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p-ADIC ORDINARY HECKE ALGEBRAS FOR GL(2)

by
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In this paper, we first give a full proof of the control theorem of the universal nearly ordinary Hecke algebra for $G = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GL}(2)$ (Theorem 3.2) for an arbitrary number field F, which is announced in [H5] without proof. As a result of this, we can define the space of ordinary p-adic (cohomological) modular forms as the p-adic dual of the Hecke algebra. The group $\mathbb{G}$ which controls the algebra is isogenous to $T(\mathbb{Z}_p)/\overline{\mathfrak{r}^\times}$ for the maximal F-split torus T of G and the integer ring $\mathfrak{r}$ of F. Then the subspace of the space of p-adic ordinary modular forms on which $\mathbb{G}$ acts via an algebraic character κ of T trivial on units is the space of ordinary modular forms of weight κ for a given level N. By a motivic and also an analytic reason, if F has at least one complex place, the Hecke algebra is of torsion over the Iwasawa algebra Λ of the torsion free part of $\mathbb{G}$. In Section 4, we study the CM component of the Hecke algebra and clarify the relation between the annihilator in Λ of the CM component and the p-adic closure of the unit group of the corresponding quadratic extension of F (Theorem 4.1 and Proposition 4.2). Then we will make a conjecture which implies that the codimension of the image of the spectrum of the Hecke algebra in $\mathrm{Spec}(\Lambda)$ is equal to the number of complex places r_2 of F (Conjecture 4.3). If one applies the conjecture

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to a CM component of the Hecke algebra, the conjecture implies the Leopoldt conjecture for the quadratic extension corresponding to the CM component. Thus the conjecture may be viewed as a non-abelian analogue of the Leopoldt conjecture. In Sections 5 and 6, we study more closely (than in [H1]) p-ordinary parabolic cohomology groups in the simplest case where F has only one complex place. In this case, we will show that the Pontryagin dual of the p-ordinary parabolic cohomology group of level p^∞ is a torsion Λ-module of homological dimension 1 (Theorems 5.2 and 6.2). This implies that the module has no non-trivial pseudo null submodules. The characteristic power series in Λ of this cohomology group is divisible by that of the nearly ordinary Hecke algebra, and we expect that these two characteristic power series are very close to each other. In particular, we determine the characteristic power series of CM irreducible components when F has only one complex place (Theorem 5.3).

1.

In this section, we recall the definition of cohomological Hecke operators. We use the same notation introduced in [H1] (see corrections listed at the end of this paper). Thus F denotes a number field. Let B be a quaternion algebra over F. We consider the algebraic group $G = B^\times$ over $\mathbb{Q}$. Thus $G(A) = (B \otimes_{\mathbb{Q}} A)^\times$ for each $\mathbb{Q}$-algebra A. We suppose that

$$(\mathrm{Sp}_p) \qquad G(\mathbb{Q}_p) \cong \mathrm{GL}_2(F_p) \quad \text{for} \quad F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

Since the case where B is definite is already treated in [H2], we assume that:

$$(\mathrm{Sp}_\infty) \qquad \begin{cases} B \otimes_{\mathbb{Q}} \mathbb{R} \text{ is indefinite, that is, it has at least one simple} \\ \text{component isomorphic to either } M_2(\mathbb{R}) \text{ or } M_2(\mathbb{C}). \end{cases}$$

We write r (resp. r_2) for the number of simple components of $B \otimes_{\mathbb{Q}} \mathbb{R}$ isomorphic to $M_2(\mathbb{R})$ (resp. $M_2(\mathbb{C})$). We fix a maximal order R of B and an identification $R_{\mathfrak{l}} \cong M_2(\mathfrak{r}_{\mathfrak{l}})$ as $\mathfrak{r}_{\mathfrak{l}}$-algebras for primes $\mathfrak{l}$ of F whenever possible. We put $\widehat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. For each open compact subgroup U of $\widehat{R}^\times$, we consider

$$Y(U) = G(\mathbb{Q})\backslash G(\mathbb{A})/UF_\infty^\times C_{\infty+},$$

where $C_{\infty+}$ is the standard maximal compact subgroup of the connected component $G_+(\mathbb{R})$ of the identity of $G(\mathbb{R})$. Then $Y(U)$ is a Riemannian manifold of dimension $2r+3r_2$ if U is sufficiently small. Then we take a finite extension $K/\mathbb{Q}_p$ with p-adic integer ring $\mathcal{O}$ such that $R\otimes_{\mathbb{Z}}\mathcal{O}\subset M_2(\mathcal{O})^I$, where I is the set of embeddings of F into $\overline{\mathbb{Q}}$. We also fix embeddings $\overline{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}}\to\mathbb{C}$. We consider the R_p-module $L(n,v;A)$ for each $\mathcal{O}$-module A. Here $0\le n\in\mathbb{Z}[I]$ and $v\in\mathbb{Z}[I]$, and $L(n,v;A)$ is the A-module of polynomials in $(X_\sigma,Y_\sigma)_{\sigma\in I}$ with coefficients in A homogeneous of degree n_σ for each $\sigma\in I$. We let $\widehat{R}$ act on it via

$$\alpha P\big((X_\sigma,Y_\sigma)\big)=\nu(\alpha_p)^v P\big((X_\sigma,Y_\sigma)^t\alpha_p^{\sigma\iota}\big)$$

for $\alpha\in\widehat{R}$, where $\alpha^\iota=\nu(\alpha)\alpha^{-1}$ for the reduced norm map $\nu:B\to F$. For any R_p-module M, we can think of the quotient space $\underline{M}$ given by

$$G(\mathbb{Q})\backslash G(\mathbb{A})\times M/UF_\infty^\times C_{\infty+}$$

via an action which is given by

$$\alpha(g,m)u=(\alpha gu,u_p^\iota m).$$

We then consider the sheaf of locally constant sections of $\underline{M}$, which we again write as $\underline{M}$. When $M=L(n,v;A)$, we simply write $\mathcal{L}(n,v;A)$ for the sheaf $\underline{M}$. Now we specify open compact subgroups of $G(\mathbb{A}^{(\infty)})$. We put, writing $\mathfrak{r}_N=\prod_{\mathfrak{l}\mid N}\mathfrak{r}_\mathfrak{l}$,

$$(1.1)\quad \begin{cases} U(N)=\Big\{x\in\widehat{R}^\times; x_N=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix} \bmod NM_2(\mathfrak{r}_N) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } u\Big\},\\ U_1(N)=\Big\{x\in\widehat{R}^\times; x_N=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} v & u\\ 0 & 1\end{pmatrix} \bmod NM_2(\mathfrak{r}_N) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } u \text{ and } v\Big\},\\ U_0(N)=\Big\{x\in\widehat{R}^\times; x_N=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \text{ with } c\in N\mathfrak{r}_N\Big\}. \end{cases}$$

We fix an open compact subgroup S of $\widehat{R}^\times$ such that $S=S_pS^{(p)}$ and $S_p=R_p^\times$, where

$$S^{(p)}=\big\{x\in S\mid x_p=1\big\}.$$

We write N for the level of S (that is, the ideal N in $\mathfrak{r}$ which is maximal with respect to the property that $x-1\in N\widehat{R}\Rightarrow x\in S$). Then we put

$$S_1(p^\alpha)=S\cap U_1(p^\alpha),\quad S_0(p^\alpha)=S\cap U_0(p^\alpha),\quad S(p^\alpha)=S\cap U(p^\alpha),$$

and we see that

$$S_0(p^\alpha)/S(p^\alpha) \cong (\mathfrak{r}/p^\alpha\mathfrak{r})^\times \times (\mathfrak{r}/p^\alpha\mathfrak{r})^\times.$$

We define $\mathbb{G}^\alpha = S_0(p^\alpha)\mathfrak{r}^\times/S(p^\alpha)\mathfrak{r}^\times$. Then $\mathbb{G}^\alpha$ naturally acts on various cohomology groups

$$H^q\big(Y(S(p^\alpha)), \mathcal{L}(n,v;A)\big),$$
$$H^q_c\big(Y(S(p^\alpha)), \mathcal{L}(n,v;A)\big),$$
$$H^q_P\big(Y(S(p^\alpha)), \mathcal{L}(n,v;A)\big)$$

introduced in [H1] 1.8 and §5. We hereafter write H^q_* for any one of these three types of cohomology groups. When $Y(S)$ is compact ($\Leftrightarrow B \neq M_2(F)$), we regard all these cohomology groups are equal to the usual cohomology groups. Throughout this paper, we assume the following condition:

(TF) $\qquad Y(S)$ is smooth.

Then $Y(U)$ is smooth for any open subgroup U of S. The condition (TF) is satisfied by sufficiently small S.

Let ε be a character of $\mathbb{G}^\alpha$. For each $s \in S_0(p^\alpha)$, we write $\varepsilon(s)$ for the value of ε of the class of s in $\mathbb{G}^\alpha$. Then we twist the action of $S_0(p^\alpha)$ on $L(n,v;A)$ by ε, and the resulting module we write as $L(n,v,\varepsilon;A)$. We put

$$\mathbb{G} = \mathbb{G}_S = \varprojlim_\alpha \mathbb{G}^\alpha.$$

Then we see:

$$\mathbb{G}^\alpha = S_0(p^\alpha)\mathfrak{r}^\times/S(p^\alpha)\mathfrak{r}^\times = S_0(p^\alpha)/\big(S_0(p^\alpha)\cap S(p^\alpha)\mathfrak{r}^\times\big). \tag{1.2}$$

By definition, we see that $S_0(p^\alpha)\cap S(p^\alpha)\mathfrak{r}^\times = (S\cap F^\times)S(p^\alpha)$. Thus writing $S\cap F^\times$ as E_S, via $(a,d)\mapsto(a^{-1}d,a)$, we have

$$\mathbb{G}^\alpha \cong \big((\mathfrak{r}/p^\alpha\mathfrak{r})^\times \times (\mathfrak{r}/p^\alpha\mathfrak{r})^\times\big)/E_S \cong (\mathfrak{r}/p^\alpha\mathfrak{r})^\times \times \big((\mathfrak{r}/p^\alpha\mathfrak{r})^\times/E_S\big). \tag{1.3}$$

This shows that

$$\mathbb{G}_S = (\mathfrak{r}_p^\times \times \mathfrak{r}_p^\times)/\overline{E}_S = \mathfrak{r}_p^\times \times (\mathfrak{r}_p^\times/\overline{E}_S),$$

where $\overline{E}_S$ is the closure of E_S in $\mathfrak{r}_p^\times$. We write Z_S for $(\mathfrak{r}_p^\times/\overline{E}_S)$. We may regard $\mathbb{G}_S$ as a quotient of $T(\mathbb{Z}_p)$ for the $\mathfrak{r}_p$-split standard torus T, that is, the algebraic subgroup of G given by:

$$T(A) = \left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}; a,d \in (\mathfrak{r}_p\otimes_{\mathbb{Z}_p}A)^\times\right\}. \tag{1.4}$$

Anyway, the character group of T is given by $X(T) = \mathbb{Z}[I] \times \mathbb{Z}[I]$. We normalize the identification so that $(n, v) \in \mathbb{Z}[I] \times Z[I]$ gives rise to the character: $(a, d) \mapsto a^{-n}(ad)^{-v}$. Thus the group $X(\mathbb{G}_S)$ of algebraic characters of $\mathbb{G}_S$ is:

$$X(\mathbb{G}_S) = \{(n, v) \in X(T); n + 2v \text{ is trivial on } E_S\}. \tag{1.5}$$

For $j, j' \in \mathbb{Z}[I]$, we write:

$$j \approx j' \quad \begin{array}{l}\text{if } \varepsilon^{j-j'} = 1 \text{ for all units in a sufficiently} \\ \text{small subgroup of finite index in } \mathfrak{r}^{\times}.\end{array} \tag{1.6}$$

If S is sufficiently small, the condition that $n + 2v$ is trivial on E_S becomes independent of S. It is equivalent to

$$n + 2v \approx 0.$$

We call a character κ of the compact group $\mathbb{G}_S$ *arithmetic* if its restriction to an open neighborhood of the identity coincides with an element of $X(\mathbb{G}_S)$. We write $\mathcal{A}(\mathbb{G}_S)$ for the group of arithmetic characters. For each $\kappa \in \mathcal{A}(\mathbb{G}_S)$, we write $(n(\kappa), v(\kappa))$ for the character in $X(\mathbb{G}_S)$ it gives on a small neighborhood of 1. Then we define ε_κ, a finite order character of $\mathbb{G}_S$, by

$$\varepsilon_\kappa(g) = \kappa(g) g^{-(n(\kappa), v(\kappa))}.$$

We simply write $\mathcal{L}(\kappa; A)$ for $\mathcal{L}(n(\kappa), v(\kappa), \varepsilon_\kappa; A)$. As explained in [H3], §4 and [H4], §6, we have an integral operator $(S(p^\alpha)xS(p^\alpha))$ acting on $H^q_*(Y(S(p^\alpha)), \mathcal{L}(\kappa; A))$ for each x in the following semi-group:

$$D = D_0(p^\alpha) = \left\{\delta \in G(\mathbb{A}^\infty) \mid \delta_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathfrak{r}_p), \right. \\ \left. a_p \in \mathfrak{r}_p^\times,\ c \in p^\alpha \mathfrak{r}_p \right\}. \tag{1.7}$$

Here we let $D^\iota = \{\delta^\iota = n(\delta)\delta^{-1}; \delta \in D\}$ act on $L(\kappa; A)$ by

$$\delta^\iota P(X_\sigma, Y_\sigma) = P\big((X_\sigma, Y_\sigma)^t \delta_p^\sigma\big).$$

The action of $(U\delta U)$ for $U = S(p^\alpha)$ is given as follows: we define

$$[\delta] : \underline{L}(\kappa; A)_{/Y(W)} \to \underline{L}(\kappa; A)_{/Y(W^\delta)}$$

by $(g, P) \mapsto (g\delta, \delta_p^\iota P)$ for $W = U \cap \delta U \delta^{-1}$ and $W^\delta = \delta^{-1} W \delta$. Then for each section ℓ in $\mathcal{L}(\kappa; A)_{/Y(W)}$, $[\delta](\ell)(g\delta) = \delta_p^\iota \ell(g)$, and we have a morphism

$$[\delta] : H^q_*(Y(W), \mathcal{L}(\kappa; A)) \longrightarrow H^q_*\big(Y(W^\delta), \mathcal{L}(\kappa; A)\big).$$

Now we define

$$(1.8)\qquad (U\delta U)=\mathrm{Tr}_{Y(W^\delta)/Y(U)}\circ[\delta]\circ \mathrm{res}_{Y(W)/Y(U)}\,.$$

We write, for $z\in F_{pN}^\times\cap\mathfrak{r}_{pN}$,

$$\mathbb{T}(z)=\left(U\begin{pmatrix}1&0\\0&z\end{pmatrix}U\right).$$

Let $\nu:B\to F$ be the reduced norm map. For each ideal $\mathfrak{n}$ outside Np, we decompose the subset of $\widehat{R}$ made of x with $\nu(x)\hat{\mathfrak{r}}=\mathfrak{n}\hat{\mathfrak{r}}$ and $x_{pN}\in U_{pN}$ into a disjoint union $\coprod_\delta U\delta U$ with $\delta_{Np}=1$, and we put

$$T(\mathfrak{n})=\sum_\delta(U\delta U).$$

This operator only depends on the ideal $\mathfrak{n}$. Instead of using $[\delta]$, we could have used $z_p^v\left[\begin{pmatrix}1&0\\0&z\end{pmatrix}\right]$. The operator obtained will be written as $T(z)$. This operator is integral only when $v\geq 0$ and $T(z)=\nu(z_p)^v\mathbb{T}(z)$.

