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A splitting theorem for the Kupka component of a foliation of $\mathbb{C}P^n$, $n \geq 6$. Addendum to a paper by O. Calvo-Andrade and N. Soares


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A SPLITTING THEOREM FOR THE KUPKA COMPONENT OF A FOLIATION OF CP\(^n\), \(n \geq 6\).

ADDENDUM TO A PAPER BY O. CALVO-ANDRADE AND N. SOARES

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A codimension one singular holomorphic foliation \(F\) of \(\mathbb{CP}^n\) is given by \(\omega \in H^0(\mathbb{CP}^n, \Omega(k))\) (for some \(k\)) with \(\omega \neq 0\), \(\omega\) not vanishing on a hypersurface. The Kupka subset \(K(F) := \{P \in \mathbb{CP}^n : \omega(P) = 0, d\omega(P) \neq 0\}\) of the singular set \(S(F) := \{P \in \mathbb{CP}^n : \omega(P) = 0\}\) of \(F\) has remarkable properties (e.g. if not empty it is a smooth submanifold of pure codimension 2 with strong stability properties with respect to deformations of \(F\)). For much more on this topic, see [GLM] and [CS]. Let \(K \neq \emptyset\) be a Kupka component of \(F\), i.e. ([CS]) a connected component of \(K(F)\). It was proved in [CL] that if \(K\) is a complete intersection, then \(F\) has a meromorphic first integral. Motivated by this result in [CS] it was conjectured and proved in some cases that every Kupka component is a complete intersection. Here we prove the following result.

Theorem. — Let \(F\) be a codimension 1 singular holomorphic foliation of \(\mathbb{CP}^n\), \(n \geq 6\), induced by \(\omega \in H^0(\mathbb{CP}^n, \Omega^1(k))\) and such that the codimension 2 component of the singular set of \(F\) consists of a single compact Kupka component \(K\) with \(\deg(K) \neq k^2/4\). Then \(K\) is a complete intersection.

Key words: Singular foliations – Codimension 1 foliations – Kupka component – Complete intersection – Unstable vector bundle – Rank 2 vector bundle – Splitting of a vector bundle – Meromorphic first integral.
The proof of this result uses in an essential way the results proven in [CS] and [GML]. We consider this paper as an addendum to [CS] and we invite the reader to turn to [GML] and [CS] for background, motivations, several results used here, and so on. For the results used on vector bundles and codimension 2 submanifolds of $\mathbb{CP}^n$, see [OSS], [FL] and [CS].

Assume that $F$ is induced by $\omega \in H^0(\mathbb{CP}^n, \Omega^1(k))$. Let $N_K$ be the normal bundle of $K$ in $\mathbb{CP}^n$. By [CS], Corollary 3.5, $N_K$ is the restriction $E|K$ to $K$ of a rank 2 vector bundle $E$ on $\mathbb{CP}^n$. $K$ is a complete intersection if and only if $E$ is the direct sum of two line bundles ([OSS]). If $n \geq 6$ every line bundle on $K$ is the restriction of a line bundle on $\mathbb{CP}^n$ (see [FL]). Hence, by a very nice result of Faltings ([F]) if $n \geq 6$ and $N_K$ is the direct sum of two line bundles, $K$ is a complete intersection. By [CS], Cor. 4.5 (2), we may assume $k > 0$. By [CS], Th. 3.4 (2) to prove our result we may distinguish two cases, according to the transversal type of $K$. First assume that the transversal type of $K$ is given by $\eta = px dy - qy dy$ with $p$, $q$ positive relatively prime integers. Look at [GML], Th. 2.3 and its proof at page 321 (in particular the two lines before eq. (2.6)) and use that $K$ is simply connected if $n \geq 6$ ([FL], Cor. 6.3). The quoted result [GML], Th. 2.3, was the essential input for the proof of [CS], Th. 3.4; then [CS], Th. 3.4, and the calculations in [CS], §4, on the applications of the Baum-Bott formulas to $K$ gave the proof of [CS], Cor. 4.5. By [GML], page 321, $N_K$ is in this case the direct sum of two line bundles. Hence our theorem is proved in this case. Now assume that the transversal type of $K$ is given by $\eta = px dy - qy dy$ with $p = q = 1$. By [CS], Th. 4.2, we have $\deg(K) = k^2/4$. Hence our theorem is proved even in this case.

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