ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 44, nº 4 (1994), p. 1181-1209

<http://www.numdam.org/item?id=AIF_1994__44_4_1181_0>

Ann. Inst. Fourier, Grenoble
44, 4 (1994), 1181-1209

AN INVERSE PROBLEM FOR THE EQUATION $\Delta u = - cu - d$

by Michael VOGELIUS

Contents.

1. Introduction.

Consider the semilinear elliptic boundary value problem

$$\triangle u = -f(u) \text{ in } \Omega\ , \quad u = 0 \text{ on } \partial\Omega. \tag{1}$$

A very interesting problem concerns the extent to which one may determine the function $f(u)$ from knowledge of the outward normal derivative $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$ corresponding to a nontrivial solution u. This inverse problem arises in several contexts, for instance in plasma physics in connection with the modelling of Tokamaks, [6]. For planar domains with corners some partial answers to this problem are already known (*cf.* [3], [4]). Roughly described

Key words : Inverse problems – Radon transform – Stationary phase – Schiffer conjecture.
A.M.S. Classification : 35J05 – 35R30.

it is shown in these papers that if f is sufficiently regular and if one requires that $-f(u) \le 0$, then complete knowledge of $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$ near a corner determines the value of f and all its derivatives at 0.

For smooth domains the situation is much more complicated. Consider for instance a ball : due to the symmetry result of Gidas, Ni and Nirenberg, [10], any positive solution to $\Delta u = -f(u)$ in Ω, $u = 0$ on $\partial\Omega$ is a function of radius alone (assuming f is Lipschitz). The function $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$ is therefore constant, and it is completely impossible to determine $f(u)$ (even the value of f at 0). There is however reason to believe that, at least in the case of positive solutions, balls are the only simply connected domains for which recovery of $f(u)$ fails so miserably. A result of Serrin, [14], asserts that balls are the only domains for which positive solutions of (1) can have a constant normal derivative $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$ (again assuming f is Lipschitz).

In this paper we shall study a very simplified version of this inverse problem for smooth, planar domains. We shall take $f : \mathbb{R} \to \mathbb{R}$ to be an affine function and examine to what extent knowledge of the normal derivative $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$ of a solution to $\Delta u = -cu - d$ in Ω, $u = 0$ on $\partial\Omega$ permits us to determine the constants c and d.

To see how degenerate the inverse problem is on a disk, even for affine f, consider $\Omega = D = \{r \le 1\} \subset \mathbb{R}^2$. Let J_0 and J_1 denote the Bessel functions of the first kind of order 0 and 1 respectively. Let I_0 and I_1 denote the modified Bessel functions corresponding to J_0 and J_1 (*i.e.*, $I_\ell(z) = e^{-\ell\pi i/2} J_\ell(iz)$). The functions J_0 and I_0 solve the equations

$$\frac{d^2}{dz^2} J_0 + \frac{1}{z}\frac{d}{dz} J_0 + J_0 = 0, \tag{2a}$$

and

$$\frac{d^2}{dz^2} I_0 + \frac{1}{z}\frac{d}{dz} I_0 - I_0 = 0, \tag{2b}$$

respectively. The functions of order 0 and order 1 are related by

$$\frac{d}{dz} J_0 = -J_1 \ , \text{ and } \ \frac{d}{dz} I_0 = I_1. \tag{3}$$

For values on the positive real axis the four functions J_0, J_1, I_0 and I_1 are all real valued. The two functions I_0 and I_1 are furthermore positive on the positive real axis. Let $\sqrt{\mu_1}$ denote the smallest positive zero of J_0, then the functions J_0 and J_1 are both positive on the interval $(0, \sqrt{\mu_1})$. If $\sqrt{c}$ is a zero of J_0 then c is an eigenvalue for $-\Delta$ with Dirichlet boundary condition and $J_0(\sqrt{c}\, r)$ is a corresponding eigenfunction. If $\sqrt{c}$ is a zero of J_1 then c is an eigenvalue for $-\Delta$ with Neumann boundary condition and $J_0(\sqrt{c}\, r)$ is a corresponding eigenfunction. The eigenvalues one obtains that way are exactly those corresponding to which there exists a radial eigenfunction. For any $c \in \mathbb{R}$ for which $\sqrt{c}$ is not a zero of J_1 we define the function

$$u_c(r) = \frac{1}{\sqrt{c}} \left(\frac{J_0(\sqrt{c}\, r) - J_0(\sqrt{c})}{J_1(\sqrt{c})} \right). \tag{4}$$

For $c < 0$ the square root is defined as $\sqrt{c} = i\sqrt{|c|}$; in this case it may be more natural to use the expression

$$-\frac{1}{\sqrt{|c|}} \left(\frac{I_0(\sqrt{|c|}\, r) - I_0(\sqrt{|c|})}{I_1(\sqrt{|c|})} \right),$$

for u_c. From the definition of u_c and the equations (2a), (2b) and (3) it follows that u_c is a solution to the overdetermined Cauchy problem

$$\begin{aligned} &\Delta u_c = -cu_c - d_c \quad \text{in } D, \\ &u_c = 0 \ \text{ and } \ \frac{\partial}{\partial \nu} u_c = -1 \quad \text{on } \partial D, \end{aligned} \tag{5}$$

with the constant d_c given by

$$d_c = \frac{\sqrt{c} J_0(\sqrt{c})}{J_1(\sqrt{c})}, \quad c > 0$$

and

$$d_c = \frac{\sqrt{|c|} I_0(\sqrt{|c|})}{I_1(\sqrt{|c|})}, \quad c < 0.$$

The formula (4) has a removable singularity at $c = 0$, and consequently the Cauchy problem (5) also has a solution for $c = 0$; the corresponding solution is $u_0 = (1 - r^2)/2$ and $d_0 = 2$. It follows immediately that there is a continuum of coefficient pairs $(c, d_c) \in \mathbb{R}^2$, and therefore a continuum of affine functions, which give rise to the same normal derivative on the boundary (one affine function corresponding to each value of $c \in \mathbb{R}$ for which $\sqrt{c}$ is not a zero of J_1, as well as one corresponding to $c = 0$).

The smallest positive zero for J_0, $\sqrt{\mu_1}$, is also the square root of the principal eigenvalue for $-\Delta$ with Dirichlet boundary condition. Using the remarks which we made above concerning the sign of the four functions J_0, J_1, I_0 and I_1 it is not difficult to see that for $c \leq \mu_1$ the function u_c is indeed a solution to

$$\Delta u_c = -cu_c - d_c \leq 0 \quad \text{in } D, \tag{6a}$$

$$u_c = 0 \ \text{ and } \ \frac{\partial}{\partial \nu} u_c = -1 \quad \text{on } \partial D, \tag{6b}$$

i.e., the same Cauchy problem as before, but with a sign condition on the right hand side. The inverse problem with a sign imposed on the right hand side therefore also possesses a continuum of affine solutions. It is easy to see that there are no solutions to (6a), (6b) corresponding to $c > \mu_1$.

The results which we prove in section 3 of this paper show that for "most" smooth, planar domains and "most" normal derivative data the inverse problem is not nearly as degenerate as seen above. Indeed there exist at most finitely many pairs of coefficients that give rise to solutions with the same non-zero normal derivative. In the case where we consider solutions to the boundary value problem with a sign imposed on the right hand side it suffices to assume that Ω is not a disk to obtain such a result (*cf.* Theorem 3.1). The strong version of the Maximum Principle makes any assumptions on the normal derivative data superfluous. For the case of solutions without the sign condition imposed we must impose extra conditions on the domain and conditions on the normal derivative. The conditions on the normal derivative are quite simple, the conditions on the domain are more subtle; they are for instance satisfied for convex domains which have maximal and minimimal diametrical thicknesses that are well separated and whose boundary curvature is only zero at a countable number of points (*cf.* Theorem 3.2). One of the extra conditions is that the domain have the socalled Schiffer property. A simply connected $C^{2,\alpha}$ domain Ω is said to have the Schiffer property if (for any c) the only solution to the overdetermined boundary value problem

$$\begin{aligned} &\Delta v = -cv - d \quad \text{in } \Omega, \\ &v = 0\ , \ \ \frac{\partial}{\partial \nu} v = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{7}$$

is the trivial solution $v = 0$ (corresponding to $d = 0$). Here and in the following when we talk about solutions to a problem like (7) we always mean classical solutions in the strong sense that $v \in C^2(\overline{\Omega})$. It is possible to define a nontrivial notion of Schiffer property for domains that are less smooth than $C^{2,\alpha}$ (for instance for domains with corners) but then it is in general necessary to relax the notion of classical solution to the problem (7). It is clear that there is some connection between the inverse problem we study and other studies related to the Schiffer property. It is well known that disks do not have the Schiffer property, indeed for any $\lambda \neq 0$ for which $J_1(\sqrt{\lambda}) = 0$, the function

$$v_\lambda(r) = \frac{1}{\lambda}\left(\frac{J_0(\sqrt{\lambda}\, r)}{J_0(\sqrt{\lambda})} - 1\right),$$

is a solution to (7) with $c = \lambda$ and $d = 1$.

