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SHARP L^p-WEIGHTED SOBOLEV INEQUALITIES(*)

by Carlos PÉREZ

1. Introduction and statements of the results.

The purpose of this paper is to prove sharp weighted inequalities of the form

$$(1) \qquad \int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \leq C \int_{\mathbf{R}^n} |p(D)(f)(x)|^2 N(v)(x)dx,$$

for homogeneous differential operators $p(D)$. N will be an appropriate maximal type operator related to the order of the differential operator. We say that N "controls" the differential operator $p(D)$. These inequalities will be derived from corresponding weighted inequalities for fractional integrals or Riesz potentials similar to those obtained in [P2] for singular integral operators. The approach for potentials is direct and does not rely upon the duality argument used in [P2].

A model example for (1) is related to the theory of Schrödinger operators. This theory has recently received a lot of attention after the work by C. Fefferman and D. H. Phong described in [F]. In that paper the following problem is proposed. Let v be a nonnegative, locally integrable function on R^n, and consider its associated Schrödinger operator $L = -\Delta - v$. Then, integration by parts says that L is a positive operator whenever the "uncertainty principle" holds, namely

$$\int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \leq \int_{\mathbf{R}^n} |\nabla f(x)|^2 dx,$$

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for smooth functions f. One is thus led to consider conditions on v which would imply weighted Sobolev inequalities of the form

$$\int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \le C_v \int_{\mathbf{R}^n} |\nabla f(x)|^2 dx. \tag{2}$$

A sufficient condition for (2) is given when the potential v satisfies for some $r > 1$

$$\sup_Q |Q|^{2/n} \left(\frac{1}{|Q|} \int_Q v(y)^r dy \right)^{1/r} < \infty, \tag{3}$$

where the supremum is taken over all the cubes in $\mathbf{R}^n$. This is the so called C. Fefferman-Phong condition obtained in [F]. Observe that we can rephrase this condition by saying that the potential belongs to the Morrey space $L^{r,n-2r}$. The case $r = 1$ is necessary but not sufficient.

We shall consider inequalities of the form

$$\int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \le C \int_{\mathbf{R}^n} |\nabla f(x)|^2 N(v)(x)dx, \tag{4}$$

with C independent of f and v, such that we can recover conditions like (3) or similar when assuming that $N(v) \in L^\infty$.

One way to prove (4) is by means of the inequality

$$|f(x)| \le c I_1(|\nabla f|)(x) \tag{5}$$

which follows from the classical Sobolev integral representation ([Ma], [St1]). Here I_α, $0<\alpha<n$, denotes the fractional integral of order α on $\mathbf{R}^n$ or Riesz potentials defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Therefore one is now led to consider for $p > 1$ weighted inequalities of the form

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p v(x)dx \le C \int_{\mathbf{R}^n} |f(x)|^p N(v)(x)dx. \tag{6}$$

Perhaps the first result of this kind was obtained by D. Adams [A]: let $r > 1$ then

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p v(x)dx \le C \int_{\mathbf{R}^n} |f(x)|^p M_{\alpha pr}(v^r)(x)^{1/r} dx, \tag{7}$$

where M_β, $\beta \ge 0$, denotes the Marcinkiewicz or fractional maximal operators

$$M_\beta f(x) = \sup_{x \in Q} \frac{|Q|^{\beta/n}}{|Q|} \int_Q |f(y)|dy.$$

Observe that (7) with $\alpha = 1$ and $p = 2$ yields the C. Fefferman–Phong condition since (3) is equivalent to $M_{2r}(v^r)^{1/r} \in L^\infty$.

Our goal will be to sharpen (7) by replacing $M_{\alpha pr}(v^r)(x)^{1/r}$ by appropriate pointwise smaller operators. In fact our estimates are closely related to the work done in [CWW] by A. Chang, M. Wilson, and T. Wolff in the case $p = 2$, and for general $p > 1$ by S. Chanillo and R. Wheeden in [CW]. In these papers the sufficient condition (3) is sharpened in a very nice way. We recall the result for $p = 2$. Let $\varphi : [0, \infty) \to [1, \infty)$ increasing and doubling such that

$$\int_1^\infty \frac{1}{\varphi(t)} \frac{dt}{t} < \infty.$$

Then a sufficient condition is given by

$$\frac{|Q|^{2/n}}{|Q|} \int_Q v(y)\varphi(v(y)|Q|^{2/n})dy \leq C. \tag{8}$$

Observe that the case $\varphi(t) = t^{r-1}$, $r > 1$ corresponds to (3), but a more interesting example is obtained from $\varphi(t) = (1 + \log^+ t)^{1+\varepsilon}$ with $\varepsilon > 0$. We shall point out in Remark 1.5 that this condition is related to iterations of the Hardy–Littlewood maximal function M at least when ε is a positive integer, and for "fractional iterations" of M in the general case (see also Remark 1.4).

