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MOTIVES OVER TOTALLY REAL FIELDS AND p-ADIC L-FUNCTIONS

by Alexei A. PANCHISHKIN

0. Introduction.

Let p be a prime number. In this paper we are interested in the arithmetic of special values of various twisted zeta functions of the type

$$\sum_{n=1}^{\infty} \chi(n)a(n)n^{-s}$$

where χ is a varying Dirichlet character. In last decades many results were found on p-adic properties of such special values. It is essential for certain arithmetical problems to study more general special values which have the form

$$\sum_{\mathfrak{n}} \chi(\mathfrak{n})a(\mathfrak{n})\mathcal{N}\mathfrak{n}^{-s}$$

where summation is taken over integral ideals $\mathfrak{n}$ of the maximal order of a number field F, χ is a Hecke character of finite order, and $\mathcal{N}\mathfrak{n}$ is the norm of $\mathfrak{n}$. General examples of Dirichlet series with Euler product of the above type come from motives over F and their Galois twists. It turns out that in the case of a totally real field F there exist many analogies between arithmetical objects over F and over $\mathbf{Q}$. In particular this analogy extends to properties of the above special values. For example, the special

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values at negative integers of the Dedekind zeta function $\zeta_F(s)$ are known to be rational numbers due to H. Klingen [Kl1], [Kl2], the fact which was established by Riemann for $F = \mathbf{Q}$. More recent arithmetical examples are related to proofs of Iwasawa conjecture and Brumer conjecture over totally real fields by A. Wiles [Wi] using arithmetic of the special values at negative integers of the Hecke L-functions. Their p-adic interpolation was built earlier by P. Deligne and K. Ribet [DeRi], D. Barsky [Ba], and P. Cassou-Noguès [Cass-N].

This paper has two purposes. First we describe some algebraic properties of the critical special values with varying Hecke character χ using Deligne's conjecture on special values [De3]. Also we describe the p-adic interpolation of such values using recent works of J. Coates and B. Perrin-Riou [CoPe-Ri], [Co]. We try to touch the delicate case of motives M over F which are not p-ordinary in the sense of [CoPe-Ri]. As was pointed out to the author by M. Harris, there exist examples of non-p-ordinary motives, for which the existence of bounded p-adic L-functions can be proven. In particular this is the case for the Katz p-adic L-functions for CM-fields at certain primes. We formulate a general criterion for the existence of bounded p-adic L-functions in terms of the coincidence of the Newton and the Hodge polygon of a motive M at the point $d^+(M)$ (the dimension of Deligne's subspace M^+ of the Betti realization of M). In a more general situation we formulate a conjecture on the existence of p-adic L-functions of logarithmic growth using the technique of admissible measures of Višik-Amice-Vélu (see [AmV], [V1]). Following an observation of A. Dabrowski we use the difference between the Newton and the Hodge polygons at the point $d^+(M)$. We give some examples of L-series for which these conjectural algebraic and p-adic properties were recently proved.

Content of the paper. — In Section 1 we recall some properties and definitions on the motives over number fields and their L-functions. Then in Section 2 we describe a factorization of Deligne's periodes $c^{\pm}(M)$ into a product whose factors $c^{\pm}(\sigma, M)$ indexed by the (real) embeddings σ of the ground field F. Then we recall in Section 3 the definition by J. Coates of the modified L-function of a motive M over F and we state there a modified period conjecture which gives a description of the critical special values of arbitrary twists $M(\chi)$ with Hecke characters χ of finite order in terms of the factorization of Deligne's periods of the original motive M. In Section 4 we recall the notion of an h-admissible measure over a Galois group and properties of its Mellin transform. Then in Section 5 we discuss p-ordinary and admissible motives over F using the Newton polygons and the Hodge polygons of them, and formulate in Section 6 a general conjecture on p-adic L-functions of such motives. Various examples are given in Section 8, after recalling in Section 7 some basic properties of Hilbert modular forms and their zeta functions.

Throught the paper we fix embeddings

$$i_\infty : \overline{\mathbf{Q}} \to \mathbf{C}, \quad i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p,$$

and we shall often regard algebraic numbers (via these embeddings) as both complex and p-adic numbers, where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ is the Tate field (the completion of a fixed algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$), which is endowed with a unique norm $|\cdot|_p$ such that $|p|_p = p^{-1}$.

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1. Motives over number fields and their L-functions.

By a motive M over a number field F of degree $n = [F : \mathbf{Q}]$ with coefficients in another number field T we shall mean a collection of the following objects:

$$M_{B,\sigma} = M_\sigma, \ M_{DR}, \ M_\lambda, \ I_{\infty,\sigma}, \ I_{\lambda,\sigma},$$

where σ runs over the set J_F of all complex embeddings of F,

M_σ is the Betti realization of M (with respect to the embedding $\sigma \in J_F$) which is a vector space over T of dimension d endowed for real $\sigma \in J_F$ with a T-rational involution ρ_σ;

M_{DR} is the de Rham realization of M, a free $T \otimes F$-module of rank d, endowed with a decreasing filtration $\{F^i_{DR}(M) \subset M_{DR} \mid i \in \mathbf{Z}\}$ of $T \otimes F$-modules (which may not be free in some cases when $F \neq \mathbf{Q}$);

M_λ is the λ-adic realization of M at a finite place λ of the coefficient field T (a T_λ-vector space of degree d over T_λ, a completion of T at λ) which is a Galois module over $G_F = \mathrm{Gal}(\overline{F}/F)$ so that we a have a compatible system of λ-adic representations denoted by

$$r_{M,\lambda} = r_\lambda : G_F \xrightarrow{\sim} GL(M_\lambda).$$

Also,

$$I_{\infty,\sigma} : M_\sigma \underset{\mathbf{Q}}{\otimes} \mathbf{C} \xrightarrow{\sim} M_{DR} \underset{F,\sigma}{\otimes} \mathbf{C}$$

is the complex comparison isomorphism of $T \otimes \mathbf{C}$-modules for each $\sigma \in J_F$,

$$I_{\lambda,\sigma} : M_\sigma \underset{T}{\otimes} T_\lambda \to M_\lambda$$

is the λ-adic comparison isomorphism of T_λ-vector spaces. It is assumed in the notation that the complex vector space $M_\sigma \underset{\mathbf{Q}}{\otimes} \mathbf{C}$ is decomposed in the Hodge bigraduation

$$M_\sigma \underset{\mathbf{Q}}{\otimes} \mathbf{C} = \underset{i,j}{\oplus} M_\sigma^{i,j}$$

in which $\rho_\sigma(M_\sigma^{i,j}) \subset M_\sigma^{j,i}$ for real $\sigma \in J_F$ and the Hodge numbers

$$h(i,j) = h(i,j,M) = \dim_{\mathbf{C}} M_\sigma^{i,j}$$

do not depend on σ. Moreover,

$$I_{\infty,\sigma}\Big(\underset{i' \geq i}{\oplus} M_\sigma^{i',j}\Big) = F_{DR}^i(M) \underset{F,\sigma}{\otimes} \mathbf{C}.$$

Also, $I_{\lambda,\sigma}$ takes ρ_σ to the r_λ-image of the Galois automorphism which is denoted by the same symbol $\rho_\sigma \in G_F$ and corresponds to the complex conjugation of $\mathbf{C}$ under an embedding of $\overline{F}$ to $\mathbf{C}$ extending σ. We assume that M is pure of weight w (i.e. $i + j = w$).

The L-function $L(M,s)$ of M is defined as the following Euler product (which takes values in $T \otimes \mathbf{C}$):

$$L(M,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M, \mathcal{N}\mathfrak{p}^{-s}),$$

extended over all maximal ideals $\mathfrak{p}$ of the maximal order $\mathcal{O}_F$ of F and where

$$\begin{aligned} L_{\mathfrak{p}}(M,X)^{-1} &= \det(1 - X \cdot r_\lambda(Fr_{\mathfrak{p}}^{-1}) \mid M_\lambda^{I_{\mathfrak{p}}}) \\ &= (1 - \alpha^{(1)}(\mathfrak{p})X) \cdot (1 - \alpha^{(2)}(\mathfrak{p})X) \cdot \ldots \cdot (1 - \alpha^{(d)}(\mathfrak{p})X) \\ &= 1 + A_1(\mathfrak{p})X + \ldots + A_d(\mathfrak{p})X^d; \end{aligned}$$

here $\mathcal{N}\mathfrak{p}$ is the norm of $\mathfrak{p}$ and $Fr_{\mathfrak{p}} \in G_F$ is the Frobenius element at $\mathfrak{p}$, defined modulo conjugation and modulo the inertia subgroup $I_{\mathfrak{p}} \subset G_{\mathfrak{p}} \subset G_F$ of the decomposition group $G_{\mathfrak{p}}$ (of any extension of $\mathfrak{p}$ to $\overline{F}$). We assume the standard hypothesis that the coefficients of $L_{\mathfrak{p}}(M,\, X)^{-1}$ belong to T, and that they are independent of λ coprime with $\mathcal{N}\mathfrak{p}$. Therefore we can and we shall regard this polynomial over the ring $T \otimes \mathbf{C}$ so that

$$L_{\mathfrak{p}}(M,X)^{-1} = (L_{\mathfrak{p}}^{(\tau)}(M,X)^{-1})_{\tau \in J_T} = (1 + A_1(\mathfrak{p})^{\tau} X + ... + A_d(\mathfrak{p})^{\tau} X^d)_{\tau}.$$

We shall need the following linear algebra operations on motives which are defined by means of their realizations and can be obviously expressed in terms of the corresponding L-functions:

$M^{\vee}$ (dual motive): its λ-adic representation is contragredient to that of M;

$M_1 \oplus M_2$ (direct sum of motives M_1, M_2): its λ-adic representation is the direct sum of those for M_1, M_2, and the corresponding L-function is the product of those of M_1 and M_2;

$M_1 \underset{F}{\otimes} M_2$ (tensor product of motives over F): its λ-adic representation is the tensor product of those for M_1, M_2, and the corresponding L-function is a kind of a "multiplicative convolution" of the L-functions of M_1 and M_2;

$R_{F/F'}M$ (restriction of scalars to a subfield F' of F) is a motive over the smaller field F' whose λ-adic representation is obtained by inducing that of M from the subgroup $G_F \subset G_{F'}$ (its L-function coincides with that of M).

$M \otimes K$ (extension of the ground field (base change), K/F being a finite extension) is a motive over K whose λ-adic representation is obtained from that of M by restriction to the open subgroup $G_K \subset G_F$.

The important examples of motives are: the cyclotomic (Tate) motive $F(1)$ and the motive $[\chi]$ associated with a Hecke character of finite order $\chi : \mathbf{A}_F^{\times}/F^{\times} \to \mathbf{C}^{\times}$. The λ-adic representation of $F(1)$ is defined by the action of G_F on the l-power roots of unity (where l is the characteristic of λ) so that $Fr_{\mathfrak{p}}$ acts as a scalar $\mathcal{N}\mathfrak{p}$ and $L(F(1),\, s) = \zeta_F(s+1)$, $\zeta_F(s)$ being the Dedekind zeta function of F. Also $F(m)$ will denote the m-th tensor power of $F(1)$ if $m \geq 0$ and the $-m$-th tensor power of $F(-1) = F(1)^{\vee}$ if $m < 0$. The λ-adic representation of $[\chi]$ is given by class field theory so that $L([\chi], s)$ coincides with the Hecke L-function $L(s, \chi^{-1})$ of the character χ^{-1}.