The Schiffer conjecture asserts that in any dimension balls are the only simply connected $C^{2,\alpha}$ domains for which (7) has a nontrivial solution for even a single value of c. It has been shown in ([15]) that for simply connected $C^{2,\alpha}$ domains the possession of the Schiffer property is equivalent to the possession of the socalled Pompeiu property. We shall not here define what is the Pompeiu property, instead we refer the reader to the paper by Brown, Schreiber and Taylor, [8]. In that same paper it was proven that any convex planar domain with a corner has the Pompeiu property. Subsequently it was shown by Williams, [16], that any simply connected Lipschitz domain, the boundary of which is not real analytic, has the Pompeiu property; a simply connected $C^{2,\alpha}$ domain, the boundary of which is not real analytic, therefore has the Schiffer property. In a sense the result of Brown, Schreiber and Taylor is similar in spirit to the uniqueness result we established in ([3]) and ([4]) for the inverse problem for analytic f. Recently large classes of planar real analytic domains with the Schiffer property have been exhibited ([9]).

The result related to the Schiffer conjecture which falls closest to the results we prove here is that of Berenstein ([1]). He shows that for any $C^{2,\alpha}$ planar domain which is not a disk there are at most finitely many values of c for which (7) has a solution for $d \neq 0$ (the example given above shows that for a disk there are infinitely (countably) many such c's). Indeed the technique we use to prove our results about the inverse problem applies to give a very direct and elementary proof of Berenstein's result for convex domains Ω, for which the curvature of the boundary is not too degenerate. Since we find this of independent interest we start the paper by giving the details of this proof.

2. A result due to Berenstein.

In this section we prove in a very simple fashion a result due to Berenstein [1], related to the Schiffer conjecture. The technique of proof we develop has two main components – the formation of appropriate line integrals (a Radon Transform) and a subsequent asymptotic analysis by means of stationary phase. In the section following this we use the same technique to obtain a similar "finiteness" result concerning the inverse problem.

THEOREM 2.1. — *Assume that Ω is a bounded, convex $C^{2,\alpha}$ domain in $\mathbb{R}^2$ such that the curvature of $\partial\Omega$ vanishes at most at a countable set of points. Assume that there exist infinitely many different λ for which the Cauchy problem*

$$\triangle u = -\lambda u - 1 \textit{ in } \Omega \ , u = 0 \textit{ and } \frac{\partial}{\partial \nu} u = 0 \textit{ on } \partial\Omega, \tag{8}$$

has a solution. Then Ω is a disk.

Note. — For (8) to have a solution it is necessary that λ be positive; indeed λ must be a nontrivial eigenvalue for $-\triangle$ with Neumann boundary condition.

The theorem as stated in [1] does not only pertain to domains that are convex, nor does it require that the curvature vanish at most at a countable set of points. The requirement there is that Ω be simply connected. We restrict attention to convex domains with the curvature condition because this permits us to give a proof which does not depend on the (highly nontrivial) result due to S. Williams [16], asserting that any domain which does not possess the Schiffer property has a real analytic boundary. The technique we develop here could in combination with that result also be used to prove the more general result. In subsequent work Berenstein and Yang have extended the "finiteness" result to higher dimensions. Even though we have not carried out the analysis we suspect that the technique we present here may also be applied to higher dimensions. □

Proof. — Fix a coordinate system and let ξ denote an arbitrary, but fixed, unit vector. We introduce the notation

$$m_\xi = \inf_{x\in\Omega} \xi \cdot x \ , \quad M_\xi = \sup_{x\in\Omega} \xi \cdot x.$$

Corresponding to the numbers m_ξ and M_ξ there exist points on the boundary, x_1 and x_2, such that

$$m_\xi = \xi \cdot x_1 \ , \quad M_\xi = \xi \cdot x_2,$$

(ξ is a normal vector to $\partial\Omega$ at such points x_1 and x_2). Since Ω is strictly convex (due to the curvature condition) it follows immediately that the points x_1 and x_2 where $\xi \cdot x$ attains its extremal values are unique for any vector $\xi \in S^1$. Let $R_\xi(s)$, $m_\xi \leq s \leq M_\xi$ denote the function

$$R_\xi(s) = \int_{\xi \cdot x = s} u,$$

where $u \in C^2(\overline{\Omega})$ is a solution to (8). The integral is taken over that part of the line $\xi \cdot x = s$ which lies inside Ω. The functions R_ξ for all $\xi \in S^1$ represent the Radon Transform of u. It is not difficult to see that R_ξ is twice continuously differentiable in (m_ξ, M_ξ). Let $\xi^\perp$ denote the vector obtained by rotation of ξ $\pi/2$ radian counterclockwise. Let $\partial/\partial\xi$ and $\partial/\partial\xi^\perp$ denote the derivatives in the direction ξ and $\xi^\perp$ respectively. Since $\dfrac{\partial}{\partial \xi^\perp} u$ vanishes on the boundary it follows immediately that

$$\text{(9)} \quad \int_{\xi \cdot x = s} \Delta u = \int_{\xi \cdot x = s} \left(\frac{\partial}{\partial \xi}\right)^2 u + \int_{\xi \cdot x = s} \left(\frac{\partial}{\partial \xi^\perp}\right)^2 u = \int_{\xi \cdot x = s} \left(\frac{\partial}{\partial \xi}\right)^2 u.$$

A simple computation yields that

$$\text{(10)} \qquad \left(\frac{d}{ds}\right)^2 \int_{\xi \cdot x = s} u = \int_{\xi \cdot x = s} \left(\frac{\partial}{\partial \xi}\right)^2 u.$$

Here we have used that u as well as $\dfrac{\partial}{\partial \xi} u$ vanish on $\partial\Omega$. We recall that the function u satisfies $\Delta u = -\lambda u - 1$ in Ω; upon integration along the line $\xi \cdot x = s$ and use of the identities (9) and (10) this yields the following equation for R_ξ

$$\left(\frac{d}{ds}\right)^2 R_\xi = -\lambda R_\xi - L_\xi(s),$$

where $L_\xi(s)$ denotes the length of the line segment $\{\xi \cdot x = s\} \cap \Omega$. Since u and $(\partial/\partial\nu)u$ vanish on $\partial\Omega$ the function R_ξ satisfies the following set of boundary conditions

$$R_\xi(s) = \left(\frac{d}{ds}\right) R_\xi(s) = 0 \text{ at } s = m_\xi \text{ and } s = M_\xi.$$

The two conditions at $s = m_\xi$ together with the equation for R_ξ imply that

$$R_\xi(s) = -\frac{1}{\sqrt{\lambda}} \int_{m_\xi}^{s} L_\xi(t) \sin\sqrt{\lambda}(s-t)\ dt.$$

The two conditions at $s = M_\xi$ imply that

$$\int_{m_\xi}^{M_\xi} L_\xi(t) \sin\sqrt{\lambda}(M_\xi - t)\ dt = \int_{m_\xi}^{M_\xi} L_\xi(t) \cos\sqrt{\lambda}(M_\xi - t)\ dt = 0,$$

which is equivalent to

$$\int_{m_\xi}^{M_\xi} L_\xi(t) e^{\pm i\sqrt{\lambda}\, t}\ dt = 0. \tag{11}$$

It is not difficult to see that $L_\xi(t)$ is twice continously differentiable in (m_ξ, M_ξ). We shall initially only consider ξ which are regular in the sense that they belong to the set

$$\begin{aligned} \mathcal{S} = \{\xi \in S^1\ :\ &\text{the curvatures at points on } \partial\Omega \text{ where } \xi \cdot x \text{ attains} \\ &\text{its extremal values are nonzero}\}. \end{aligned} \tag{12}$$

For $\xi \in \mathcal{S}$ one obtains, as $t \searrow m_\xi$, the asymptotics

$$\begin{aligned} L_\xi(t) \approx 2C(t - m_\xi)^{1/2},\ L'_\xi(t) \approx C(t - m_\xi)^{-1/2} \\ \text{and } L''_\xi(t) \approx -\frac{C}{2}(t - m_\xi)^{-3/2}, \end{aligned} \tag{13}$$

where the constant C is given by $C = \sqrt{2/K_{\partial\Omega}(x_1)}$, $K_{\partial\Omega}(x_1)$ denoting the curvature of $\partial\Omega$ at x_1. Here we have used the notation $g(t) \approx h(t)$ to signify that $g(t)/h(t) \to 1$ as $t \searrow m_\xi$. Similarly, as $t \nearrow M_\xi$, one obtains

$$\begin{aligned} L_\xi(t) \approx 2D(M_\xi - t)^{1/2},\ L'_\xi(t) \approx -D(M_\xi - t)^{-1/2} \\ \text{and } L''_\xi(t) \approx -\frac{D}{2}(M_\xi - t)^{-3/2}, \end{aligned} \tag{14}$$

with $D = \sqrt{2/K_{\partial\Omega}(x_2)}$. Integrating the left hand side of the identity (11) by parts and using the fact that L_ξ vanishes at the endpoints we obtain

$$\int_{m_\xi}^{M_\xi} L'_\xi(t) e^{\pm i\sqrt{\lambda}\, t}\ dt = 0. \tag{15}$$

Assuming there exist infinitely many λ for which the problem (8) has a solution, the identity (15) is satisfied for the same infinitum of λ's. It is easy to see that these λ's must necessarily form a sequence of positive numbers whose only limit point is $+\infty$ (for any such λ the pair $\left(\lambda,\ u + \frac{1}{\lambda}\right)$ represents a nonzero eigenvalue and a corresponding eigenvector for the operator $-\Delta$ with Neumann boundary condition). Since $\sqrt{\lambda}$ and L'_ξ are real, the two signs in (15) just correspond to complex conjugation. It is to the identity (15) that we shall apply the method of stationary phase.

