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THE MARTIN COMPACTIFICATION OF A PLANE DOMAIN

by Nikolai S. NADIRASHVILI

In this note we prove the following

THEOREM. — *The Martin compactification of a plane domain is homeomorphic to a subset of the two-dimensional sphere.*

ASSUMPTIONS. — *If Ω be a plane domain and $\mathbb{R}^2\backslash\Omega$ is polar then any positive harmonic function on Ω is a constant. In this case we define the Martin compactification of Ω as a one point set. So we assume from now on that $\mathbb{R}^2\backslash\Omega$ is non-polar. We may also assume without loss of generality that $\bar{B}_1 \subset \Omega$ where B_1 is the unit disk in $\mathbb{R}^2$ with the center at* 0.

Remark. — If a simply connected domain is a proper subset of the plane then by Riemann mapping theorem its Martin compactification is homeomorphic to a closed disk.

CONJECTURE 1. — *The Martin compactification of a subdomain of a compact Riemannian surface is homeomorphic to a subset of this surface.*

CONJECTURE 2. — *Any compact metrizable space can be represented as the Martin boundary of a certain (generaly of infinite genus) Riemannian surface.*

1. *The Martin compactification.*

Let $G(x,y)$ be the Green function of the Dirichlet Laplacian on Ω, with the pole at x. Let us denote $g_x(y) = G(x,y)/G(x,0)$ for $x \neq 0$ and

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$g_0 \equiv 0$. Let $\tilde{g}_x(y)$ be the restriction of the function $g_x(y)$ on $y \in B_1$. So we have a map

$$\gamma : x \to \tilde{g}_x \in L^2(B_1).$$

The Martin metric on Ω can be defined as the metric inducted on Ω by the map $\gamma : \Omega \to L^2(B_1)$, (cf. [1]). Compactification of Ω in the Martin metric we denote as Ω^M.

Canonical map.

We set

$$f : x \to \nabla_y g_x(0)$$

and $f(0) = \infty$ by the definition. We claim that the introduced canonical map f has the uniformly continuous inverse map from $f(\Omega)$ to Ω^M.

Proof of the theorem.

1.1. Let $G \subset \mathbb{R}^2$ be a domain and Q a disk such that $\bar{Q} \subset G$. Also, let $a_i \in \partial Q, i = 1, ..., 2n$, be distinct points on ∂Q. We assume that the a_i are indexed in the order in which they are encountered when traversing ∂Q. Let f be a continuous function in $G \backslash Q$ such that $f(a_i)f(a_{i+1}) < 0$ for all $i = 1, ..., 2n-1$. We denote by $G_i \subset G \backslash \bar{Q}$ the domain where f does not change sign, such that $a_i \in \bar{G}_i$.

Lemma ([2]). — *At least $n+1$ of the domains G_i, $i = 1, ..., 2n$, are distinct.*

1.2. Let $x_1, x_2 \in \Omega$. We prove that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Let $u = g_{x_1} - g_{x_2}$. Then $\Delta u = 0$ in $\Omega \backslash (\{x_1\} \cup \{x_2\})$, $u(0) = \nabla u(0) = 0$. Let Γ be the nodal set of u, $\Gamma = \{x \in \Omega, u(x) = 0\}$. If $u \not\equiv 0$ then in a neighborhood of 0, Γ consists of n smooth curves intersected at the point 0, where n is an order of vanishing of the function u at 0 (cf. [2]). By Lemma Γ splits the domain Ω at least on three distinct subdomains. By maximum principle each of those subdomains should contain a pole of the function u. Since function u has only two poles x_1, x_2, it follows that $u \equiv 0$.

2. Now we prove that the map

$$F : z = f(x) \in f(\Omega) \to \tilde{g}_x$$

is uniformly continuous.

2.1. Let $\bar{B}_1 \subset B$, $\bar{B} \subset \Omega$. By Harnak inequality for any $x \in \Omega \backslash B$, $\tilde{g}_x < C$, where $C > 0$ is some constant.