The twist operation: for an arbitrary motive M over F with coefficients in T an integer m and a Hecke character χ of finite order one can define the twist $N = M(m)(\chi)$ which is again a motive over F with the coefficient field $T(\chi)$ of the same rank d and weight w so that we have

$$L(N,s)=\prod_{\mathfrak{p}} L_{\mathfrak{p}}(M,\chi^{-1}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s-m}).$$

Conjecturally the function $L(M,s)$ can be analytically continued to the entire complex plane and it satisfies the functional equation of the type

$$\Lambda(M,s)=\varepsilon(M,s)\Lambda(M^{\vee}(1),-s)$$

where $\Lambda(M,s) = L_\infty(M,s)L(M,s)$, $L_\infty(M,s)$ is the Γ-factor, which is completely determined by the Hodge structure of $RM = R_{F/\mathbf{Q}}M, \varepsilon(M,s)$ is a certain ε-factor, which can be decomposed into a product of local factors $\varepsilon_v(M,\ s)$(v runs over places of F). Moreover, $\Lambda(M,\ s)$ is entire, unless the weight w is even and $\mathbf{Q}(-w/2)$ is a direct summand of RM.

Conversely, if we have a zeta function $Z(s)$ satisfying a functional equation of the above type, and if we know that $Z(s) = L(M,\ s)$ for a motive M then we can use this functional equation in order to determine the Hodge type of the motive.

2. Factorization of Deligne's periods over a totally real field.

2.1. According to the famous conjecture of Deligne on critical values for a motive M over $\mathbf{Q}$ the values of $L(M,\ s)$ at some points (the critical values) can be described modulo $T^\times \subset (T\otimes\mathbf{C})^\times$ in terms of periods $c^\pm(M)\in(T\otimes\mathbf{C})^\times$ which are defined (modulo $T^\times$) so that the quantity

$$c^\pm(M)=(c^\pm(M)^{(\tau)})_{\tau\in J_T}$$

may be regarded as a set of non zero complex constants $c^\pm(M)^{(\tau)}$ which are defined modulo $\tau(T)^\times\subset\mathbf{C}^\times$. Then Deligne's conjecture on critical values states that if $s=0$ is critical for a motive M over $\mathbf{Q}$ then

$$L(M,0)c^+(M)^{-1}\in T\subset T\otimes\mathbf{C}.$$

Using periods $c^\pm(M)$ and the Gauss sums one can also describe the critical values of the twisted motive $M(m)(\chi)$ where χ is a Dirichlet character and m is an integer such that $M(m)(\chi)$ is critical at $s=0$.

If M is a motive over F then we have that

$$L(M,s)=L(RM,s),\quad RM=R_{F/\mathbf{Q}}M$$

hence the above description can be used for a motive M over an arbitrary number field F. However we wish to extend this description to arbitrary twists $M(\chi)$ with Hecke characters χ of finite order in terms of the original motive M.

In order to do this we introduce first a factorization of periods of a motive M over a totally real field F. For each (real) embedding σ consider the eigenspaces M_σ^+ and M_σ^- of the T-linear involution ρ_σ with eigenvalues 1 and -1 respectively. Assume that ρ_σ acts on $M_\sigma^{w/2,w/2}$ as a scalar $(-1)^\varepsilon$ which does not depend on σ. This will automatically implied by our assumption made later that M is critical at $s = 0$. Note that under this assumption the dimensions $d_\sigma^\pm(M) = d^\pm(M)$ do not depend on σ. It follows also that one can choose appropriate terms F_{DR}^+M and F_{DR}^-M of the de Rham filtration such that

$$\dim(M_{DR}/F_{DR}^+M) = d^+(M),\ \dim(M_{DR}/F_{DR}^-M) = d^-(M)$$

and that the comparison map $I_{\infty,\sigma}$ induces isomorphisms

$$I_{\infty,\sigma}^\pm : M_\sigma^\pm \underset{\mathbf{Q}}{\otimes} \mathbf{C} \xrightarrow{\sim} M_{DR}^\pm \underset{F,\sigma}{\otimes} \mathbf{C},$$

where $M_{DR}^\pm = M_{DR}/F_{DR}^\pm M$. In order to define σ-periods we put

$$c^\pm(\sigma, M) = (c^\pm(\sigma, M)^{(\tau)}) = \det(I_{\infty,\sigma}^\pm) \in (T \otimes \mathbf{C})^\times,$$

where the right hand side denotes the determinants of matrices representing the maps relative to some T-rational bases of the sourse and target. Note that the quantities $c^\pm(\sigma, M)$ are defined modulo the multiplicative subgroup $(T \otimes \sigma(F))^\times$ of $(T \otimes \mathbf{C})^\times$ because in the $(T \otimes \mathbf{C})$-module $M_{DR}^\pm \underset{F,\sigma}{\otimes} \mathbf{C}$ an obvious rational structure is defined only over $T \otimes \sigma(F)$ so that the quantity $c^\pm(\sigma, M)^{(\tau)}$ is a complex constant which is a priori defined modulo $(\tau(T)\sigma(F))^\times$, where $\tau(T)\sigma(F)$ is the composite of the subfields $\tau(T)$ and $\sigma(F)$ in $\mathbf{C}$.

Now let us fix a basis $\{\alpha_i\}$ of $M_{DR} \otimes \mathbf{C}$ as a free $(T \otimes F \otimes \mathbf{C})$-module. Note that $(R_{F/\mathbf{Q}}M)_{DR}$ coincides with M_{DR} as T-module (forgetting the F-structure). Therefore for a basis $\{\beta_j\}$ of F over $\mathbf{Q}$ we have that $\{\alpha_i \otimes (1 \otimes \beta_j)\}$ form a basis of

$$(R_{F/\mathbf{Q}}M)_{DR} \underset{\mathbf{Q}}{\otimes} \mathbf{C} = \quad M_{DR} \underset{F,\sigma}{\otimes} \mathbf{C}$$

as a free $(T \otimes \mathbf{C})$-module. Using this basis let us compute the quantity $c^\varepsilon(RM)$ which is well defined modulo $T^\times$. Taking into account that $\det(\beta_j^\sigma) \sim D_F^{1/2} \bmod \mathbf{Q}^\times$ we see that

$$c^\varepsilon(RM) \sim (1 \otimes D_F^{1/2})^{d^\varepsilon(M)} \prod_\sigma c^\varepsilon(\sigma, M) \bmod T^\times.$$

We can easily compute σ-periods of the twisted motive $M(m)$: for $\nu = \operatorname{sgn}(-1)^m$

$$c^\varepsilon(\sigma, M(m)) = (2\pi i)^{d^{\varepsilon\nu} m} c^{\varepsilon\nu}(\sigma, M) \bmod (\sigma(F)T)^\times.$$

Note that the periods of the twisted motive $M(m)$ are given by:

$$c^\varepsilon(M(m)) = (2\pi i)^{d^{\varepsilon\nu} nm} c^{\varepsilon\nu}(M) \bmod T^\times,$$

where $(-1)^m = \nu \cdot 1$, $(\nu = \pm)$.

2.2. *Periods of the twist with a Hecke character of finite order* $\chi : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$.

We shall denote by the same letter χ the corresponding character of the group $I(\mathfrak{c})$ of ideals of F prime to the conductor $\mathfrak{c}$ of χ, and we shall identify χ with the character of the Galois group $\operatorname{Gal}(F^{ab}/F)$ such that $\chi(Fr_\mathfrak{p}^{-1}) = \chi(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ which do not divide $\mathfrak{c}$. Let $\operatorname{Sgn}_F \subset F_\infty^\times$ denote the group of signs of F (elements of order 2 in $F_\infty^\times$), where $F_\infty = F \otimes \mathbf{R} \cong \mathbf{R}^n$. If we define the sign $\operatorname{sgn}(\chi)$ of χ by

$$\operatorname{sgn}(\chi) = \operatorname{sgn}(\chi_\infty) = (\varepsilon_\sigma) = (\varepsilon_\sigma(\chi)) \in \operatorname{Sgn}_F$$

then we see that $d^\varepsilon(M(\chi)) = d^{\varepsilon\varepsilon(\chi)}(M)$ and

$$c^\varepsilon(M(\chi)) = (G(\chi)^{-1} D_F^{-1/2})^{d^{\varepsilon\varepsilon(\chi)}(M)} \prod_\sigma c^{\varepsilon\varepsilon_\sigma(\chi)}(\sigma, M) \in (T \otimes \mathbf{C})^\times \bmod T(\chi)^\times$$

where $\varepsilon = \pm$, $\varepsilon(\chi)$ is any of $\varepsilon_\sigma(\chi)$, $G(\chi)$ is the Gauss sum (see [Shi6], [De3]):

$$G(\chi) = \sum_{x \in \mathfrak{d}^{-1}\mathfrak{c}^{-1}/\mathfrak{d}} \chi((x)\mathfrak{c}\mathfrak{d}) \underset{\mathbf{Q}}{\otimes} e(\operatorname{Tr}(x)) \in \mathbf{Q}(\chi) \otimes \mathbf{C},$$

($\mathfrak{d}$ is the different of F, $\operatorname{Tr}(x) = x^{\sigma_1} + \ldots + x^{\sigma_n}$, $e(x) = \exp(2\pi i x)$). Note that in the case when $M_\sigma^{k,k} \neq \{0\}$ the involution ρ_σ acts on $M_\sigma^{k,k}$ as a scalar independent of σ, hence $\varepsilon_\sigma(\chi)$ does not depend on σ, and if $M_\sigma^{k,k} = \{0\}$ then the dimensions $d^{\varepsilon\varepsilon_\sigma(\chi)} = \dim_{T(\chi)} M(\chi)^\varepsilon = d^+ = d^- = d/2$ also do not depend on σ.

Combining the above equalities we get the following general formula for the periods of the twist $M(m)(\chi)$:

$$c^\varepsilon(M(\chi)(m)) = (G(\chi)^{-1}(1 \otimes D_F^{1/2}(2\pi i)^{nm}))^{d^{\varepsilon\nu\varepsilon(\chi)}(M)} \prod_\sigma c^{\varepsilon\varepsilon_\sigma(\chi)\nu}(\sigma, M) \in (T \otimes \mathbf{C})^\times \bmod T(\chi)^\times.$$

Starting from this formula we may hope to determine the σ-factors $c^\varepsilon(\sigma, M)$ more delicately, namely, modulo the group $T^\times$. Following a suggestion of A.A. Beilinson we formulate this hope as the following conjecture.

