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# A NEW PROOF OF MULTISUMMABILITY OF FORMAL SOLUTIONS OF NON LINEAR MEROMORPHIC DIFFERENTIAL EQUATIONS

by J.-P. RAMIS and Y. SIBUYA

## 0. Introduction.

In 1978-80 the first author introduced the notion of $k$-summability of formal power series expansions. This notion is based on fundamental works of Borel, Watson, Nevanlinna. He proved that power series solutions of *sufficiently generic* linear meromorphic differential equations are $k$-summable for some $k > 0$ depending on the equation. He remarked also that unfortunately there exists some formal power series solutions of some linear meromorphic differential equations which are *not* $k$-summable for *any* $k > 0$ (cf. J.-P. Ramis and Y. Sibuya [10] for an example). J.-P. Ramis proved also a *factorization theorem* of formal solutions (J.-P. Ramis [9], Y. Sibuya [12] : Theorem 4.2.3, p. 237) which implies that every formal power series $\hat{f}$ solution of a linear meromorphic differential equation can (non uniquely) be written as a sum of products of $k$-summable power series, where the occuring $k$'s belong to a finite set depending on the equation. Then it is possible to get a *multisum* $f$ of $\hat{f}$ in a given generic direction $d$. Later J.-P. Ramis proved that this multisum does not depend on the decomposition and that, if $d$ is fixed, the map $\hat{f} \mapsto f$ is Galois. (W. Balser obtained also independently the factorization theorem as a byproduct of his treatment of his *first level formal solutions.*)

The major inconvenient of this approach is that the factorization theorem of formal solutions is an existence theorem and is not truly

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effective. So it does not give an *explicit* way to compute the sum. More recently J. Ecalle found an explicit way to compute the sum (using analytic continuation and integral formulas). His method is based on his *acceleration theory* (which is connected with previous works of G.H. Hardy and his student Good). He named *accelerosummation* the very general corresponding process of summation [4]. For a restricted class of functions the, at this time yet unpublished, ideas of J. Ecalle were *exposed* by J. Martinet and J.-P. Ramis in the first part of [8]. In the same paper they proved the *multisummability* in Ecalle's sense of formal power series solutions of linear meromorphic differential equations : this multisummability property is a *trivial* consequence of Ecalle's theory and of Ramis factorization theorem. Afterwards other proofs of multisummability of formal power series solutions of linear meromorphic differential equations were given by W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya [1], B.L.J. Braaksma [2] and B. Malgrange and J.-P. Ramis [7]. The first two papers used Ecalle's definition of multisummability. In the third paper the authors introduced a new equivalent definition of multisummability based on cohomological ideas (and in particular on a relative version of Watson Lemma due to B. Malgrange).

In the *non-linear* situation multisummability of formal power series solutions was independently conjectured by J. Ecalle and the first author, but the first complete proof was only very recently given by B.L.J. Braaksma [3]. In his proof he uses Ecalle's definition of multisummability. Afterwards in [11] the second author outlined another proof based on the cohomological definition of multisummability (cf. B. Malgrange and J.-P. Ramis [7] and W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya [1]). In this paper we shall present a complete version of this analysis. As we mentioned in [11], the main problem is to prove Theorem 2.1 of §2, since multisummability of formal power series solutions can be derived from Theorem 2.1 in a manner similar to the proof of Theorem 4.1 of [1] based on Lemma 7.1 of [1]. We shall present a complete proof of Theorem 2.1. Our proof is based on the methods due to M. Hukuhara [5], M. Iwano [6], and J.-P. Ramis and Y. Sibuya [10]. We shall also derive explicitely multisummability of formal power series solutions using Malgrange-Ramis definition of multisummability.

Our proof is quite different from Braaksma's. We do not use Laplace transform, acceleration and convolution products. The idea is to perform a sort of analytic continuation across an *infinitesimal neighborhood*. We get in a finite number of steps a more and more precise estimate of the sum, which is defined up to exponentially small corrections of some order which increases at each step. Our proof is not simpler than the elegant proof of Braaksma but we hope that it will shed a new light on the problem. In particular our approach is based on a very detailed analysis of formal normal forms (in relation with resonances) and Stokes phenomena for non linear systems of differential equations. This analysis extends some works of M. Hukuhara [5] and M. Iwano [6]. It has certainly an independant interest and it would be interesting to investigate more deeply such questions.