We now interpret our operator introduced in terms of classical Hecke operators. Choose a representative set $T=T(U)$ for $G(\mathbb{Q})\backslash G(\mathbb{A})/UG_+(\mathbb{R})$. We assume that t_p commutes with δ_p and $t_\infty=1$ for all $t\in T$. Then for $t\in T$,

$$(tg_\infty,P)=(\gamma t g'_\infty u,u_p^\iota P)$$

($u\in U, g_\infty, g'_\infty\in G_+(\mathbb{R})$ and $\gamma\in G(\mathbb{Q})$) implies that $tg_\infty=\gamma t g'_\infty u$. Thus $\gamma=tg_\infty\, u^{-1}g'^{-1}_\infty t^{-1}s$ and $\gamma\in\Gamma_t=tUG_+(\mathbb{R})t^{-1}\cap G(\mathbb{Q})$. Moreover $\gamma_p=t_pu_p^{-1}t_p^{-1}$ and $\gamma_\infty=g_\infty g'^{-1}_\infty$, and we get

$$\left(\gamma_\infty g'_\infty, t_p^{-1}(\gamma_p^{-1})^\iota t_pP\right)=(g_\infty,P).$$

Since the center has to act trivially on $L(\kappa;A)$ to get non-trivial cohomology, we have $(\gamma_\infty g'_\infty,(\gamma_p^{-1})^\iota t_pP)=(\gamma_\infty g'_\infty,\gamma_pt_pP)$. Thus writing $\mathcal{Z}$ for the symmetric space $G_+(\mathbb{R})/F_\infty^\times C_{\infty+}$, we see that, via $(tg_\infty,P)\mapsto(g_\infty(z_0),t_pP)$,

$$\underline{L}(\kappa;A)_{/Y(U)}\cong\coprod_{t\in T(U)}\Gamma_t\backslash\left(\mathcal{Z}\times t_pL(\kappa;A)\right),$$

where Γ_t acts on the product $\mathcal{Z}\times L(\kappa;A)$ by $\gamma(z,P)=(\gamma_\infty(z),\gamma_pP)$ and z_0 is the fixed point in $\mathcal{Z}$ of $C_{\infty+}$. Now suppose $\delta\in D$ and

$$G(\mathbb{Q})_tUG_+(\mathbb{R})\delta=G(\mathbb{Q})_tUG_+(\mathbb{R}).$$

Then $tg_\infty\delta = \gamma tug'_\infty$. Thus $g_\infty = \gamma_\infty g'_\infty$, $\delta^{(\infty)} = \gamma^{(\infty)}tut^{-1}$ and

$$\delta_p t_p u_p^{-1} t_p^{-1} = t_p \delta_p t_p^{-1} t_p u_p^{-1} t_p^{-1} = \gamma_p.$$

That is, $\nu(\delta)\mathfrak{r} = \nu(\gamma)\mathfrak{r}$ and $\gamma^{-1}\delta \in \Gamma_t$. We then see that:

$$(U\delta U) = \bigoplus_{t\in D(U)} (\Gamma_t\gamma\Gamma_t).$$

Here $(\Gamma_t\gamma\Gamma_t)$ is defined as follows. The map: $(z,P) \mapsto (\gamma(z), \gamma^\iota P)$ induces a morphism:

$$[\gamma] : \Phi\backslash(\mathcal{Z} \times L(\kappa; A)) \longmapsto \gamma^{-1}\Phi\gamma\backslash(\mathcal{Z} \times L(\kappa; A).$$

We apply this construction to $\Phi = \gamma\Gamma_t\gamma^{-1} \cap \Gamma_t$. Then we have a morphism

$$[\gamma] : H^q_*(Y(\Phi), \mathcal{L}(\kappa; A)) \longrightarrow H^q_*(Y(\Phi^\gamma), \mathcal{L}(\kappa; A)),$$

where $Y(\Gamma) = \Gamma\backslash\mathcal{Z}$. Now we define:

$$(\Gamma_t\gamma\Gamma_t) = \mathrm{Tr}_{Y(\Phi^\gamma)/Y(\Gamma_t)} \circ [\gamma] \circ \mathrm{res}_{Y(\Phi)/Y(\Gamma_t)}. \tag{1.9}$$

If Γ is sufficiently small, the fundamental group of $Y(\Gamma)$ is given by

$$\overline{\Gamma} = \Gamma/\Gamma \cap F^\times.$$

Abusing the notation, we write $H^q_*(\overline{\Gamma}, L(\kappa; A))$ for $H^q_*(Y(\Gamma), \mathcal{L}(\kappa; A))$ in this paper.

2.

We now lift the control theorem in [H1] for $\mathrm{SL}(2)_{/F}$ to $\mathrm{GL}(2)_{/F}$. We choose $T(S(p^\alpha))$ as follows: first we fix a decomposition

$$G(\mathbb{A}) = \coprod_{i=1}^h G(\mathbb{Q})t_i S G_+(\mathbb{R})$$

so that $(t_i)_{p\infty} = 1$. Since $\nu(S) = \nu(S_0(p^\alpha))$ for all α, we have from the strong approximation theorem that

$$G(\mathbb{A}) = \coprod_{i=1}^h G(\mathbb{Q})t_i S_0(p^\alpha) G_+(\mathbb{R}). \tag{2.1}$$

Now we see

$$\begin{aligned}
&G(\mathbb{Q})\backslash G(\mathbb{Q})t_iS_0(p^\alpha)G_+(\mathbb{R})/S(p^\alpha)G_+(\mathbb{R})\\
&\quad\cong t_i^{-1}G(\mathbb{Q})t_i\cap S_0(p^\alpha)G_+(\mathbb{R})\backslash S_0(p^\alpha)G_+(\mathbb{R})/S(p^\alpha)G_+(\mathbb{R})\ (\text{via } t_is\mapsto s)\\
&\quad\cong \big\{(S(p^\alpha)G_+(\mathbb{R})\backslash S(p^\alpha)G_+(\mathbb{R})(t_i^{-1}G(\mathbb{Q})t_i\cap S_0(p^\alpha))G_+(\mathbb{R})\big\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\backslash S_0(p^\alpha)G_+(\mathbb{R})/S(p^\alpha)G_+(\mathbb{R}).
\end{aligned}$$

Put:

$$(2.2)\quad X=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in S_0(p^\alpha)\mid ad\equiv\varepsilon \text{ for some totally positive unit } \varepsilon \text{ of } \mathfrak{r}\right\}.$$

Then again by the strong approximation theorem, the finite group $X/S(p^\alpha)$ coincides with $\{(S(p^\alpha)\backslash S(p^\alpha)(t_i^{-1}G(\mathbb{Q})t_i\cap S_0(p^\alpha))\}$, and we have

$$(2.3)\qquad G(\mathbb{Q})\backslash G(\mathbb{Q})t_iS_0(p^\alpha)G_+(\mathbb{R})/S(p^\alpha)G_+(\mathbb{R})\cong X\backslash S_0(p^\alpha)/S(p^\alpha)\quad (\text{via}\, t_is\mapsto s).$$

We choose a complete representative set S_α of $X\backslash S_0(p^\alpha)/S(p^\alpha)$. We may assume that S_α is made of diagonal matrices at p. Then

$$T\big(S(p^\alpha)\big)=\big\{t_is;s\in S_\alpha\big\}.$$

In particular, t_is commutes with $\begin{pmatrix}1&0\\0&z\end{pmatrix}$ for $z\in F_p^\times$. Note that $t_isS(p^\alpha)s^{-1}t_i^{-1}=t_iS(p^\alpha)t_i^{-1}$. Thus writing $\Gamma^i(p^\alpha)$ for Γ_{t_i} for $U=S(p^\alpha)$, we have

$$\begin{aligned}
(2.4)\quad H^q_*\big(Y(S(p^\alpha)),\mathcal{L}(\kappa;A)\big)&\cong\left\{\bigoplus_{i=1}^h H^q_*\big(Y(\Gamma^i(p^\alpha)),\mathcal{L}(\kappa;A)\big)\right\}^{S_\alpha}\\
&\cong\left\{\bigoplus_{i=1}^h H^q_*\big(\overline{\Gamma}^i(p^\alpha),L(\kappa;A)\big)\right\}^{S_\alpha},
\end{aligned}$$

where $\overline{\Gamma}=\Gamma/\Gamma\cap F^\times$. Note that

$$\mathbb{E}^\alpha=\overline{\Gamma}^i_0(p^\alpha)/\overline{\Gamma}^i(p^\alpha)\cong XS(p^\alpha)\mathfrak{r}^\times/S(p^\alpha)\mathfrak{r}^\times\subset\mathbb{G}^\alpha,$$

where

$$\Gamma^i_0(p^\alpha)=t_iS_0(p^\alpha)t_i^{-1}G_+(\mathbb{R})\cap G(\mathbb{Q}),\quad \overline{\Gamma}^i_0(p^\alpha)=\Gamma^i_0(p^\alpha)/\Gamma^i_0(p^\alpha)\cap F^\times.$$

We put

$$\mathbb{E}=\varprojlim_\alpha \mathbb{E}^\alpha$$

as a subgroup of $\mathbb{G}$.

Thus we can rewrite (2.4) as:

$$H_*^q\big(Y(S(p^\alpha)),\mathcal{L}(\kappa;A)\big)\cong \mathrm{Ind}_{\mathbb{E}^\alpha}^{\mathbb{G}^\alpha}\Big\{\bigoplus_{i=1}^{h} H_*^q\big(\overline{\Gamma}^i(p^\alpha),L(\kappa;A)\big)\Big\} \quad \text{as } \mathbb{G}^\alpha\text{-modules.} \tag{2.5}$$

We now put

$$\begin{aligned} H_*^q\big(Y(S(p^\infty)),\mathcal{L}(\kappa;A)\big) &\cong \varinjlim_\alpha \mathrm{Ind}_{\mathbb{E}^\alpha}^{\mathbb{G}^\alpha}\Big\{\bigoplus_{i=1}^{h} H_*^q\big(\overline{\Gamma}^i(p^\alpha),L(\kappa;A)\big)\Big\} \\ &\cong \mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big\{\varinjlim_\alpha \bigoplus_{i=1}^{q} H_*^q\big(\overline{\Gamma}^i(p^\alpha),L(\kappa;A)\big)\Big\} \\ &\cong \mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big\{\bigoplus_{i=1}^{h} H_*^q\big(\overline{\Gamma}^i(p^\infty),L(\kappa;A)\big)\Big\}, \end{aligned} \tag{2.6}$$

where for each right $\mathbb{E}$-module M, $\mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}} M$ is the space of locally constant functions on G satisfying $f(g\varepsilon)=f(g)\varepsilon$. We let $\mathbb{G}$ act on $\mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}} M$ by $f \mid g(x)=f(gx)$.

Let Γ be one of $\Gamma_0^i(p^\alpha)$ and $\Gamma^i(p^\alpha)$. We put $\Delta=(\mathfrak{r}^\times\Gamma)\cap G^1(\mathbb{Q})$, where G^1 is the algebraic subgroup of G made of norm 1 elements. We have a natural injection: $\overline{\Delta}\to\overline{\Gamma}$ whose cokernel is finite and of exponent 2. Thus if $p>2$, we have by using $\mathrm{Tr}_{\Gamma/\Delta}$ and $\mathrm{res}_{\Gamma/\Delta}$ that

$$H_*^q\big(\overline{\Gamma},L(\kappa;A)\big)\cong H^0\big(\overline{\Gamma}:\overline{\Delta},H_*^q(\overline{\Delta},L(\kappa;A))\big),$$

and in general, we have a natural morphism

$$H_*^q\big(\overline{\Gamma},L(\kappa;A)\big)\longrightarrow H^0\big(\overline{\Gamma}/\overline{\Delta},H_*^q(\overline{\Delta},L(\kappa;A))\big)$$

with finite kernel and cokernel of exponent at most 2. The group $\overline{\Gamma}/\overline{\Delta}$ is isomorphic to $(\mathfrak{r}^\times)^2\nu(\Gamma)/(\mathfrak{r}^\times)^2\cong\nu(\Gamma)/\nu(\Gamma)\cap(\mathfrak{r}^\times)^2$. Since the left-hand side is a subgroup of $E_+/(\mathfrak{r}^\times)^2$, which is a finite group of exponent 2, it stabilizes as α grows. Here E_+ is the group of all totally positive units in $\mathfrak{r}^\times$. As is well known, $\nu(G(\mathbb{Q}))$ is made of elements in $F^\times$ which are positive at all real places where B ramifies. Then for any $\varepsilon\in\nu(UG_+(\mathbb{R}))\cap F^\times$, we can find an element $\beta\in G(\mathbb{Q})$ with $\nu(\beta)=\varepsilon$. By the strong approximation theorem, for $u\in UG_+(\mathbb{R})$ with $\nu(u)=\varepsilon$, we can find $\gamma\in G^1(\mathbb{Q})$ such that $u\beta^{-1}\gamma^{-1}=v\in G^1(\mathbb{A})\cap U$. In other words, we find $\beta\gamma\in\Gamma_U=UG_+(\mathbb{R})\cap G(\mathbb{Q})$ such that $\nu(\beta\gamma)=\varepsilon$. This shows that $\nu(\Gamma_U)=\nu(UG_+(\mathbb{R}))\cap F^\times$. Thus $\nu(\Gamma)$ is independent of i. We write this stabilized group $(\mathfrak{r}^\times)^2\nu(\Gamma)/(\mathfrak{r}^\times)^2$ for large α as Θ.

