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FOLIATIONS ON THE COMPLEX PROJECTIVE PLANE WITH MANY PARABOLIC LEAVES

by Marco BRUNELLA

Let $\mathcal{F}$ be a holomorphic foliation with isolated singularities on the complex projective plane $\mathbf{C}P^2$. The leaves of $\mathcal{F}$ are non compact Riemann surfaces whose structure can be very complicated and may be analyzed from various points of view. From the function theoretic point of view, there is a classical distinction between parabolic and hyperbolic (=not parabolic) Riemann surfaces [Tsu] : a Riemann surface R is said to be *parabolic* if it does not admit a positive non constant superharmonic function, or equivalently if it does not admit a Green function. Parabolic Riemann surfaces have a "small" ideal boundary and are, in some sense, the simplest Riemann surfaces; examples are closed surfaces minus a finite set. On the other hand, their topology can be arbitrary and so their simplicity is only from the analytical point of view.

Aim of this paper is to prove that a foliation with "sufficiently many" parabolic leaves must have a very simple dynamics. We need the following concept [Suz]. Let $V \subset \mathbf{C}P^2 \setminus \mathrm{Sing}(\mathcal{F})$ be a $\mathcal{F}$ - invariant set, then we say that V has *positive capacity* if there exists a disc T transverse to $\mathcal{F}$ such that $T \cap V \subset T$ has positive (logarithmic) capacity [Tsu]. We also recall that a singularity $p \in \mathrm{Sing}(\mathcal{F})$ is *hyperbolic* if the linear part at p of a vector field which generates $\mathcal{F}$ near p has eigenvalues different from zero and with non real ratio. See [Arn], §36, for the topology of this kind of singularities. Clearly, a generic foliation has only hyperbolic singularities.

THEOREM. — *Let $\mathcal{F}$ be a foliation with hyperbolic singularities on $\mathbf{C}P^2$. Assume that the set of parabolic leaves*

$$\mathcal{P} = \{p \in \mathbf{C}P^2 \setminus \mathrm{Sing}(\mathcal{F}) \mid L_p \textit{ is parabolic}\}$$

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has positive capacity. Then $\mathcal{F}$ is a linear hyperbolic foliation, i.e. there is an affine chart $\mathbf{C}^2 \subset \mathbf{C}P^2$ such that $\mathcal{F}|_{\mathbf{C}^2}$ is given by $xdy + \lambda ydx = 0$, for some $\lambda \in \mathbf{C} \setminus \mathbf{R}$.

Let us observe that some hypothesis on $\mathrm{Sing}(\mathcal{F})$ is necessary to obtain the conclusion of the theorem. For example, a foliation with a meromorphic first integral or a foliation which is the pull-back of a linear foliation by a rational map of $\mathbf{C}P^2$ (of degree ≥ 2) has all its leaves parabolic, but also has non hyperbolic singularities.

The proof of the theorem is in two steps : first we prove the existence of an algebraic leaf $L \in \mathcal{F}$ (i.e., a leaf whose closure is an algebraic curve), using the theory of harmonic measures [Gar], then we analyze the structure of L in order to apply the constructions of [CLS2].

1. Existence of an algebraic leaf.

We firstly recall the (well-known) fact that there is only a countable set of leaves of $\mathcal{F}$ with nontrivial holonomy. This is a consequence of the finite generation of the holonomy pseudogroup of $\mathcal{F}$, together with the property that nontrivial fixed points of one dimensional biholomorphisms are isolated.

For every $L \in \mathcal{F}$ we denote by $\hat{L}$ its holonomy covering, i.e. the covering of L associated to the kernel of the holonomy representation of L. Because a countable set has zero capacity, the hypothesis that $\mathcal{P}$ has positive capacity means that

$$\hat{\mathcal{P}} = \{p \in \mathbf{C}P^2 \setminus \mathrm{Sing}(\mathcal{F}) \mid \hat{L}_p \text{ is parabolic}\}$$

has also positive capacity.

Take now the restriction $\mathcal{F}^*$ of $\mathcal{F}$ to any affine chart $\mathbf{C}^2 \subset \mathbf{C}P^2$, with $\mathbf{C}P^2 \setminus \mathbf{C}^2$ not $\mathcal{F}$- invariant. To every leaf $L \in \mathcal{F}$ there corresponds a leaf $L^* \in \mathcal{F}^*$ obtained from L by deleting a discrete set, so that the parabolicity of L or $\hat{L}$ is equivalent to that of L^* or $\hat{L^*}$ [Tsu]. The foliation $\mathcal{F}^*$ is defined on a Stein manifold and the leaves with parabolic holonomy covering form a set of positive capacity. Hence we may apply theorem I of [Suz] to conclude that every leaf of $\mathcal{F}^*$ has parabolic holonomy covering. It follows that *every leaf of $\mathcal{F}$ has parabolic holonomy covering* (in particular, *every leaf of $\mathcal{F}$ is parabolic*).