## 1. Preliminaries.

As in [11], throughout this paper we shall use the following notations :

1) $\lambda(x)$ is an $n \times n$ diagonal matrix :

$$\lambda(x) = \begin{bmatrix} \lambda_1(x) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2(x) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3(x) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n(x) \end{bmatrix}, \tag{1.1}$$

where either $\lambda_j(x) = 0$ identically or

$$\lambda_j(x) = \sum_{\ell=1}^{N_j} \lambda_{j,\ell}\, x^{-\nu_\ell}; \tag{1.2}$$

here

$$\begin{cases} 0 < \nu_1 < \nu_2 < \cdots < \nu_{\tilde N}, \\ 1 \leqslant N_j \leqslant \tilde N, \\ \lambda_{j,\ell} \in C, \\ \lambda_{j,N_j} \neq 0 . \end{cases} \tag{1.3}$$

2)

$$A_0 = \begin{bmatrix} \mu_1 & \delta_1 & 0 & \cdots & 0 & 0 \\ 0 & \mu_2 & \delta_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mu_{n-1} & \delta_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu_n \end{bmatrix}, \tag{1.4}$$

where

(i) the $\mu_j$ are complex numbers such that the differences $\mu_j - \mu_h$ are not equal to nonzero integers if $\lambda_j(x) = \lambda_h(x)$;

(ii) the $\delta_j$ are complex numbers such that $\lambda_j(x) + \mu_j = \lambda_{j+1}(x) + \mu_{j+1}$ if $\delta_j \neq 0$ ;

(iii) $\Re(\mu_j) < 0 \quad (j = 1, 2, \cdots, n)$.

3) The quantity $\mu$ is a positive real number such that $\mu + \Re(\mu_j - \mu_h) > 0$ $(j, h = 1, 2, \cdots, n)$ .

4) As in [11] $\mathcal{R}$ denotes the set of all $j$ such that $\lambda_j(x)$ is not identically equal to zero; i.e.

$$\mathcal{R} = \{j \ ; \ \lambda_j(x) \neq 0\}. \tag{1.5}$$

5) We set

$$\Lambda_j(x) = \begin{cases} 0 & \text{if } \ j \notin \mathcal{R}, \\ \displaystyle\sum_{\ell=1}^{N_j} \frac{-\lambda_{j,\ell}}{\nu_\ell} \, x^{-\nu_\ell} & \text{if } \ j \in \mathcal{R}, \end{cases} \tag{1.6}$$

and

$$\tau_j = \nu_{N_j} \qquad \text{for} \quad j \in \mathcal{R}. \tag{1.7}$$

6) Let

$$0 < k_p < k_{p-1} < \cdots < k_2 < k_1 < +\infty \tag{1.8}$$

be all of the distinct real numbers in the set $\{ \ \tau_j \ ; \ j \in \mathcal{R} \ \}$; i.e.

$$\{k_1, \cdots, k_p\} = \{\tau_j \ ; \ j \in \mathcal{R}\}.$$

7) We fix an integer $q$ such that $2 \leqslant q \leqslant p$ and set

$$k = k_q, \qquad k' = k_{q-1}. \tag{1.9}$$

8) We also set

$$\mathcal{R}_k = \{j \in \mathcal{R} \; ; \; \tau_j = k\}. \tag{1.10}$$

Throughout this paper, all sectorial domains are considered on the Riemann surface of $\log x$. Also, for an $m$-vector $\vec{y} = \begin{bmatrix} y_1 \\ \cdots \\ y_m \end{bmatrix}$, we define a norm $|\vec{y}|$ by

$$|\,\vec{y}\,| = \max_{1 \leqslant j \leqslant m} |y_j|. \tag{1.11}$$

Furthermore, for every $\wp = (p_1, \cdots, p_m)$ where the $p_\ell$ are nonnegative integers, we set

$$|\wp| = p_1 + p_2 + \cdots + p_m, \qquad \vec{y}^{\wp} = y_1^{p_1} y_2^{p_2} \cdots y_m^{p_m}. \tag{1.12}$$

## 2. Main problem.

As in [11], we consider a differential equation :

$$x \frac{d\vec{y}}{dx} = \vec{G}_0(x) + [\, \lambda(x) + A_0 \,] \, \vec{y} + x^{\mu} \, \vec{G}(x, \vec{y}), \tag{2.1}$$

where

(I) the $n$-vector $\vec{G}_0(x)$ is holomorphic in an open sector $\mathcal{D}(a, b, r_0) = \{x \; ; \; a < \arg x < b, \; 0 < |\, x \,| < r_0\}$;

(II) for every closed subsector $\mathcal{D}[\alpha, \beta, r] = \{x \; ; \; \alpha \leqslant \arg x \leqslant \beta, \; 0 < |\, x \,| \leqslant r\}$ of $\mathcal{D}(a, b, r_0)$, there exists a positive number $\rho(\alpha, \beta, r)$ such that the power series

$$\vec{G}(x, \vec{y}) = \sum_{|\wp| \geqslant 1} \vec{y}^{\wp} \vec{G}_{\wp}(x)$$

is uniformly convergent for

$$x \in \mathcal{D}[\alpha, \beta, r], \qquad |\vec{y}| \leqslant \rho(\alpha, \beta, r), \tag{2.2}$$

where the coefficients $\vec{G}_{\wp}$ are holomorphic and bounded in $\mathcal{D}(a, b, r_0)$.