It is not difficult to check that, due to (13) and (14), the function $\phi(t) = L'_\xi(t)$ satisfies the prerequisites of Lemma 4.1. As a consequence we conclude from (15) that

$$\frac{e^{\pi i/4}}{\sqrt{K_{\partial\Omega}(x_1)}} e^{i\sqrt{\lambda}\ m_\xi} - \frac{e^{-\pi i/4}}{\sqrt{K_{\partial\Omega}(x_2)}} e^{i\sqrt{\lambda}\ M_\xi} = o(1) \tag{16}$$

as λ approaches $+\infty$ along a particular sequence of real values. Here and in the following $o(1)$ denotes a term that converges to zero as λ approaches infinity. It is important to note that the sequence of λ's is independent of $\xi \in \mathcal{S}$ – it consists of exactly those values for which (8) has a solution. The equation (16) may be rewritten

$$1 + ie^{i\sqrt{\lambda}\ (M_\xi - m_\xi)} \sqrt{\frac{K_{\partial\Omega}(x_1)}{K_{\partial\Omega}(x_2)}} = o(1). \tag{17}$$

Consider the set

$$\mathcal{M} = \{M_\xi - m_\xi\ : \xi \in S^1\} \subset \mathbb{R}.$$

Since Ω is strictly convex it is not difficult to see that the mapping $\mathcal{D}:\ S^1 \ni \xi \to M_\xi - m_\xi$ is continuous (convexity alone does not suffice to guarantee continuity). It is therefore clear that $\mathcal{M}$ is connected. We now proceed to show that $\mathcal{M}$ consists indeed of a single value. Due to $\mathcal{M}$'s connectivity we can accomplished this by showing that $\mathcal{M}$ is at most countable. The set $\mathcal{M}$ may be decomposed as

$$\mathcal{M} = \mathcal{D}(\mathcal{S}) \cup \mathcal{D}(S^1 \setminus \mathcal{S}).$$

Since the set of points on $\partial\Omega$ where the curvature vanishes is at most countable, the set $\mathcal{D}(S^1 \setminus \mathcal{S})$ is at most countable. Because of the continuity of the curvature the set $\mathcal{S}$ is open. Let

$$\mathcal{S} = \bigcup_{k=1}^{\infty} \omega_k$$

be the decomposition of $\mathcal{S}$ as a union of its (at most) countably many connected components. We now show that $\mathcal{D}$ is constant on each of the connected components, from which it will directly follow that $\mathcal{D}(\mathcal{S})$ is at most countable. Let ω be anyone of the connected components (for simplicity we drop the index k). To reach a contradiction let us assume that $\mathcal{D}$ attains two different values

$$\mathcal{D}(\xi_0) < \mathcal{D}(\xi_1)$$

at two points ξ_0 and ξ_1 in ω. Select a subsequence $\{\lambda_n\}$ from those λ for which the problem (8) has a solution, with the property that $(n+1)\sqrt{\lambda_n} < \sqrt{\lambda_{n+1}}$. We now apply Lemma 6.1 with

$$a = \mathcal{D}(\xi_0),\ b = \mathcal{D}(\xi_1),\ L = 2\pi,\ c_n = n+1,\ \mu_n = \sqrt{\lambda_n},\ \text{and } t = 0.$$

This lemma asserts the existence of an $\mathcal{D}(\xi_0) < s < \mathcal{D}(\xi_1)$ such that

$$\sqrt{\lambda_n} s \to 0 \text{ modulo } 2\pi.$$

Since ω is connected it follows that there exists $\xi^* \in \omega$ such that $\mathcal{D}(\xi^*) = s$. We have therefore found $\xi^* \in \omega$ such that

$$\sqrt{\lambda_n}(M_{\xi^*} - m_{\xi^*}) = \sqrt{\lambda_n}\mathcal{D}(\xi^*) \to 0 \text{ modulo } 2\pi, \tag{18}$$

i.e. such that

$$e^{i\sqrt{\lambda_n}(M_{\xi^*} - m_{\xi^*})} \to 1 \text{ as } n \to \infty.$$

It follows now directly from the formula (17), with $\xi = \xi^*$, that

$$1 + i\sqrt{\frac{K_{\partial\Omega}(x_1^*)}{K_{\partial\Omega}(x_2^*)}} = 0,$$

which represents an obvious contradiction. We therefore conclude that $\mathcal{D}$ is constant on ω. This completes the proof of the fact that $\mathcal{M}$ is at most countable, and thus shows that $\mathcal{M}$ consists of a single value. In summary there exists D_0 such that

$$M_\xi - m_\xi = D_0 \quad \forall \xi \in S^1. \tag{19}$$

Let (x_1, x_2) be a set of extremal points corresponding to any $\xi \in S^1$, *i.e.*, $m_\xi = \inf_{x\in\Omega} \xi \cdot x = \xi \cdot x_1$ and $M_\xi = \sup_{x\in\Omega} \xi \cdot x = \xi \cdot x_2$. Let t_{x_1} and t_{x_2} be the tangent lines to $\partial\Omega$ through x_1 and x_2 respectively. The distance between these two parallel lines is $M_\xi - m_\xi = D_0$. Consider the line ℓ that goes through x_1 and x_2. This line ℓ must be orthogonal to t_{x_1} and t_{x_2}; because if not, then that segment of this line that lies between t_{x_1} and t_{x_2} will have length greater than D_0 and it follows immediately that $M_\eta - m_\eta > D_0$, where η denotes a unit vector parallel to the line ℓ. This however is a contradiction to (19). If ℓ_x denotes the line through $x \in \partial\Omega$, orthogonal to $\partial\Omega$, and if $\operatorname{diam}(x)$ is defined by

$$\operatorname{diam}(x) = \sup_{y \in \ell_x \cap \Omega} |x - y|,$$

we therefore have established that

$$\begin{aligned} &\operatorname{diam}(x_1) = D_0, \quad \ell = \ell_{x_1} = \ell_{x_2}, \quad \text{and} \quad x_2 \quad \text{is the} \\ &\text{unique point on } \ell \cap \partial\Omega \quad \text{with } |x_1 - x_2| = \operatorname{diam}(x_1). \end{aligned} \tag{20}$$

It is quite clear that

$$\sqrt{\lambda}\, D_0 \to \pi/2 \quad \text{modulo } \pi, \tag{21}$$

as λ approaches infinity along the sequence of values for which (8) has a solution. If not, one could find a subsequence such that $e^{i\sqrt{\lambda}\, D_0}$ approaches some number with real part $\neq 0$, and according to (17) this is clearly impossible. From insertion of (21) into (17) we get the limiting statement

$$1 \pm \sqrt{\frac{K_{\partial\Omega}(x_1)}{K_{\partial\Omega}(x_2)}} = 0. \tag{22}$$

It is obvious that the above equality with $+$ is impossible (*i.e.* we must really have $\sqrt{\lambda}\, D_0 \to \pi/2$ modulo 2π). The remaining identity (corresponding to $-$) implies that

$$K_{\partial\Omega}(x_1) = K_{\partial\Omega}(x_2), \tag{23}$$

for any extremal pair (x_1, x_2) corresponding to any $\xi \in \mathcal{S}$. The mapping which given x_1 selects a ξ for which it is extremal (say $\xi \cdot x_1 = \inf_{x \in \Omega} \xi \cdot x$) and then assigns x_2 (the other extremal point) is a continuous mapping. It therefore follows immediately that (23) holds for the extremal pair corresponding to any $\xi \in S^1$. The statements (20) and (23) imply, according to Lemma 5.1, that Ω is a disk of radius $D_0/2$. $\square$

3. Two results for the inverse problem.

In this section we proceed to show how the technique developed in the previous section applies to the inverse problem. We start by considering solutions corresponding to nonpositive right hand sides.