2.3. CONJECTURE ON FACTORIZATION OF THE PERIODS. — *Assume that there exists an integer m such that $M(m)$ is critical at $s = 0$. Then there exist constants*

$$\tilde{c}^{\varepsilon}(\sigma, M) \in (T \otimes \mathbf{C})^{\times} \quad (\varepsilon = \pm)$$

which are defined modulo $T^{\times}$ such that

$$c^{\varepsilon}(\sigma, M) = \tilde{c}^{\varepsilon}(\sigma, M) \bmod \sigma(F)^{\times} T^{\times}$$

and if we put for a Hecke character χ, $\operatorname{sgn}(\chi) = (\varepsilon_{\sigma})$ and an integer $m, \nu = \operatorname{sgn}(-1)^m$ such that $M(\chi)(m)$ is critical at $s = 0$,

$$\tilde{c}^{+}(M(\chi)(m)) = (G(\chi)^{-1}(1 \otimes (D_F^{1/2}(2\pi i)^{nm}))^{d^{\nu\varepsilon(\chi)}(M)} \prod_{\sigma} \tilde{c}^{\varepsilon_{\sigma}(\chi)\nu}(\sigma, M)$$
$$\in (T \otimes \mathbf{C})^{\times} \bmod T(\chi)^{\times}$$

then

$$L(M(\chi)(m), 0)\tilde{c}^{+}(M(\chi)(m))^{-1} \in T(\chi).$$

3. Modified L-function of a motive over F.

3.1. Following J. Coates we shall formulate this modified period conjecture in a form appropriate for further use in a p-adic construction. First we multiply $L(M, s)$ by an appropriate factor at infinity and define

$$\Lambda_{(\infty)}(M,\ s)^{(\tau)} = E_{\infty}(M,\ s)L(M, s)^{(\tau)}$$

as $\Lambda_{(\infty)}(\tau,\ R_{F/\mathbf{Q}}M,\ \rho, s)$ in the notation of J. Coates [Co] with $\rho = i$ so that $E_{\infty}(M, s) = E_{\infty}(\tau, R_{F/\mathbf{Q}}M, \rho, s)$ is the modified Γ-factor at infinity which actually does not depend on τ. Also we put

$$\Lambda_{(\infty)}(M,\ s) = (\Lambda_{(\infty)}(M,\ s)^{(\tau)})_{\tau \in J_T}$$

for the modified L-function with values at $T \otimes \mathbf{C}$ and put

$$\Omega^{\nu}(M) = (\Omega^{\nu}(M)^{(\tau)}) = c^{\nu}(RM)(2\pi i)^{r(RM)} \in (T \otimes \mathbf{C})^{\times}$$

where

$$\nu = (-1)^m,\ \ r(RM) = \sum_{j<0} jh(i, j, R_{F/\mathbf{Q}}M) = \sum_{j<0} jh(i, j, M),\ \ n = [F : \mathbf{Q}],$$

$c^\nu(RM) = c^\nu(R_{F/\mathbf{Q}}M)$ is the period of $R_{F/\mathbf{Q}}M$. Then the period conjecture can be stated in the following convenient form: if $s = 0$ is critical for M then for any m such that $M(m)$ is critical at $s = 0$ we have that there exists an element α of T such that

$$\frac{\Lambda_{(\infty)}(M(m),0)^{(\tau)}}{\Omega^\nu(M)^{(\tau)}} = \tau(\alpha),$$

i.e. that

$$\Lambda_{(\infty)}(M(m),0)\Omega^\nu(M)^{-1} \in T \subset T \otimes \mathbf{C},$$

where $\nu = \operatorname{sgn}((-1)^m) = \pm$.

In order to deduce this statement from the original conjecture on critical values we can use the same arguments as in J. Coates's work [Co], where it was shown that

$$E_\infty(M,\ 0) \sim (2\pi i)^{r(RM)} \bmod \mathbf{Q}^\times,$$

and it follows that

$$E_\infty(M(m),\ 0) \sim (2\pi i)^{r(RM)-md^\varepsilon(RM)} = (2\pi i)^{n(r(M)-md^\varepsilon(M))} \bmod \mathbf{Q}^\times,$$

where $\varepsilon = +$ if $j < 0$ and $\varepsilon = -$ if ≥ 0 for $j = w/2$.

If we combine this fact with the equivalence

$$c^+(M(m)) \sim (2\pi i)^{d^\nu nm} c^\nu(M) \bmod T^\times$$

we deduce from the above form of the conjecture that

$$\Lambda_{(\infty)}(M(m),\ 0) \sim (2\pi i)^{n(r(M)-md^\varepsilon(M)+md^\nu(M))} c^\nu(M).$$

Note that in our situation we have that $d^\varepsilon(M) = d^\nu(M)$ because both M and $M(m)$ are critical at $s = 0$: we have that $\nu = +$ only for $j - m < 0$ because $M(m)$ is critical but according to Lemma 3 in [Co] the condition $j < 0$ is equivalent in this situation to $j - m < 0$.

Taking into account the conjecture on factorization of the periods of Section 2, we now can state the following:

3.2. MODIFIED CONJECTURE ON THE CRITICAL VALUES. — *Assume that M is critical at $s = 0$. Then there exist constants $c^{\varepsilon_\sigma}(\sigma, M) \in (T \otimes \mathbf{C})^\times (\varepsilon_\sigma = \pm)$ defined modulo $T^\times$ such that if we put for a given sign $\varepsilon_0 = (\varepsilon_{0,\sigma}) \in \mathrm{Sgn}_F$*

$$\Omega(\varepsilon_0, M) = (1 \otimes (2\pi i))^{nr(M)} \prod_\sigma c^{\varepsilon_{0,\sigma}}(\sigma, M)$$

with $r(M) = \sum_{j<0} jh(i, j, M)$ *then for any integer* m *and Hecke character* χ *such that* $M(\chi)(m)$ *is critical at* $s = 0$ *and* $\varepsilon_\sigma(\chi)\nu = \varepsilon_{0,\sigma}$ *we have that*

$$\Lambda_{(\infty)}(M(\chi)(m),\ 0)((G(\chi)^{-1}(1 \otimes D_F^{1/2}))^{d^{\varepsilon_0}(M)}\Omega(\varepsilon_0, M))^{-1} \in T(\chi)$$

where $\nu = \operatorname{sgn}((-1)^m) = \pm$.

We recall that by definition

$$E_\infty(M, s) = E_\infty(\tau, R_{F/\mathbf{Q}}M, \rho, s) = E_\infty(U, \rho, s),$$

where U runs over direct summands of the Hodge decomposition, $\rho = i$ and $E_\infty(U, \rho, s)$ is given by:

(a) If $U = M^{j,k} \oplus M^{k,j}$ with $j < k$, then $E_\infty(U, \rho, s) = \Gamma_{\mathbf{C},\rho}(s - j)^{h(j,k)}$;

(b) If $U = M^{k,k}$ with $k \geq 0$, then $E_\infty(U, \rho, s) = 1$;

(c) If $U = M^{k,k}$ with $k < 0$, then $E_\infty(U, \rho, s) = R_\infty(U, \rho, s)$.

Here

$$\rho^{-s} = \exp(-\rho\pi s/2), \quad \Gamma_{\mathbf{C},\rho}(s) = \rho^{-s}\Gamma_{\mathbf{C}}(s),$$

$$\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s), \quad \Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$

$$R_\infty(U, \rho, s) = L_\infty(\tau, U,\ s)/(\varepsilon_\infty(\tau, U, \rho, s)L_\infty(\tau,\ U^\vee(1),\ -s))$$

with L-and ε-factors described in [De3] on p. 329, so that we have in case (c)

$$R_\infty(U, \rho, s) = \frac{\Gamma_{\mathbf{R}}(s - k + \delta)}{i^\delta \Gamma_{\mathbf{R}}(1 - s + k - \delta)} = \frac{2\Gamma(s - k + \delta)\cos(\pi(s - k + \delta))/2}{i^\delta (2\pi)^{s-k+\delta}}$$

where $\delta = 0, 1$ is chosen according with the sign of the scalar action of ρ_σ on $U = M_\sigma^{k,k}$ so that ρ_σ acts as $(-1)^{k+\delta}$.

4. Non-Archimedean integration and admissible measures.

4.1. In this section we recall the notion of the h-admissible measures over a Galois group and properties of their Mellin transform. This Mellin transform is a certain p-adic analytic function on the $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_p = \operatorname{Hom}_{\mathrm{contin}}(\mathrm{Gal}_p,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Galois group Gal_p, where

$$\mathrm{Gal}_p = \mathrm{Gal}(F_{p,\infty}^{ab}/F)$$

denotes the Galois group of the maximal abelian extension $F^{ab}_{p,\infty}$ of F unramified out side primes of F above p and ∞. Recall that by class field theory the group Gal_p can be described as the projective limit

$$\mathrm{Gal}_p = \varprojlim_{\mathfrak{m}} H(\mathfrak{m}), \quad H(\mathfrak{m}) = I(\mathfrak{m})/P(\mathfrak{m}),$$

where $\mathfrak{m}$ runs over ideals of F with the support in the set of prime divisors $\mathfrak{p}$ of p in F, $I(\mathfrak{m})$ denotes the group of ideals prime to $\mathfrak{m}$,

$$P(\mathfrak{m}) = \{(\alpha) \in I(\mathfrak{m}) \mid \alpha \gg 0, \ \alpha \equiv 1 \bmod \mathfrak{m}\}.$$

There is the natural exact sequence

$$1 \to \mathrm{Gal}^0_p \to \mathrm{Gal}_p \to G_1 \to 1$$

where $G_1 = \mathrm{Gal}(F(\infty)/F)$, $F(\infty)$ is the maximal abelian extension of F ramified only at ∞, $\mathrm{Gal}^0_p = \mathcal{O}_p^\times/\mathrm{clos}(\mathcal{O}_+^\times)$, $\mathcal{O}_p = \mathcal{O}_F \otimes \mathbf{Z}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_\mathfrak{p}$, $\mathrm{clos}(\mathcal{O}_+^\times)$ denotes the closure of the group of all totally positive units $\mathcal{O}_+^\times$ in F. The canonical $\mathbf{C}_p$-analytic structure on $\mathcal{X}_p$ is obtained by shifts from the obvious $\mathbf{C}_p$-analytic structure on the group

$$\mathrm{Hom}_{\mathrm{contin}}((\mathcal{O}_F \otimes \mathbf{Z}_p)^\times/\mathrm{clos}(\mathcal{O}_+^\times), \ \mathbf{C}_p^\times).$$

We regard the elements of finite order $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ as Hecke characters of finite order whose conductor $\mathfrak{c}(\chi)$ may contain only primes $\mathfrak{p}$ of F lying above p, by means of the decomposition

$$\chi : \mathbf{A}_F^\times/F^\times \xrightarrow{\text{class field theory}} \mathrm{Gal}_p \to \mathbf{Q}^\times \xrightarrow{i_\infty} \mathbf{C}^\times,$$

where i_∞ is the fixed embedding. The characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ form a discrete subgroup $\mathcal{X}_p^{\mathrm{tors}} \subset \mathcal{X}_p$. We shall need also the following natural homomorphism

$$\mathcal{N}x_p : \mathrm{Gal}_p \to \mathrm{Gal}(\mathbf{Q}^{ab}_{p,\infty}/\mathbf{Q}) \cong \mathbf{Z}_p^\times \to \mathbf{C}_p^\times, \ \mathcal{N}x_p \in \mathcal{X}_p.$$

Recall that a p-adic measure on Gal_p may be regarded as a bounded $\mathbf{C}_p$-linear form μ on the space $\mathcal{C}$ (Gal_p) of all continuous $\mathbf{C}_p$-valued functions

$$\varphi \to \mu(\varphi) = \int_{\mathrm{Gal}_p} \varphi d\mu \in \mathbf{C}_p, \ \varphi \in \mathcal{C}(\mathrm{Gal}_p),$$

which is uniquely determined by its restriction to the subspace $\mathcal{C}^1$ (Gal_p) of locally constant functions. We denote by $\mu(a + (\mathfrak{m}))$ the value of μ on the characteristic function of the set

$$a + (\mathfrak{m}) = \left\{x \in \ \mathrm{Gal}^0_p \mid x \equiv a \bmod \mathfrak{m}\right\} \subset \mathrm{Gal}^0_p.$$

The Mellin transform L_μ of μ is a bounded analytic function

$$L_\mu : \mathcal{X}_p \to \mathbf{C}_p,\ L_\mu(\chi) = \int \chi d\mu \in \mathbf{C}_p,\ \chi \in \mathcal{X}_p,$$

on $\mathcal{X}_p$, which is uniquely determined by its values $L_\mu(\chi)$ for the characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$.