Let $p_1, ..., p_k \in \mathbf{C}P^2$ be the (hyperbolic) singularities of $\mathcal{F}$. For every $j = 1, ..., k$ let U_j be a small round neighborhood of p_j, with $S_j = \partial U_j$ real analytic and transverse to $\mathcal{F}$. Let $\mathcal{G}$ be the foliation obtained by doubling the foliation $\mathcal{F}|_{\mathbf{C}P^2 \setminus \cup_{j=1}^k U_j}$; it is a transversely holomorphic foliation without singularities and defined on some compact manifold M (which is not complex). Moreover, the leaves of $\mathcal{G}$ have a natural complex structure, obtained by doubling the complex structures of the leaves of $\mathcal{F}|_{\mathbf{C}P^2 \setminus \cup_{j=1}^k U_j}$ (Schwarz's reflection principle). A theorem of Kusunoki and Mori [KM] asserts that if R is a parabolic Riemann surface and $S \subset R$ is a subsurface with real analytic boundary then the double of S is still a parabolic Riemann surface. Hence we deduce that *every leaf of $\mathcal{G}$ is a parabolic Riemann surface.*

In order to continue the proof, we need a few facts about harmonic measures [Gar], [Ghy]. Fix a Riemannian metric g on M, such that the restriction of g to every leaf of $\mathcal{G}$ is hermitian. Let $\Delta^{\mathcal{G}}$ denote the foliated laplacian associated to $(\mathcal{G}, g)$. A probability measure μ on M is said to be *harmonic* if its foliated laplacian is zero, i.e. $\mu(\Delta^{\mathcal{G}} f) = 0$ for every function f continuous on M and of class C^2 along the leaves (we are considering a measure as a linear functional on the space of continuous functions). It is established in [Gar] that the set of harmonic measures is non empty; more precisely, if $K \subset M$ is a compact $\mathcal{G}$ - saturated set then there exists a harmonic measure whose support is contained in K [Ghy].

The simplest example of harmonic measure is constructed by combining a holonomy invariant measure (if it exists) with the area form on leaves of $\mathcal{G}$. Conversely, it is proven in [Ghy] that if μ is a harmonic measure such that for μ - almost every $p \in M$ the leaf $L_p \in \mathcal{G}$ does not possess a positive non constant harmonic function, then μ can be decomposed as a product of a holonomy invariant measure with the area form on leaves.

All these properties of harmonic measures hold for any foliation on a compact Riemannian manifold; but in our case the leaves are parabolic Riemann surfaces and so they do not have positive non constant (super)harmonic functions (recall that the metric g is hermitian on leaves). Hence a consequence of the previous theory is that *for every compact $\mathcal{G}$-saturated set $K \subset M$ there exists a holonomy invariant measure whose support is contained in K.*

Let now $K \subset M$ be a minimal set of $\mathcal{G}$ (possibly $K = M$) and let μ be a holonomy invariant measure with $\text{supp}(\mu) \subset K$, that is $\text{supp}(\mu) = K$. Denote still by $S_1, ..., S_k \subset M$ the 3-spheres arising from the boundary of

$\mathbf{C}P^2 \setminus \bigcup_{j=1}^{k} U_j$. If $K \cap \bigcup_{j=1}^{k} S_j = \varnothing$, then K corresponds to a compact minimal set of $\mathcal{F}|_{\mathbf{C}P^2\setminus\mathrm{Sing}(\mathcal{F})}$; but this is not possible because such a minimal set cannot support a holonomy invariant measure [CLS1]. Hence K intersects some sphere S_i, and $K \cap S_i$ is a compact set saturated by the foliation $\mathcal{G}|_{S_i}$, which is a one - dimensional foliation with two hyperbolic closed leaves as limit set. The measure μ induces a holonomy invariant measure ν for $\mathcal{G}|_{S_i}$ with $\mathrm{supp}(\nu) = K \cap S_i$, it follows that $K \cap S_i$ is contained in the union of the two closed leaves and hence K is a compact leaf of $\mathcal{G}$.

Returning to $\mathcal{F}$, we see that *there exists an algebraic leaf* $L \in \mathcal{F}$. The closure $\bar{L}$ is an algebraic curve with normal crossings, obtained adding to L some singular points of $\mathcal{F}$.

2. Holonomy of L and linearization of $\mathcal{F}$.

Because $\bar{L}$ contains singularities (all hyperbolic), if L is uniformized by $\mathbf{C}$ then L is isomorphic to $\mathbf{C}^*$ and $\bar{L}$ either is smooth (diffeomorphic to a sphere) and contains two singularities, or contains only one singularity which is a normal crossing. In particular, in both cases the holonomy of L is cyclic (because $\pi_1(L)$ is) with a hyperbolic generator (because singularities are hyperbolic).