THEOREM 3.1. — *Assume that Ω is a bounded, strictly convex $C^{3,\alpha}$ domain in $\mathbb{R}^2$, which is not a disk. Given any $\psi \in C^1(\partial\Omega)$, which is not identically zero, there exist at most finitely many different pairs of coefficients $(c_k, d_k) \in \mathbb{R}^2$ such that the Cauchy problem*

$$\Delta v = -c_k v - d_k \leq 0 \text{ in } \Omega, \quad v = 0 \text{ and } \frac{\partial}{\partial \nu} v = \psi \text{ on } \partial\Omega, \tag{24}$$

has a solution.

Note. — The case $\psi = 0$ is very special. It is not hard to see that the only solution that can satisfy (24) with $\psi = 0$ is the trivial solution $v = 0$ (see the beginning of the proof of Lemma 3.1). This solution corresponds to $d = 0$ and arbitrary c. □

Before we give the proof of Theorem 3.1 we shall show that the sign condition on the right hand side of the P.D.E. in itself guarantees that corresponding to a fixed c there is at most one d for which (24) has a solution.

LEMMA 3.1 — *Assume that Ω is a bounded C^2 domain in $\mathbb{R}^2$. Given any $\psi \in C^1(\partial\Omega)$, and given any $c \in \mathbb{R}$ there exists at most one $d \in \mathbb{R}$ such that the Cauchy problem*

$$\Delta v = -cv - d \leq 0 \text{ in } \Omega, \quad v = 0 \text{ and } \frac{\partial}{\partial \nu} v = \psi \text{ on } \partial\Omega, \tag{25}$$

has a solution.

Proof. — The Maximum Principle asserts that any solution of (25) is either strictly positive inside Ω or constantly equal to zero. The strong version of the Maximum Principle, frequently referred to as the Hopf Lemma, furthermore asserts that either $\frac{\partial}{\partial \nu} v = \psi$ is strictly negative on all of $\partial\Omega$ or v is constantly zero (*cf.* [13]). If the function ψ is equal to zero it therefore follows that the only possible solution is $v = 0$, which necessarily corresponds to $d = 0$ (for any value of c). We proceed to consider the case that ψ is not identically zero.

If c is *not* an eigenvalue of $-\Delta$ with Dirichlet boundary condition, then the uniqueness of d is clear. If such a d exists it must necessarily be positive (since ψ is not identically zero) and be given by

$$d = \frac{\int_{\partial\Omega} \psi \, ds}{\int_{\partial\Omega} \frac{\partial w}{\partial \nu} \, ds}, \tag{26}$$

where w is the unique solution to

$$\Delta w = -cw - 1 \text{ in } \Omega \,, \quad w = 0 \text{ on } \partial\Omega. \tag{27}$$

Note that the fact that there exists $d > 0$ for which (25) has a solution implies that the right hand side of (27), $-cw-1$, is nonpositive. The strong version of the Maximum Principle, implies that the functions ψ and $\dfrac{\partial w}{\partial \nu}$ are both strictly negative on all of $\partial\Omega$. As a consequence the denominator as well as the numerator of (26) are both strictly negative.

It remains to consider those c's that are eigenvalues for $-\Delta$ with Dirichlet boundary condition. Let μ_1 and ϕ_1 denote the principal eigenvalue and the principal eigenfunction of $-\Delta$ with Dirichlet boundary condition. The eigenvalue μ_1 is simple and the eigenfunction ϕ_1 is of one sign; we shall take it to be positive inside Ω. By integration by parts it follows immediately that

$$\mu_1 \int_\Omega v\phi_1 \, dx = -\int_\Omega \Delta v \phi_1 \, dx = c \int_\Omega v\phi_1 \, dx + d \int_\Omega \phi_1 \, dx,$$

i.e.,

$$(\mu_1 - c) \int_\Omega v\phi_1 \, dx = d \int_\Omega \phi_1 \, dx. \tag{28}$$

If there exists a solution to (25) then we necessarily have $v > 0$ in Ω (since we at this point assume that ψ is not identically zero). For $c = \mu_1$ the left hand side of (28) becomes zero and the only possibility for d is therefore $d = 0$. For $c > \mu_1$ the left hand side of (28) would be negative, since $\int_\Omega v\phi_1 \, dx > 0$. The right hand side would be nonnegative, since $d \geq 0$. This shows that (25) has no solution for any $c > \mu_1$ (eigenvalue or not). In combination with the result for $c = \mu_1$ this in particular gives the uniqueness of d for any c which is an eigenvalue of $-\Delta$ with Dirichlet boundary condition. □

We are now ready for

Proof of Theorem 3.1. — As noted in the previous proof it follows directly from the strong version of the Maximum Principle (and the fact that ψ is not identically zero) that when (24) has a solution then we must necessarily have

$$\psi < 0 \ \text{ on } \partial\Omega.$$

To arrive at a contradiction let us now assume that (24) has a solution corresponding to a sequence of infinitely many different pairs $(c_k, d_k) \in \mathbb{R}^2$. We denote by v_k a solution corresponding to the pair (c_k, d_k). From Lemma 3.1 it follows immediately that there must be infinitely many of the constants c_k that are different. By extraction of a subsequence, if necessary, we may assume that *all* the constants c_k are different and nonzero. By subtraction we arrive at the functions $w_k = v_k - v_1$ which satisfy

$$\triangle w_k + c_k w_k = (c_1 - c_k)v_1 + (d_1 - d_k) \text{ in } \Omega\ , \quad w_k = 0 \text{ and } \frac{\partial}{\partial\nu} w_k = 0 \text{ on } \partial\Omega.$$

Using the exact same approach as in the derivation of the identity (11) we now get

$$(29)\quad (c_1 - c_k) \int_{m_\xi}^{M_\xi} V_\xi(t) e^{\pm i\sqrt{c_k}\ t}\, dt + (d_1 - d_k) \int_{m_\xi}^{M_\xi} L_\xi(t) e^{\pm i\sqrt{c_k}\ t}\, dt = 0,$$

for any $\xi \in S^1$. Here $L_\xi(t)$ is the length of the line segment $\{\xi \cdot x = t\} \cap \Omega$ and $V_\xi(t)$ is defined by

$$V_\xi(t) = \int_{\xi\cdot x = t} v_1 .$$

We already know that for $\xi \in \mathcal{S}$

$$(30)\quad \begin{aligned} L_\xi(t) &\approx 2C(t - m_\xi)^{1/2} \text{ as } t \searrow m_\xi \ \text{ and} \\ L_\xi(t) &\approx 2D(M_\xi - t)^{1/2} \text{ as } t \nearrow M_\xi, \end{aligned}$$

including derivatives of order ≤ 2 (cf. (13) and (14)). The constants C and D are given by

$$C = \sqrt{2/K_{\partial\Omega}(x_1)} \ \text{ and } \ D = \sqrt{2/K_{\partial\Omega}(x_2)}.$$

Similarly it is very easy to see that for $\xi \in \mathcal{S}$

$$(31)\quad V_\xi(t) \approx -\frac{4}{3} C \frac{\partial v_1}{\partial \nu}(x_1)(t - m_\xi)^{3/2} \text{ as } t \searrow m_\xi, \ \text{ and}$$

$$(32)\quad V_\xi(t) \approx -\frac{4}{3} D \frac{\partial v_1}{\partial \nu}(x_2)(M_\xi - t)^{3/2} \text{ as } t \nearrow M_\xi\, ,$$

including derivatives of order ≤ 3. Integrating the terms on the left hand side of (29) by parts (twice and once respectively) and using the vanishing of the boundary contributions, we get

$$-\frac{(c_1 - c_k)}{c_k} \int_{m_\xi}^{M_\xi} V_\xi''(t) e^{\pm i\sqrt{c_k}\, t}\, dt$$

$$-\frac{(d_1 - d_k)}{\pm i\sqrt{c_k}} \int_{m_\xi}^{M_\xi} L_\xi'(t) e^{\pm i\sqrt{c_k}\, t}\, dt = 0. \tag{33}$$

Since there are no solutions to (24) for $\mu_1 < c_k$ (see the end of the proof of Lemma 3.1) there are now two possibilities :

(34a) there exists a subsequence $\{c_{k_\ell}\}$ such that $c_{k_\ell} \to -\infty$ as $\ell \to \infty$,

(34b) the entire sequence $\{c_k\}$ is bounded.