It should be mentioned that E. Sawyer obtained in [S2] a full characterization of the two weight problem for I_α

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p v(x)dx \leq C \int_{\mathbf{R}^n} |f(x)|^p u(x)dx.$$

However, such condition is difficult to test and in particular we do not know how to recover our results (part (A) of the Theorem). Also, R. Long and F. Nie [LN] have given a complete characterization of the two weight problem for the gradient, namely

$$\int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \leq C_v \int_{\mathbf{R}^n} |\nabla f(x)|^2 u(x)dx.$$

The condition for this problem is less restrictive than Sawyer's condition but still difficult to verify. We do not pursue in this direction and remit to [SW] for more information related to both problems.

As we mentioned above our results for potentials are related to the work in [P2] (also [Wil]), where the main result is the following. Let T be a classical singular integral operator (see [GCRdF]) and let $1 < p < \infty$. Then there exists a constant C such that for each weight w

$$\int_{\mathbf{R}^n} |Tf(x)|^p w(x)dx \leq C \int_{\mathbf{R}^n} |f(x)|^p M^{[p]+1} w(x)dx. \tag{9}$$

Here $M^k = M\circ \overset{(k)}{\cdots} \circ M$ denotes the Hardy–Littlewood maximal operator M iterated k times. This result was first obtained by M. Wilson in [Wil] in the range $1 < p \leq 2$. The estimate is sharp since it does not hold for $M^{[p]}$. The method used in [P2] is different from the one in [Wil] and it relies upon a duality argument combined with sharp weighted estimates for M from [P1].

As we said we are going to consider a different approach for the potentials. We shall treat them directly by writing down the operator as a sum of pieces after breaking down the kernel appropriately. Then, we shall combine ideas from [SW] and [P3] to sum up the pieces to get the desired estimates. In fact, this is one of the main points of the paper, namely that we can get optimal weighted inequalities for potential operators by using size estimates only. In this fashion we avoid the duality argument in [P2].

Our method is flexible enough to produce also sharp weighted estimates of the form

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p v(x)dx \leq C \int_{\mathbf{R}^n} M_\alpha(f)(x)^p N(v)(x)dx. \tag{10}$$

THEOREM 1.1. — *Suppose that $0 < \alpha < n$ and that v is a nonnegative measurable function on $\mathbf{R}^n$.*

(A) *Let $1 < p < \infty$. Then there exists a constant $C = C_{n,p}$ such that*

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p v(x)dx \leq C \int_{\mathbf{R}^n} |f(x)|^p M_{\alpha p}(M^{[p]}v)(x)dx. \tag{11}$$

(B) *Let $1 < p < \infty$. Then there exists a constant C_n such that*

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p v(x)dx \leq C \int_{\mathbf{R}^n} M_\alpha(f)(x)^p M^{[p]+1}(v)(x)dx. \tag{12}$$

When $p = 1$ we can find a constant C such that

$$\int_{\mathbf{R}^n} |I_\alpha f(x)| v(x)dx \leq C \int_{\mathbf{R}^n} M_\alpha f(x) Mv(x)dx. \tag{13}$$

All these estimates are sharp since we cannot replace $[p]$ by $[p]-1$. Some counterexamples will be given in §5.

That these estimates are sharper than (7) can be seen from

Remark 1.2. — Let $k = 1, 2, 3, \cdots$ and $r > 1$. Then there exists a constant $C = C_{n,r,k}$ such that for all non negative functions v

$$M_\beta(M^k(v))(x) \leq C M_{\beta r}(v^r)(x)^{1/r}. \tag{14}$$

This follows from standard arguments and the fact that

$$M^k v(x) \leq C^{k-1} M(v^r)(x)^{1/r}, \tag{15}$$

where $C = C_{n,r}$ is the best constant in the inequality $M(M(v^r)^{1/r}) \leq CM(v^r)^{1/r}$ (see [GCRdF], p. 158).

