following result.

**Theorem.** Let \( F \) be a codimension 1 singular holomorphic foliation of \( \mathbb{CP}^n \), \( n \geq 6 \), induced by \( \omega \in H^0(\mathbb{CP}^n, \Omega^1(k)) \) and such that the codimension 2 component of the singular set of \( F \) consists of a single compact Kupka component \( K \) with \( \deg(K) \neq k^2/4 \). Then \( K \) is a complete intersection.
The proof of this result uses in an essential way the results proven in [CS] and [GML]. We consider this paper as an addendum to [CS] and we invite the reader to turn to [GML] and [CS] for background, motivations, several results used here, and so on. For the results used on vector bundles and codimension 2 submanifolds of $\mathbb{CP}^n$, see [OSS], [FL] and [CS].

Assume that $F$ is induced by $\omega \in H^0(\mathbb{CP}^n, \Omega^1(k))$. Let $N_K$ be the normal bundle of $K$ in $\mathbb{CP}^n$. By [CS], Corollary 3.5, $N_K$ is the restriction $E|K$ to $K$ of a rank 2 vector bundle $E$ on $\mathbb{CP}^n$. $K$ is a complete intersection if and only if $E$ is the direct sum of two line bundles ([OSS]). If $n \geq 6$ every line bundle on $K$ is the restriction of a line bundle on $\mathbb{CP}^n$ (see [FL]). Hence, by a very nice result of Faltings ([F]) if $n \geq 6$ and $N_K$ is the direct sum of two line bundles, $K$ is a complete intersection. By [CS], Cor. 4.5 (2), we may assume $k > 0$. By [CS], Th. 3.4 (2) to prove our result we may distinguish two cases, according to the transversal type of $K$. First assume that the transversal type of $K$ is given by $\eta = px dy - qy dy$ with $p, q$ positive relatively prime integers. Look at [GML], Th. 2.3 and its proof at page 321 (in particular the two lines before eq. (2.6)) and use that $K$ is simply connected if $n \geq 6$ ([FL], Cor. 6.3). The quoted result [GML], Th. 2.3, was the essential input for the proof of [CS], Th. 3.4; then [CS], Th. 3.4, and the calculations in [CS], §4, on the applications of the Baum-Bott formulas to $K$ gave the proof of [CS], Cor. 4.5. By [GML], page 321, $N_K$ is in this case the direct sum of two line bundles. Hence our theorem is proved in this case. Now assume that the transversal type of $K$ is given by $\eta = px dy - qy dy$ with $p = q = 1$. By [CS], Th. 4.2, we have $\deg(K) = k^2/4$. Hence our theorem is proved even in this case.

The author was partially supported by MURST and GNSAGA of CNR (Italy).

BIBLIOGRAPHY


A SPLITTING THEOREM FOR THE KUPKA COMPONENT


Edoardo BALLICO,
Dept. of Mathematics
University of Trento
38050 Povo (TN) (Italie).