For each linear operator T on an $\mathcal{O}$-module M of finite type, we can define a projector e_T in $\mathrm{End}_{\mathcal{O}}(M)$ so that T is invertible on $e_T M$ and topologically nilpotent on $(1-e_T)M$ (see [H1], §1.11). This definition of e_T extends to $\mathcal{O}$-modules of co-finite type by the Pontryagin duality. Since $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ for $z \in F_p^\times$ commutes with $t_i s$, the operator $\mathbb{T}(p)$ coincides with the operator $T(p)$ introduced in [H1], §1.10 on $H_*^q(\overline{\Delta}, L(\kappa; A))$ under the above morphisms. Thus we can think of the nearly ordinary part

$$H^q_{*,n\text{-ord}}\big(Y(S(p^\alpha)), \mathcal{L}(\kappa; A)\big) = \bigoplus_{i=1}^{h} H^q_{*,n,\mathrm{ord}}\big(\overline{\Gamma}^i(p^\alpha), L(\kappa; A)\big)$$

(that is the image of $e_{\mathbb{T}(p)}$) and compare it with the ordinary part

$$H^q_{*,\mathrm{ord}}\big(\overline{\Delta}^i(p^\alpha), L(\kappa; A)\big)$$

(that is, the image of $e_{T(p)}$) studied in [H1], where $\Delta^i(p^\alpha) = \Gamma^i(p^\alpha) \cap G^1(\mathbb{Q})$. We now put:

$$H^q_{*,n\text{-ord}}\big(Y(S(p^\infty)), \mathcal{L}((\kappa; A)\big) = \varinjlim_{\alpha} H^q_{*,n\text{-ord}}\big(Y(S(p^\alpha)), \mathcal{L}(\kappa; A)\big). \tag{2.7}$$

Then we have, if $p > 2$

$$\begin{aligned} &H^q_{*,n\text{-ord}}\big(Y(S(p^\infty)), \mathcal{L}(\kappa; A)\big) \\ &\quad = \mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big\{\bigoplus_{i=1}^{h} H^q_{*,n\text{-ord}}\big(\overline{\Gamma}^i(p^\infty), L(\kappa; A)\big)\Big\} \\ &\quad \cong \mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big\{H^0\big(\Theta, \bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}\big(\overline{\Delta}^i(p^\infty), L(\kappa; A)\big)\big)\Big\}. \end{aligned}$$

If $p = 2$, we have a natural morphism

$$\begin{aligned} &H^q_{*,n\text{-ord}}\big(Y(S(p^\infty)), \mathcal{L}(\kappa; A)\big) \\ &\qquad \longrightarrow \mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big\{H^0\big(\Theta, \bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}\big(\overline{\Delta}^i(p^\infty), L(\kappa; A)\big)\big)\Big\} \end{aligned}$$

whose kernel and cokernel are killed by 2. As shown in [H1], Thm I and Prop. 2.2, we have a canonical isomorphism

$$\iota : H^q_{*,\mathrm{ord}}\big(\overline{\Delta}^i(p^\infty), L(\kappa; K/\mathcal{O})\big) \cong H^q_{*,\mathrm{ord}}\big(\overline{\Delta}^i(p^\infty), K/\mathcal{O}\big),$$

which is an isomorphism of Hecke modules and satisfies $\kappa(e)\iota \circ e = e \circ \iota$ for $e \in \mathbb{E}$. By induction, we got, if $p > 2$

$$\begin{aligned} I : H^q_{*,n\text{-ord}}(Y(S(p^\infty)), \mathcal{L}(\kappa; K/\mathcal{O})) &\cong \mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big\{H^0(\Theta, \bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i(p^\infty), L(\kappa; K/\mathcal{O})))\Big\} \\ &\cong \mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big\{H^0(\Theta, \bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i(p^\infty), K/\mathcal{O}))\Big\} \\ &\cong H^q_{*,n\text{-ord}}(Y(S(p^\infty)), K/\mathcal{O}), \end{aligned}$$

which satisfies $\kappa(g)I \circ g = g \circ I$ for $g \in \mathbb{G}$. When $p = 2$, the above argument only supplies us with a morphism

$$I : H^q_{*,n\text{-ord}}(Y(S(p^\infty)), \mathcal{L}(\kappa; K/\mathcal{O})) \longrightarrow H^q_{*,n\text{-ord}}(Y(S(p^\infty)), K/\mathcal{O}),$$

whose kernel and cokernel are killed by 2. However, we can check that I is actually an isomorphism by the same argument in [H2]. Thus we have:

Proposition 2.1. — *We have an isomorphism of $\mathcal{O}$-modules for every prime p*

$$I : H^q_{*,n\text{-ord}}(Y(S(p^\infty)), \mathcal{L}(\kappa; K/\mathcal{O})) \cong H^q_{*,n\text{-ord}}(Y(S(p^\infty)), K/\mathcal{O}),$$

which satisfies, for $g \in \mathbb{G}$,

$$I \circ T(\mathfrak{n}) = T(\mathfrak{n}) \circ I, \quad I \circ \mathbb{T}(z) = \mathbb{T}(z) \circ I, \quad \kappa(g)I \circ g = g \circ I.$$

Now we fix an arithmetic character $\kappa : \mathbb{G} \to \overline{\mathbb{Q}}_p^\times$. Suppose that ε_κ factors through $\mathbb{G}^\alpha$. We assume that $q = r + r_2$. Hereafter we assume $H^q_* = H^q$ or H^q_P. It is shown in [H1], Thm II and Thm 5.1 that there is a natural morphism:

$$\bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i_0(p^\alpha), \mathcal{L}(\kappa; K/\mathcal{O})) \longrightarrow H^0\Big(\mathbb{E}, \bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i(p^\infty), \mathcal{L}(\kappa, K/\mathcal{O}))\Big)$$

whose kernel and cokernel are finite. Note that

$$\mathrm{res} : H^q_{*,n\text{-ord}}(Y(S_0(p^\alpha)), \mathcal{L}(\kappa; K/\mathcal{O})) \longrightarrow \Big\{H^0(\Theta, \bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i_0(p^\alpha), L(\kappa; K/\mathcal{O})))\Big\}$$

has finite kernel and cokernel. Now for any $\mathbb{G}$-module or $\mathbb{E}$-module M, let $M[\chi]$ denotes the subspace on which $\mathbb{G}$ or $\mathbb{E}$ acts via the character χ. By the definition of the induced module, we have:

$$\Big\{\mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}\Big(H^0(\Theta,\bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i(p^\infty),L(\kappa;K/\mathcal{O}))\Big)\Big\}[\mathrm{id}]$$
$$= H^0\Big(\mathbb{E}_\alpha, H^0(\Theta,\bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i(p^\infty),L(\kappa;K/\mathcal{O}))\Big).$$

On the other hand, from [H1], Thms II and 5.1, we see that the natural morphism

$$H^0\Big(\Theta,\bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}_0^i(p^\alpha),L(\kappa;K/\mathcal{O}))\Big)$$
$$\longrightarrow H^0\Big(\mathbb{E}_\alpha, H^0(\Theta,\bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}^i(p^\infty),L(\kappa;K/\mathcal{O}))\Big)$$

has finite kernel and cokernel. Thus the induced morphism

$$H^q_{*,n\cdot\mathrm{ord}}\big(Y(S_0(p^\alpha)),\mathcal{L}(\kappa;K/\mathcal{O})\big)$$
$$\longrightarrow H^0\Big(\Theta,\bigoplus_{i=1}^{h} H^q_{*,\mathrm{ord}}(\overline{\Delta}_0^i(p^\alpha),L(\kappa K/\mathcal{O})))\Big)$$
$$\longrightarrow \Big\{\mathrm{Ind}_{\mathbb{E}}^{\mathbb{G}}(H^0(\Theta,\bigoplus_{i=1}^{h} H^h_{*,\mathrm{ord}}(\overline{\Delta}^i(p^\infty),L(\kappa;K/\mathcal{O})))\Big\}[\mathrm{id}]$$
$$\cong H^q_{*,n\cdot\mathrm{ord}}\big(Y(S(p^\infty)),\mathcal{L}(\kappa;K/\mathcal{O})\big)[\mathrm{id}]$$

has finite kernel and cokernel. Then we conclude

$$H^q_{*,n\cdot\mathrm{ord}}\big(Y(S_0(p^\alpha)),\mathcal{L}(\kappa;K/\mathcal{O})\big)$$

is isogenous to

$$H^q_{*,n\cdot\mathrm{ord}}\big(Y(S(p^\infty)),\mathcal{L}(\kappa;K/\mathcal{O})\big)[\mathrm{id}]$$

and hence is isogenous to

$$H^q_{*,n\cdot\mathrm{ord}}\big(Y(S(p^\infty)),K/\mathcal{O}\big)[\kappa].$$

Thus we have:

THEOREM 2.2. — *Suppose that $q = r + r_2$ and $\kappa \in \mathcal{A}(\mathbb{G}_S)$ with $n(\kappa) \geq 0$. We denote by H^q_* either the usual cohomology or parabolic cohomology. Then we have a natural homomorphism of $\mathcal{O}$-modules*

$$I_\kappa : H^q_{*,n\cdot\mathrm{ord}}\big(Y(S_0(p^\alpha)),\mathcal{L}(\kappa;K/\mathcal{O})\big) \longrightarrow H^q_{*,n\cdot\mathrm{ord}}\big(Y(S(p^\infty)),K/\mathcal{O}\big)[\kappa]$$

whose kernel and cokernel are finite and which is equivariant under Hecke operators $T(\mathfrak{n})$ for all $\mathfrak{n}$ prime to pN and $\mathbb{T}(z)$ for all $e \in \mathfrak{r}_{pN} \cap F^\times_{pN}$.

3.

We define the Hecke algebras $\boldsymbol{h}_\kappa(S(p^\alpha);\mathcal{O})$ and $\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S(p^\alpha);\mathcal{O})$ by the $\mathcal{O}$-subalgebras of

$$\mathrm{End}_{\mathcal{O}}\big(H_P^q(Y(S(p^\alpha)),\mathcal{L}(\kappa;K/\mathcal{O}))\big),$$
$$\mathrm{End}_{\mathcal{O}}\big(H_{P,n\cdot\mathrm{ord}}^q(Y(S(p^\alpha)),\mathcal{L}(\kappa;K/\mathcal{O}))\big)$$

generated by Hecke operators $T(\mathfrak{n})$, $\mathbb{T}(z)$ and the action of $\mathbb{G}$. Then we take projective limits:

$$\boldsymbol{h}_\kappa\big(S(p^\infty);\mathcal{O}\big)=\varprojlim_\alpha \boldsymbol{h}_\kappa\big(S(p^\alpha);\mathcal{O}\big),$$
$$\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}\big(S(p^\infty);\mathcal{O}\big)=\varprojlim_\alpha \boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}\big(S(p^\alpha);\mathcal{O}\big).$$

We have the following fact by Proposition 2.1:

THEOREM 3.1. — *Suppose that κ is arithmetic and $n(\kappa)\geq 0$. Then we have an isomorphism of compact $\mathcal{O}$-algebras: $\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S(p^\infty);\mathcal{O})\cong \boldsymbol{h}_{\mathrm{id}}^{n\cdot\mathrm{ord}}(S(p^\infty);\mathcal{O})$ taking $T(\mathfrak{n})$ to $T(\mathfrak{n})$ and $\mathbb{T}(z)$ to $\mathbb{T}(z)$.*

We identify the nearly ordinary part of the Hecke algebra by the above isomorphism for all κ and write it as $\boldsymbol{h}^{n\cdot\mathrm{ord}}=\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$. Hereafter until the end of this section, we assume that $B=M_2(F)$. Let

$$M=\mathrm{Hom}_{\mathcal{O}}\big(H_{P,n\cdot\mathrm{ord}}^q(Y(S(p^\infty)),K/\mathcal{O}),K/\mathcal{O}\big),$$

that is, the Pontryagin dual module of $H_{P,n\cdot\mathrm{ord}}^q(Y(S(p^\infty)),K/\mathcal{O})$. For each $\mathcal{O}[[\mathbb{G}]]$-module X, we write X^* for its Pontryagin dual, which is again an $\mathcal{O}[[\mathbb{G}]]$-module. Then M is an $\mathcal{O}[[\mathbb{G}]]$-module. Let $\kappa:\mathbb{G}\to\overline{\mathbb{Q}}_p^\times$ be an arithmetic character. We use the same symbol $\kappa:\mathcal{O}[[\mathbb{G}]]\to\overline{\mathbb{Q}}_p$ for the algebra homomorphism induced. Then we write $\mathcal{P}_\kappa=\mathrm{Ker}(\kappa)$. We suppose that $n(\kappa)\geq 0$. Then by Theorem 2.2, we have a natural map

$$I_\kappa^*:M/\mathcal{P}_\kappa M\longrightarrow H_{P,n\cdot\mathrm{ord}}^q\big(Y(S_0(p^\alpha)),\mathcal{L}(\kappa;K/\mathcal{O})\big)^*,$$

which has finite kernel and cokernel. Thus by localizing at $\mathcal{P}=\mathcal{P}_\kappa$, we have:

$$I_\kappa^*:M_{\mathcal{P}}/\mathcal{P}_\kappa M_{\mathcal{P}}\cong H_{P,n\cdot\mathrm{ord}}^q\big(Y(S_0(p^\alpha)),\mathcal{L}(\kappa;K/\mathcal{O})\big)^*\otimes_{\mathcal{O}}K.$$

It is known (see [H3], §§8.2a–b) that $H_{P,n\cdot\mathrm{ord}}^q(Y(S_0(p^\alpha)),\mathcal{L}(\kappa;K))$ is free of rank 2^{r_1} over $\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha);K)$, where r_1 is the number of real places of F and α is given by the integer such that ε_κ factors through $\mathbb{G}^\alpha$. Let C_∞ be the standard maximal compact subgroup of $G(\mathbb{R})$ containing $C_{\infty+}$. As shown in [H3], §4 and [H4], p. 307 and §7, there is a natural action of $C_\infty/C_{\infty+} \cong \{\pm 1\}^{\Sigma(\mathbb{R})}$ on M which commutes with the Hecke operators, where $\Sigma(\mathbb{R})$ is the set of real places of F. For each character ι of $\{\pm 1\}^{\Sigma(\mathbb{R})}$, taking the ι-eigenspace $M[\iota]$ of M, we see

$$M_{\mathcal{P}} = \bigoplus_{\iota} M_{\mathcal{P}}[\iota].$$

It is also known that

$$I_\kappa^* : M_{\mathcal{P}}[\iota]/\mathcal{P}_\kappa M_{\mathcal{P}}[\iota] \cong \boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha); K)$$

as Hecke modules for all ι (see [H3], §§8.2a–b). Let $\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}$ be the localization of $\boldsymbol{h}^{n\cdot\mathrm{ord}}$ at $\mathcal{P}$. Fixing one ι and choose an element $m \in M_{\mathcal{P}[\iota]}$ such that $I_{k^*}(m) \bmod \mathcal{P}M_{\mathcal{P}} = 1$, we get a morphism $\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}$ into $M_{\mathcal{P}}[\iota]$:

$$i : \boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}} \longrightarrow M_{\mathcal{P}}[\iota] \quad \text{given by } h \mapsto hm.$$

This i induces a homomorphism:

$$i \bmod \mathcal{P} : \boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}/\mathcal{P}\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}} \longrightarrow M_{\mathcal{P}}[\iota]/\mathcal{P}_\kappa M_{\mathcal{P}}[\iota].$$

Since $\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}/\mathcal{P}\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}$ projects down surjectively to $\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha); K)$, $i \bmod \mathcal{P}$ is surjective. As shown in [H1], Thm 2.3, M is of finite type over $\mathcal{O}[[\mathbb{G}]]$. Then by Nakayama's lemma, we know that i is surjective. Thus $M_{\mathcal{P}[\iota]} \cong \boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}/\mathfrak{a}_\iota$ for an ideal $\mathfrak{a}_\iota \subset \mathcal{P}$ as $\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}$-modules. By definition, $\boldsymbol{h}^{n\cdot\mathrm{ord}}$ acts faithfully on M and hence $\bigcap_\iota \mathfrak{a}_\iota = \{0\}$. Anyway, for all κ,

$$\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}}/\mathcal{P}\boldsymbol{h}_{\mathcal{P}}^{n\cdot\mathrm{ord}} \cong M_{\mathcal{P}}[\iota]/\mathcal{P}M_{\mathcal{P}}[\iota] \cong \boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha); K).$$

THEOREM 3.2. — *Suppose that $B = M_2(F)$. Let $\kappa : \mathbb{G} \to \overline{\mathbb{Q}}_p^\times$ be an arithmetic character with $n(\kappa) \geq 0$. We have an isomorphism*

$$\boldsymbol{h}_{\mathcal{P}_\kappa}^{n\cdot\mathrm{ord}}(S)/\mathcal{P}_\kappa \boldsymbol{h}_{\mathcal{P}_\kappa}^{n\cdot\mathrm{ord}}(S) \cong \boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha); K),$$

where α is the integer such that ε_κ factors through $\mathbb{G}^\alpha$. In other words, the natural algebra homomorphism

$$\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)/\mathcal{P}_\kappa \boldsymbol{h}^{n\cdot\mathrm{ord}}(S) \to \boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha); \mathcal{O})$$

has finite kernel and cokernel. If $r_1 = 0$, $M_{\mathcal{P}_\kappa}$ is free of rank 1 over $\boldsymbol{h}_{\mathcal{P}_\kappa}^{n\cdot\mathrm{ord}}(S)$.