Some of the consequences for differential operators are as follows.

COROLLARY 1.3.

(A) *There exists a dimensional constant* C *such that*

$$\int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \leq C \int_{\mathbf{R}^n} |\nabla f(x)|^2 M_2(M^2(v))(x)dx \tag{16}$$

for all $f \in C_0^\infty(\mathbf{R}^n)$.

(B) *There exists a dimensional constant* C *such that*

$$\int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \leq C \int_{\mathbf{R}^n} |\Delta f(x)|^2 M_4(M^2(v))(x)dx \tag{17}$$

for all $f \in C_0^\infty(\mathbf{R}^n)$.

Part (A) follows from (5) and part (B) follows from the fact that $f = I_2(\Delta f)$. Part (B) is related to the work done in [CR] and [CS].

Remark 1.4. — If we look at the proof of the theorem we have the more precise estimate

$$\int_{\mathbf{R}^n} |f(x)|^2 v(x)dx \leq C \int_{\mathbf{R}^n} |\nabla f(x)|^2 M_2(M_{L(\log L)^\varepsilon}(v))(x)dx, \tag{18}$$

and the result is false for $\varepsilon = 0$. $M_{L(\log L)^\varepsilon}$ is a maximal type operator which can be seen as a fractional iteration of M (cf. next section for the precise definition). A corresponding result holds for Δf.

Remark 1.5. — As we mentioned above inequality (16) (and also (18)) is related to condition (8) of Chang, Wilson, and Wolff. Indeed if the potential v satisfies that

$$M_2(M^2(v)) \in L^\infty$$

then for some constant C and for all cubes

$$|Q|^{2/n} \frac{1}{|Q|} \int_Q M^2 v(y)dy \leq C.$$

Let $\varphi(t) = (1 + \log t)^2$. Then by homogeneity and Stein's inequality [St1] (see (33) below) we get

$$\frac{1}{|Q|} \int_Q v(y)|Q|^{2/n} \varphi(v(y)|Q|^{2/n})dy \leq \frac{1}{|Q|} \int_Q M^2(v|Q|^{2/n})(y)dy$$

$$= \frac{|Q|^{2/n}}{|Q|} \int_Q M^2(v)(y)dy \leq C,$$

yielding (8).

Another antecedent of the inequality in part (A) of the theorem is the following generalization due to E. Sawyer [S1] of the celebrated weighted inequality of C. Fefferman and E. Stein [FS1]

$$\int_{\mathbf{R}^n} |M_\alpha f(x)|^p w(x)dx \le C \int_{\mathbf{R}^n} |f(x)|^p M_{\alpha p}(w)(x)dx.$$

Observe that this estimate combined with (12) does not yield inequality (11). We loose one iteration by doing this.

2. Some preliminaries and notation.

As usual, a function $B : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex and increasing satisfying $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. We define the B–average of a function f over a cube Q by means of the Luxemburg norm

$$\|f\|_{B,Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\Big(\frac{|f(y)|}{\lambda}\Big)dy \le 1\Big\}. \tag{19}$$

The following generalized Hölder's inequality holds:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \le \|f\|_{B,Q}\|g\|_{\bar{B},Q}, \tag{20}$$

where $\bar{B}$ is the complementary Young function associated to B.

We define a natural maximal operator associated to the Young function associated to B.

Definition 2.1. — *For each locally integrable function f the maximal operator M_B is defined by*

$$M_B f(x) = \sup_{x\in Q} \|f\|_{B,Q},$$

where the supremum is taken over all the cubes containing x.

The main examples that we are going to be using are $B(t) = t(1 + \log^+ t)^m$, $m = 1,2,3,\cdots$, with maximal function denoted by $M_{L(\log L)^m}$. The complementary Young function is given by $\bar{B}(t) \approx e^{t^{1/m}}$ with corresponding maximal function denoted by $M_{(\exp L)^{1/m}}$.

The relevant class of Young functions is the following.

DEFINITION 2.2. — *Let $1 < p < \infty$. We say that a doubling Young function B satisfies the B_p condition if there is a positive constant c such that*

$$\int_c^\infty \frac{B(t)}{t^p}\frac{dt}{t} < \infty.$$

This condition gives a characterization of those maximal operators M_B which are bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$. In fact, we have the following.