A more delicate notion of an h-admissible measure was introduced by Amice-Vélu and Višik (see [AmV], [V1]). Let $\mathcal{C}^h$ (Gal_p) denote the space of $\mathbf{C}_p$-valued functions which can be locally represented by polynomials of degree less than a natural number h of the variable $\mathcal{N}x_p \in \mathcal{X}_p$ introduced above.

4.2. Definition. — *A $\mathbf{C}_p$-linear form*

$$\mu : \mathcal{C}^h(\mathrm{Gal}_p) \to \mathbf{C}_p$$

is called h-admissible measure if for all $a \in \mathrm{Gal}_p^0$ and for all $r = 0, 1, \cdots, h-1$ the following growth condition is satisfied

$$\sup_{a \in \mathrm{Gal}_p^0} \left| \int_{a+\mathfrak{m}} (\mathcal{N}x_p - \mathcal{N}a_p)^r d\mu \right| = o(|\mathfrak{m}|_p^{r-h}).$$

4.3. Note that the notion of a bounded measure is covered by the case $h = 1$, but the set of 1-admissible measures is bigger: it consists of the so called measures of bounded growth [Ma4], [V1], which grow on the open compact sets slower as $o(|\mathfrak{m}|_p^{-1})$. We know (essentially due to Amice-Vélu and Višik) that each h-admissible measure can be uniquely extended to a linear form on the $\mathbf{C}_p$-space of all locally analytic functions so that one can associate to its Mellin transform

$$L_\mu : \mathcal{X}_p \to \mathbf{C}_p,\ L_\mu(\chi) = \int_{\mathrm{Gal}_p} \chi d\mu \in \mathbf{C}_p,\ \chi \in \mathcal{X}_p,$$

which is a $\mathbf{C}_p$-analytic function on $\mathcal{X}_p$ of the type $o(\log_p^h(\cdot))$. Moreover, the measure μ is uniquely determined by the special values of the type

$$L_\mu(\chi \mathcal{N}x_p^r) \quad (\chi \in \mathcal{X}_p^{\mathrm{tors}},\ r = 0,\ 1,\ \cdots,\ h-1).$$

5. The Newton polygon and the Hodge polygon of a motive; p-ordinary and p-admissible motives.

5.1. We shall formulate in the next section a general conjecture on p-adic L-functions of motives in terms of the existence of certain h-admissible measures, where the quantity h is defined in terms of the Newton polygon and the Hodge polygon of a motive. Properties of these polygons are closely related to the notions of a p-ordinary and a p-admissible motive; such motives will correspond to the case $h = 1$.

From now on we fix an embedding $\tau : T \to \overline{\mathbf{Q}}$ in order to deal with p-adic L-functions. It is often convenient to omit the symbol (τ) from the notation

$$L(M,s)^{(\tau)},\ \Lambda_{(\infty)}(M,s)^{(\tau)},\ c^{\varepsilon}(\sigma,M)^{(\tau)},\ \Omega(\varepsilon_0,M)^{(\tau)}$$

viewing these quantities as complex numbers. Then for a motive M over F with coefficients in T under the assumptions of the period conjecture of Section 3 the algebraic number

$$\Lambda_{(\infty)}(M(\chi)(m),0)(G(\chi)^{-1}D_F^{1/2})^{d^{\varepsilon_0}(M)}\Omega(\varepsilon_0,M)^{-1} \in T(\chi)$$

can be regarded via $i_\infty : \overline{\mathbf{Q}} \to \mathbf{C}$, $i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$ as an element of both $\mathbf{C}$ and $\mathbf{C}_p$. Also, with an embedding $\sigma \in J_F$ one can associate the embeddings $F \to \overline{\mathbf{Q}}$, $F \to \mathbf{C}_p$ and define a prime divisor $\mathfrak{p} = \mathfrak{p}(\sigma)$ of p in F attached to σ (keeping in mind that the same $\mathfrak{p}$ might be attached to different σ).

We are going now to define the Newton polygon $P_{\mathrm{Newton},\sigma}(u) = P_{\mathrm{Newton},\sigma}(u,M)$ and the Hodge polygon $P_{\mathrm{Hodge},\sigma}(u) = P_{\mathrm{Hodge},\sigma}(u,M)$ attached to M, σ (and to the fixed embedding $\tau \in J_T$). First for $\mathfrak{p} = \mathfrak{p}(\sigma)$ we consider (using i_∞) the local $\mathfrak{p}$-polynomial

$$\begin{aligned} L_{\mathfrak{p}}(M,X)^{-1} &= 1 + A_1(\mathfrak{p})X + \cdots + A_d(\mathfrak{p})X^d \\ &= (1-\alpha^{(1)}(\mathfrak{p})X)\cdot(1-\alpha^{(2)}(\mathfrak{p})X)\cdot\ldots\cdot(1-\alpha^{(d)}(\mathfrak{p})X), \end{aligned}$$

and we assume that its inverse roots are indexed in such a way that

$$\mathrm{ord}_p\alpha^{(1)}(\mathfrak{p}) \le \mathrm{ord}_p\alpha^{(2)}(\mathfrak{p}) \le \cdots \le \mathrm{ord}_p\alpha^{(d)}(\mathfrak{p}).$$

5.2. DEFINITION. — *The Newton polygon $P_{\mathrm{Newton},\sigma}(u)(0 \le u \le d)$ of M at $\mathfrak{p} = \mathfrak{p}(\sigma)$ is the convex hull of the points $(i, \mathrm{ord}_p A_i(\mathfrak{p}))$ $(i = 0,1,\ldots,d)$.*

The important property of the Newton polygon is that the length the horizontal segment of slope i is equal to the number of the inverse roots $\alpha^{(j)}(\mathfrak{p})$ such that $\mathrm{ord}_p\alpha^{(j)}(\mathfrak{p}) = i$ (note that the number i may not necessarily be integer but this will be the case for p-ordinary motives below).

5.3. *The Hodge polygon* $P_{\mathrm{Hodge},\sigma}(u)$ $(0 \le u \le d)$ of M at σ is defined using the Hodge decomposition of the d-dimensional $\mathbf{C}$-vector space

$$M_\sigma^{(\tau)} = M_\sigma \underset{T,\tau}{\otimes} \mathbf{C} = \underset{i,j}{\oplus} M_\sigma^{(\tau)i,j}$$

where we keep τ fixed and regard $M_\sigma^{(\tau)i,j} = M_\sigma^{i,j} \underset{T,\tau}{\otimes} \mathbf{C}$ as the $\mathbf{C}$-subspace of $M_\sigma^{i,j}$ on which T acts via $\tau \in J_T$. Note that the dimension $h_\sigma^{(\tau)}(i, j) = \dim_{\mathbf{C}} M_\sigma^{(\tau)i,j}$ may depend on τ and σ (but in case $F = \mathbf{Q}$ this number is independent of τ, see Deligne [De3]). The Hodge polygon $P_{\mathrm{Hodge},\sigma}(u)$ by definition passes through the points

$$(0,0),\ \dots,\ \Big(\sum_{i'\le i} h_\sigma^{(\tau)}(i',j),\ \sum_{i'\le i} i' h_\sigma^{(\tau)}(i',j)\Big), \dots,\ \Big(d, \sum_{i'\le d} i' h_\sigma^{(\tau)}(i',j)\Big),$$

so that the length of the horizontal segment of slope i is equal to the dimension $h_\sigma^{(\tau)}(i,j)$.

5.4. Now we recall the definition of a p-ordinary motive in the simpliest case $F = T = \mathbf{Q}$ (see [Co], [CoPe-Ri]). We assume that M is pure of weight w and rank d. Let $G_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the decomposition group (of a place in T over p) and

$$\psi_p : G_p \to \mathbf{Z}_p^\times$$

be the cyclotomic character of G_p. Then M is called p ordinary at p if the following conditions are satisfied:

(i) The inertia group $I_p \subset G_p$ acts trivially on each of the l-adic realizations M_l for $l \ne p$;

(ii) There exists a decreasing filtration $F_p^i V$ on $V = M_p = M_B \otimes \mathbf{Q}_p$ of $\mathbf{Q}_p$-subspaces which are stable under the action of G_p such that for all $i \in \mathbf{Z}$ the group G_p acts on $F_p^i V / F_p^{i+1} V$ via some power of the cyclotomic character, say $\psi_p^{-e_i}$. Then

$$e_1(M) \ge \cdots \ge e_t(M)$$

and the following properties take place:

(a)

$$\dim_{\mathbf{Q}_p} F_p^i V / F_p^{i+1} V = h(e_i, w - e_i);$$

(b) The Hodge polygon and the Newton polygon of M coincide:

$$P_{\text{Newton}}(u) = P_{\text{Hodge}}(u).$$

If furthermore M is critical at $s = 0$ then it is easy to verify that the number d_p of the inverse roots $\alpha^{(j)}(p)$ with

$$\text{ord}_p \alpha^{(j)}(p) < 0 \text{ is equal to } d^+ = d^+(M).$$

In the general case (of a motive M over F with coefficients in T) the notion of a p-ordinary motive can be defined using the restriction of the ground field F to $\mathbf{Q}$ and the restriction of the coefficient field T to $\mathbf{Q}$ (the last operation corresponds to forgetting the T-module structure on the realizations of M). In this way we get a motive M' over $\mathbf{Q}$ with coefficients in $\mathbf{Q}$ of the same weight w and the rank

$$\text{rk}(M') = [F : \mathbf{Q}][T : \mathbf{Q}] \cdot d.$$

For p-ordinary motives over $\mathbf{Q}$ and their Dirichlet twists J. Coates and B. Perrin-Riou [CoPe-Ri] have formulated a general conjecture on the existence of bounded p-adic measures attached to such motives. However it turns out that such bounded measures can exist even for certain non-p-ordinary motives, which can be characterized by the following simple condition:

5.5. DEFINITION. — *The motive M over F with coefficients in T is called admissible at p if for all $\sigma \in J_F$ we have that*

$$P_{\text{Newton},\sigma}(d^+) = P_{\text{Hodge},\sigma}(d^+)$$

here $d^+ = d^+(M)$ is the dimension of M_σ^+ which under our earlier assumptions is independent of $\sigma \in J_F$.

On the other hand, in a number of cases when M is not p-ordinary and even when M is not admissible at p one can prove the existence of the corresponding (growing) h-admissible measures. One can show that all these cases admit a unified description if we use the following positive integer h which is defined in terms of the difference between the Newton polygon and the Hodge polygon of M:

$$h = \max_{\sigma \in J_F}[(P_{\text{Newton},\sigma}(d^{\pm}) - P_{\text{Hodge},\sigma}(d^{\pm}))] + 1.$$

Note the following important properties of the quantity h:

(i) $h = h(M)$ does not change if we replace M by its Tate twist.

(ii) $h = h(M)$ does not change if we replace M by its twist $M = M(\chi)$ with a Hecke character χ of finite order of conductor prime to p.

(iii) $h(M) = h_p(M^\vee)$.

In the next section we state in terms of this quantity a general conjecture on p-adic L-functions.