We start by considering the case (34a) and for simplicity we also denote the subsequence by $\{c_k\}$. As the square root (of a negative real number c_k) we select $\sqrt{c_k} = i\sqrt{|c_k|}$. Let us first suppose that the sequence $\left\{\frac{(d_1 - d_k)}{i\sqrt{c_k}}\right\}$ stays bounded. Select a subsequence (also indexed by k) so that

$$\frac{(d_1 - d_k)}{i\sqrt{c_k}} = -\frac{(d_1 - d_k)}{\sqrt{|c_k|}} \to \tilde{d} \text{ as } k \to \infty, \quad \tilde{d} \in \mathbb{R}.$$

If we apply Lemma 4.2 to (33), say with the $+$ sign, we now get

$$-e^{-\sqrt{|c_k|} m_\xi} \frac{1}{\sqrt{K_{\partial\Omega}(x_1)}} (\psi(x_1) + \tilde{d} + o(1)) = 0 \tag{35}$$

as c_k approaches $-\infty$ along a particular sequence of real values (for any $\xi \in \mathcal{S}$). This implies that $\psi(x_1) = -\tilde{d}$ for any $\xi \in \mathcal{S}$. Since Ω is strictly convex it follows easily that the points x_1 corresponding to $\xi \in \mathcal{S}$ are dense on $\partial\Omega$ (these are the points x_1 for which the curvature together with the curvature at the "opposite extremal" point is nonzero). By continuity we therefore get that $\psi = -\tilde{d}$ on all of $\partial\Omega$, and due to Serrin's theorem this implies that Ω is a disk, *i.e.*, we have arrived at a contradiction. If the sequence $\left\{\frac{(d_1 - d_k)}{i\sqrt{c_k}}\right\}$ is unbounded, then we similarly arrive at a subsequence along which

$$-e^{-\sqrt{|c_k|} m_\xi} \left(\frac{1}{\sqrt{K_{\partial\Omega}(x_1)}} + o(1)\right) = 0,$$

for any $\xi \in \mathcal{S}$ as c_k approaches $-\infty$. This is clearly a contradiction. At this point we have by contradiction eliminated the case corresponding to (34a). It remains to eliminate the case (34b). By extracting a subsequence, if necessary, we may suppose that there exists $c_* \in \mathbb{R}$ such that

$$c_k \to c_* \quad \text{as } k \to \infty.$$

We may also (by extraction of a further subsequence) obtain that none of the c_k's are eigenvalues of $-\Delta$ with Dirichlet boundary condition. For any $z \in \mathbb{C}$ which is not an eigenvalue of $-\Delta$ with Dirichlet boundary condition we define w_z as being the solution to

$$\Delta w_z = -z w_z - 1 \ \text{ in } \Omega, \quad w_z = 0 \ \text{ on } \partial\Omega. \tag{36}$$

The function $z \to \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_z \, ds$ is meromorphic (and nontrivial) as follows easily from the formula

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_z \, ds = \sum_{j=1}^{\infty} \frac{\mu_j}{z - \mu_j} \left(\int_\Omega \phi_j \, dx \right)^2, \tag{37}$$

where μ_j, ϕ_j denote the eigenvalues and a complete orthonormal set of real eigenvectors for $-\Delta$ with Dirichlet boundary condition. The function $z \to \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_z \, ds$ has a pole at μ_j iff the corresponding eigenspace is not orthogonal to constants. We now define the function

$$d(z) = \frac{\int_{\partial\Omega} \psi \, ds}{\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_z \, ds}. \tag{38}$$

The function $d(z)$ is also a meromorphic function on $\mathbb{C}$. As pointed out in the proof of Lemma 3.1, the components of the sequence (c_k, d_k) are related by $d_k = d(c_k)$. Using the function d we define a meromorphic function

$$F_\xi(z) = (c_1 - z^2) \int_{m_\xi}^{M_\xi} V_\xi(t) e^{iz\,t} \, dt + (d_1 - d(z^2)) \int_{m_\xi}^{M_\xi} L_\xi(t) e^{iz\,t} \, dt.$$

Note that the function F_ξ is analytic in a neighborhood of the point z if d is analytic in a neighborhood of the point z^2. The identity (29), corresponding to the + sign, may now be expressed

$$F_\xi(\sqrt{c_k}) = 0 \ \ \forall k.$$

Since a nontrivial meromorphic function (defined on all of $\mathbb{C}$ except at its poles) cannot have a sequence of zeroes with a finite limit point, it follows that $F_\xi \equiv 0$. In particular there exists a sequence of real numbers $s_k \nearrow +\infty$ such that $-s_k^2$ are not poles of d and such that $F_\xi(is_k) = 0$ (for all ξ). The identity $F_\xi(is_k) = 0$ may also be written

$$(39)\quad (c_1 + s_k^2) \int_{m_\xi}^{M_\xi} V_\xi(t) e^{-s_k t}\, dt + (d_1 - d(-s_k^2)) \int_{m_\xi}^{M_\xi} L_\xi(t) e^{-s_k t}\, dt = 0.$$

Just as in the case corresponding to (34a) the existence of a sequence of real numbers $s_k \nearrow +\infty$ along which (39) holds leads to a contradiction. Having thus by contradiction also eliminated the case (34b), we conclude that it is impossible to have an infinite number of different pairs $(c, d) \in \mathbb{R}^2$ for which the Cauchy problem (24) has a solution. This completes the proof of Theorem 3.1. □

We now turn to the case where there are no restrictions on the sign of the right hand side of the PDE. In order to ensure that there can at most be one value d, corresponding to a fixed c, for which the Cauchy problem

$$\Delta v = -cv - d \text{ in } \Omega, \quad v = 0 \text{ and } \frac{\partial}{\partial \nu} v = \psi \text{ on } \partial\Omega,$$

has a solution, we shall assume that the domain Ω has the Schiffer property (cf. the introduction).

LEMMA 3.2. — *Assume that Ω is a bounded, simply connected $C^{2,\alpha}$ domain in $\mathbb{R}^2$, which has the Schiffer property. Given any $\psi \in C^1(\partial\Omega)$, and given any $c \in \mathbb{R}$ there exist at most one $d \in \mathbb{R}$ such that the Cauchy problem*

$$(40)\qquad \Delta v = -cv - d \text{ in } \Omega, \quad v = 0 \text{ and } \frac{\partial}{\partial \nu} v = \psi \text{ on } \partial\Omega,$$

has a solution.

Proof. — Assume, to arrive at a contradiction, that there exists a c for which (40) has a solution for two different values d_1 and d_2. We denote by v_1 and v_2 two solutions corresponding to d_1 and d_2 respectively. The function $w = v_1 - v_2$ satisfies

$$(40)\qquad \Delta w = -cw + (d_2 - d_1) \text{ in } \Omega, \quad w = 0 \text{ and } \frac{\partial}{\partial \nu} w = 0 \text{ on } \partial\Omega.$$

The existence of w contradicts the fact that Ω has the Schiffer property (since $d_1 - d_2 \neq 0$). This completes the proof of the lemma. □

At this point it is convenient to introduce a couple of definitions. Given a point $x \in \partial\Omega$ let ℓ_x denote the line through x, orthogonal to $\partial\Omega$. As in the proof of Theorem 2.1 we define

$$\operatorname{diam}(x) = \sup_{y \in \ell_x \cap \Omega} |x - y|.$$

In the following we shall always by x' denote a point on $\ell_x \cap \partial\Omega$ such that $\operatorname{diam}(x) = |x - x'|$. For convex $C^{2,\alpha}$ domains it is very easy to see that one has the following equivalence

$$\operatorname{diam}(x) = D_0 \quad \text{and} \quad \ell_x = \ell_{x'} \quad \forall x \in \partial\Omega \Leftrightarrow M_\xi - m_\xi = D_0 \ \forall \xi \in S^1.$$

The $\Leftarrow$ part of this statement was already proven in the proof of Theorem 2.1. The $\Rightarrow$ part follows directly from the observation that x' must be unique (given x) and that the tangents to $\partial\Omega$ at x and x' must be parallel.

We say that a convex $C^{2,\alpha}$ domain Ω has *constant diametrical thickness* (D_0) if $\operatorname{diam}(x) = D_0$ and $\ell_x = \ell_{x'}$ for all $x \in \partial\Omega$ (or equivalently if $M_\xi - m_\xi = D_0 \ \forall \xi \in S^1$). We shall say that a bounded, convex $C^{2,\alpha}$ domain Ω is *non-degenerate* if

Ω does not have constant diametrical thickness, Ω possesses the Schiffer property, and there is at most a countable set of points where $\partial\Omega$ has zero curvature.

It is well known that any bounded, convex $C^{2,\alpha}$ domain for which

$$2 \min_{\xi \in S^1} (M_\xi - m_\xi) \leq \max_{\xi \in S^1} (M_\xi - m_\xi) \tag{41}$$

has the Schiffer property ([7]). It therefore follows that any bounded, convex $C^{2,\alpha}$ domain which satisfies (41), and which has at most a countable set of points where the curvature of $\partial\Omega$ vanishes, is non-degenerate in the sense defined above. We shall say that a function $\psi \in C^1(\partial\Omega)$ (the candidate for the normal derivative) is *non-degenerate* if

ψ is not identically constant, and ψ has at most countably many zeroes.