When F is totally real and $B = M_2(F)$, we have the notion of p-adic modular forms. Here we briefly recall its definition. For each weight κ, we consider the automorphic factor

$$J_\kappa(g,z) = \det(g)^{v-t}(cz+d)^{n+2t} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_+(\mathbb{R}),$$

where $t = \sum_{\sigma \in I} \sigma$, and z is a variable on the product of copies indexed by I of the upper half complex plane. Then for $S = U_1(N)$, we consider the space $\mathcal{S}_\kappa(S_1(p^\alpha);\mathbb{C})$ of holomorphic cusp forms $f : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ such that, for $u \in S_1(p^\alpha)F_\infty^\times C_{\infty+}$,

$$f(xu) = \varepsilon_\kappa(u_p)f(x)J_\kappa(u_\infty,\mathbf{i})$$

for $\mathbf{i} = \sqrt{-1},\ldots,\sqrt{-1}$ (see [H4], §2 for a more detailed definition). Then as shown in [H7], §1, if f has algebraic Fourier expansion, we can associate f a q-expansion

$$f(q) = \sum_{0 \ll \xi \in F} \mathbf{a}_p(\xi y d; f) q^\xi \in \overline{\mathbb{Q}}_p[[q]],$$

which is actually a function on $F_A^\times$ with values in the ring of q-expansions. Thus we have a well defined function: $y \mapsto \mathbf{a}_p(y; f)$. Then $\mathcal{S}_\kappa(S_1(p^\alpha);\mathcal{O})$ is defined to be the p-adic completion of the space of algebraic f with $\mathbf{a}_p(y,f) \in \mathcal{O}$ for all $y \in F_A^\times$. Then the following facts are known (see [H7], §3):

(i) The p-adic completion $\mathcal{S}(S;\mathcal{O})$ of $\bigcup_\alpha \mathcal{S}_\kappa(S_1(p^\alpha);\mathcal{O})$ is independent of κ;

(ii) The space $\mathcal{S}(S;\mathcal{O})$ is canonically isomorphic to $\mathrm{Hom}_{\mathcal{O}}(h,\mathcal{O})$ as Hecke modules, where h is the algebra in $\mathrm{End}(\mathcal{S}(S;\mathcal{O}))$ generated topologically by Hecke operators. Since there is no natural q-expansion for cohomological automorphic forms over non-real field F, it would be natural to define the space $\mathcal{S}^{n\cdot\mathrm{ord}}(S;O)$ of p-adic nearly ordinary cusp forms on G by $\mathcal{S}^{n\cdot\mathrm{ord}}(S;\mathcal{O}) = \mathrm{Hom}_O(\boldsymbol{h}^{n\cdot\mathrm{ord}}(S),\mathcal{O})$. This at least gives an analogue of the space of nearly ordinary p-adic modular forms defined for totally real F. Anyway by duality, the above theorem implies

$$\begin{aligned}\mathcal{S}^{n\cdot\mathrm{ord}}(S;\mathcal{O})[\kappa] &= \left\{f \in \mathcal{S}^{n\cdot\mathrm{ord}}(S;\mathcal{O}); f \mid z = \kappa(z)f \text{ for all } z \in \mathbb{G}\right\} \\ &\cong \mathrm{Hom}_{\mathcal{O}}\left(\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}\left(S_0(p^\alpha);\mathcal{O}\right),\mathcal{O}\right).\end{aligned}$$

Although we do not have q-expansions for non-real F, if $B = M_2(F)$, we can define even for non-real F, using the Fourier expansion relative to the standard Whittaker function, the space $\mathcal{S}_\kappa(S_0(p^\alpha);A)$ of A-integral cusp forms for $S_0(p^\alpha)$ and κ for any valuation ring A in $\mathbb{C}$ containing all conjugates of r (see [H3], §6). Then we just put

$$\mathcal{S}_\kappa\big(S_0(p^\alpha);\mathcal{O}\big) = \mathcal{S}_\kappa\big(S_0(p^\alpha);A\big) \otimes_A \mathcal{O}$$

for $A = \mathcal{O} \cap \overline{\mathbb{Q}}$. It is known (see [H3], Thm 6.4) that

$$\mathrm{Hom}_{\mathcal{O}}\big(\boldsymbol{h}_\kappa\big(S_0(p^\alpha);\mathcal{O}\big),\mathcal{O}\big) \cong \mathcal{S}_\kappa\big(S_0(p^\alpha);\mathcal{O}\big)$$

canonically as Hecke modules.

Each linear form $\phi : \boldsymbol{h}_\kappa(S_0(p^\alpha);\mathcal{O}) \to \mathcal{O}$ gives a modular form whose Fourier coefficient at $\mathfrak{n}$ is given by $\phi(T(\mathfrak{n}))$ (see [H3]). Here we note that the above duality is not known for $S_1(p^\alpha)$. Anyway we have:

COROLLARY 3.3. — *Suppose that $B = M_2(F)$. Then for all arithmetic character κ with $n(\kappa) \geq 0$, we have $\mathcal{S}^{n\cdot\mathrm{ord}}(S;\mathcal{O})[\kappa] \cong S_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha);\mathcal{O})$ canonically as Hecke modules.*

4.

Let us now construct some irreducible components of $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$ by means of Hecke characters of quadratic extensions of F. For the moment, we assume that $B = M_2(F)$. We now take $S = U_1(N)$ for an ideal N prime to p. In this case, we write $\boldsymbol{h}^{n\cdot\mathrm{ord}}(N)$ in place of $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$ and $\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(Np^\alpha;\mathcal{O})$ in place of $\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha);\mathcal{O})$. Similarly we write $Y_0(Np^\alpha)$ for $Y(S_0(Np^\alpha))$. Let L be a quadratic extension of F. We assume that L contains a CM field. Let L_{CM} be the maximal CM subfield of L. Let φ be a Hecke character of $L_A^\times/L^\times$ whose infinity type is $-j$. That is, $\varphi_\infty(x) = x^{-j}$. Then as is well known, there is a rank 1 motive $M(\varphi)$ defined over L with coefficients in $\mathbb{Q}(\varphi)$ whose L-function is given by $L(s,\varphi)$ (e.g. [H3], §1). Here $\mathbb{Q}(\varphi)$ is the subfield of $\overline{\mathbb{Q}}$ generated by $\varphi(\mathfrak{n})$ for all ideals $\mathfrak{n}$. Then the Hodge type of $M(\varphi)$ at $\sigma \in I_L$ is given by $(j_\sigma, j_{\sigma c})$ for complex conjugation c. Now we consider the base change of $M(\varphi)$ from L to F. We write the resulted motive as $M_F(\varphi)$ which is a rank 2 motive with coefficients in $\mathbb{Q}(\varphi)$. Then the Hodge type of $M_F(\varphi)$ at $\sigma \in I$ is given by $(j_\rho, j_{\rho c})$ and $(j_\tau, j_{\tau c})$ for ρ and τ whose restrictions to F are $\sigma \in I$. We write w for the weight of $M(\varphi)$. Thus $j_\rho + j_{\rho c} = w = j_\tau + j_{\tau c}$. We assume that $M_F(\varphi)$ is regular, that is,

$$\text{(R)} \qquad (j_\rho, j_{\rho c}) \neq (j_\tau, j_{\tau c}).$$

We write $p_\sigma = \max(j_\rho, j_\tau)$, $v_{\sigma c} = w - p_\sigma$ and $v_\sigma = \min(j_\sigma, j_\tau)$. Then the Hodge type of $M_F(\varphi)$ is rewritten as $(p_\sigma, v_{\sigma c})$ and $(v_\sigma, p_{\sigma c})$ for each $\sigma \in I$. Since $p_\sigma > v_\sigma$, we can write $n_\sigma + 1 = p_\sigma - v_\sigma$ with $n_\sigma \geq 0$. Since $p_\sigma + v_{\sigma c} = w = v_\sigma + p_{\sigma c}$, we have:

$$n_\sigma + 1 = p_\sigma - v_\sigma = p_{\sigma c} - v_{\sigma c} = n_{\sigma c} + 1.$$

Thus $n = nc$ for $n = \sum_{\sigma \in I} n_\sigma \sigma$. Let D be the relative discriminant of L/F and C be the conductor of φ. Then it is well known by Langlands' theory that there exists $\theta(\varphi) \in H^q_{\text{cusp}}(Y_0(DN_{L/F}(C)), \mathcal{L}(\kappa^\circ; \mathbb{C}))$ such that

$$\theta(\varphi) \mid T(\mathfrak{n}) = \Big\{\sum_{\mathfrak{b}} \varphi(\mathfrak{b})\Big\}\theta(\varphi),$$

where $\mathfrak{b}$ runs over integral ideals in L whose norm are equal to $\mathfrak{n}$, $n(\kappa^\circ) = n$, $v(\kappa^\circ) = v$ and $\{\kappa^\circ\}(z) = \varphi(z)N(z)^{-1}$ (with $z \in Z_S$) for the norm character $N : Z_S \to \mathbb{Z}_p^\times$. In other words, we have an algebra homomorphism $\lambda^\circ_\varphi : \boldsymbol{h}_\kappa(DN_{L/F}(C); O) \to \mathbb{Q}_p(\varphi)$ such that:

$$\sum_{\mathfrak{n}} \lambda_\varphi \circ \big(T(\mathfrak{n})\big) N(\mathfrak{n})^{-s} = L(s, \varphi).$$

Let N be the prime-to-p part of $DN_{L/F}(C)$. To have p-adic unit eigenvalue $\lambda_\varphi(T(p))$, we need to assume several conditions. The first one is

(Sp) all the prime factors of p in F are split in L.

We now fix an embedding $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and $i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$. Then writing

$$\Sigma = \big\{\sigma \in I_L \mid j_\sigma = v_\sigma\big\},$$

we have that $I_L = \Sigma \coprod \Sigma c$. This set can be rewritten as:

$$\Sigma = \big\{\sigma \in I_L \mid j_\sigma \leq j_{\sigma c}\big\}.$$

Thus Σ has to be a CM type in the sense of [H3], (1.9). Let I_{L_p} be the set of p-adic places of L, and regard $\Sigma_p = i_p \circ \Sigma$ as a subset of I_{L_p}. We now need to modify $\theta(\varphi)$ to get $\Theta(\varphi)$ which is nearly ordinary. For that, we need to assume that

$$\left|\Big\{\prod_{\sigma \in \Sigma} \varpi^{-\sigma j_\sigma}\Big\} \varphi_p(\varpi)\right|_p = 1,$$

where ϖ is an element in $F_p^\times$ such that $\varpi\mathfrak{r}_p$ is the intersection of all the maximal ideals of $\mathfrak{r}_p = \mathfrak{r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This is in turn equivalent to the following condition, for the generator ρ of $\mathrm{Gal}(L/F)$:

$$(\mathrm{Ord}_p) \qquad I_{L_p} = \Sigma_p \coprod \Sigma_p c = \Sigma_p \coprod \Sigma_p \rho.$$

It is certainly possible that $\sigma \in I_L$ and $\sigma c \in I_L$ induce the same p-adic valuation on F. Thus if $\sigma \in \Sigma$, the above condition tells us that the prime p in F corresponding to $i_p \circ \sigma$ is split in L into $\mathcal{P}$ and $\mathcal{P}^\rho$ so that $\mathcal{P}$ corresponds to σ and $\rho\sigma c$ and $\mathcal{P}^\rho$ corresponds to σc and $\rho\sigma$. Let $\mathcal{R}$ be the integer ring of L. By (Ord_p),

$$\mathcal{R}_p = \mathcal{R}_{\Sigma_p} \times \mathcal{R}_{\Sigma_p c},$$

where $\mathcal{R}_{\Sigma_p}$ is the Σ_p-adic completion of $\mathcal{R}$. Let $\widehat{\varphi}$ be the p-adic avatar of φ. We write $\widehat{\varphi}_{\Sigma_p}$ for the restriction of $\widehat{\varphi}$ to $\mathcal{R}_{\Sigma_p}^\times$. Since we can identify $\mathfrak{r}_p = \mathfrak{r} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ naturally with $\mathcal{R}_{\Sigma_p}$, we regard $\widehat{\varphi}_{\Sigma_p}$ as a character of $\mathfrak{r}_p^\times$. Let κ be such that $n(\kappa) = n$, $v(\kappa) = v$ and $\kappa(a, z) = \varphi_{\Sigma_p}(a)\{\kappa^0\}(z)$. As seen in [HT], §6, we can modify $\theta(\varphi)$ in the representation space spanned by $\theta(\varphi)$ under $G(\mathbb{A}^{(\infty)})$, and find $\Theta(\varphi)$ in $H^q_{\mathrm{cusp}}(Y_0(DN_{L/F}(C)p), \mathcal{L}(\kappa; \mathbb{C}))$ whose eigenvalue for $\mathbb{T}(\varpi)$ is given by

$$\lambda_\varphi(\mathbb{T}(\varpi)) = \varpi^{-v}T(\varpi) = \varpi^{-v}\varphi_p(\varpi) = \Big\{\prod_{\sigma\in\Sigma} \varpi^{-\sigma j_\sigma}\Big\}\varphi_p(\varpi).$$

Thus $\Theta(\varphi)$ is an element of $H^q_{P,n\text{-}\mathrm{ord}}(Y_0(DN_{L/F}(C)p), \mathcal{L}(\kappa; \mathbb{C}))$.