THEOREM 2.3. — *Let $1 < p < \infty$. Suppose that B is a Young function. Then the following are equivalent:*

i)

$$B \in B_p; \tag{21}$$

ii) *there is a constant c such that*

$$\int_{\mathbf{R}^n} M_B f(y)^p dy \leq c \int_{\mathbf{R}^n} f(y)^p dy \tag{22}$$

for all nonnegative functions f;

iii) *there is a constant c such that*

$$\int_{\mathbf{R}^n} M_B f(y)^p w(y) dy \leq c \int_{\mathbf{R}^n} f(y)^p Mw(y) dy \tag{23}$$

for all nonnegative functions f and w;

iv) *there is a constant c such that*

$$\int_{\mathbf{R}^n} Mf(y)^p \frac{w(y)}{[M_{\bar{B}}(u^{1/p})(y)]^p} dy \leq c \int_{\mathbf{R}^n} f(y)^p \frac{Mw(y)}{u(y)} dy, \tag{24}$$

for all nonnegative functions f, w and u.

In the proof of Theorem 1.1 and for $p > 1$ we shall be working with Young functions of the form $B(t) \approx t^p(\log t)^{-1-\varepsilon}$ which satisfies the B_p condition and therefore their associated maximal operators $M_{L^p(\log L)^{-1-\varepsilon}}$ are bounded on $L^p(\mathbf{R}^n)$.

The proof of this result can be found in [P1].

3. Basic lemma.

LEMMA 3.1. — *Let f and g be L^∞ functions with compact support, and let μ be a nonnegative measure finite on compact sets. Let $a > 2^n$, then there exists a family of cubes $Q_{k,j}$ and a family of pairwise disjoint subsets $E_{k,j}$, $E_{k,j} \subset Q_{k,j}$, with*

$$|Q_{k,j}| < \frac{1}{1-\frac{2^n}{a}} |E_{k,j}| \tag{25}$$

for all k, j, and such that

$$\begin{aligned} &\int_{\mathbf{R}^n} I_\alpha f(x) g(x) d\mu(x) \\ &\qquad \le C \sum_{k,j} \frac{|Q_{k,j}|^{\alpha/n}}{|Q_{k,j}|} \int_{3Q_{k,j}} f(y)dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y)d\mu(y) |E_{k,j}|. \end{aligned} \tag{26}$$

Proof. — Observe first that $I_\alpha f(x) < \infty$ for all $x \in \mathbf{R}^n$. Then, we discretize the operator $I_\alpha f$ as follows:

$$\begin{aligned} I_\alpha f(x) &= \sum_{\nu \in \mathbf{Z}} \int_{2^{-\nu-1} < |x-y| \le 2^{-\nu}} \frac{f(y)}{|x-y|^{n-\alpha}} dy \\ &= \sum_{\nu \in \mathbf{Z}} \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) = 2^{-\nu}}} \chi_Q(x) \int_{\ell(Q)/2 < |x-y| \le \ell(Q)} \frac{f(y)}{|x-y|^{n-\alpha}} dy \\ &\le C \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{|y-x| \le \ell(Q)} f(y) dy \chi_Q(x). \end{aligned}$$

$\mathcal{D}$ denotes the family of dyadic cubes on $\mathbf{R}^n$. Since the ball $B(x, \ell(Q))$ is contained in the cube $3Q$ when $x \in Q$ we have

$$I_\alpha f(x) \le \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f(y) dy \chi_Q(x),$$

and then

$$\int_{\mathbf{R}^n} I_\alpha f(x) g(x) d\mu(x) \le c \sum_{Q \in \mathcal{D}} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f(y) dy \int_Q g(x) d\mu(x).$$

Recall that $a > 2^n$ and consider for each integer k the set

$$D_k = \{x \in \mathbf{R}^n : M^d(g d\mu)(x) > a^k\},$$

where M^d is the usual dyadic Hardy–Littlewood maximal operator. Then it follows that if D_k is not empty there exists some dyadic cube Q with

$$a^k < \frac{1}{|Q|}\int_Q g(y)d\mu(y), \tag{27}$$

then Q is contained in one dyadic cube satisfying this condition and maximal with respect to inclusion. Thus for each k we can write $D_k = \bigcup_j Q_{k,j}$ where the cubes $\{Q_{k,j}\}$ are nonoverlapping, they satisfy (27), and by maximality we also get

$$a^k < \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} g(y)d\mu(y) \le 2^n a^k. \tag{28}$$