6. A conjecture on p-adic L-functions of motives over totally real fields.

6.1. In order to formulate precisely a general conjecture on p-adic L-functions of a motive M over a totally real field F with coefficients in T we suppose that M is pure of weight w, M has the rank d, and there exists an integer m such that the motive $M(m)$ is critical at $s = 0$. Then we put

$$m_* = \min\left\{m \mid \exists\ \chi \in \mathcal{X}_p^{\mathrm{tors}} \text{ such that } \ N = M(\chi)(m) \text{ is critical at } s = 0\right\}$$
$$m^* = \max\left\{m \mid \exists \chi \in \mathcal{X}_p^{\mathrm{tors}} \text{ such that } \ N = M(\chi)(m) \text{ is critical at } s = 0\right\}.$$

Then the number $m^* - m_* + 2$ coincides with the width of the critical strip of our motive M. The integers m_* and m^* can be characterized in terms of the Hodge decomposition of M:

$$m_* = \max\{j \mid \exists\ j, k, j < k \text{ such that } h(j,k,M) \neq 0\} + 1,$$
$$m^* = \min\{j \mid \exists\ j, k, j > k \text{ such that } h(j,k,M) \neq 0\}.$$

Furthermore for an integer m, a Hecke character χ of finite order and of the conductor $\mathfrak{c}(\chi)$ we put $N = M(\chi)(m)$ and define the $\mathfrak{p}$-factors:

$$A_{\mathfrak{p}}(M(\chi), s) = \begin{cases} \prod\limits_{i=d^++1}^{d} (1-\chi(\mathfrak{p})\alpha^{(i)}(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-s}) \prod\limits_{i=1}^{d^+} (1-\chi^{-1}(\mathfrak{p})\alpha^{(i)}(\mathfrak{p})^{-1}\mathcal{N}\mathfrak{p}^{s-1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \mathfrak{p} \nmid \mathfrak{c}(\chi) \\ \prod\limits_{i=1}^{d^+} \left(\dfrac{\mathcal{N}\mathfrak{p}^s}{\alpha^{(i)}(\mathfrak{p})}\right)^{\mathrm{ord}_{\mathfrak{p}}\, \mathfrak{c}(\chi)}, \quad \text{otherwise.} \end{cases}$$

Let us fix a sign $\varepsilon_0 = \{\varepsilon_{0,\sigma}\} \in \mathrm{Sgn}_F = \{\pm 1\}^n$. Assuming that the modified period conjecture of Section 3 is true we have that there exist constants

$$\tilde{c}^{\varepsilon_\sigma}(\sigma, M) \in (T \otimes \mathbf{C})^\times \quad (\varepsilon_\sigma = \pm)$$

defined modulo $T^\times$ such that if we put for $\varepsilon_0 = (\varepsilon_{0,\sigma}) \in \mathrm{Sgn}_F$

$$\Omega(\varepsilon_0, M) = (1 \otimes (2\pi i))^{nr(M)} \prod_\sigma \tilde{c}^{\varepsilon_{0,\sigma}}(\sigma, M)$$

with $r(M) = \sum_{j<0} jh(j,k,M)$, then for any integer m and Hecke character χ such that $M(\chi)(m)$ is critical at $s = 0$ and $\varepsilon_\sigma(\chi)\nu = \varepsilon_{0,\sigma}$ we have that

$$\Lambda_{(\infty)}(M(\chi)(m),\ 0)((G(\chi)^{-1}(1 \otimes D_F^{1/2}))^{d^{\varepsilon_0}(M)}\Omega(\varepsilon_0, M))^{-1} \in T(\chi)$$

where $\nu = \mathrm{sgn}((-1)^m) = \pm$.

6.2. CONJECTURE. — *For each sign $\varepsilon_0 = \{\varepsilon_{0,\sigma}\} \in \mathrm{Sgn}_F$ there exists a $\mathbf{C}_p$-analytic function $L_{(p)}^{(\varepsilon_0)}$ on $\mathcal{X}_p$ with the properties:*

(i) *for all but a finite number of pairs $(m,\ \chi)$ such that the motive $N = M(\chi)(m)$ is critical at $s = 0$ and for $\varepsilon_{0,\sigma} = \varepsilon_\sigma(\chi)\nu$ we have that*

$$L_{(p)}^{(\varepsilon_0)}(\chi \mathcal{N} x_p^m) = \frac{D_F^{md^{\varepsilon_0}(M)/2}}{G(\chi)^{d^{\varepsilon_0}(M)}} \prod_{\mathfrak{p} \nmid p} A_\mathfrak{p}(N) \frac{\Lambda_{(\infty)}(M(\chi)(m),\ 0)}{\Omega(\varepsilon_0, M)};$$

(ii) *the function $L_{(p)}^{(\varepsilon_0)}$ is holomorphic on $\mathcal{X}_p$ if $M^{k,k} = 0$; otherwise there exists a finite set $\Xi \subset \mathcal{X}_p$ of p-adic characters and positive integers $n(\xi)$ (for $\xi \in \Xi$) such that for any $g_0 \in \mathrm{Gal}_p$ we have that the function*

$$\prod_{\xi \in \Xi} (x(g_0) - \xi(g_0))^{n(\xi)} L_{(p)}^{(\varepsilon_0)}(x)$$

is holomorphic on $\mathcal{X}_p$;

(iii) *the holomorphic function in (ii) is bounded if*

$$P_{\mathrm{Newton},\sigma}(d^+) = P_{\mathrm{Hodge},\sigma}(d^+) \text{ for all } \sigma \in J_F;$$

(iv) *in the general case the holomorphic function in (ii) belongs to the type*

$$o(\log_p^h(\cdot))$$

and it can be represented as the Mellin transform of an h-admissible measure with h defined in the end of the previous section;

(v) *if $h \le m^* - m_* + 1$ then the function $L_{(p)}^{(\varepsilon_0)}$ is uniquely determined by the above conditions (i)-(ii).*

Note that the last statement follows from the properties of h-admissible measures (see Section 4).

6.3. *Remark.* — It would be interesting to rewrite the right hand side of the equality in (i) in a more invariant form using the ε-factors of certain complex representations of the Weil-Deligne groups W'_p which was done by J. Coates for p-ordinary motives over $\mathbf{Q}$ [Co]. These representations can probably occur in the complexification $Y = M_\lambda \underset{T_\lambda,\tau}{\otimes} \mathbf{C}$ of the λ-adic realization M_λ of M, where for $\tau \in J_T$ the same symbol τ denotes an embedding $T_\lambda \to \mathbf{C}$ extending $\tau: T \to \mathbf{C}$.

7. Hilbert modular forms and motives associated with them.

We use the notation of Shimura [Shi6], [Shi10] and we regard the group $GL_2(F)$ as the group $G_\mathbf{Q}$ of all $\mathbf{Q}$-rational points of a certain $\mathbf{Q}$-subgroup $G \subset GL_{2n}$. Then Hilbert modular forms will be regarded as complex functions on the adelic group $G_\mathbf{A} = G(\mathbf{A})$ which is apparently identified with the product

$$GL_2(F_\mathbf{A}) = G_\infty \times G_{\widehat{\mathbf{Q}}}$$

where

$$G_\infty = GL_2(F_\infty) \cong GL_2(\mathbf{R})^n,\ G_{\widehat{\mathbf{Q}}} = GL_2(\widehat{F}),$$

$\mathbf{A}$, $F_\mathbf{A}$ denote the rings of finite adeles of $\mathbf{Q}$ and F respectively.

The subgroup

$$G_\infty^+ = GL_2^+(F_\infty) \cong GL_2^+(\mathbf{R})^n$$

consists of all elements

$$\alpha = (\alpha_1, \cdots, \alpha_n), \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $\det \alpha_\nu > 0$, $\nu = 1, 2, \cdots, n$. Every element $\alpha \in G_\infty^+$ acts on the product $\mathfrak{H}^n$ of n copies of the upper half plane according to the formula

$$\alpha(z_1,\ \cdots,\ z_n) = (\alpha_1(z_1),\ \cdots,\ \alpha_n(z_n)),$$

where

$$\alpha_\nu(z_\nu) = (a_\nu z_\nu + b_\nu)/(c_\nu z_\nu + d_\nu) \quad (\nu = 1,\ 2,\ ...,\ n).$$

For $z = (z_1, z_2, \cdots, z_n) \in \mathfrak{H}^n$ we put $e_F(z) = e(\{z\}), \{z\} = z_1 + ... + z_n$ and $e(x) = \exp(2\pi i x)$ and we use the notations $\mathcal{N}z = z_1 \cdot ... \cdot z_n$, and $\mathbf{i} = (i,$ $..., i)$. For $\alpha \in G_\infty^+$, an integer n-tuple $k = (k_1, \cdots, k_n)$ and an arbitrary function $f: \mathfrak{H}^n \to \mathbf{C}$ we use the notation

$$(f \mid_k \alpha)(z) = \prod_\nu (c_\nu z_\nu + d_\nu)^{-k_\nu} f(\alpha(z)) \det(\alpha_\nu)^{k_\nu/2}.$$

Let $\mathfrak{c} \subset \mathcal{O}_F$ be an integral ideal, $\mathfrak{c}_\mathfrak{p} = \mathfrak{c}\mathcal{O}_\mathfrak{p}$ its $\mathfrak{p}$-part, $\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\mathcal{O}_\mathfrak{p}$ the local different. We shall need the open compact subgroups $W = W_\mathfrak{c} \subset G_\mathbf{A}$ defined by

$$W_\mathfrak{c} = G_\infty^+ \times \prod_\mathfrak{p} W_\mathfrak{c}(\mathfrak{p}),$$

$$W_\mathfrak{c}(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\mathfrak{p}) \mid b \in \mathfrak{d}_\mathfrak{p}^{-1} \ c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p}, \ a, d \in \mathcal{O}_\mathfrak{p}, ad - bc \in \mathcal{O}_\mathfrak{p}^\times \right\}.$$

By a Hilbert automorphic form of the weight $k = (k_1, \cdots, k_n)$, the level $\mathfrak{c}$, and the Hecke character ψ we mean a function

$\mathbf{f} : G_\mathbf{A} \to \mathbf{C}$ satisfying the following conditions (7.1)–(7.3):

(7.1) $$\mathbf{f}(s\alpha x) = \psi(s)\mathbf{f}(x) \text{ for all } x \in G_\mathbf{A}, \quad s \in F_\mathbf{A}^\times \text{ (the center of } G_\mathbf{A}), \text{ and } \alpha \in G_\mathbf{Q}.$$

We let $\psi_0 : (\mathcal{O}/\mathfrak{c})^\times \to \mathbf{C}^\times$ denote the $\mathfrak{c}$-part of the character ψ and the extend the definition of ψ over the group $W_\mathfrak{c}$ by the formula

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \psi_0(a_\mathfrak{c} \bmod \mathfrak{c})$$

then for all $x \in G_\mathbf{A}$

(7.2) $$\mathbf{f}(xw) = \psi(w^\iota)\mathbf{f}(x) \text{ for } w \in W_\mathfrak{c} \text{ with } w_\infty = 1,$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If

$$w = w(\theta) \text{ where } w(\theta) = (w_1(\theta_1), \ \cdots, \ w_n(\theta_n)),$$

$$w_\nu(\theta_\nu) = \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu \\ \sin\theta_\nu & \cos\theta_\nu \end{pmatrix},$$

then

$$\mathbf{f}(xw(\theta)) = \mathbf{f}(x)\exp(-i(k_1\theta_1 + \ldots + k_n\theta_n)).$$

An automorphic form $\mathbf{f}$ is called a cusp form if

(7.3) $$\int_{F_\mathbf{A}/F} \mathbf{f}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\right) dt = 0 \text{ for all } g \in G_\mathbf{A}.$$

The vector space $\mathcal{M}_k(\mathfrak{c}, \psi)$ of Hilbert automorphic forms of holomorphic type is defined as the set functions satisfying (7.1)–(7.3) and the following holomorphy condition (7.4): for any $x \in G_\mathbf{A}$ with $x_\infty = 1$ there exists a holomorphic function $g_x : \mathfrak{H}^n \to \mathbf{C}$ such that for all $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+$ we have that

$$\mathbf{f}(xw) = (g_x \mid_k w)(\mathbf{i}) \tag{7.4}$$

(in the case $F = \mathbf{Q}$ we must also require that the function g_x is holomorphic at the cusps). Let $\mathcal{S}_k(\mathfrak{c}, \psi) \subset \mathcal{M}_k(\mathfrak{c}, \psi)$ be the subspace of cusp forms.