Let X be a number field, and for an ideal $\mathfrak{a}$ of X, we write $\mathrm{Cl}_X(\mathfrak{a})$ for the (strict) ray class group modulo $\mathfrak{a}$. Abusing the notation, we also write

$$\mathrm{Cl}_X(\mathfrak{a}p^\infty) = \varprojlim_\alpha \mathrm{Cl}_X(\mathfrak{a}p^\alpha).$$

We regard the class group $\mathrm{Cl}_X(\mathfrak{a})$ as a quotient group of the idele group of X so that an idele outside $\mathfrak{a}$ corresponds to the class of the ideal associated. The inclusion $\mathfrak{r} \subset \mathcal{R}$ induces two isomorphisms:

$$\mathfrak{r}_p \cong \mathcal{R}_{\Sigma_p} \quad \text{and} \quad \mathfrak{r}_p \cong \mathcal{R}_{\Sigma_p c}.$$

Regarding $(\mathcal{R}/Cp^\alpha\mathcal{R})^\times$ as a subgroup of $\mathrm{Cl}_L(Cp^\alpha)$, we have a natural morphism:

$$\mathfrak{r}_p^\times \times \mathfrak{r}_p^\times = \mathcal{R}_{\Sigma_p} \times \mathcal{R}_{\Sigma_p c} = \mathcal{R}_p^\times \longrightarrow \mathrm{Cl}_L(Cp^\infty).$$

The kernel of this map contains E_S and thus induces a morphism $\mathbb{G}_S \to \mathrm{Cl}_L(C_p^\infty)$. There is another description of this morphism. We have a morphism

$$\mathrm{Cl}_F(Np^\infty) \longrightarrow \mathrm{Cl}_L(Cp^\infty)$$

induced by the inclusion $F \subset L$. This map induces a homomorphism $Z_S \to \mathrm{Cl}_L(Cp^\infty)$. Under the identification $(a,d) \mapsto (a^{-1}d, a)$ in (1.3), the morphism

$$\mathbb{G}_S = \mathfrak{r}_p^\times \times Z_S \longrightarrow \mathrm{Cl}_L(Cp^\infty)$$

given by identifying $\mathfrak{r}_p^\times = \mathcal{R}_{\Sigma_p}^\times$ coincides with the map already constructed. Then the correspondence of characters: $\varphi \mapsto \kappa$ is induced by the twist by N of the morphism: $\mathbb{G}_S \to \mathrm{Cl}_L(Cp^\infty)$, and Θ induces an $\mathcal{O}[[\mathbb{G}_S]]$-algebra homomorphism

$$\Theta : \boldsymbol{h}^{n\cdot\mathrm{ord}}(N) \to \mathcal{O}\big[\big[\mathrm{Cl}_L(Cp^\infty)\big]\big].$$

Let $\boldsymbol{\mu}$ be the maximal finite subgroup of $\mathrm{Cl}_L(Cp^\infty)$, and put:

$$\mathbb{W} = \mathrm{Cl}_L(Cp^\infty)/\boldsymbol{\mu}.$$

Then $\mathbb{W} \cong \mathbb{Z}_p^m$ for a suitable m. Applying to the present situation the argument given for totally real base field in [HT], §6 to prove Prop. 6.3 there, we conclude that $\mathrm{Coker}(\Theta)$ is a torsion $\mathcal{O}[[\mathbb{W}]]$-module. Actually writing

$$\mathbb{G} = \mathrm{Cl}_L(Cp^\infty) \quad \text{and} \quad \mathbb{H} = \big\{uu^{-\rho} \mid u \in \mathcal{R}_{\Sigma_p\rho}^\times\big\},$$

we know from the proof of [HT], Prop. 6.3 that $\mathrm{Coker}(\Theta)$ is a surjective image of $\mathcal{O}[\mathbb{G}/\mathbb{H}]$. Since $\dim_{\mathbb{Q}_p} \mathbb{H} \otimes_{\mathbb{Z}} \mathbb{Q} > 0$, $\mathcal{O}[\mathbb{G}/\mathbb{H}]$, is a torsion $\mathcal{O}[[\mathbb{W}]]$-module. Thus Θ is generically surjective. Each character $\psi : \boldsymbol{\mu} \to \mathcal{O}^\times$ induces a surjective algebra homomorphism

$$\Psi : \mathcal{O}[[\mathrm{Cl}_L(Cp^\infty)]] \longrightarrow \mathcal{O}[[\mathbb{W}]]$$

such that $\Psi(\mu) = \psi(\mu)$ for $\mu \in \boldsymbol{\mu}$. We can regard Ψ as a character of $\mathrm{Cl}_L(Cp^\infty)$ having values in $\mathcal{O}[[\mathbb{W}]]^\times$. Then pulling it back to $L_A^\times$, we have a decomposition

$$\Psi = \prod_{\mathcal{L}} \Psi_{\mathcal{L}}$$

over places $\mathcal{L}$ of L. Now we consider the case of a general quaternion algebra $B_{/F}$. We write Ξ for the set of primes $\mathfrak{l}$ of F for which $B \otimes_F F_{\mathfrak{l}}$ is a division algebra. We assume the following two conditions:

(C_1) For every $\mathfrak{l} \in \Xi$, the prime $\mathfrak{l}$ either ramifies or remains prime in L;

(C_2) For every $\mathfrak{l} \in \Xi$, $\Psi_{\mathfrak{l}} \circ \rho \neq \Psi_{\mathfrak{l}}$,

where ρ is the non-trivial automorphism of L/F. Since Ξ is outside p, $\Psi_{\mathfrak{l}}$ for $\mathfrak{l} \in \Xi$ induces a finite order character on the inertia group at $\mathfrak{l}$. Thus $\Psi_{\mathfrak{l}} \circ \rho \not\equiv \Psi_{\mathfrak{l}} \bmod P$ for any prime ideal P in $O[[\mathbb{W}]]$ with characteristic 0 residue field. Then by the Jacquet-Langlands correspondence (see, for example, [H6], 7.1.2–3), for all arithmetic characters φ which is a specialization of Ψ, the $\theta(\varphi)$ and $\Theta(\varphi)$ exist as a cohomological modular form on G. Then we have (cf. [HT], §6):

THEOREM 4.1. — *Let the notation be as above. Suppose that* (C_1), (C_2), (Ord_p) *and that L contains a* CM *field. Then there exists a suitable open compact subgroup S of $\widehat{R}^\times$ such that $S \supset R_p^\times$, $S_{\mathfrak{l}} = U_1(N_{L/F}(C)D)_{\mathfrak{l}}$ if $\mathfrak{l} \notin \Xi$, and we have an $\mathcal{O}[[\mathbb{G}_S]]$-algebra homomorphism $\Theta_\Psi : \boldsymbol{h}^{n\cdot\mathrm{ord}}(S) \to \mathcal{O}[[\mathbb{W}]]$ given by*

$$\Theta_\Psi(T(\mathfrak{n})) = \sum_{\mathfrak{b}:N_{L/F}(\mathfrak{b})=\mathfrak{n}} \Psi([\mathfrak{b}])$$

for the Artin symbol $[\mathfrak{b}]$ in $\mathrm{Cl}_L(Cp^\infty)$. Here we agree to assume $[\mathfrak{b}] = 0$ if $\mathfrak{b}$ has a non-trivial common divisor with C. The morphism Θ_Ψ is generically surjective.

We now study the homomorphism $\iota : \mathbb{G}_S \to \mathrm{Cl}_L(Cp^\infty)$. As already described, we identify $\mathcal{R}_{\Sigma_p}$ and $\mathcal{R}_{\Sigma_p c}$ with $\mathfrak{r}_p$ via natural inclusion. Then $\iota(x,y)$ for $x, y \in \mathfrak{r}_p^\times$ is the class of $(x,y) \in \mathcal{R}_p^\times$ in $\mathrm{Cl}_L(Cp^\infty)$. Write:

$$E_C = \{\varepsilon \in \mathcal{R}^\times \mid \varepsilon \equiv 1 \bmod C\}.$$

Note that $\Theta(x,y) = \iota(x,y)N(x)^{-1}$ in $\mathcal{O}[[\mathrm{Cl}_L(Cp^\infty)]]$. Note that for $\varepsilon \in \mathcal{R}^\times$, $\Theta(\varepsilon) = \iota(\varepsilon)\varepsilon^\Sigma$, where $\varepsilon^\Sigma = \prod_{\sigma\in\Sigma} \varepsilon^\sigma$. Thus Θ kills E_C. By definition, the image of $\mathbb{G}_S$ is the image of $\mathcal{R}_p^\times$ in $\mathrm{Cl}_L(Cp^\infty)$. Thus we see that

$$\text{the image of } \mathbb{G}_S \text{ in } \mathrm{Cl}_L(Cp^\infty) = \mathcal{R}_p/\overline{E}_C = (\mathcal{R}_{\Sigma_p}^\times \times \mathcal{R}_{\Sigma_p c}^\times)/\overline{E}_C,$$

where the closure of E_C is taken in $\mathcal{R}_p^\times$. Assuming the Leopoldt conjecture, we conclude, for $g = [F : \mathbb{Q}]$,

$$\dim_{Q_p} \mathrm{Cl}_L(Cp^\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = g + 1.$$

On the other hand, we have:

$$\dim_{\mathbb{Q}_p} \mathrm{Cl}_F(Np^\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = r_2 + 1.$$

Thus the reasonable number of independent variables of CM factors of $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$ should be:

$$g + 1 - (r_2 + 1) = r_1 + r_2 = \#\text{ of infinite places of } F.$$

Identifying $\mathcal{R}_{\Sigma_p} \cong \mathfrak{r}_p \cong \mathcal{R}_{\Sigma_{pc}}$, we get an exact sequence

$$\mathfrak{r}_p^\times/\overline{E}_S \longrightarrow (\mathcal{R}_{\Sigma_p}^\times \times \mathcal{R}_{\Sigma_{pc}}^\times)/\overline{E}_C \longrightarrow \mathcal{R}_p^\times/\overline{(E_C^{(1)})^2} \to 1,$$

where

$$E_C^{(1)} = \{\varepsilon \in E_C; N_{L/F}(\varepsilon) = 1\}.$$

The last map is given by $(a,d) \mapsto ad^{-1}$, and the first map is the diagonal map. Thus as already mentioned, the map $\mathbb{G}_S \to \mathrm{Cl}_L(Cp^\infty)$ is realized by sending $\mathbb{G}_S = (\mathfrak{r}_p^\times \times \mathfrak{r}_p^\times)/\overline{E}_S$ onto $(\mathcal{R}_{\Sigma_p}^\times \times \mathcal{R}_{\Sigma_{pc}}^\times)/\overline{E}_C$ componentwisely identifying $\mathcal{R}_{\Sigma_p} \cong \mathfrak{r}_p \cong \mathcal{R}_{\Sigma_{pc}}$. Since E_C is isogenous with $E_C^{(1)} \times E_S$, via $(a,d) \mapsto (ad^{-1}, ad)$,

$$\text{the image of } \mathbb{G}_S \text{ is isogenous to } \mathfrak{r}_p^\times/\overline{(E_C^{(1)})^2} \times Z_S.$$

Thus we have:

Proposition 4.2. — *Let the assumption and notation be as above and as in Theorem 4.1. Then the image of $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$ under Θ_Ψ is torsion-free over $\mathcal{O}[[\mathbb{W}]]$. In particular its relative dimension over $\mathcal{O}$ is equal to $[F:\mathbb{Q}] + 1 + \delta_L$, where δ_L is the defect for the Leopoldt conjecture for L.*

Thus it is reasonable to speculate:

Conjecture 4.3. — *If $\boldsymbol{h}_\kappa^{n\cdot\mathrm{ord}}(S_0(p^\alpha); K) \neq 0$ for some κ and α, then*

$$\dim_{\mathcal{O}} \boldsymbol{h}^{n\cdot\mathrm{ord}}(S) = [F:\mathbb{Q}] + 1.$$

Of course this conjecture implies the Leopoldt conjecture for L as in Theorem 4.1. Note that

$$\dim_{\mathcal{O}} \Lambda = [F:\mathbb{Q}] + r_2 + 1 + \delta_F \geq [F:\mathbb{Q}] + r_2 + 1.$$

Thus the image of $\mathrm{Spec}(\boldsymbol{h}^{n\cdot\mathrm{ord}}(S))$ in $\mathrm{Spec}(\Lambda)$ is conjectured to be of codimension r_2.

Let ρ denote the non-trivial automorphism of L over F. Then for $\sigma \in \Sigma$ which induces a real place of F, the Hodge type of $M_F(\varphi)$ at σ is given by $(j_\sigma, j_{\rho\sigma})$ and $(j_{\rho\sigma}, j_\sigma)$. For $\sigma \in \Sigma$ inducing a complex place of F, the Hodge type is given by $(j_\sigma, j_{\sigma c})$ and $(j_{\rho\sigma}, j_{\rho\sigma c})$. This tells us that if $\sigma \in \Sigma$ is complex, then $\rho\sigma c \in \Sigma$ inducing σc on F. In particular, $\Sigma_{|F} = I$. Write $\Sigma(\mathbb{R})$ (resp. $(\Sigma(\mathbb{C}))$ for the set of elements in Σ which induces a real (resp. complex) place of F. Then anyway φ restricted to the subgroup

$$\mathbb{G}_- = \left\{(a, a^{-1}); a \in \mathfrak{r}_p^\times\right\}$$

is given on a small neighborhood of 1 by

$$\begin{aligned} a \longmapsto & \prod_{\sigma\in\Sigma(\mathbb{R})} a^{-\sigma(j_\sigma - j_{\rho\sigma})} \times \prod_{\sigma\in\Sigma(\mathbb{C})} a^{-\sigma(j_\sigma - j_{\rho\sigma})} a^{-\sigma c(j_{\rho\sigma c} - j_{\sigma c})} \\ & = \prod_{\sigma\in\Sigma(\mathbb{R})} a^{\sigma(n_\sigma+1)} \times \prod_{\sigma\in\Sigma(\mathbb{C})} a^{\sigma(n_\sigma+1)} a^{\sigma c(n_\sigma+1)} = a^{n(\Sigma)+\Sigma}. \end{aligned}$$

Here we have lifted n to $\mathbb{Z}[I_L]$ by inflation. We still use the same symbol for the inflated image, and we put:

$$n(\Sigma) = \sum_{\sigma\in\Sigma} n_\sigma \sigma.$$

We write

$$\xi \sim 0 \quad (\text{resp. } \xi \approx 0)$$

for $\xi \in \mathbb{Z}[I_L]$ if $\varepsilon^{\xi m} = 1$ for all $\varepsilon \in E_C^{(1)}$ (resp. $\varepsilon \in \mathcal{R}^\times$) for a sufficiently large integer m. Restricting φ to the cyclotomic line

$$\mathbb{G}_+ = \left\{(a, a); a \in \mathfrak{r}_p^\times\right\},$$

we get:

$$\begin{aligned} a \longmapsto & \prod_{\sigma\in\Sigma(\mathbb{R})} a^{-\sigma(j_\sigma + j_{\rho\sigma})} \times \prod_{\sigma\in\Sigma(\mathbb{C})} a^{-\sigma(j_\sigma + j_{\rho\sigma})} a^{-\sigma c(j_{\rho\sigma c} + j_{\sigma c})} \\ & = \prod_{\sigma\in\Sigma(\mathbb{R})} a^{-\sigma(n_\sigma+1+2v_\sigma)} \times \prod_{\sigma\in\Sigma(\mathbb{C})} a^{-\sigma(n_\sigma+1+2v_\sigma)} a^{-\sigma c(n_\sigma+1+2v_{\sigma c})} \\ & = a^{-(n+2v+t)}. \end{aligned}$$