We also need the following property. For all integers k, j we let $E_{k,j} = Q_{k,j} - Q_{k,j} \cap D_{k+1}$. Then $\{E_{k,j}\}$ is a disjoint family of sets which satisfy

$$|Q_{k,j} \cap D_{k+1}| < \frac{2^n}{a}|Q_{k,j}|, \tag{29}$$

and therefore

$$|Q_{k,j}| < \frac{1}{1-\frac{2^n}{a}}|E_{k,j}|. \tag{30}$$

Indeed, by standard properties of the dyadic cubes we can compute what portion of $Q_{k,j}$ is covered by D_{k+1} as in [GCRdF], p. 398:

$$\begin{aligned}\frac{|Q_{k,j} \cap D_{k+1}|}{|Q_{k,j}|} &= \sum_i \frac{|Q_{k,j} \cap Q_{k+1,i}|}{|Q_{k,j}|} = \sum_{i:Q_{k+1,i}\subset Q_{k,j}} \frac{|Q_{k+1,i}|}{|Q_{k,j}|} \\ &< \frac{4^n}{a^{k+1}} \sum_{i:Q_{k+1,i}\subset Q_{k,j}} \frac{1}{|Q_{k,j}|}\int_{Q_{k+1,i}} g(y)d\mu(y) \\ &\le \frac{4^n}{a^{k+1}} \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}\cap \cup_i Q_{k+1,i}} g(y)d\mu(y) \le \frac{2^n}{a}.\end{aligned}$$

This gives (29).

We continue with the proof of the lemma by adapting some ideas from [SW]. For each integer k we let

$$\mathcal{C}^k = \{Q \in \mathcal{D} : a^k < \frac{1}{|Q|}\int_Q g(y)d\mu(y) \le a^{k+1}\}.$$

Every dyadic cube Q for which $\int_Q g(y)d\mu(y) \neq 0$ belongs to exactly one

$\mathcal{C}^k$. Furthermore, if $Q \in \mathcal{C}^k$ it follows that $Q \subset Q_{k,j}$ for some j. Then

$$\begin{aligned}
\int_{\mathbf{R}^n} I_\alpha f(x) g(x) d\mu(x) &\leq \sum_{k\in\mathbf{Z}} \sum_{Q\in\mathcal{C}^k} \frac{|Q|^{\alpha/n}}{|Q|} \int_{3Q} f(y)dy \int_Q g(y)d\mu(y) \\
&\leq a \sum_{k\in\mathbf{Z}} a^k \sum_{j\in\mathbf{Z}} \sum_{\substack{Q\in\mathcal{C}^k \\ Q\subset Q_{k,j}}} |Q|^{\alpha/n} \int_{3Q} f(y)dy \\
&\leq C \sum_{k,j} \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y)d\mu(y) |Q_{k,j}|^{\alpha/n} \int_{3Q_{k,j}} f(y)dy.
\end{aligned}$$

The last inequality follows from the fact that for each dyadic cube P

$$\sum_{\substack{Q\in\mathcal{D} \\ Q\subset P}} |Q|^{\alpha/n} \int_{3Q} f(y)dy \leq C|P|^{\alpha/n} \int_{3P} f(y)dy.$$

Indeed, if $\ell(P) = 2^{-\nu_0}$

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{D} \\ Q\subset P}} |Q|^{\alpha/n} \int_{3Q} f(y)dy &= C \sum_{\nu\geq\nu_0} \sum_{\substack{Q\in\mathcal{D}:Q\subset P \\ \ell(Q)=2^{-\nu}}} 2^{-\nu\alpha} \int_{3Q} f(y)dy \\
&= C \sum_{\nu\geq\nu_0} 2^{-\nu\alpha} \sum_{\substack{Q\in\mathcal{D}:Q\subset P \\ \ell(Q)=2^{-\nu}}} \int_{3Q} f(y)dy \\
&\leq C|P|^{\alpha/n} \sum_{\substack{Q\in\mathcal{D}:Q\subset P \\ \ell(Q)=2^{-\nu}}} \int_{3Q} f(y)dy \\
&\leq C|P|^{\alpha/n} \int_{3P} f(y)dy
\end{aligned}$$