THEOREM 3.2. — *Assume that Ω is a bounded, convex, non-degenerate $C^{3,\alpha}$ domain in $\mathbb{R}^2$. Given any $\psi \in C^1(\partial\Omega)$, which is also non-degenerate, there exist at most finitely many different pairs of coefficients $(c_k, d_k) \in \mathbb{R}^2$ such that the Cauchy problem*

$$\Delta v = -c_k v - d_k \text{ in } \Omega, \quad v = 0 \text{ and } \frac{\partial}{\partial \nu} v = \psi \text{ on } \partial\Omega, \tag{42}$$

has a solution.

Proof. — To arrive at a contradiction let us assume that (42) has a solution corresponding to a sequence of infinitely many different pairs $(c_k, d_k) \in \mathbb{R}^2$. From Lemma 3.2 it follows immediately that there must be infinitely many of the constants c_k that are different. By extraction of a subsequence, if necessary, we may assume that *all* the constants c_k are different and nonzero. Let v_k denote a solution corresponding to the pair (c_k, d_k). As in the proof of Theorem 3.1 we have the identity

$$-\frac{(c_1 - c_k)}{c_k}\int_{m_\xi}^{M_\xi} V_\xi''(t)e^{\pm i\sqrt{c_k}\ t}\ dt$$

$$-\frac{(d_1 - d_k)}{\pm i\sqrt{c_k}}\int_{m_\xi}^{M_\xi} L_\xi'(t)e^{\pm i\sqrt{c_k}\ t}\ dt = 0, \tag{43}$$

where $L_\xi(t)$ denotes the length of the line segment $\{\xi\cdot x = t\}\cap\Omega$ and $V_\xi(t)$ is defined by

$$V_\xi(t) = \int_{\xi\cdot x = t} v_1.$$

There are now two possibilities :

(44a) there exists a subsequence $\{c_{k_\ell}\}$ such that $c_{k_\ell} \to \pm\infty$ as $\ell \to \infty$,

(44b) the entire sequence $\{c_k\}$ is bounded.

We start by considering the case (44a) and for simplicity we also denote the subsequence by $\{c_k\}$. If $c_k \to -\infty$ we may proceed exactly as in the proof of Theorem 3.1 , eventually showing that $\psi = -\tilde{d}$ (a constant) on all of $\partial\Omega$. This represents a contradiction to the fact that ψ is non-degenerate.

The case $c_k \to +\infty$ gives rise to slightly different asymptotics, much like that exhibited in the proof of Theorem 2.1. There are two possible scenarios : 1) the sequence $\left\{\frac{(d_1 - d_k)}{\sqrt{c_k}}\right\}$ is bounded, or 2) the sequence $\left\{\frac{(d_1 - d_k)}{\sqrt{c_k}}\right\}$ is unbounded. Let us first assume that 1) is the case, and select a subsequence (also indexed by k) such that

$$\frac{(d_1 - d_k)}{\sqrt{c_k}} \to \tilde{d}\ \text{ as } k \to \infty, \quad \tilde{d} \in \mathbb{R}.$$

Using the known asymptotic behaviour of the functions L_ξ and V_ξ near the points m_ξ and M_ξ (the statements (30), (31), and (32)) we apply the method of stationary phase to (43), say with the + sign. After some simple manipulations Lemma 4.1 leads to the asymptotic identity

$$-(\psi(x_1) - i\,\tilde{d})\frac{e^{\pi i/4}}{\sqrt{K_{\partial\Omega}(x_1)}}e^{i\sqrt{c_k}\ m_\xi}$$

$$-(\psi(x_2) + i\,\tilde{d})\frac{e^{-\pi i/4}}{\sqrt{K_{\partial\Omega}(x_2)}}e^{i\sqrt{c_k}\ M_\xi} = o(1) \tag{45}$$

as c_k approaches $+\infty$ along a particular sequence of real values, for any $\xi \in \mathcal{S}'$. Here the set $\mathcal{S}'$ is given by

$$\mathcal{S}' = \mathcal{S} \cap \{\xi \in S^1 \ :\ \psi \text{ is nonzero at the points on } \partial\Omega \text{, where } \xi \cdot x \text{ attains its extremal values}\}.$$

The formula (45) may be restated

$$(\psi(x_1) - i\,\tilde{d}) - i\sqrt{\frac{K_{\partial\Omega}(x_1)}{K_{\partial\Omega}(x_2)}}e^{i\sqrt{c_k}\ (M_\xi - m_\xi)}(\psi(x_2) + i\,\tilde{d}) = o(1). \tag{46}$$

If $\tilde{d} = 0$ then we proceed along the same line as in the proof of Theorem 2.1, the only essential difference being that the set $\mathcal{S}$ gets replaced by $\mathcal{S}'$. The set of values $M_\xi - m_\xi$ corresponding to $\xi \in S^1 \setminus \mathcal{S}'$ rests at most countable, and as in the proof of Theorem 2.1 we can, based on (46), prove that $M_\xi - m_\xi$ is constant on any connected component of $\mathcal{S}'$. As in the proof of Theorem 2.1 it follows that

$$M_\xi - m_\xi = D_0 \ \ \forall \xi \in S^1\ , \quad \text{and} \tag{47a}$$

$$\frac{\psi(x_1)^2}{K_{\partial\Omega}(x_1)} = \frac{\psi(x_2)^2}{K_{\partial\Omega}(x_2)}, \tag{47b}$$

for any extremal pair (x_1, x_2) corresponding to any $\xi \in \mathcal{S}'$. The identity (47a) clearly represents a contradiction to the fact that Ω does not have constant diametrical thickness. If $\tilde{d}$ of (46) is not zero, then we may again proceed as in the proof of Theorem 2.1 provided we make an additional change. To show that $M_\xi - m_\xi = D_0$ we use Lemma 6.1 with $t = -\pi/2$ instead of $t = 0$. This way (if $M_\xi - m_\xi$ is not constant) we find $\xi^* \in \mathcal{S}'$ so that $\sqrt{c_k}\ \mathcal{D}(\xi^*) \to -\pi/2 \bmod 2\pi$, *i.e.*, such that

$$e^{i\sqrt{c_k}\ (M_{\xi^*} - m_{\xi^*})} \to -i \text{ as } c_k \to +\infty.$$

Inserted into (46) this gives

$$(\psi(x_1^*) - i\,\tilde{d}) - \sqrt{\frac{K_{\partial\Omega}(x_1^*)}{K_{\partial\Omega}(x_2^*)}}(\psi(x_2^*) + i\,\tilde{d}) = 0,$$

which is a contradiction (since $\tilde{d} \neq 0$). As in the proof of Theorem 2.1 this shows that

$$M_\xi - m_\xi = D_0 \ \ \forall \xi \in S^1. \tag{48a}$$

We also immediately get that

$$\frac{\psi(x_1)^2 + \tilde{d}^2}{K_{\partial\Omega}(x_1)} = \frac{\psi(x_2)^2 + \tilde{d}^2}{K_{\partial\Omega}(x_2)}, \tag{48b}$$

for any extremal pair (x_1, x_2) corresponding to any $\xi \in \mathcal{S}'$. Again, the identity (48a) represents a contradiction to the fact that Ω does not have constant diametrical thickness. Suppose on the other hand that the sequence $\left\{ \frac{(d_1 - d_k)}{\sqrt{c_k}} \right\}$ is unbounded. Application of the method of stationary phase (Lemma 4.1) to (43) then leads to the asymptotic identity

$$\frac{e^{\pi i/4}}{\sqrt{K_{\partial\Omega}(x_1)}} e^{i\sqrt{c_k}\ m_\xi} - \frac{e^{-\pi i/4}}{\sqrt{K_{\partial\Omega}(x_2)}} e^{i\sqrt{c_k}\ M_\xi} = o(1)$$

as c_k approaches $+\infty$ along a particular sequence of real values (for any $\xi \in \mathcal{S}'$). By use of the exact same argument as in the proof of Theorem 2.1 it now follows that

$$M_\xi - m_\xi = D_0 \ \ \forall \xi \in S^1, \ \text{ and} \tag{49a}$$

$$\frac{1}{\sqrt{K_{\partial\Omega}(x_1)}} = \frac{1}{\sqrt{K_{\partial\Omega}(x_2)}}, \tag{49b}$$

for any extremal pair (x_1, x_2) corresponding to any $\xi \in \mathcal{S}'$. Just as in the proof of Theorem 2.1 this implies that Ω is a disk, which clearly represents a contradiction.