From these arguments, we conclude that the allowable weights (n, v) are given by

$$\Xi_\Sigma = \left\{(n, v) \in \mathbb{Z}[I] \mid n + 2v \approx 0 \text{ and } n(\Sigma) + \Sigma \sim 0\right\}.$$

If F is totally real, then $E_C^{(1)}$ is trivial (because L is a CM field in this case), and hence the condition: $n(\Sigma) + \Sigma \sim 0$ does not impose any restriction. However, if F is not totally real, this condition impose a strong restriction, although we always have $\Xi_\Sigma \neq \emptyset$. Actually $(mt, \ell t) \in \Xi_\Sigma$ for all integer m and ℓ. Let L_{CM} be the maximal CM subfield of L. Then $\Sigma = \mathrm{Inf}_{L/L_{\mathrm{CM}}} \Sigma_0$ for a CM type Σ_0 of L_{CM}. If $L_{\mathrm{CM}} \subset F$, then $\sigma \in \Sigma$ implies $\rho\sigma c \in \Sigma$, which is impossible because $\rho\sigma c_{|F} = \sigma c$. Thus in this case, there are no CM-types Σ which give rise to cohomological modular forms. This shows that we have actually assumed

(NCM) $\qquad L_{\mathrm{CM}} \not\subset F.$

Thus we write $F_0 = L_{\mathrm{CM}} \cap F$ which is a proper subfield of L. To determine the condition $\xi \sim 0$ explicitly, we look at the torus $T_L(A) = (A \otimes_{\mathbb{Q}} L)^\times$ over $\mathbb{Q}$ and the connected component T_1 of the Zariski closure of $E_C^{(1)}$ in T_L. The character group of $T^1 = T_L/T_1$ is given by

$$X(T^1) = \{\xi \in \mathbb{Z}[I_L]; \xi \sim 0\}.$$

The group $X(T^1)$ is spanned by all the CM-types and $1 + \rho$, because the $1 + \rho$ is only the relation we imposed. CM-types span a rank $g + 1$ submodule if $2g = [L_{\mathrm{CM}} : \mathbb{Q}]$, and if $g > 1$, $1 + \rho$ is not in the linear span of CM-types. Thus we have $\mathrm{rank}_{\mathbb{Z}} X(T^1) = g + 2$. Then it is clear that

$$\Xi_\Sigma = \{(n, v) \in \mathbb{Z}[I]^2; n + 2v \approx 0 \text{ and } n(\Sigma) + \Sigma \sim 0\}$$

$$= \{(n, v) \in \mathbb{Z}[I]^2; n \in \mathbb{Z}t \text{ and } n + 2v \approx 0\}$$

if $g > 1$ and F is not totally real. Now suppose that F is an imaginary quadratic field. In this case, $L_{\mathrm{CM}} = L$ by (NCM). Since L contains a real quadratic field L_+, L is a biquadratic extension. Thus $\Sigma = \{\sigma, \rho\sigma c\}$ and hence $\Sigma_{|L_+} = \{\iota, \rho\}$, where we identified $\mathrm{Gal}(L/F)$ with $\mathrm{Gal}(L_+/\mathbb{Q})$ by restriction. Then it is obvious that

$$n(\Sigma) \sim 0 \iff n(\Sigma) \in \mathbb{Z}\Sigma \iff n_\sigma = n_{\sigma c}.$$

Now we assume that F has only one complex place and $[F : \mathbb{Q}] > 2$. Then $L = F(\sqrt{-d})$ for a positive integer $d \in \mathbb{Z}$. In this case, $E_C^{(1)}$ is rank 1 over $\mathbb{Z}$. We write J for the Galois group of $M/\mathbb{Q}$ for $M = \mathbb{Q}(\sqrt{-d})$. Then naturally $I_L \cong I \times J$ via $\sigma \mapsto (\sigma_{|F}, \sigma_{|M})$. We write ρ for the generator of J. Then $E_C^{(1)} = \mathrm{Ker}(1 + \rho)$ in E_C. To have non-trivial Σ satisfying (Ord_p), we need to assume that p splits in M. We have two choices of Σ. One is that $\Sigma = I$ in I_L, and the other is ρ_I. Anyway in this case, if $j = m\Sigma + \ell\rho\Sigma$, then

$$n = (\ell - m - 1)t \quad \text{and} \quad v = mt \quad (t = \sum_{\sigma \in I} \sigma).$$

This shows that $\Xi_\Sigma = \mathbb{Z}t \times \mathbb{Z}t$.

5.

We study more closely the ordinary cohomology group in [H1] when $r_2 = 1$ and B ramifies at all real places. We have

$$G^1(A) = \{x \in (B \otimes_{\mathbb{Q}} A)^\times; \nu(x) = 1\}$$

for each $\mathbb{Q}$-algebra A, where $\nu : B \to F$ is the reduced norm map. Let Γ be a torsion-free congruence subgroup of $G^1(\mathbb{Q})$ (that is, $\overline{\Gamma} = \Gamma$). Then $Y(\Gamma) = \Gamma\backslash\mathcal{Z}$ is a 3 (real) dimensional Riemannian manifold. We have fixed a maximal order R of B and an isomorphism $i_p : R_p \cong M_2(r_p)$. For an integral ideal N *prime* to p, we consider a torsion-free subgroup Γ in $R^\times \cap G^1(\mathbb{Q})$ *containing* $R^\times(N) = \{\gamma \in R^\times \mid \gamma - 1 \in NR,\ \nu(\gamma) = 1\}$. Then we put:

$$(5.1)\qquad \begin{cases} D = D_0(p^\alpha) = \left\{\gamma \in R_p \mid i_p(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \in p^\alpha \mathfrak{r}_p\right\}, \\ \Gamma_0(p^\alpha) = \left\{\gamma \in \Gamma \mid i_p(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \in p^\alpha \mathfrak{r}_p\right\} \\ \Gamma_1(p^\alpha) = \left\{\gamma \in \Gamma \mid i_p(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \in p^\alpha \mathfrak{r}_p \right. \\ \qquad\qquad\qquad\qquad\qquad\qquad \left. \text{and } d - 1 \in p^\alpha \mathfrak{r}_p\right\}. \end{cases}$$

Then we consider the space X of column vectors ${}^t(x, y)$ with $x \in \mathfrak{r}_p^\times$ and $y \in p\mathfrak{r}_p$. We let the semi-group $D_0(p^\alpha)_p$ act on X by $x\gamma = \gamma^\iota x$. We consider the space $\mathcal{C} = \mathcal{C}(K/\mathcal{O})$ of continuous functions on X having values in the discrete module $K/\mathcal{O}$. Then $\mathcal{C}$ is naturally a left $D_0(p)_p$-module by $\gamma\phi(x) = \phi(x\gamma)$. Similarly, we consider the measure space $\mathcal{M} = \mathcal{M}(O)$ of p-adic bounded measures on X having values in $\mathcal{O}$. We let $D_0(p^\alpha)_p$ act on $\mathcal{M}$ by $\gamma\mu(x) = \mu(\gamma x)$. Then $\mathcal{M}$ is the Pontryagin dual module of $\mathcal{C}$. Naturally $\mathfrak{r}_p^\times$ acts on X by scalar multiplication. This action commutes with the action of $D_0(p)_p$. We identify X with $\mathfrak{r}_p^\times \times p\mathfrak{r}_p$ by ${}^t(x, y) \mapsto (x, u)$ for $u = y/x$. Then we see that

$$\gamma * \mu(x, u) = \mu((a + bu)x, \gamma(u)),$$

where $\gamma(u) = (c + du)/(a + bu)$. Thus writing $\mathbf{A} = \mathcal{O}[[r_p^\times]]$, $\mathcal{M}$ is an $\mathbf{A}$-torsion-free module. Actually $\mathcal{M} \cong \mathbf{A}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[[p\mathfrak{r}_p]]$ as $\mathbf{A}$-modules. The pairing $[\ ,\ ] : \mathcal{M} \times \mathcal{C} \to K/\mathcal{O}$ induces by cup product the perfect Pontryagin duality pairing:

$$\langle\ ,\ \rangle : H^q_c(Y(\Gamma), \underline{\mathcal{M}}) \times H^{3-q}(Y(\Gamma), \underline{\mathcal{C}}) \longrightarrow K/\mathcal{O}.$$

We have the ordinary idempotent e acting on these cohomology groups (see [H1], §1.11). We write $H^q_{*,\mathrm{ord}}$ for eH^q_*, where H^q_* is any one of H^q, H^q_p and H^q_c, defined in [H1]. As shown in [H1], Prop. 2.1 and §3, we have a canonical isomorphism

$$\begin{aligned} H^q_{*,\mathrm{ord}}(Y(\Gamma_0(p)),\underline{\mathcal{C}}) &\cong H^q_{*,\mathrm{ord}}(Y(\Gamma_1(p^\infty)),\underline{K/\mathcal{O}}) \\ &= \varinjlim_{\alpha} H^q_{*,\mathrm{ord}}(Y(\Gamma_1(p)),\underline{K/\mathcal{O}}). \end{aligned}$$

Thus the study of the right-hand side is reduced to the study of cohomology groups with coefficients in $\mathcal{C}$. We know from [H4], Lemma 9.2 that $H^0_{\mathrm{ord}}(Y(\Phi),\underline{M}) = 0$ and $H^3_{c,\mathrm{ord}}(Y(\Phi),\underline{M}) = 0$ for any Φ-submodule M of $\mathcal{C}$, where $\underline{M}$ is the locally constant sheaf associated with M. Let W be the torsion-free part of $\mathfrak{r}_p^\times$. We write $\mathbf{\Lambda} = \mathcal{O}[[W]]$. Then $\mathbf{\Lambda}$ is a regular local ring of dimension $d+1$ for $d = [F:\mathbb{Q}]$. Actually $\mathbf{\Lambda} \cong \mathcal{O}[[X_1,\dots,X_d]]$ for a suitable choice of X_i. Let (T,T') be a regular sequence in the maximal ideal of $\mathbf{\Lambda}$. We have a long exact sequence for $M = \mathcal{C}$ or $\mathcal{C}[T']$:

$$\begin{aligned} H^q_{\mathrm{ord}}(Y(\Phi),\underline{M}[T^n]) &\longrightarrow H^q_{\mathrm{ord}}(Y(\Phi),\underline{M}) \\ &\xrightarrow{T^n} H^q_{\mathrm{ord}}(Y(\Phi),\underline{M}) \longrightarrow H^{q+1}_{\mathrm{ord}}(Y(\Phi),\underline{M}[T^n]) \\ &\qquad \longrightarrow H^{q+1}_{\mathrm{ord}}(Y(\Phi),\underline{M}) \xrightarrow{T^n} H^{q+1}_{\mathrm{ord}}(Y(\Phi),\underline{M}). \end{aligned}$$

This implies that the following sequence is exact:

$$\begin{aligned} (5.2)\quad 0 \to H^q_{\mathrm{ord}}(Y(\Phi),\underline{M}\otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}/T^n\mathbf{\Lambda}) &\longrightarrow H^{q+1}_{\mathrm{ord}}(Y(\Phi),\underline{M}[T^n]) \\ &\longrightarrow H^{q+1}_{\mathrm{ord}}(Y(\Phi),\underline{M})[T^n] \to 0. \end{aligned}$$

Applying (5.2) to $q = 0$, we get the control theorem:

$$(5.3)\qquad \begin{cases} H^1_{\mathrm{ord}}(Y(\Phi),\underline{\mathcal{C}}[T]) \cong H^1_{\mathrm{ord}}(Y(\Phi),\underline{\mathcal{C}})[T], \\ H^1_{\mathrm{ord}}(Y(\Phi),\underline{\mathcal{C}}[T,T']) \cong H^1_{\mathrm{ord}}(Y(\Phi),\underline{\mathcal{C}})[T,T']. \end{cases}$$

We suppose hereafter in this section that

(D) B is a division algebra.

Then $Y(\Phi)$ is compact. We then know from [H1], Thm 5.2 that, for arithmetic points P of $\mathrm{Spec}(\mathbf{\Lambda})$ corresponding to characters $\mathfrak{r}_p^\times \to \mathcal{O}^\times$ of the form $x \mapsto x^n$ with $n \in \mathbb{Z}[I]$, the groups $H^q_{\mathrm{ord}}(Y(\Phi),\underline{\mathcal{C}})[P]$ (for $q = 1,2$) are finite if $n \neq nc$. This shows that $H^q_{\mathrm{ord}}(Y(\Phi),\underline{\mathcal{C}})^*$ is of $\mathbf{\Lambda}$-torsion.

Lemma 5.1. — *The cohomology group $H^2_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})$ is $\mathbf{\Lambda}$-divisible. Similarly, $H^2_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}[T_1, \dots, T_r])$ is $\mathbf{\Lambda}/(T_1, \dots, T_r)\mathbf{\Lambda}$-divisible for any regular sequence $(T_1, \dots, T_r)$ in the maximal ideal $\mathfrak{m}$ of $\mathbf{\Lambda}$. In other words, for any non-zero-divisor $T' \in \mathbf{\Lambda}/(T_1, \dots, T_r)\mathbf{\Lambda}$, the multiplication by T' is surjective on $H^2_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}[T_1, \dots, T_r])$. In particular, we have the vanishing: $H^2_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}) = 0$ and $H^2_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}[T_1, \dots, T_r]) = 0$ if $(T_1, \dots, T_r) \notin \mathrm{Supp}(H^1_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})^*)$.*

Proof. — Let M be an $\mathbf{\Lambda}$-module of finite type. Suppose that for a regular sequence $(T_1, \dots, T_{d+1})$ of Λ in $\mathfrak{m}$, $(T_1, \dots, T_{d+1})$ is regular for M. Then M is $\mathbf{\Lambda}$-free. This follows from the fact

$$\mathrm{hd}_{\mathbf{\Lambda}}(M) + \mathrm{depth}_{\mathbf{\Lambda}}(M) = \dim(\mathbf{\Lambda}) = d + 1$$

and $\mathrm{hd}_{\mathbf{\Lambda}}(M) = 0 \Leftrightarrow M$ is $\mathbf{\Lambda}$-free, where $\mathrm{hd}_{\mathbf{\Lambda}}(M)$ is the homological dimension of M. The Pontryagin dual version is that M^* is $\mathbf{\Lambda}$-injective if and only if

$$\begin{cases} T_i : M^*[T_1, \dots, T_{i-1}] \to M^*[T_1, \dots, T_{i-1}] \\ \text{is surjective for all } i = 1, 2, \dots, d+1. \end{cases} \tag{5.4}$$

By assumption, (5.4) is satisfied by $\mathcal{C}$. Thus we have an exact sequence

$$0 \to \mathcal{C}[T_1, \dots, T_i] \longrightarrow \mathcal{C}[T_1, \dots, T_{i-1}] \xrightarrow{T_i} \mathcal{C}[T_1, \dots, T_{i-1}] \to 0.$$

Then we have the long exact sequence

$$H^2_{\mathrm{ord}}\big(Y(\Phi), \mathcal{C}[T_1, \dots, T_{i-1}]\big) \xrightarrow{T_i} H^2_{\mathrm{ord}}\big(Y(\Phi), \mathcal{C}[T_1, \dots, T_{i-1}]\big) \longrightarrow H^3_{\mathrm{ord}}\big(Y(\Phi), \mathcal{C}[T_1, \dots, T_i]\big).$$