since the overlap is finite. Hence

$$\begin{aligned}
&\int_{\mathbf{R}^n} I_\alpha f(x) g(x) d\mu(x) \\
&\quad\leq C \sum_{k,j} \frac{|3Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y)dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y)d\mu(y) |Q_{k,j}| \\
&\quad\leq C \sum_{k,j} \frac{|3Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|} \int_{3Q_{k,j}} f(y)dy \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} g(y)d\mu(y) |E_{k,j}|,
\end{aligned}$$

by (30). $\square$

4. Proof of the theorem.

Proof of (A). — Since $p>1$, it is enough to show that there exists a positive constant C such that

$$\begin{aligned}(I) &= \int_{\mathbf{R}^n} I_\alpha f(x) v(x)^{1/p} g(x)dx \\ &\le C\left[\int_{\mathbf{R}^n} f(x)^p M_{\alpha p}(M^{[p]}v)(x)dx\right]^{1/p}\left[\int_{\mathbf{R}^n} g(x)^{p'}dx\right]^{1/p'}\end{aligned}$$

for all nonnegative, bounded functions f and g with compact support. Now, by Lemma 3.1 with $d\mu(x)=v(x)^{1/p}dx$ we have that

$$(I)\le C\sum_{k,j}\frac{|3Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|}\int_{3Q_{k,j}} f(y)dy\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} v(y)^{1/p}g(y)dy|E_{k,j}|.$$

Next we apply the generalized Hölder's inequality (20) with respect to the associated spaces $L^p(\log L)^{(p-1)(1+\varepsilon)}$ and $L^{p'}(\log L)^{-1-\varepsilon}$, $\varepsilon>0$, (see [O] for instance). After that we apply Hölder's inequality at the discrete level with exponents p and p'. We can follow the estimate with

$$\begin{aligned}&C\sum_{k,j}\frac{|3Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|}\int_{3Q_{k,j}} f(y)dy\|v^{1/p}\|_{L^p(\log L)^{(p-1)(1+\varepsilon)}}|E_{k,j}|^{1/p} \\ &\qquad\qquad\times\|g\|_{L^{p'}(\log L)^{-1-\varepsilon}}|E_{k,j}|^{1/p'} \\ &\le C\left[\sum_{k,j}\left(\frac{|Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|}\int_{3Q_{k,j}} f(y)dy\right)^p\|v^{1/p}\|^p_{L^p(\log L)^{(p-1)(1+\varepsilon)},}|E_{k,j}|\right]^{1/p} \\ &\qquad\qquad\times\left[\sum_{k,j}\|g\|^{p'}_{L^{p'}(\log L)^{-1-\varepsilon}}|E_{k,j}|\right]^{1/p'}.\end{aligned}$$

Now, if $\varepsilon=\dfrac{[p]}{p-1}-1>0$, then

$$\|v^{1/p}\|^p_{L^p(\log L)^{(p-1)(1+\varepsilon)}}=\|v\|_{L(\log L)^{[p]}}. \tag{31}$$

Now, we need the following lemma

LEMMA 4.1. — *If $k=1,2,3,\cdots$, then there exists a constant $C=C_n$ such that for all bounded functions f with support contained in Q*

$$\|f\|_{L(\log L)^k,Q}\le\frac{C}{|Q|}\int_Q M^k f(y)dy. \tag{32}$$

Indeed, by homogeneity we can assume that the right hand side is equal to C. Then by the definition of the Luxemburg norm it is enough to prove

$$\frac{1}{|Q|}\int_Q f(y)(1+\log^+(f(y)))^k dy \leq C,$$

but this is a consequence of iterating the following inequality of E.M. Stein:

$$\int_Q f(y)(1+\log^+(f(y)))^k dy \leq C\int_Q Mf(y)(1+\log^+(Mf(y)))^{k-1}dy, \tag{33}$$

with $k=1,2,3,\cdots$.