2.2. Let x_n, $z_n \in \Omega$, $n = 1, 2, \dots$, and $g_{x_n} \to h_1$, $g_{z_n} \to h_2$ on any compact in Ω as $n \to \infty$, $h_1 \not\equiv h_2$. Its required to prove that $\nabla h_1(0) \neq \nabla h_2(0)$. Let us assume the contrary, namely that $\nabla h_1(0) = \nabla ! h_2(0)$. We denote $h = h_1 - h_2$ and let k be an order of vanishing of the function h at 0, $k \geq 2$.

2.3. Let Γ be the nodal set of the function h. There exists such a small $\rho > 0$ that on $S_\rho = \partial B_\rho$, $|\nabla h| > 0$ and the cardinality of the set $S_\rho \cap \Gamma$ is equal to $2k$.

2.4. We prove the existence of two bounded non-constant harmonic functions v_1, v_2 in Ω, such that $\nabla v_1(0) \neq 0$, $\nabla v_2(0) \neq 0$, $\nabla v_1(0) \neq a \nabla v_2(0)$, for any $a \in \mathbb{R}$.

Let us choose discs $D_1, D_2, D_3 \subset \mathbb{R}^2$ such that $D_i \backslash \Omega$ non-polar, $i = 1, 2, 3$, and for any points $x_i \in D_i$ the quadrangle $0, x_1, x_2, x_3$ is convex. Let μ_i be a probability measure on $D_i \backslash \Omega$ such that the convolution $ln\,|x| * \mu_i$ is bounded from below. We set $v_1 = ln\,|x| * (\mu_1 - \mu_2)$, $v_2 = ln\,|x| * (\mu_3 - \mu_2)$. Then v_1, v_2 are bounded harmonic functions in Ω and the $\nabla v_1(0), \nabla v_2(0)$ have the required property.

For any $\alpha \in \mathbb{R}^2$ there exists a unique linear combination

$$w_\alpha = \beta_1 v_1 + \beta_2 v_2 - \beta_1 v_1(0) - \beta_2 v_2(0)$$

such that $\nabla w_\alpha(0) = \alpha$, $w_\alpha(0) = 0$. Further, if $|\alpha| \to 0$ then $|w_\alpha| \to 0$ uniformly in Ω.

2.5. Let us denote

$$\nabla g_{x_n}(0) - \nabla g_{z_n}(0) = \alpha_n,$$

$$q_n = g_{x_n} - g_{z_n} - w_{\alpha_n}.$$

Then $q_n(0) = \nabla q_n(0) = 0$ for all $n = 1, 2, \dots$.

From (2.1), (2.2), (2.4) it follows that $q_n \to h$ in B_1 and hence also $q_n \to h$ in $C^1(B_\rho)$ as $n \to \infty$. Therefore, if Γ_n is a nodal set of q_n then for a sufficiently large $n \geq N$, $S_\rho \backslash \Gamma_n$ is a union of $2k$ distinct intervals $I_n^1, \ldots, I_n^{2k}$ and

$$\sup_{n \geq N} \inf_{1 \leq j \leq 2k} \sup_{I_n^j} q_n > a > 0$$

with some constant a. Since $w_{\alpha_n} \to 0$ uniformly in Ω as $n \to \infty$ then for sufficiently large $n \geq N' \geq N$, $|w_{\alpha_n}| < a$ in Ω. Hence $|q_n| < a$ on $\partial\Omega$ for $n \geq N'$.

2.6. Since $q_n(0) = \nabla q_n(0) = 0$ then by Lemma the set $\Omega \backslash \Gamma_n$ contains at least three components G_1, G_2, G_2 such that $0 \in \bar{G}_i$, $i = 1, 2, 3$. From (2.5) it follows that for $n \geq N'$ and $i = 1, 2, 3$

$$\sup_{G_i \cap B_\rho} |q_n| > \sup_{\partial G_i} |q_n| \, .$$

By the maximum principle from the last inequality it follows that any of the domains G_i, $i = 1, 2, 3$ contains a pole of function q_n. Since the function q_n has only two poles we get a contradiction which proves the theorem.

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