Hecke operators which act on $\mathcal{S}_k(\mathfrak{c}, \psi)$ and $\mathcal{M}_k(\mathfrak{c}, \psi)$ are defined by means of the double cosets of the type WyW for y in the semigroup

$$Y_\mathfrak{c} = G_\mathbf{A} \cap (G_\infty^\times \prod_\mathfrak{p} Y_\mathfrak{c}(\mathfrak{p})),$$

where

$$Y_\mathfrak{c}(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F_\mathfrak{p}) \mid b \in \mathfrak{d}_\mathfrak{p}^{-1}, c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p}, a, d \in \mathcal{O}_\mathfrak{p}, a\mathcal{O}_\mathfrak{p} + \mathfrak{c}_\mathfrak{p} = \mathcal{O}_\mathfrak{p} \right\}.$$

The Hecke algebra $\mathcal{H}_\mathfrak{c}$ consists of all formal finite sums of the type $\sum_y c_y WyW$, with the multiplication in $\mathcal{H}_\mathfrak{c}$ defined by a standard rule. By definition $T_\mathfrak{c}(m)$ is the element of $\mathcal{H}_\mathfrak{c}$ obtained by taking the sum of all different WwW with $w \in Y_\mathfrak{c}$ such that $\operatorname{div}(\det(y)) = \mathfrak{m}$. Let

$$T_\mathfrak{c}(\mathfrak{m})' = \mathcal{N}(\mathfrak{m})^{(k_0-2)/2} T_\mathfrak{c}(\mathfrak{m})$$

be the normalized Hecke operator, where k_0 denotes the maximal component of the weight k. Suppose that $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ is an eigenform of all $T_\mathfrak{c}(\mathfrak{m})'$ with the eigenvalues $C(\mathfrak{m}, \mathbf{f})$. Then there is the following Euler product expansion:

$$L(s, \mathbf{f}) = \sum_\mathfrak{n} C(\mathfrak{n}, \mathbf{f}) \mathcal{N}\mathfrak{n}^{-s} = \prod_\mathfrak{p} (1 - C(\mathfrak{p}, \mathbf{f}) \mathcal{N}\mathfrak{p}^{-s} + \psi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{k_0-1-2s})^{-1}.$$

All of the numbers $C(\mathfrak{n}, \mathbf{f})$ are known to be algebraic integers.

Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp eigenform. In this case the numbers $C(\mathfrak{n}, \mathbf{f})$ can be regarded as the normalized Fourier coefficients of $\mathbf{f}$. The important analytic property of the corresponding L-function $L(\mathbf{f}, s)$ (see [Shi6], p. 655) is that it admits a holomorphic analytic continuation onto the entire complex plane, and if we set

$$\Lambda(\mathbf{f}, s) = \prod_{i=1}^n \Gamma_\mathbf{C}(s - (k_0 - k_i)/2) L(\mathbf{f}, s)$$

then $\Lambda(\mathbf{f}, s)$ satisfies a functional equation, which expresses $\Lambda(\mathbf{f}, s)$ in terms of the function $\Lambda(\mathbf{f}^\rho, k_0 - s)$. According to a general conjecture on analytic properties of the L-functions of motives we may suggest that $\mathbf{f}$ should correspond to a motive $M = M(\mathbf{f})$ over F of rank 2, weight k_0, with coefficients in a field T containing all $C(\mathfrak{n}, \mathbf{f})$ such that

$$L(M,s) = L(s,\mathbf{f}),\ \Lambda(M,s) = \Lambda(s,\mathbf{f})$$

and for fixed embeddings $\tau \in J_T$ and $\sigma = \sigma_i \in J_F$ the Hodge decomposition of M_{σ_i} is given by

$$\begin{aligned} M_{\sigma_i}^{(\tau)} &= M_{\sigma_i} \underset{T,\tau}{\otimes} \mathbf{C} \\ &= M_{\sigma_i}^{(\tau)(k_0-k_i)/2,(k_0+k_i)/2-1} \oplus M_{\sigma_i}^{(\tau)(k_0+k_i)/2-1,(k_0-k_i)/2} \end{aligned} \tag{7.5}$$

where k_i is the component of the weight k, attached to the fixed embedding σ_i (as was mentioned above this decomposition may depend on τ and σ_i). It is obvious from (7.5) that if such motive exists then the weight k must satisfy the condition $k_1 \equiv k_2 \equiv \ldots \equiv k_n \bmod 2$.

There are several confirmation of the conjecture [BlRo]. First of all it is known in the elliptic modular case $F = \mathbf{Q}$ due to U. Jannsen and A.J. Scholl [Ja], [Scho]; the existence of the Galois representations of $\mathrm{Gal}(\overline{F}/F)$ corresponding to λ-adic realizations of these motives was discovered earlier by Deligne [Del]. If we restrict such a representation to the subgroup $G_{F'}$ corresponding to a totally real extension F'/F we obtain the L-function of a certain Hilbert modular form of the same weight which is the Doi-Naganuma lift (or "base change") of the original elliptic cusp form. In the general case the existence of Galois representations attached to Hilbert modular forms was established by Rogawski-Tunnell [RoTu] and Ohta [Oh] (n odd) (under a local hypothesis) and by R. Taylor [Ta] in the general case. Also a number of results on special values of the function $L(s,\mathbf{f})$ is known, which math the above conjectures on the critical values and on the p-adic L-functions [Shi1], [Man], [Ka1]. As in the elliptic modular case there is a conjectural link between motives of the type $M(\mathbf{f})$ and the cohomology of certain Kuga-Shimura variety (fiber product of several copies of the universal Hilbert-Blumenthal abelian variety with a fixed level structure and and endomorphisms): namely, for the decomposition $R_{F/\mathbf{Q}}M = \overset{n}{\underset{i=1}{\otimes}} M^{\sigma_i}$ the tensor product $\overset{n}{\underset{i=1}{\otimes}} M^{\sigma_i}$ is a motive over $\mathbf{Q}$ of rank 2^n which conjecturally lies in the above cohomology, see the interesting discussion of this link in [Ha2], [Oda]. In case $k_1 = \ldots = k_n = 2$ the motives have the Hodge type $H^{0,1} \oplus H^{1,0}$. In some cases (e.g. when n is odd) the motives M^{σ_i} can be realized as factors of Jacobians of Shimura curves corresponding to quaternion algebras, which split at one fixed infinite place σ_i and ramified at all other infinite places $\sigma_j (j \neq i)$ ([Shi7]; see also recent work of M. Harris [Ha3]).

8. Examples.

8.1. *Periods of Hilbert cusp forms.*

Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp eigenform which is supposed to be "motivic" in the sense of the previous section, and let

$$\begin{aligned} L(s, \mathbf{f}(\chi)) &= \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, \mathbf{f}) \mathcal{N}\mathfrak{n}^{-s} \\ &= \prod_{\substack{\mathfrak{p} \\ \mathfrak{p}+\mathfrak{c}=\mathcal{O}_F}} \left(1 - \chi(\mathfrak{p}) C(\mathfrak{p}, \mathbf{f}) \mathcal{N}\mathfrak{p}^{-s} + \chi^2(\mathfrak{p}) \psi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{k_0-1-2s}\right)^{-1}. \end{aligned}$$

Then the critical strip of $L(s, \mathbf{f}(\chi))$ is given by $m_* \leq m \leq m^*$,

$$m_* = \max\{(k_0 - k_i)/2\} + 1, \quad m^* = \min\{(k_0 + k_i)/2\} - 1.$$

Using the Rankin-Selberg method G. Shimura proved [Sh6] that there exist constants

$$u(\varepsilon, \mathbf{f}) \in (T \otimes \mathbf{C})^\times \text{ for } \varepsilon \in \operatorname{Sgn} F$$

defined modulo $T^\times$ such that if we put $\varepsilon = \varepsilon_\chi(-1)^m$ for $m \in \mathbf{Z}$ and for a Hecke character χ of finite order and then define

$$c^\varepsilon(\chi, \mathbf{f}) = D_F^{-1/2} G(\chi) u(\varepsilon, \mathbf{f}),$$

then for all $m \in \mathbf{Z}$, $m_* \leq m \leq m^*$ we have that

$$\frac{i^{nm} \Lambda(m, \ \mathbf{f}(\chi))}{c^\varepsilon(\chi, \mathbf{f})} \in T(\chi).$$

This statement perfectly matches the modified period conjecture 3.2, if we take for $c^\pm(\sigma, M(\mathbf{f}))$ the quantities $c^\pm(\sigma, \mathbf{f})$.

In order to formulate the results on p-adic L-functions, put

$$1 - C(\mathfrak{p}, \mathbf{f}) X + \psi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{k_0-1} X^2 = (1 - \alpha(\mathfrak{p}) X)(1 - \alpha'(\mathfrak{p}) X) \in \mathbf{C}_p[X]$$

where $\alpha(\mathfrak{p})$, $\alpha'(\mathfrak{p})$ are the inverse roots of the Hecke polynomial assuming that

$$\operatorname{ord}_p \alpha(\mathfrak{p}) \leq \operatorname{ord}_p \alpha'(\mathfrak{p}).$$

Note that in the p-ordinary case we should have

$$\operatorname{ord}_p \alpha(\mathfrak{p}) = (k_0 - k_i)/2, \quad \operatorname{ord}_p \alpha'(\mathfrak{p}) = (k_0 + k_i)/2 - 1$$

for the prime $\mathfrak{p} = \mathfrak{p}_i = \mathfrak{p}(\sigma_i)$ attached to an embedding σ_i (see Section 5).