Then using (31) and (32) we get

$$\begin{aligned}
(I) &\leq C\left[\sum_{k,j}\left(\frac{|Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|}\int_{3Q_{k,j}} f(y)dy\right)^p \frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} M^{[p]}v(y)dy|E_{k,j}|\right]^{1/p} \\
&\qquad \times\left[\sum_{k,j}\|g\|^{p'}_{L^{p'}(\log L)^{-1-\varepsilon}}|E_{k,j}|\right]^{1/p'} \\
&\leq C\left[\sum_{k,j}\int_{E_{k,j}}\left(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f(y)M_{\alpha p}(M^{[p]}v)(y)^{1/p}dy\right)^p dx\right]^{1/p} \\
&\qquad \times\left[\sum_{k,j}\int_{E_{k,j}} M_{L^{p'}(\log L)^{-1-\varepsilon}}g(x)^{p'}dx\right]^{1/p'} \\
&\leq C\left[\sum_{k,j}\int_{E_{k,j}} M(fM_{\alpha p}(M^{[p]}v)^{1/p})(x)^p dx\right]^{1/p} \\
&\qquad \times\left[\int_{\mathbf{R}^n} M_{L^{p'}(\log L)^{-1-\varepsilon}}g(x)^{p'}dx\right]^{1/p'} \\
&\leq C\left[\int_{\mathbf{R}^n} M(fM_{\alpha p}(M^{[p]}v)^{1/p})(x)^p dx\right]^{1/p}\times\left[\int_{\mathbf{R}^n} g(x)^{p'}dx\right]^{1/p'} \\
&\approx \left[\int_{\mathbf{R}^n} f(x)^p M_{\alpha p}(M^{[p]}v)(x)dx\right]^{1/p}\times\left[\int_{\mathbf{R}^n} g(x)^{p'}dx\right]^{1/p'}.
\end{aligned}$$

We used first that the sets in the family $E_{k,j}$ are pairwise disjoint. Also it is used that M is a bounded operator on $L^p(\mathbf{R}^n)$ and that the maximal operator $M_{L^{p'}(\log L)^{-1-\varepsilon}}$ is bounded on $L^{p'}(\mathbf{R}^n)$ by Theorem 2.3 since the Young function $B(t)\approx t^{p'}(\log t)^{-1-\varepsilon}$ satisfies the $B_{p'}$ condition.

Proof of (B). — Consider first the case $p > 1$. Hence as above we prove that

$$(I) = \int_{\mathbf{R}^n} I_\alpha f(x) v(x)^{1/p} g(x)dx$$
$$\leq C\left[\int_{\mathbf{R}^n} M_\alpha(f)(x)^p M^{[p]+1}(v)(x)dx\right]^{1/p}\left[\int_{\mathbf{R}^n} g(x)^{p'}dx\right]^{1/p'}$$

for all nonnegative, bounded functions f and g with compact support. Now, by Lemma 3.1 with $d\mu(x) = v(x)^{1/p}dx$ and by the disjoint property of the sets $E_{k,j}$ we have

$$(I) \leq C\sum_{k,j}\frac{|3Q_{k,j}|^{\alpha/n}}{|3Q_{k,j}|}\int_{3Q_{k,j}} f(y)dy\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} v(y)^{1/p}g(y)dy|E_{k,j}|$$
$$\leq \sum_{k,j}\int_{E_{k,j}} M_\alpha(f)(x)M(v^{1/p}g)(x)dx$$
$$\leq \int_{\mathbf{R}^n} M_\alpha(f)(x)M(v^{1/p}g)(x)dx.$$

Using again that the spaces $L^p(\log L)^{(p-1)(1+\varepsilon)}$ and $L^{p'}(\log L)^{-1-\varepsilon}$, $\varepsilon > 0$, are associated one to each other we can apply the generalized Hölder's inequality (20)

$$\leq C\int_{\mathbf{R}^n} M_\alpha(f)(x)M_{L^p(\log L)^{(p-1)(1+\varepsilon)}}(v^{1/p})(x)M_{L^{p'}(\log L)^{-1-\varepsilon}}(g)(x)dx$$
$$\leq C\left[\int_{\mathbf{R}^n} M_\alpha(f)(x)^p M_{L^p(\log L)^{(p-1)(1+\varepsilon)}}(v^{1/p})(x)^p dx\right]^{1/p}$$
$$\times\left[\int_{\mathbf{R}^n} M_{L^{p'}(\log L)^{-1-\varepsilon}}(g)(x)^{p'}dx\right]^{1/p'}.$$

As above taking $\varepsilon = \dfrac{[p]}{p-1} - 1 > 0$ and using Lemma 4.1 we get

$$(I) \leq C\left[\int_{\mathbf{R}^n} M_\alpha(f)(x)^p M^{[p]+1}(v)(x)dx\right]^{1/p}\times\left[\int_{\mathbf{R}^n} g(x)^{p'}dx\right]^{1/p'}$$

since as above $B \in B_{p'}$ and we apply again Theorem 2.3. □

5. Examples.

We show in this section that the results in Theorem 1.1 are sharp. For part (A) we show that