At this point we have by contradiction eliminated the case corresponding to (44a). It remains to eliminate the case (44b). By extracting a subsequence, if necessary, we may suppose that there exists $c_* \in \mathbb{R}$ such that

$$c_k \to c_* \ \text{ as } k \to \infty.$$

As before we define the meromorphic function

$$d(z) = \frac{\int_{\partial\Omega} \psi \, ds}{\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_z \, ds}, \tag{50}$$

where for any $z \in \mathbb{C}$, which is not an eigenvalue of $-\Delta$ with Dirichlet boundary condition, w_z is defined as being the solution to

$$\Delta w_z = -z w_z - 1 \text{ in } \Omega, \quad w_z = 0 \text{ on } \partial\Omega. \tag{51}$$

By extracting a subsequence, if necessary, we may suppose that none of the c_k's are eigenvalues of $-\Delta$ with Dirichlet boundary condition, nor zeros of the function $z \to \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_z \, ds$. The components of the sequence (c_k, d_k) are then related by $d_k = d(c_k)$. Using the function d we define a meromorphic function

$$F_\xi(z) = (c_1 - z^2) \int_{m_\xi}^{M_\xi} V_\xi(t) e^{iz\ t}\, dt + (d_1 - d(z^2)) \int_{m_\xi}^{M_\xi} L_\xi(t) e^{iz\ t}\, dt.$$

Note that the function F_ξ is analytic in a neighborhood of the point z if d is analytic in a neighborhood of the point z^2. After undoing the integration by parts we may express the identity (43) (with the + sign) as

$$F_\xi(\sqrt{c_k}) = 0 \ \ \forall k.$$

Since a nontrivial meromorphic function (defined on all of $\mathbb{C}$ except at its poles) cannot have a sequence of zeroes with a finite limit point, it follows that $F_\xi \equiv 0 \ \forall \xi \in S^1$. This yields a contradiction in exactly the same way as in the proof of Theorem 3.1. We have thus also excluded the case (44b), and we may finally conclude that there are at most finitely many different pairs of coefficients $(c_k, d_k) \in \mathbb{R}^2$ such that the Cauchy problem (42) has a solution. This completes the proof of the theorem. $\square$

4. Asymptotic integral formulas.

Let $\phi(t)$, $m \leq t \leq M$ be a real valued function which is smooth in the open interval (m, M) but which has singularities at $t = m$ and $t = M$. In a way to be made specific just below we shall assume that the singularities at $t = m$ and $t = M$ are of the types $(t-m)^{-1/2}$ and $(M-t)^{-1/2}$ respectively. The first task of this section is to find an asymptotic formula for the integral

$$I(\lambda) = \int_m^M \phi(t) e^{i\sqrt{\lambda}\ t}\, dt \tag{52}$$

as the real parameter λ approaches $+\infty$. Formulas like this are well known and date back to Kelvin. They are commonly referred to as approximations of stationary phase. We shall assume that

i) $\phi(t)$ is continuously differentiable in (m, M),

ii) $\phi(t) \approx C_m(t-m)^{-1/2}$, and $\phi'(t) \approx -\frac{C_m}{2}(t-m)^{-3/2}$ as $t \searrow m$,

iii) $\phi(t) \approx C_M(M-t)^{-1/2}$, and $\phi'(t) \approx \frac{C_M}{2}(M-t)^{-3/2}$ as $t \nearrow M$.

Here we have used the notation $\phi(t) \approx \psi(t)$ to signify that $\phi(t)/\psi(t) \to 1$ as $t \searrow m$ (or $t \nearrow M$). The constants C_m and C_M are nonzero.

LEMMA 4.1. — *If ϕ satisfies the conditions i), ii) and iii) above, then*

$$I(\lambda) = \lambda^{-1/4}\left(C_m e^{\pi i/4}\sqrt{\pi}e^{i\sqrt{\lambda}\ m} + C_M e^{-\pi i/4}\sqrt{\pi}e^{i\sqrt{\lambda}\ M}\right) + o(\lambda^{-1/4}),$$

as the real parameter λ approaches $+\infty$.

Proof. — This result follows immediately from Theorem 13.1 on page 101 of [11] by considering the endpoint singularities one at a time. In adapting the formulas given in [11] we have used the fact that $\Gamma(1/2) = \sqrt{\pi}$.

□

The second task is to find an asymptotic formula for the integral (52) as the real parameter λ approaches $-\infty$, *i.e.*, we shall seek asymptotic formulas for the integrals

$$J_{\pm}(\mu) = \int_m^M \phi(t)e^{\pm\sqrt{\mu}\ t}\,dt$$

as μ approaches $+\infty$. Formulas like this are also well known, they date back to Laplace.

LEMMA 4.2. — *If ϕ satisfies condition i) and the first part of conditions ii) and iii), then*

$$J_+(\mu) = \mu^{-1/4}e^{\sqrt{\mu}\ M}\left(C_M\sqrt{\pi} + o(1)\right)$$
$$\text{and}\quad J_-(\mu) = \mu^{-1/4}e^{-\sqrt{\mu}\ m}\left(C_m\sqrt{\pi} + o(1)\right),$$

as the real parameter μ approaches $+\infty$.

Proof. — By the change of variables, $s = M - t$ and $s = t - m$ respectively, we get

$$J_+(\mu) = e^{\sqrt{\mu}\ M}\int_0^{M-m} \phi\,(M-s)\,e^{-\sqrt{\mu}\ s}\,ds \quad\text{and}$$
$$J_-(\mu) = e^{-\sqrt{\mu}\ m}\int_0^{M-m} \phi\,(m+s)\,e^{-\sqrt{\mu}\ s}\,ds.$$

The asymptotic results stated follow now directly from Theorem 7.1 on page 81 of ([11]). □

5. A geometric lemma.

In this section we shall state and prove a geometric lemma which we have already used in the proofs of Theorems 2.1, and 3.2. Before we state the lemma we shall need a little bit of notation. Let Ω be a bounded C^2 domain in $\mathbb{R}^2$. For any $x \in \partial\Omega$ we denote by $K_{\partial\Omega}(x)$ the curvature of $\partial\Omega$ at x. This is defined as $\|f''\|$ where f is a parametrization of $\partial\Omega$ by arclength. In terms of an arbitrary parametrization, g, it is given by $|g_1''g_2' - g_1'g_2''|/\|g'\|^3$, (see for instance [5] page 290). In geometric terms the curvature may also be expressed as

$$K_{\partial\Omega} = \|\frac{\partial}{\partial\tau}\nu\| = |(\frac{\partial}{\partial\tau}\nu)\cdot\tau|, \tag{53}$$

where ν and τ denote a unit normal- and unit tangent-vector fields to $\partial\Omega$, respectively. For any $x \in \partial\Omega$, let ℓ_x denote the line orthogonal to $\partial\Omega$ through x. We define

$$\operatorname{diam}(x) = \sup_{y\in\ell_x\cap\Omega} |x-y|.$$

The following lemma provides a somewhat unorthodox characterization of a disk. It is a special case of a characterization due to Berenstein and Yang (*cf.* [2]). For the convenience of the reader we provide a simple proof.

LEMMA 5.1. — *Assume that Ω is a bounded C^2 domain in $\mathbb{R}^2$ with the following two properties :*

i) *there exists a constant D_0 so that* $\operatorname{diam}(x) = D_0$ *for all $x \in \partial\Omega$,*

ii) *for any $x \in \partial\Omega$ one has $K_{\partial\Omega}(x) = K_{\partial\Omega}(x')$, and $\ell_x = \ell_{x'}$ where x' denotes any point on $\ell_x \cap \partial\Omega$, with* $|x - x'| = \operatorname{diam}(x)$.

Then Ω is a disk of radius $D_0/2$.

Proof. — It is quite easy to see that given any $x \in \partial\Omega$ there is only one point $x' \in \ell_x \cap \partial\Omega$ with $|x - x'| = \operatorname{diam}(x)$ $(= D_0)$. To verify this let x' be on $\ell_x \cap \partial\Omega$ with $|x - x'| = \operatorname{diam}(x)$. If there were a second point x'' on $\ell_x \cap \partial\Omega$ with $|x - x''| = \operatorname{diam}(x)$ it now follows that $x'' \in \ell_x = \ell_{x'}$ and $|x' - x''| = 2D_0$ so that $\operatorname{diam}(x') \geq 2D_0$, a contradiction.