8.2. THEOREM. — *Put $h = [\max(\mathrm{ord}_p(\alpha(\mathfrak{p}(\sigma_i)) - (k_0 - k_i)/2)] + 1$. Then for each sign $\varepsilon_0 = \{\varepsilon_{0,\sigma}\} \in \mathrm{Sgn}_F$ there exists a $\mathbf{C}_p$-analytic function $L_{(p)}^{(\varepsilon_0)}$ on $\mathcal{X}_p$ of the type $o(\log_p^h(\cdot))$ with the properties:*

(i) *for all $m \in \mathbf{Z}$, $m_* \le m \le m^*$, and for all Hecke characters of finite order $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ with $\nu\varepsilon_\sigma(\chi) = \varepsilon_{0,\sigma}$ $(\sigma \in J_F)$ the following equality holds*

$$L_{(p)}^{(\varepsilon_0)}(\chi \mathcal{N} x_p^m) = \frac{D_F^m i^{mn}}{G(\chi)} \prod_{\mathfrak{p}|p} A_{\mathfrak{p}}(\mathbf{f}(\chi), m) \cdot \frac{\Lambda(\mathbf{f}(\chi), m)}{\Omega(\varepsilon_0,\ \mathbf{f})},$$

where

$$A_{\mathfrak{p}}(\mathbf{f}(\chi), m) = \begin{cases} (1 - \chi(\mathfrak{p})\alpha'(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-m})(1 - \chi^{-1}(\mathfrak{p})\alpha(\mathfrak{p})^{-1}\mathcal{N}\mathfrak{p}^{m-1}), & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathfrak{p} \mid \mathfrak{c}(\chi) \\ \left(\dfrac{\mathcal{N}\mathfrak{p}^m}{\alpha(\mathfrak{p})}\right)^{\mathrm{ord}_{\mathfrak{p}} \mathfrak{c}(\chi)}, & \text{if } \mathfrak{p} \nmid \mathfrak{c}(\chi) \end{cases}$$

and the constant $\Omega(\varepsilon_0,\ \mathbf{f})$ is given by

$$\Omega(\varepsilon_0,\ \mathbf{f}) = (2\pi i)^{-nm_*} \cdot D_F^{1/2} \cdot \prod_{\sigma} c^{\varepsilon_{0,\sigma}}(\sigma, \mathbf{f}),$$

(ii) *If $h \le m^* - m_* + 1$ then the function $L_{(p)}^{(\varepsilon_0)}$ on $\mathcal{X}_p$ is uniquely determined by (i).*

(iii) *If*

$$\max(\mathrm{ord}_p(\alpha(\mathfrak{p}(\sigma_i)) - (k_0 - k_i)/2) = 0$$

then the function $L_{(p)}^{(\varepsilon_0)}$ is bounded on $\mathcal{X}_p$.

In the p-ordinary case this theorem was established by Yu.I. Manin [Man6] (in a less explicit form) using the theory of generalized modular symbols on Hilbert-Blumenthal modular varieties. The non-p-ordinary case was treated only for $F = \mathbf{Q}$ by Višik [V1] and Amice-Vélu [AmVe]. For an arbitrary totally real field F one can use the Rankin method and the techniques of Shimura's work [Shi6], see more details in [Da].

8.3. *The Rankin convolution and the tensor product of motives.*

Let us consider the Rankin convolution

$$L(s, \mathbf{f}, \mathbf{g}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathbf{f})\, C(\mathfrak{n}, \mathbf{g}) \mathcal{N}(\mathfrak{n})^{-s} \tag{8.1}$$

attached to two Hilbert modular forms $\mathbf{f}$, $\mathbf{g}$ over a totally real field F of degree $n = [F : \mathbf{Q}]$, where $C(\mathfrak{n}, \mathbf{f})$, $C(\mathfrak{n}, \mathbf{g})$ are normalized "Fourier coefficients" of $\mathbf{f}$ and $\mathbf{g}$, indexed by integral ideals $\mathfrak{n}$ of the maximal order $\mathcal{O}_F \subset F$ (see §7). We suppose that $\mathbf{f}$ is a primitive cusp form of vector weight $k = (k_1, \cdots, k_n)$, and $\mathbf{g}$ a primitive cusp form of weight $l = (l_1, \cdots, l_n)$. We assume that for a decomposition of J_F into a disjoint union $J_F = J \cup J'$ the following condition is satisfied

$$k_i > l_i \text{ (for } \sigma_i \in J\text{)}, \quad \text{and} \quad l_i > k_i \text{ (for } \sigma_i \in J'\text{)}. \tag{8.2}$$

Also, assume that

$$k_1 \equiv k_2 \equiv \cdots \equiv k_n \bmod 2, \tag{8.3}$$

and

$$l_1 \equiv l_2 \equiv \cdots \equiv l_n \bmod 2. \tag{8.4}$$

Let $\mathfrak{c}(\mathbf{f}) \subset \mathcal{O}_F$ denote the conductor and ψ the character of $\mathbf{f}$ and $\mathfrak{c}(\mathbf{g})$, ω denote the conductor and the character of $\mathbf{g}$ ($\psi, \omega : \mathbf{A}_F^\times / F^\times \to \mathbf{C}^\times$ being Hecke characters of finite order).

The essential property of the convolution

$$L(s, \mathbf{f}, \mathbf{g}(\chi)) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, \mathbf{f})\, C(\mathfrak{n}, \mathbf{g}) \mathcal{N}(\mathfrak{n})^{-s}$$

(twisted with a Hecke character χ of finite order) is the following Euler product decomposition

$$\begin{aligned} &L_{\mathfrak{c}}(2s+2-k_0-l_0, \psi\omega\chi^2) L(s, \mathbf{f}, \mathbf{g}(\chi)) \\ &= \prod_{\mathfrak{q}} \big((1-\chi(\mathfrak{q})\alpha(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi(\mathfrak{q})\alpha(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}) \\ &\quad \times (1-\chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi(\mathfrak{q})\alpha'(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})\big)^{-1}, \end{aligned} \tag{8.5}$$

where the numbers $\alpha(\mathfrak{q})$, $\alpha'(\mathfrak{q})$, $\beta(\mathfrak{q})$, and $\beta'(\mathfrak{q})$ are roots of the Hecke polynomials

$$X^2 - C(\mathfrak{q}, \mathbf{f})X + \psi(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k_0-1} = (X - \alpha(\mathfrak{q}))(X - \alpha'(\mathfrak{q})),$$

and

$$X^2 - C(\mathfrak{q}, \mathbf{g})X + \omega(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{l_0-1} = (X - \beta(\mathfrak{q}))(X - \beta'(\mathfrak{q})).$$

The decomposition (8.5) is not difficult to deduce from the following elementary lemma on rational functions, applied to each of the Euler $\mathfrak{q}$-factors: if

$$\sum_{i=0}^{\infty} A_i X^i = \frac{1}{(1-\alpha X)(1-\alpha' X)}, \quad \sum_{i=0}^{\infty} B_i X^i = \frac{1}{(1-\beta X)(1-\beta' X)},$$

then

$$\sum_{i=0}^{\infty} A_i B_i X^i = \frac{1-\alpha\alpha'\beta\beta' X^2}{(1-\alpha\beta X)(1-\alpha\beta' X)(1-\alpha'\beta X)(1-\alpha'\beta' X)}. \tag{8.6}$$

Assume that there exist motives $M(\mathbf{f})$ and $M(\mathbf{g})$ associated with $\mathbf{f}$ and $\mathbf{g}$. Then

$$L_{\mathfrak{c}}(2s+2-k-l, \psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)) = L(M(\chi), s)$$

where $M = M(\mathbf{f}) \underset{F}{\otimes} M(\mathbf{g})$ is the tensor product of motives over F with coefficients in some common number field T. Using the Hodge decompositions for $M(\mathbf{f})$ and $M(\mathbf{g})$ and the Künneth formula for $M = M(\mathbf{f}) \underset{F}{\otimes} M(\mathbf{g})$ we see that under our assumption the motive M has $d = 4$, $w = k_0 + l_0 - 2$, and the following Hodge type:

$$\begin{aligned} M_{\sigma_i} \otimes \mathbf{C} \cong \bigoplus_{\tau \in J_T} \Big(& M_{\sigma_i}^{(k_0+l_0-k_i^\tau-l_i^\tau)/2,(k_0+l_0+k_i^\tau+l_i^\tau)/2-2)} \\ & \oplus M_{\sigma_i}^{(k_0+l_0+k_i^\tau+l_i^\tau)/2-2,(k_0+l_0-k_i^\tau-l_i^\tau)/2} \\ & \oplus M_{\sigma_i}^{(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1} \\ & \oplus M_{\sigma_i}^{(k_0+l_0+|k_i^\tau-l_i^\tau|)/2-1,(k_0+l_0-|k_i^\tau-l_i^\tau|)/2-1} \Big). \end{aligned}$$

Moreover,

$$\begin{aligned} \Lambda(M(\chi^{-1}), s) &= \Lambda(s, \mathbf{f}, \mathbf{g}(\chi)) \\ &= \prod_{i=1}^{n} \left(\Gamma_{\mathfrak{c}}(s-(k_0+l_0-k_i-l_i)/2)\Gamma_{\mathfrak{c}}(s-(k_0+l_0-|k_i-l_i|)/2+1)\right) \\ &\quad \times L_{\mathfrak{c}}(2s+2-k_0-l_0, \psi\omega\chi^2)L(s,\mathbf{f},\mathbf{g}(\chi)), \end{aligned}$$

and this function satisfies a functional equation of the type $s \mapsto k_0+l_0-2-s$.

8.4. *The critical values of the Rankin convolution.*

Let us now set

$$m_* = \max_i((k_0 + l_0 - |k_i - l_i|)/2 - 1) + 1, \quad m^* = k_0 + l_0 - 2 - m_*.$$

The periods $c^{\pm}(\sigma, M)$ can be easily computed in terms of $c^{\pm}(\sigma, M)$ (as in the elliptic modular case; see a more general calculation in [Bl2]). The validity of the conjecture on factorizaton of Deligne's periods in this case was proved recently by H. Yoshida [Yo], 4.6, see also [Ha3]. As a result one gets that the quantity

$$c^{\pm}(\sigma, M) = c(\sigma, M) = (c^{\pm}(\sigma, M)^{(\tau)})_\tau \in (T \otimes \mathbf{C})^\times$$

does not depend on the sign $\pm$, and is given by

$$c^{\pm}(\sigma, M)^{(\tau)} = \begin{cases} c^{+}(\sigma, \mathbf{f})^{(\tau)} c^{-}(\sigma, \mathbf{f})^{(\tau)} \delta(\sigma, \mathbf{g})^{(\tau)}, & \text{if } \sigma^{-1}\tau \in J \\ c^{+}(\sigma, \mathbf{g})^{(\tau)} c^{-}(\sigma, \mathbf{g})^{(\tau)} \delta(\sigma, \mathbf{f})^{(\tau)}, & \text{if } \sigma^{-1}\tau \in J'. \end{cases}$$

Moreover,

$$c^{\pm}(M(\chi^{-1})) = G(\chi) c^{\pm}(M).$$

Let us apply the modified conjecture on special values to the L-function

$$\Lambda(M(\chi^{-1}), s) = \Lambda(s, \mathbf{f}, \mathbf{g}(\chi)),$$

and set $c(\mathbf{f}, \mathbf{g}) = \prod_{\sigma} c^{+}(\sigma, M)$,

$$c(J, \mathbf{f}) = \prod_{\sigma \in J} c^{+}(\sigma, \mathbf{f}) c^{-}(\sigma, \mathbf{f}), \; c(J', \mathbf{g}) = \prod_{\sigma \in J'} c^{+}(\sigma, \mathbf{g}) c^{-}(\sigma, \mathbf{g}),$$

and

$$\delta(J, \mathbf{f}) = \prod_{\sigma \in J} \delta(\sigma, \mathbf{f}), \; \delta(J', \mathbf{g}) = \prod_{\sigma \in J'} \delta(\sigma, \mathbf{g}).$$