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p w(x)dx \leq C\int_{\mathbf{R}^n} |f(x)|^p M_{\alpha p}(M^{[p]-1}w)(x)dx$$

is false in general. By duality, this inequality is equivalent to

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^{p'} M_{\alpha p}(M^{[p]-1}w)(x)^{1-p'}dx \leq C \int_{\mathbf{R}^n} |f(x)|^{p'} w(x)^{1-p'}dx.$$

Let $f = w = \chi_B$ where $B = B(0,1)$ is the unit ball centered at the origin so that the right hand side is finite. For the left hand side we get

$$\begin{aligned}
&\int_{\mathbf{R}^n} I_\alpha f(x)^{p'} M_{\alpha p}(M^{[p]-1}f)(x)^{1-p'}dx \\
&\qquad \geq \int_{\mathbf{R}^n} M_\alpha f(x)^{p'} M_{\alpha p}(M^{[p]-1}f)(x)^{1-p'}dx \\
&\qquad \geq \int_{|x|>e} M_\alpha f(x)^{p'} M_{\alpha p}(M^{[p]-1}f)(x)^{1-p'}dx.
\end{aligned}$$

Now for $|x| > 1$, $0 < \alpha < n$, $k = 0, 1, 2, \cdots$ we have

$$M_\alpha(M^k f)(x) \approx \frac{(\log|x|)^k}{|x|^{n-\alpha}}.$$

Then

$$\begin{aligned}
&\int_{|x|>e} M_\alpha f(x)^{p'} M_{\alpha p}(M^{[p]-1}f)(x)^{1-p'}dx \\
&\qquad \approx \int_{|x|>e} \frac{1}{|x|^{(n-\alpha)p'}} \left(\frac{(\log|x|)^{[p]-1}}{|x|^{n-\alpha p}}\right)^{1-p'} dx \\
&\qquad \approx \int_{|x|>e} (\log|x|)^{([p]-1)(1-p')}dx \\
&\qquad \approx \int_1^\infty t^{([p]-1)(1-p')}dt = \infty.
\end{aligned}$$

Finally we also show that

$$\int_{\mathbf{R}^n} |I_\alpha f(x)|^p w(x)dx \leq C \int_{\mathbf{R}^n} M_\alpha(f)(x)^p M^{[p]}(w)(x)dx$$

is false in general. We cannot use duality since we do not have the dual space of the Banach space $\{f : \int_{\mathbf{R}^n} M_\alpha(f)(x)^p v(x)dx < \infty\}$.

Let $n = 1$, $0 < \alpha < 1$, and $1 < p < 2$. Consider $f(y) = \chi_{(0,1)}(y)|y|^{-\alpha}$. Then we have

$$I_\alpha f(x) \geq C \log\frac{1}{|x|} \quad 0 < x < 1,$$

and

$$M_\alpha f(x) \leq C_n.$$

For $\varepsilon > 0$ we let $w(x) = \chi_{(0,\frac{1}{e})}(x)\frac{1}{x\left(\log\frac{1}{x}\right)^{\varepsilon+2}}$. Then

$$M(w)(x) \approx w(x)\log\frac{1}{x} \quad 0 < x < 1.$$

Hence

$$\begin{aligned}\int_{\mathbf{R}} I_\alpha f(x)^p w(x)dx &\geq C\int_0^{1/e}\left(\log\frac{1}{x}\right)^p \frac{1}{x\left(\log\frac{1}{x}\right)^{\varepsilon+2}}dx \\ &= \int_1^\infty \frac{1}{t^{1+\varepsilon-p}}\frac{dt}{t} = \infty\end{aligned}$$

if we choose $0 < \varepsilon \leq p-1$. On the other hand

$$\begin{aligned}&\int_{\mathbf{R}} M_\alpha f(x)^p Mw(x)dx \\ &\quad\leq C^p\int_{|x|\leq 1/e} Mw(x)dx + \int_{|x|\geq 1/e}\frac{C}{x^{(1-\alpha)p}}Mw(x)dx \\ &\quad\leq C\int_0^{1/e}\frac{1}{\left(\log\frac{1}{x}\right)^{\varepsilon+1}}\frac{dx}{x} + C\int_{1/e}^\infty \frac{1}{x^{(1-\alpha)p}}\frac{dx}{x} < \infty.\end{aligned}$$

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