Now assume that L is uniformized by $\mathbf{D}$, i.e. $L = \mathbf{D}/\Gamma$ where $\Gamma \simeq \pi_1(L)$ is a finitely generated Fuchsian group. Observe that Γ contains parabolic elements, because L is a closed Riemann surface minus a finite non empty set. The holonomy covering $\hat{L}$ of L is isomorphic to $\mathbf{D}/\hat{\Gamma}$, where $\hat{\Gamma}$ is the normal subgroup of Γ corresponding to paths with trivial holonomy. The holonomy group G of L is then identifiable (as an abstract group) with $\Gamma/\hat{\Gamma}$.

The parabolic elements of Γ corresponding to paths freely homotopic to the ends of L have infinite order in the quotient $\Gamma/\hat{\Gamma}$, because the singularities in $\bar{L}$ are hyperbolic and hence their separatrices have hyperbolic ($\Rightarrow$ infinite order) holonomy.

Recall that $\hat{L}$ is a parabolic Riemann surface, and that this is equivalent to say that $\hat{\Gamma}$ is a Fuchsian group of divergent type [Tsu]. We can apply a theorem of Varopoulos [Var] asserting that (because Γ has parabolic elements of infinite order in $\Gamma/\hat{\Gamma}$) the group $\Gamma/\hat{\Gamma}$ is a finite extension of $\mathbf{Z}$.

In fact, *the holonomy group* $G \simeq \Gamma/\hat{\Gamma}$ *is abelian*. To see this, assume that it is not; then G contains a hyperbolic germ h and also a germ g tangent to but different from the identity, and clearly $\mathrm{id} \neq h^n \neq g^m \neq \mathrm{id}$ $\forall n, m \in \mathbf{Z} \setminus \{0\}$. But this is impossible, because a finite extension of $\mathbf{Z}$ cannot contain two infinite cyclic subgroups H_1, H_2 with $H_1 \cap H_2 = \{\mathrm{id}\}$.

Because G is abelian and contains hyperbolic germs (even when $L \simeq \mathbf{C}^*$), we may apply the methods of [CLS2] to construct in a neighborhood of $\bar{L}$ a closed meromorphic 1-form ω which defines $\mathcal{F}$, and then to extend ω to all of $\mathbf{C}P^2$ by a Levi type theorem. The polar divisor $(\omega)_\infty$ of ω is composed by irreducible algebraic curves $\Gamma_1, \dots, \Gamma_n \subset \mathbf{C}P^2$, and the hyperbolicity of $\mathrm{Sing}(\mathcal{F})$ implies that $\bigcup_{j=1}^{n} \Gamma_j$ is an algebraic curve with normal crossings (singularities of $\bigcup_{j=1}^{n} \Gamma_j$ are also singularities of $\mathcal{F}$).

Fix an affine chart $\mathbf{C}^2 \subset \mathbf{C}P^2$ such that $\mathbf{C}P^2 \setminus \mathbf{C}^2$ does not contain singularities of the foliation. Let $f_1, \dots, f_n$ be irreducible polynomials which define $\Gamma_1, \dots, \Gamma_n$ in $\mathbf{C}^2$, then we have [CLS2]

$$\omega|_{\mathbf{C}^2} = \sum_{j=1}^{n} \lambda_j \frac{df_j}{f_j}$$

where the complex numbers λ_j are related to the eigenvalues at the singularities and satisfy $\sum_{j=1}^{n} \lambda_j \deg(f_j) = 1$. Again the hyperbolicity of the singularities (λ_j's are pairwise rationally independent) implies that every Γ_j is smooth (i.e. without selfintersections), and that $n \geq 3$.

The foliation $\mathcal{F}$ does not have singular points outside $\bigcup_{j=1}^{n} \Gamma_j$, because outside this set ω is holomorphic (and closed). Hence the cardinality of $\mathrm{Sing}(\mathcal{F})$ is given by

$$\sharp\mathrm{Sing}(\mathcal{F}) = \sum_{i>j} \deg(f_i)\deg(f_j) = \frac{1}{2}\left[\left(\sum_{j=1}^{n} \deg(f_j)\right)^2 - \sum_{j=1}^{n} (\deg(f_j))^2\right].$$

On the other hand, $\mathcal{F}|_{\mathbf{C}^2}$ is generated by a polynomial vector field of degree $d = \left[\sum_{j=1}^{n} \deg(f_j)\right] - 1$ (≥ 2) and hence (because the line at infinity is not invariant)

$$\sharp\mathrm{Sing}(\mathcal{F}) = (d-1)^2 + (d-1) + 1 = d^2 - d + 1.$$

Comparing the two expressions :

$$\sum_{j=1}^{n}(\deg(f_j))^2 = -d^2 + 4d - 1$$

and, taking into account that $d \geq 2$ and $n \geq 3$, the only possibility is $d = 2$, $n = 3$, $\deg(f_j) = 1\ \forall j$, from which it follows easily the conclusion of the theorem.

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