The vanishing $H^3_{\mathrm{ord}}(Y(\Phi), \underline{M}) = H^3_{\mathrm{ord}}(Y(\Phi), \underline{M}) = 0$ then shows the divisibility. We have an exact sequence:

$$0 \to H^1_{\mathrm{ord}}\big(Y(\Phi), \mathcal{C}[T_1, \dots, T_{i-1}]\big) \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}/T_i\mathbf{\Lambda} \longrightarrow H^2_{\mathrm{ord}}\big(Y(\Phi), \mathcal{C}[T_1, \dots, T_i]\big) \longrightarrow H^2_{\mathrm{ord}}\big(Y(\Phi), \mathcal{C}[T_1, \dots, T_{i-1}]\big)[T_i] \to 0.$$

Since $H^2_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})$ is $\mathbf{\Lambda}$-divisible and of $\mathbf{\Lambda}$-co-torsion at the same time, $H^2_{\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}) = 0$. Thus for any T,

$$H^2_{\mathrm{ord}}\big(Y(\Phi), \underline{\mathcal{C}}[T]\big) = H^1_{\mathrm{ord}}\big(Y(\Phi), \mathcal{C}\big) \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}/T\mathbf{\Lambda}.$$

Thus:

$$\mathrm{Supp}\big(H^2_{\mathrm{ord}}(Y(\Phi),\mathcal{C}[T])^*\big) \subset \mathrm{Supp}\big(H^1_{\mathrm{ord}}(Y(\Phi),\mathcal{C})^*\big).$$

By the assumption, taking $T = T_1$, the group $H^2_{\mathrm{ord}}(Y(\Phi),\mathcal{C}[T])$ is of $\mathbf{\Lambda}/T\mathbf{\Lambda}$-co-torsion. Thus the $\mathbf{\Lambda}/T\mathbf{\Lambda}$-divisibility of $H^2_{\mathrm{ord}}(Y(\Phi),\mathcal{C}[T])$ shows the vanishing $H^2_{\mathrm{ord}}(Y(\Phi),\mathcal{C}[T]) = 0$. If the ideal (T,T') $(T' = T_2)$ is outside the support of $H^1_{\mathrm{ord}}(Y(\Phi),\mathcal{C})^*$, we may assume that (T) is outside the support. Then we have from the above exact sequence,

$$H^1_{\mathrm{ord}}\big(Y(\Phi),\mathcal{C}[T]\big) \otimes_{\mathbf{\Lambda}} \mathbf{\Lambda}/T'\mathbf{\Lambda} \cong H^2_{\mathrm{ord}}\big(Y(\Phi),\mathcal{C}[T,T']\big).$$

Thus

$$\begin{aligned}\mathrm{Supp}\big(H^2_{\mathrm{ord}}(Y(\Phi),\mathcal{C}[T,T'])^*\big) &\subset \mathrm{Supp}(H^1_{\mathrm{ord}}\big(Y(\Phi),\mathcal{C}[T])^*\big) \\ &\subset \mathrm{Supp}\big(H^1_{\mathrm{ord}}(Y(\Phi),\mathcal{C})^*\big).\end{aligned}$$

Since (T,T') is outside the support of $H^1_{\mathrm{ord}}(Y(\Phi),\mathcal{C})^*$, $H^2_{\mathrm{ord}}(Y(\Phi),\mathcal{C}[T,T'])^*$ is of $\mathbf{\Lambda}/(T,T')$-torsion. However at the same time, it is $\mathbf{\Lambda}/(T,T')$-torsion-free and hence is trivial. We continue this type of argument to reach the vanishing of $H^2_{\mathrm{ord}}(Y(\Phi),\underline{C}[T_1,\ldots,T_r])$ if $(T_1,\ldots,T_r)$ is not an element of $\mathrm{Supp}(H^1_{\mathrm{ord}}(Y(\Phi),\underline{C})^*)$.

Suppose that $(T_1,\ldots,T_r,T')$ is a regular sequence. Let:

$$V = \mathrm{Supp}(H^1_{P,\mathrm{ord}}\big(Y(\Phi),\underline{C})^*\big).$$

If $(T_1,\ldots,T_r) \notin V$, we get for any choice of T'

$$\text{(5.5a)}\quad \begin{aligned}&H^1_{\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)[T_1,\ldots,T_r] \otimes \mathbf{\Lambda}/T'\mathbf{\Lambda} \\ &\qquad\cong H^2_{\mathrm{ord}}\big(Y(\Phi),\underline{C}[T_1,\ldots,T_r,T']\big).\end{aligned}$$

By taking the dual, let $H = H^1_{\mathrm{ord}}(Y(\Phi),\underline{C})^*$. If $(T_1,\ldots,T_r)$ is not an element of V, then we have,

$$\text{(5.5b)}\qquad \big\{H/(T_1,\ldots,T_r)H\big\}[T'] \cong H^2_{\mathrm{ord}}\big(Y(\Phi),\underline{C}[T_1,\ldots,T_r,T']\big)^*$$

for any choice of T'. If H has a pseudo-null submodule, we can find T' such that $H[T']$ is non-zero but of $\mathbf{\Lambda}/T'\mathbf{\Lambda}$-torsion (i.e. killed by a non-zero-divisor of $\mathbf{\Lambda}/T'\mathbf{\Lambda}$). Then it will be contained in $H^2_{\mathrm{ord}}(Y(\Phi),\underline{C}[T'])^*$, which is $\mathbf{\Lambda}/T'\mathbf{\Lambda}$-torsion-free. This is impossible. Thus H has no-pseudo-null submodules. Similarly, $H/(T_1,\ldots,T_r)H$ does not have pseudo-null $(\mathbf{\Lambda}/(T_1,\ldots,T_r)\mathbf{\Lambda})$-submodules, if $(T_1,\ldots,T_r) \notin V$. Now we pick a regular sequence $\{T_1, \ldots, T_d, T'\}$ so that $(T_1, \ldots, T_r)$ for $r = 1, \ldots, d$ outside V but T' kills H. Then we know that

$$H/(T_1, \ldots, T_r)H = \{H/(T_1, \ldots, T_r)H\}[T']$$

is isomorphic to the group $H^2_{\mathrm{ord}}(Y(\Phi), \underline{C}[T_1, \ldots, T_r, T'])^*$, and hence it is $\mathbf{\Lambda}/(T_1, \ldots, T_r, T')$-torsion-free. Thus in particular, the multiplication by

$$T_{r+1} : H/(T_1, \ldots, T_r)H \longrightarrow H/(T_1, \ldots, T_r)H$$

is injective as long as $r \leq d-1$. Thus $\{T_1, \ldots, T_d\}$ is a regular sequence in $\mathfrak{m}$ for H. Furthermore, $H/(T_1, \ldots, T_d)H$ is finite. Thus we know that:

$$\begin{aligned}
&\mathrm{hd}_{\mathbf{\Lambda}}\big(H/(T_1, \ldots, T_d)H\big) = d+1,\\
&\mathrm{hd}_{\mathbf{\Lambda}}(H) + d = \mathrm{hd}_{\mathbf{\Lambda}}\big(H/(T_1, \ldots, T_d)H\big) = d+1,\\
&\mathrm{hd}_{\mathbf{\Lambda}}(H) = 1.
\end{aligned}$$

Therefore there exists an exact sequence:

$$0 \to \mathbf{\Lambda}^r \longrightarrow \mathbf{\Lambda}^r \xrightarrow{\ \pi\ } H \to 0.$$

THEOREM 5.2. — *Let Γ be a torsion-free congruence subgroup of $G^1(\mathbb{Q})$ and $H = H^1_{\mathrm{ord}}(Y(\Phi), \underline{C})^*$. Then for $\Phi = \Gamma_0(p^\alpha)$, the $\mathbf{\Lambda}$-module H is killed by a non-zero element in $\mathbf{\Lambda}$ and has homological dimension 1 over $\mathbf{\Lambda}$. In other words, there is a regular sequence of length d for H in $\mathbf{\Lambda}$.*

For a while, we return to the situation in §3. We study the relation of the characteristic power series of

$$M = \mathrm{Hom}_{\mathcal{O}}\big(H^q_{P,n\text{-}\mathrm{ord}}\big(Y(S(p^\infty)), K/\mathcal{O}\big), K/\mathcal{O}\big)$$

and that of the Hecke algebra. Let $\mathbf{W}$ be the torsion free part of $\mathbb{G}_S$. By Theorem 5.2, M is of $\mathcal{O}[[\mathbf{W}]]$-torsion and has homological dimension 1 if $p > 2$ by the argument of §2. If $p = 2$, M may not be of homological dimension 1 but is of $\mathcal{O}[[\mathbf{W}]]$-torsion. Thus it is meaningful to study the characteristic power series of M and $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$. Since $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$ acts faithfully on M, we can embed $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$ into M. Since M is an $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$-module of finite type, M is a surjective image of $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)^m$ for the minimum number m of generators of M over $\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)$. Thus, writing $\mathrm{ch}(X)$ for the characteristic power series of a $\mathcal{O}[[\mathbf{W}]]$-module X, we have

$$\mathrm{ch}\big(\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)\big) \mid \mathrm{ch}(M) \mid \mathrm{ch}\big(\boldsymbol{h}^{n\cdot\mathrm{ord}}(S)\big)^m \quad \text{in } \mathcal{O}[[\mathbf{W}]]. \tag{5.6}$$

We now consider the component induced from Hecke characters of a quadratic extension Γ of F. When $[F:\mathbb{Q}] \geq 2$ (hence F is not a CM field), we have $L = F(\sqrt{-D})$ for a positive rational integer D. This description of L holds even if F is imaginary quadratic, as will be seen in the following section. Ordinary CM types exist if

(Sp) all prime factors of p in F split in L.

Supposing the above condition, we fix an ordinary CM-type Σ. We use here the notation introduced in Theorem 4.1. In our situation, $r_1 = d-2$ and $r_2 = 1$. Thus

$$\dim_{\mathbb{Q}}(E_S \otimes_{\mathbb{Z}} \mathbb{Q}) = d-2, \quad \dim_{\mathbb{Q}}(E_C \otimes_{\mathbb{Z}} \mathbb{Q}) = d-1,$$

and $E_C^{(1)}$ is of rank 1. Then the characteristic ideal of the component associated to Θ_ψ in §4 is basically given by the kernel of the natural map

$$\mathcal{O}[[\mathfrak{r}_p^\times \times \mathfrak{r}_p^\times)/\overline{E_S}\,]] = \mathcal{O}[[\mathcal{R}_{\Sigma_p}^\times \times \mathcal{R}_{\Sigma_p c}^\times)/\overline{E_S}\,]] \to \mathcal{O}[[\mathcal{R}_p^\times/\overline{E_C}\,]]$$

which is generated by $(\varepsilon - 1)$ for a generator ε of the free part of $E_C^{(1)}$. This fact follows from the following two facts:

(i) the twist (i.e. $\Theta(x,y) = \iota(x,y)N(x)^{-1}$) by the norm character to get Θ from ι does not affect the characteristic ideal because the norm character N is trivial on E_C, and

(ii) $\mathrm{Coker}(\Theta)$ is of torsion over $\mathcal{O}[[\mathbb{W}]]$ and hence is pseudo-null over $\mathcal{O}[[\mathbf{W}]]$. We have a commutative diagram defining $\theta : \mathbf{W} \to \mathbb{W}$

$$\begin{tikzcd}
(\mathfrak{r}_p^\times \times \mathfrak{r}_p^\times)/\overline{E_S} \arrow[r] \arrow[d, "\rho"'] & \mathcal{R}_p^\times/\overline{E_C} \arrow[d, "\pi"] \\
\mathbf{W} \arrow[r, "\theta"] & \mathbb{W},
\end{tikzcd}$$

where ρ and π are the two projections. Since $\mathcal{O}[[\mathbb{W}]]$ is isomorphic to $\mathcal{O}[[\theta(\mathbf{W})]]^{[\mathbb{W}:\theta(\mathbf{W})]}$ as $\mathcal{O}[[\mathbf{W}]]$-module, the characteristic power series in $\mathcal{O}[[\mathbf{W}]]$ of the irreducible component of $\mathrm{Spec}(\boldsymbol{h}^{n\cdot\mathrm{ord}}(S))$ associated with Θ_Ψ is given by $(\rho(\varepsilon)-1)^{[\mathbb{W}:\theta(\mathbf{W})]}$ for a generator ε of the free part of $E_C^{(1)}$. Thus we have:

THEOREM 5.3. — *Let the notation be as above and the assumption be as in Theorem 4.1. We assume that $r_2 = 1$ and that B ramifies at all real places of F. Let S be the open subgroup as specified in Theorem 4.1. Then the irreducible component of Θ_ψ isomorphic to* $\mathrm{Spec}(\mathcal{O}[[\mathbb{W}]])$ *lifts to a unique irreducible component of* $\mathrm{Spec}(\boldsymbol{h}^{n\cdot\mathrm{ord}}(S))$. *Moreover the characteristic power series in* $\mathcal{O}[[\mathbf{W}]]$ *of the irreducible component is* $(\rho(\varepsilon)-1)^{[\mathbb{W}:\theta(\mathbf{W})]}$ *for a generator ε of the free part of $E_C^{(1)}$.*

6.