Then we see that

$$c(J, \mathbf{f}) c(J', \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle, \quad \delta(J, \mathbf{f}) \delta(J', \mathbf{f}) = G(\psi)^{-1} (2\pi i)^{n(k_0 - 1)},$$
$$c(J, \mathbf{g}) c(J', \mathbf{g}) = \langle \mathbf{g}, \mathbf{g} \rangle, \quad \delta(J, \mathbf{f}) \delta(J', \mathbf{g}) = G(\omega)^{-1} (2\pi i)^{n(l_0 - 1)},$$

and

$$c(M(\chi^{-1})) = c^{\pm}(M(\chi^{-1})) = G(\chi)^2 c(J, \mathbf{f}) \delta(J, \mathbf{g}) c(J', \mathbf{g}) \delta(J', \mathbf{f}).$$

With this notation the modified conjecture on the critical values takes the following form: for all Hecke characters χ of finite order and $r \in \mathbf{Z}, m_* \leq r \leq m^*$ we have that

$$\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{G(\chi)^2 c(J, \mathbf{f}) \delta(J, \mathbf{g}) c(J', \mathbf{g}) \delta(J', \mathbf{f})} = \frac{\Lambda(M(\chi^{-1}), r)}{G(\chi)^2 c(M)} \in \mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi).$$

8.5. Let us consider the special case when $J' = \emptyset$, i.e. $k_i > l_i$ for all $\sigma_i \in J_F$. Then

$$c(J, \mathbf{f}) = c(J_F, \mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle, \quad \delta(J, \mathbf{g}) = \delta(J_F, \mathbf{g}) = G(\omega)^{-1} (2\pi i)^{n(l_0 - 1)},$$

and the above property transforms to the following:

$$\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{G(\chi)^2 \langle \mathbf{f}, \mathbf{f} \rangle, G(\omega)^{-1} (2\pi i)^{n(l_0 - 1)}} \in \mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi),$$

where $\mathbf{Q}(\mathbf{f}, \mathbf{g}, \chi)$ denotes the subfield of $\mathbf{C}$ generated by the Fourier coefficients of $\mathbf{f}$ and $\mathbf{g}$, and the values of χ. This algebraicity property was established by G. Shimura [Sh1] by means of a version of the Rankin-Selberg method.

In the general case the above algebraicity property was also studies by G. Shimura [Sh2], [Sh3] (for some special Hilbert modular forms, coming from quaternion algebras) and by M. Harris [Ha3] using the theory of arithmetical vector bundles on Shimura varieties. The idea of the proof was to replace the original automorphic cusp form $\mathbf{f} : G(\mathbf{A}) \to \mathbf{C}$ of holomorphic type by another cusp form $\mathbf{f}^J : G(\mathbf{A}) \to \mathbf{C}$ such that

$$\mathbf{f}^J(g_1, \cdots, g_n) = \mathbf{f}(g_1 j_1, \cdots, g_n j_n),$$

where $g_i \in \mathrm{GL}_2(\mathbf{R})$,

$$j_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } i \in J \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \text{if } i \in J'. \end{cases}$$

Then $\mathbf{f}^J$ can be described by functions $\mathbf{f}^J_\lambda$ on $\mathfrak{H}^n$, which are holomorphic in z_i $(i \in J)$ and antiholomorphic in z_i $(i \in J')$. Then the differential forms

$$\mathbf{f}^J_\lambda \wedge_{i \in J} d\bar{z}_i$$

define a certain class $cl(\mathbf{f}^J)$ of the degree $|J|$ in the coherent cohomology of the Hilbert-Blumenthal modular variety, or rather its toroidal compactification ([Ha1], [Ha2]). This space of coherent cohomology has a natural rational structure over a certain number field F^J, defined in terms of canonical models. From the theory of new forms it follows that there exist a constant $\nu(J, \mathbf{f}) \in \mathbf{C}^\times$ such that the differential form attached to $\nu(J, \mathbf{f})^{-1}\mathbf{f}^J$ is rational over the extension of F^J obtained by adjoining the Hecke eigenvalues of $\mathbf{f}$. Then the critical values of the type $\Lambda(r, \mathbf{f}, \mathbf{g})$ can be expressed in terms of a cup product of the form

$$cl(\mathbf{f}^J) \cup cl(\mathbf{g}^{J'}) \cup E,$$

where E is a (nearly) holomorphic Eisenstein series. Then the above algebraicity property can be deduced from the fact that the cup product preserves the rational structure in the coherent cohomology. However, the technical details of the proof are quite difficult.

8.6. *p-adic convolutions of Hilbert cusp forms.*

Now we give a precise description of the p-adic convolution of $\mathbf{f}$ and $\mathbf{g}$ assuming that both $\mathbf{f}$ and $\mathbf{g}$ are p-ordinary, i.e. for $\mathfrak{p}_i = \mathfrak{p}(\sigma_i)$ one has

$$\operatorname{ord}_p \alpha(\mathfrak{p}_i) = (k_0 - k_i)/2, \quad \operatorname{ord}_p \alpha'(\mathfrak{p}_i) = (k_0 + k_i)/2 - 1,$$
$$\operatorname{ord}_p \beta(\mathfrak{p}_i) = (l_0 - l_i)/2, \quad \operatorname{ord}_p \beta'(\mathfrak{p}_i) = (l_0 + l_i)/2 - 1,$$

or equivalently, $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{f}) = (k_0 - k_i)/2$, and $\mathrm{ord}_p C(\mathfrak{p}_i, \mathbf{g}) = (l_0 - l_i)/2$. We assume also that the conductors of $\mathbf{f}$ and $\mathbf{g}$ are coprime to p and we set

$$\begin{aligned}
&A_p(s, \mathbf{f}, \mathbf{g}(\chi)) \\
&= \prod_{\sigma_i \in J \setminus S(\chi)} (1-\chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1-\chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \\
&\times (1-\chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1-\chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta'(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}) \\
&\times \prod_{\sigma_i \in J' \setminus S(\chi)} (1-\chi(\mathfrak{p}_i)\alpha(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s})(1-\chi(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)\mathcal{N}\mathfrak{p}_i^{-s}) \\
&\times (1-\chi(\mathfrak{p}_i)^{-1}\alpha(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1})(1-\chi(\mathfrak{p}_i)^{-1}\alpha'(\mathfrak{p}_i)^{-1}\beta(\mathfrak{p}_i)^{-1}\mathcal{N}\mathfrak{p}_i^{s-1}).
\end{aligned}$$

Then we introduce the following constant:

$$\begin{aligned}
\Omega(\mathbf{f}, \mathbf{g}) &= c(J, \mathbf{f})\delta(J, \mathbf{g})c(J', \mathbf{g})\delta(J', \mathbf{f}) \\
&= \prod_{\sigma \in J} c^+(\sigma, \mathbf{f})c^-(\sigma, \mathbf{f})\delta(\sigma, \mathbf{g}) \prod_{\sigma \in J'} c^+(\sigma, \mathbf{g})c^-(\sigma, \mathbf{g})\delta(\sigma, \mathbf{f}).
\end{aligned}$$

8.7. CONJECTURAL DESCRIPTION OF THE p-ADIC CONVOLUTION. — *Under the conventions and notation as above there exists a bounded $\mathbf{C}_p$-valued measure $\mu = \mu_{\mathbf{f},\mathbf{g}}$ on Gal_p, which is uniquely determined by the following condition: for all Hecke characters $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ and all $r \in \mathbf{Z}$ satisfying $m_* \le r \le m^*$ the following equality holds:*

$$\begin{aligned}
&\int_{\mathrm{Gal}_p} \chi^{-1}\mathcal{N}x_p^r d\mu_{\mathbf{f},\mathbf{g}} \\
&= i_p\Big(\frac{D_F^{2r}(-1)^r}{G(\chi)^2}\frac{\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))}{\Omega(\mathbf{f}, \mathbf{g})}\prod_{\mathfrak{p}|p} A_p(r, \mathbf{f}, \mathbf{g}(\chi)) \\
&\times \prod_{\sigma_i \in J}\Big(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\alpha(\mathfrak{p}_i)^2\beta(\mathfrak{p}_i)\beta'(\mathfrak{p}_i)}\Big)^{\mathrm{ord}_{\mathfrak{p}_i} \mathfrak{c}(\chi)} \prod_{\sigma_i \in J'}\Big(\frac{\mathcal{N}\mathfrak{p}_i^{r-1}}{\beta(\mathfrak{p}_i)^2\alpha(\mathfrak{p}_i)\alpha'(\mathfrak{p}_i)}\Big)^{\mathrm{ord}_{\mathfrak{p}_i} \mathfrak{c}(\chi)}\Big),
\end{aligned}$$

and the measure $\mu_{\mathbf{f},\mathbf{g}}$ defines a bounded $\mathbf{C}_p$-analytic function

$$L_{\mathbf{f},\mathbf{g}} : \mathcal{X}_p \to \mathbf{C}_p, \quad \mathcal{X}_p \ni x \mapsto \int_{\mathrm{Gal}_p} x \, d\mu_{\mathbf{f},\mathbf{g}}$$

(the p-adic Mellin transform of $\mu_{\mathbf{f},\mathbf{g}}$), which is uniquely determined by its values on the characters $x = \chi^{-1}\mathcal{N}x_p^r \in \mathcal{X}_p$.

(Note that the above expression could be written in a slightly simplier form if we take into account the equalities:

$$\begin{aligned}
\alpha(\mathfrak{p})^2\beta(\mathfrak{p})\beta'(\mathfrak{p}) &= \alpha(\mathfrak{p})^2\omega(\mathfrak{p})\mathcal{N}\mathfrak{p}^{l_0-1}, \\
\beta(\mathfrak{p})^2\alpha(\mathfrak{p})\alpha'(\mathfrak{p}) &= \beta(\mathfrak{p})^2\psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1}.)
\end{aligned}$$

8.8. *Concluding remarks.*

The existence of the p-adic measure in 8.7 is known in the special case, and $J = \emptyset$ (see [Pa2]), where $\mathbf{f}$ and $\mathbf{g}$ are assumed to be automorphic forms of scalar weights k and l, $k > l$. One verifies easily that the description 8.7 perfectly matches with the modified period conjecture and with the general conjecture on the p-adic L-functions of Section 6. Also, this construction was recently extended by My Vinh Quang [My] to Hilbert automorphic forms $\mathbf{f}$ and $\mathbf{g}$ of arbitrary vector weights $k = (k_1, \cdots, k_n)$, and $l = (l_1, \cdots, l_n)$ such that $k_i > l_i$ for all $i = 1, \cdots, n$, and to the non-p-ordinary case. In this situation the p-adic convolution of $L_{\mathbf{f},\mathbf{g}}$ is also uniquely determined by the above condition provided that it has the prescribed logarithmic growth on $\mathcal{X}_p$ (see [V1]). More general classes of p-adic L-functions in the p-ordinary case were constructed by H. Hida [Hi3] using powerful techniques of nearly p-ordinary Hecke algebras, duality theorems, and p-adic families Hilbert modular forms and of Galois representations.

In the mixte case the proof of the algebraic properties of the Rankin convolution in [Ha3] can be used also in order to carry out a p-adic construction. First of all, one obtains an expression for complex valued distributions attached to $\Lambda(r, \mathbf{f}, \mathbf{g}(\chi))$ in terms of the cup product of certain coherent cohomology classes, and one verifies that these distributions take algebraic values. Then, integrality properties of the arithmetic vector bundles can be used for proving some generalized Kummer congruences for the values of these distributions, which is equivalent to the existence of p-adic L-functions in 8.7. However, some essential technical difficulties remain in the general case, and 8.7 can not be regarded yet as a theorem proven in full generality, although it holds in important special cases described above.

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