Let x_0 be a fixed point on $\partial\Omega$. Pick a coordinate system with origin at x_0 and first axes tangent to $\partial\Omega$. Near x_0, $\partial\Omega$ is now of the form $\{(t, h(t)) : t \in I\}$, for some interval I. For $x \in \partial\Omega$, near x_0, the points $x' \in \ell_x \cap \partial\Omega$ with $|x - x'| = D_0$ are given by

$$(t, h(t)) \pm \frac{D_0}{\sqrt{1 + h'(t)^2}}(-h'(t), 1) \quad t \in I.$$

Using the fact that there is only one point x' on $\ell_x \cap \partial\Omega$ with $|x - x'| = D_0$ one easily sees that the same sign applies throughout I. We may without loss of generality assume that the points $x' \in \ell_x \cap \partial\Omega$ with $|x - x'| = D_0$, near x'_0, are parametrized by

$$(g_1(t), g_2(t)) = (t, h(t)) + \frac{D_0}{\sqrt{1 + h'(t)^2}}(-h'(t), 1) \quad t \in I. \tag{54}$$

The above parametrization describes a proper curve, *i.e.*, $(g'_1(t), g'_2(t))$ does not vanish near $t = 0$. To see this we note that the roles of x and x' are completely reversible, since $\ell_x = \ell_{x'}$. If $g'_1(0) = g'_2(0) = 0$ this would correspond to an infinite value of the gradient for a corresponding parametrization of x near x_0. Due to the smoothness of Ω this is impossible. The unit normals (as well as the unit tangents) to the curve $\partial\Omega$ at the points $(t, h(t))$ and $(g_1(t), g_2(t))$ are the same. Let $J \subset I$ denote the parameters t for which the curvature of $\partial\Omega$ at the point $(t, h(t))$ is non-zero. From the definition (53) and the fact that the curvatures at the points $(t, h(t))$ and $(g_1(t), g_2(t))$ coincide for each $t \in J$ it now follows that the mapping

$$(t, h(t)) \to (g_1(t), g_2(t)) \quad t \in J,$$

must necessarily preserve arclength. This implies that

$$1 + h'(t)^2 = g'_1(t)^2 + g'_2(t)^2, \tag{55}$$

for all $t \in J$. By continuity it immediately follows that (55) is satisfied for all $t \in \overline{J}$. For $t \in I \setminus \overline{J}$ there exists a small open interval ω of parameters, containing t, such that the curve $(t, h(t)$ is linear. For these values the mapping $(t, h(t)) \to (g_1(t), g_2(t))$ is a translation. It follows immediately that (55) also holds in $I \setminus \overline{J}$. In summary we conclude that (55) holds in all of I. From the formula for $(g_1(t), g_2(t))$ one readily computes that

$$\begin{aligned} g'_1(t)^2 + g'_2(t)^2 &= \left(1 - \frac{D_0 h''(t)}{\sqrt{1 + h'(t)^2}^{\,3}}\right)^2 + \left(h'(t) - \frac{D_0 h'(t) h''(t)}{\sqrt{1 + h'(t)^2}^{\,3}}\right)^2 \\ &= 1 + h'(t)^2 + \frac{D_0^2 h''(t)^2}{(1 + h'(t)^2)^2} - \frac{2D_0 h''(t)}{\sqrt{1 + h'(t)^2}}, \end{aligned}$$

so that (55) is satisfied if and only if

$$D_0 h''(t)^2 = 2h''(t)\sqrt{1+h'(t)^2}^{\,3}.$$

Since the first axis of the coordinate system is tangent to $\partial\Omega$ at x_0, the initial conditions for h are $h(0) = h'(0) = 0$. The C^2 solutions to this initial value problem consist of $h(t) \equiv 0$, together with the solution to

$$D_0 h''(t) = 2\sqrt{1+h'(t)^2}^{\,3}, \quad h(0) = h'(0) = 0.$$

This latter problem has the unique solution

$$h(t) = \frac{D_0}{2} - \sqrt{\left(\frac{D_0}{2}\right)^2 - t^2},$$

with the maximal interval of definition $-D_0/2 \leq t \leq D_0/2$. It therefore follows that if the conditions i) and ii) are satisfied then $\partial\Omega$ locally around x_0 and x_0' consist of two line segments or two pieces of a circle of radius $D_0/2$. Since we require that $\partial\Omega$ be a C^2-curve (and since Ω is bounded) it now follows that Ω must necessarily be a disk of radius $D_0/2$. □

As before we shall say that the domain Ω has *constant diametrical thickness* if $\mathrm{diam}(x) = D_0$ and $\ell_x = \ell_{x'}$ $\forall x \in \partial\Omega$ (and any $x' \in \ell_x \cap \partial\Omega$ with $|x - x'| = \mathrm{diam}(x)$). There are plenty of C^2 domains which have constant diametrical thickness but which are not disks. To construct an example consider the curve σ given by

$$\sigma(t) = \rho(t)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \quad 0 \leq t \leq \pi.$$

Assume that ρ is a C^∞-function with $\rho(0) = \rho(\pi) = \frac{D_0}{2}$ and $\rho'(0) = \rho'(\pi) = 0$ and assume that ρ is sufficiently close to $\frac{D_0}{2}$ in C^2. Given any point $\sigma(t)$, the point

$$\tilde\sigma(t) = \rho(t)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} + D_0\frac{\rho'(t)\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix} - \rho(t)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}}{\sqrt{\rho'(t)^2 + \rho(t)^2}}$$

lies on the line through $\sigma(t)$ orthogonal to the curve σ (we previously called that $\ell_{\sigma(t)}$). The point $\tilde\sigma(t)$ lies at a distance $|\sigma(t) - \tilde\sigma(t)| = D_0$ from $\sigma(t)$. Furthermore the curve $\sigma \cup \tilde\sigma$ forms the boundary of a bounded set Ω. It is not difficult to see that if $\rho^{(2)}(0) = \rho^{(2)}(\pi) = 0$ then Ω is a C^1 domain.

Due to its construction it follows immediately that Ω is starshaped and has constant diametrical thickness. By further computation one verifies that Ω is a C^2 domain if additionally $\rho^{(3)}(0) = \rho^{(3)}(\pi) = 0$. One will continue to increase the regularity of Ω by imposing the condition $\rho^{(k)}(0) = \rho^{(k)}(\pi) = 0$ on derivatives of ever higher order.

6. A simple algebraic lemma.

In this section we prove a simple algebraic lemma which we have already used in the proofs of Theorems 2.1, and 3.2. We shall say that a sequence $\{t_n\}_{n=1}^{\infty}$ converges to t modulo L as n approaches infinity, if there exists a sequence of integers $\{k_n\}_{n=1}^{\infty}$ such that $t_n - k_n L \to t$ as $n \to \infty$.

LEMMA 6.1. — *Let $a < b$ and $L > 0$ be three real numbers. Let $\{c_n\}_{n=1}^{\infty}$ be a monotonely increasing sequence of positive numbers tending to infinity and starting with $1 < c_1$. Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of positive numbers which satisfy $c_n\mu_n < \mu_{n+1}$. Given any $t \in \mathbb{R}$ there exists a number $s : a < s < b$ such that*

$$\mu_n s \to t \quad \textit{modulo } L \quad \textit{as } n \to \infty.$$

Proof. — We may without loss of generality assume that

$$\mu_n(b-a) > L \tag{56}$$

for all n; if this is not apriori satisfied we just disregard those finitely many μ_n for which the inequality is not satisfied. We first select a number s_1 such that

$$a < s_1 < b \quad \text{and } s_1 \text{ satisfies } \mu_1 s_1 = t \text{ modulo } L.$$

This is possible because (56) holds for $n = 1$. Given $a < s_n < b$ we now select a closed interval I_n of length L/μ_{n+1} which is contained in (a, b) and which contains s_n. This is possible due to (56). We now select a number s_{n+1} in the closed interval I_n so that $\mu_{n+1}s_{n+1} = t$ modulo L. This again is possible due to (56). Following this iterative procedure we obtain a sequence $\{s_n\}_{n=1}^{\infty}$ with the properties that

$$\mu_n s_n = t \text{ modulo } L \quad \text{and}$$
$$|s_k - s_{k+1}| < Lc_k^{-1}\mu_k^{-1} \le Lc_n^{n-k-1}\mu_n^{-1} \quad \forall k \ge n. \tag{57}$$

Here we used the facts that $\mu_{k+1}^{-1} < c_k^{-1}\mu_k^{-1}$ and that $c_k^{-1} < c_n^{-1}$, $k > n$. Because of the second property in (57) the values s_n converge to some $s \in [a, b]$ in such a way that

$$|s - s_n| \leq L\mu_n^{-1} \sum_{k=1}^{\infty} c_n^{-k} = L\mu_n^{-1} \frac{1}{c_n - 1}.$$

Because of this estimate and the first property of (57) be get that there exist integers k_n for which

$$|\mu_n s - (t + k_n L)| = |\mu_n s - \mu_n s_n| \leq \frac{L}{c_n - 1}.$$

This immediately implies that

$$\mu_n s \rightarrow t \ \text{ modulo } L \ \text{ as } n \rightarrow \infty.$$

The s constructed here is in the interval $[a, b]$, but not necessarily inside (a, b) – this however could be guaranteed by performing the above construction using a slightly smaller interval $(a', b') \subset\subset (a, b)$. $\square$

Acknowledgments : This research was partially supported by NSF grant DMS-9202042 and by AFOSR contract 89NM605. This work was performed while the author was visiting the Université Joseph Fourier during the academic year 92/93. He would like to thank his colleagues at "Laboratoire de Modélisation et Calcul" for the hospitality extended to him. In particular he would like to thank Jacques Blum for his efforts towards making this visit possible. He would also like to express his thanks to Otared Kavian of the Université de Nancy, with whom he had very fruitful discussions regarding the results in this paper.

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Manuscrit reçu le 13 juillet 1993,
révisé le 13 juin 1994.

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