Now we assume that F is an imaginary quadratic field and $B = M_2(F)$. We look into the boundary exact sequence for $\Phi = \Gamma_0(p^\alpha)$ and for any Φ-submodule M of $\mathcal{C}$:

$$\begin{aligned} 0 &\to H^0(Y(\Phi),\underline{M}) \to H^0(\partial Y(\Phi),\underline{M}) \to H^1_c(Y(\Phi),\underline{M}) \\ &\to H^1(Y(\Phi),\underline{M}) \longrightarrow H^1(\partial Y(\Phi),\underline{M}) \to H^2_c(Y(\Phi),\underline{M}) \\ &\to H^2_c(Y(\Phi),\underline{M}) \to H^2(\partial Y(\Phi),\underline{M}) \to 0. \end{aligned}$$

By abusing the notation, here we have written $\partial Y(\Phi)$ for the boundary of the Borel-Serre compactification of $Y(\Phi)$. We know from [H4], Lemma 9.2, $H^0_{\mathrm{ord}}(Y(\Phi),\underline{M}) = H^3_{c,\mathrm{ord}}(Y(\Phi),\underline{M}) = 0$ for any Φ-submodule M of $\mathcal{C}$. We thus have another exact sequence

$$\begin{aligned} 0 &\to H^0_{\mathrm{ord}}(\partial Y(\Phi),\underline{M}) \to H^1_{c,\mathrm{ord}}(Y(\Phi),\underline{M}) \to H^1_{\mathrm{ord}}(Y(\Phi),\underline{M}) \\ &\to H^1_{\mathrm{ord}}(\partial Y(\Phi),\underline{M}) \to H^2_{c,\mathrm{ord}}(Y(\Phi),\underline{M}) \\ &\to H^2_{\mathrm{ord}}(Y(\Phi),\underline{M}) \to H^2_{\mathrm{ord}}(\partial Y(\Phi),\underline{M}) \to 0. \end{aligned}$$

By [H1], Cor. 3.14, writing $C(\Gamma)$ for the set of equivalence classes of cusps of Γ, we have

$$(6.1)\quad \begin{cases} H^0_{\mathrm{ord}}(\partial Y(\Phi),\underline{\mathcal{C}}) \cong \bigoplus_{s\in C(\Gamma)} \mathcal{C}(\mathfrak{r}_p^\times;K/\mathcal{O}); \\ H^1_{\mathrm{ord}}(\partial Y(\Phi),\underline{\mathcal{C}}) \cong \begin{cases} \bigoplus_{s\in C(\Gamma)} (\mathcal{C}(\mathfrak{r}_{\mathfrak{p}}^\times;K/\mathcal{O}) \oplus \mathcal{C}(\mathfrak{r}_{\mathfrak{p}^c}^\times;K/\mathcal{O})) & \text{if } p = \mathfrak{p}\mathfrak{p}^c \text{ with } \mathfrak{p}\neq\mathfrak{p}^c, \\ 0 & \text{otherwise;} \end{cases} \\ H^2_{\mathrm{ord}}(\partial Y(\Phi),\underline{\mathcal{C}}) \cong \bigoplus_{s\in C(\Gamma)} \mathcal{C}(\mathfrak{r}_p^\times;K/\mathcal{O}); \end{cases}$$

where $\mathcal{C}(\mathfrak{r}_p^\times; K/\mathcal{O})$ is the space of continuous functions on $\mathfrak{r}_p^\times$ having values in $K/\mathcal{O}$. We have a long exact sequence for $M = \mathcal{C}$ or $\mathcal{C}[T']$:

$$\begin{aligned} &H^q_{c,\mathrm{ord}}(Y(\Phi), \underline{M}[T^n]) \to H^q_{c,\mathrm{ord}}(Y(\Phi), \underline{M}) \\ &\qquad \xrightarrow{T^n} H^q_{c,\mathrm{ord}}(Y(\Phi), \underline{M}) \longrightarrow H^{q+1}_{c,\mathrm{ord}}(Y(\Phi), \underline{M}[T^n]) \\ &\qquad\qquad \longrightarrow H^{q+1}_{c,\mathrm{ord}}(Y(\Phi), \underline{M}) \xrightarrow{T^n} H^{q+1}_{c,\mathrm{ord}}(Y(\Phi), \underline{M}). \end{aligned}$$

This implies that the following sequence is exact:

$$\begin{aligned} (6.2)\quad &0 \to H^q_{c,\mathrm{ord}}(Y(\Phi), \underline{M}) \otimes_R R/T^n R \\ &\qquad \to H^{q+1}_{c,\mathrm{ord}}(Y(\Phi), \underline{M}[T^n]) \to H^{q+1}_{c,\mathrm{ord}}(Y(\Phi), \underline{M})[T^n] \to 0. \end{aligned}$$

Applying (6.2) to $q = 0$ we get the control theorem:

$$(6.3)\qquad \begin{cases} H^1_{c,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}[T]) \cong H^1_{c,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})[T], \\ H^1_{c,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}[T,T']) \cong H^1_{c,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})[T,T']. \end{cases}$$

Since $H^0_{\mathrm{ord}}(\partial Y(\Phi), \underline{\mathcal{C}})$ is $\mathbf{\Lambda}$-injective, we have from the exact sequence

$$0 \to H^0_{\mathrm{ord}}(\partial Y(\Phi), \underline{\mathcal{C}}) \longrightarrow H^1_{c,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}) \longrightarrow H^1_{P,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}) \to 0$$

the splitting of $\mathbf{\Lambda}$-module:

$$H^1_{c,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}) \cong H^0_{\mathrm{ord}}(\partial Y(\Phi), \underline{\mathcal{C}}) \oplus H^1_{P,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}).$$

Then by (6.3), we get the control theorem for parabolic cohomology groups:

$$(6.4)\qquad H^1_{P,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}}[P]) \cong H^1_{P,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})[P]$$

for any height 2 prime ideal P of $\mathbf{\Lambda}$.

Let W be the torsion-free part of $\mathfrak{r}_p^\times$. We write $\mathbf{\Lambda} = \mathcal{O}[[W]]$. Then $\mathbf{\Lambda}$ is a regular local ring of dimension 3. Actually $\mathbf{\Lambda} \cong \mathcal{O}[[X,Y]]$ for a suitable choice of X and Y. By (6.1), $H^1_{\mathrm{ord}}(\partial Y(\Phi), \underline{\mathcal{C}})^*$ is always of $\mathbf{\Lambda}$-torsion. Thus for (Zariski) densely populated arithmetic points P of $\mathrm{Spec}(\mathbf{\Lambda})$ corresponding to characters $\mathfrak{r}_p^\times \to \mathcal{O}^\times$ of the form $x \mapsto x^n$ with $n \in \mathbb{Z}[I]$, the group $H^1_{\mathrm{ord}}(\partial Y(\Phi), \underline{\mathcal{C}})[P]$ is finite. We also know from [H1], Thm 5.2, for densely populated such P, $H^1_{\mathrm{ord}}(\partial Y(\Phi), \underline{\mathcal{C}})[P]$ and $H^q_{P,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})[P]$ ($q = 1, 2$) are both finite. This shows that $H^1_{\mathrm{ord}}(\partial Y(\Phi), \underline{\mathcal{C}})^*$, $H^1_{P,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})^*$ and $H^2_{c,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})^*$ are of $\mathbf{\Lambda}$-torsion.

Lemma 6.1. — *Let T be a non-constant element in the maximal ideal of $\mathbf{\Lambda}$. Then $H^2_{c,\mathrm{ord}}(Y(\Phi),\underline{C})$ and $H^2_{\mathrm{ord}}(Y(\Phi),\underline{C})$ are $\mathbf{\Lambda}$-divisible. Similarly, $H^2_{c,\mathrm{ord}}(Y(\Phi),\underline{C}[T])$ and $H^2_{\mathrm{ord}}(Y(\Phi),\underline{C}[T])$ are $\mathbf{\Lambda}/T\mathbf{\Lambda}$-divisible. In particular,*

- $H^2_{c,\mathrm{ord}}(Y(\Phi),\underline{C}) = 0$ *and* $H^2_{c,\mathrm{ord}}(Y(\Phi),\underline{C}[T]) = 0$ *if (T) is not an element of* $\mathrm{Supp}(H^1_{P,\mathrm{ord}}(Y(\Phi),\underline{C})^*)$, *and*
- $H^2_{c,\mathrm{ord}}(Y(\Phi),\underline{C}[T,T']) = 0$ *if (T,T') is not an element of* $\mathrm{Supp}(H^1_{P,\mathrm{ord}}(Y(\Phi),\underline{C})^*)$ *and (T,T') is a regular sequence.*

Since the proof of this lemma is exactly the same as the one for Lemma 5.1, we leave it to the reader. Since $H^1_{P,\mathrm{ord}}(Y(\Phi),\underline{C})$ is of $\mathbf{\Lambda}$-cotorsion and $H^0_{\mathrm{ord}}(\partial Y(\Phi),\underline{C})$ is $\mathbf{\Lambda}$-injective, we get:

$$H^1_{c,\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)\otimes_{\mathbf{\Lambda}}\mathbf{\Lambda}/T\mathbf{\Lambda}\cong H^1_{P,\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)\otimes_{\mathbf{\Lambda}}\mathbf{\Lambda}/T\mathbf{\Lambda}.$$

Suppose that (T_1,T_2,T') is a regular sequence. Let:

$$V=\mathrm{Supp}(H^1_{P,\mathrm{ord}}\big(Y(\Phi),\underline{C})^*\big).$$

We get:

$$(6.5\mathrm{a})\quad\begin{cases} H^1_{P,\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)[T_1,T_2]\otimes_{\mathbf{\Lambda}}\mathbf{\Lambda}/T'\mathbf{\Lambda} \\ \qquad\cong H^2_{c,\mathrm{ord}}(Y(\Phi),\underline{C}[T_1,T_2,T']) \text{ if } (T_1,T_2)\notin V, \\ H^1_{P,\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)[T_1]\otimes_{\mathbf{\Lambda}}\mathbf{\Lambda}/T'\mathbf{\Lambda} \\ \qquad\cong H^2_{c,\mathrm{ord}}\big(Y(\Phi),\underline{C}[T_1,T']\big) \text{ if } (T_1)\notin V, \\ H^1_{P,\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)\otimes_{\mathbf{\Lambda}}\mathbf{\Lambda}/T'\mathbf{\Lambda} \\ \qquad\cong H^2_{c,\mathrm{ord}}\big(Y(\Phi),\underline{C}[T']\big) \text{ for any choice of } T'. \end{cases}$$

Here we have used the fact:

$$H^1_{P,\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)[T_1]\otimes_{\mathbf{\Lambda}}\mathbf{\Lambda}/T'\mathbf{\Lambda}\cong H^1_{c,\mathrm{ord}}\big(Y(\Phi),\underline{C}\big)[T_1]\otimes_{\mathbf{\Lambda}}\mathbf{\Lambda}/T'\mathbf{\Lambda}.$$

By taking the dual, let $H=H^1_{P,\mathrm{ord}}(Y(\Phi),\underline{C})^*$. Then we have

$$(6.5\mathrm{b})\quad\begin{cases} \{H/(T_1,T_2)H\}[T']\cong H^2_{c,\mathrm{ord}}\big(Y(\Phi),\underline{C}[T_1,T_2,T']\big)^* \text{ if } (T_1,T_2)\notin V, \\ \{H/T_1H\}[T']\cong H^2_{c,\mathrm{ord}}\big(Y(\Phi),\underline{C}[T_1,T']\big)^* \text{ if } (T_1)\notin V, \\ H[T']\cong H^2_{c,\mathrm{ord}}\big(Y(\Phi),\underline{C}[T']\big)^*, \end{cases}$$

for any choice of T'. Then in exactly the same manner as in §5, we get:

THEOREM 6.2. — *Let* Γ *be a torsion-free congruence subgroup of* $\mathrm{SL}_2(F)$. *Then for* $\Phi = \Gamma_0(p^\alpha)$, *the* $\mathbf{\Lambda}$*-module* $H = H^1_{P,\mathrm{ord}}(Y(\Phi), \underline{\mathcal{C}})^*$ *is killed by a non-zero element in* $\mathbf{\Lambda}$ *and has homological dimension* 1 *over* $\mathbf{\Lambda}$.

We now consider the CM component associated to a quadratic extension L of F. Supposing that we have an ordinary CM-type Σ. We use the notation introduced in §4. We write $\mathcal{R}_{\Sigma_p}$ for the Σ_p-adic completion of $\mathcal{R}$. Then $\mathcal{R}_{\Sigma_p} \cong r_p$. We suppose the regularity condition:

$$\text{(R)} \qquad (j_\sigma, j_{\sigma c}) \neq (j_{\rho\sigma}, j_{\rho\sigma c}) \quad \text{for all } \sigma \in I.$$

Since $j_\sigma = j_{\sigma\rho}$ if F is the maximal CM field, this eliminates the case where L is not a CM field. Thus we assume that:

$$\text{(CM)} \qquad L \text{ is a CM field.}$$

Then the existence of ordinary CM-type is equivalent to the following two conditions:

(Sp_1) all prime over p in the maximal real subfield of L split in L;

(Sp_2) all prime over p in F split in L $(\Leftrightarrow \Sigma_p\rho \cap \Sigma_p = \emptyset)$.

The conditions (Sp_1)–(Sp_2) are equivalent to:

$$(\mathrm{Sp}_3) \qquad \begin{cases} \text{The decomposition group at } p \text{ in } \mathrm{Gal}(L/\mathbb{Q}) \\ \text{does not contain either } c \text{ nor } \rho. \end{cases}$$

Under (Sp_1)–(Sp_2) and (CM), we have the CM component induced by Θ_ψ as discussed in §4. Let ε be a generator of the torsion-free part of E_C. We may assume that $N_{L/F}(\varepsilon) = 1$. Thus $\varepsilon \in \mathbb{E}_S = \mathfrak{r}_p^\times$. Defining ρ and θ as in §5, we know similarly to Theorem 5.3 that $(\rho(\varepsilon) - 1)^{[\mathbb{W}:\theta(\mathbf{W})]}$ gives the characteristic power series of the CM component attached to Θ_Ψ. We can naturally identify $\mathcal{R}_{\Sigma_p}(= \mathfrak{r}_p)$ with the p-adic integer ring of the p-adic completion of the maximal real subfield L_+ of L. Since c induces an automorphism on $\mathfrak{r}_p$, $\varepsilon\varepsilon^c = 1$. Thus the annihilator in $\mathcal{O}[[\mathbb{G}_S]]$ of each CM irreducible component is given by a prime ideal $P_{\psi N}$ associated to the norm character $\psi N : \mathbb{E}_S \to \mathbb{Z}_p[\psi]^\times$ twisted by a finite order character ψ, where $\mathbb{Z}_p[\psi]$ is the subring of $\overline{\mathbb{Q}}_p$ generated by the values of ψ. We may conjecture that the same fact might be true for all irreducible components of the nearly ordinary Hecke algebra of GL(2) over an imaginary quadratic field.

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Corrections to [H1]

Page 262, line 3:
« $H_q^{\mathrm{ord}}(\Delta_1(p^\infty), \mathrm{L}(\mathrm{n}, \varepsilon; \mathcal{O}))$ » should be « $H_q^{\mathrm{ord}}(\Delta_0(p^\alpha), \mathrm{L}(\mathrm{n}, \varepsilon; \mathcal{O}))$ ».

Page 262, line 5:
« $H^q_{\mathrm{ord}}(\Delta_1(p^\infty), \mathrm{L}(\mathrm{n}, \varepsilon; \mathcal{K}/\mathrm{O}))$ » should be « $H^q_{\mathrm{ord}}(\Delta_0(p^\alpha), \mathrm{L}(\mathrm{n}, \varepsilon; \mathcal{K}/\mathrm{O}))$ ».

Page 311, line 8: delete the phrase
« and in this case $\dim_{\mathbb{C}}(\mathrm{Im}(\mathrm{res})) = \frac{1}{2}\dim_{\mathbb{C}} H^q(\partial Y*, \mathcal{L}(\mathrm{n}, \mathbb{C}))$ ».

Page 311, line 4 from the bottom:
« $H^q_{P\cdot\mathrm{ord}}(\Delta_1(p^\alpha), \mathrm{L}(\mathrm{n}, \varepsilon; \mathrm{K}/\mathcal{O}))$ » should be « $H^q_{P\cdot\mathrm{ord}}(\Delta_0(p^\alpha), \mathrm{L}(\mathrm{n}, \varepsilon; \mathrm{K}/\mathcal{O}))$ ».

Page 312, line 18 from the bottom:
« $H^q_{c\cdot\mathrm{ord}}(Y_1(p^\alpha), \mathrm{L}(\mathrm{n}, \varepsilon; \mathrm{K}/\mathcal{O}))$ » should be « $H^q_{c\cdot\mathrm{ord}}(Y_0(p^\alpha), \mathrm{L}(\mathrm{n}, \varepsilon; \mathrm{K}/\mathcal{O}))$ ».

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