We assume that the following conditions are satisfied :

(1) there exists a direction $\arg x = d$ such that

$$a < d - \frac{\pi}{2k} < d + \frac{\pi}{2k} < b; \tag{2.3}$$

(2) for each $j \in \mathcal{R}_k$, there exists a direction arg $x = d_j$ such that

$$d - \frac{\pi}{2k} < d_j < d + \frac{\pi}{2k} \tag{2.4}$$

and that $\Re[\Lambda_j(x)]$ changes its sign across the direction arg $x = d_j$ (cf. figure 1). This means that the direction arg $x = d$ is not singular on the level $k$.

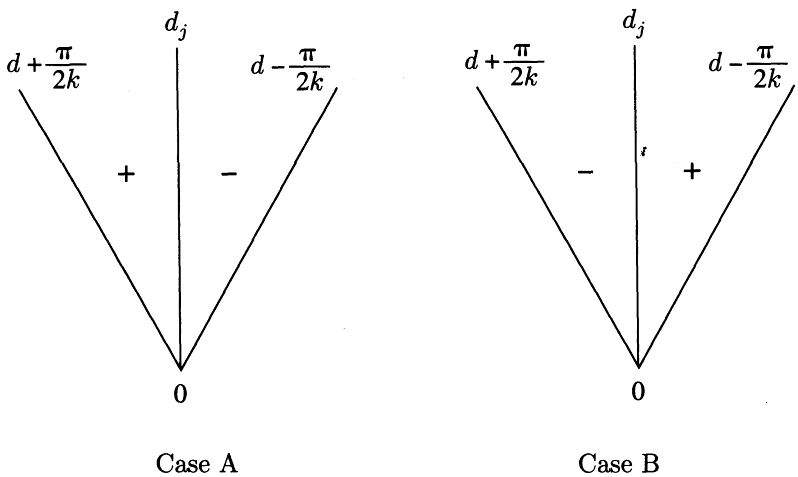


Fig. 1 : the sign of $\Re[\Lambda_j(x)]$.

We consider the following situation : for any positive number $r$, set

$$\mathcal{W}_0(r) = \{x \in C \; ; \; |\arg x - d| \leqslant \frac{\pi}{2k}, 0 < |x| < r\} \tag{2.5}$$

and let

$$\mathcal{U}_\nu(r) \;=\; \mathcal{D}(\alpha_\nu, \beta_\nu, r) \qquad (\nu = 1, 2, \cdots, N),$$

be a covering of $\mathcal{W}_0(r)$ , i.e.

$$\mathcal{W}_0(r) \subset \bigcup_{1\leqslant\nu\leqslant N} \mathcal{U}_\nu(r) \subset \mathcal{D}\left(d - \frac{\pi}{2k} - \epsilon_0, d + \frac{\pi}{2k} + \epsilon_0, r\right), \tag{2.6}$$

where $\epsilon_0$ is a sufficiently small positive number. Let us assume that the covering $\{\mathcal{U}_\nu(r)\ ;\ \nu = 1, 2, \cdots, N\}$ satisfies the following conditions :

(i) $0 < \beta_\nu - \alpha_\nu < \dfrac{\pi}{k_1} \qquad (\nu = 1, 2, \cdots, N)$;

(ii) $a < \alpha_1 < \alpha_2 < \cdots < \alpha_N$ and $\beta_1 < \beta_2 < \cdots < \beta_N < b$;

(iii)

$$\mathcal{U}_\nu(r) \cap \mathcal{U}_{\nu'}(r) \begin{cases} \neq \varnothing & \text{if } \mid \nu - \nu' \mid \leqslant 1, \\ = \varnothing & \text{if } \mid \nu - \nu' \mid \geqslant 2; \end{cases}$$

(iv) there exists a positive number $r_1$ and $N$ functions $\vec{f}_1(x), \cdots, \vec{f}_N(x)$ such that

(a) for each $\nu$, the function $\vec{f}_\nu$ is holomorphic in $\mathcal{U}_\nu(r_1)$;

(b) for each $\nu$, we have lim $\vec{f}_\nu(x) = \vec{0}$ as $x \to 0$ in $\mathcal{U}_\nu(r_1)$;

(c) for each $\nu \geqslant 2$, we have

$$\left| \vec{f}_\nu(x) - \vec{f}_{\nu-1}(x) \right| \leqslant K\, e^{-\epsilon_1 |\, x\, |^{-k}} \quad \text{in} \quad \mathcal{U}_\nu(r_1) \cap \mathcal{U}_{\nu-1}(r_1) \tag{2.7}$$

for some positive numbers $K$ and $\epsilon_1$;

(d) for each $\nu$, $\vec{f}_\nu$ is a solution of differential equation (2.1) in $\mathcal{U}_\nu(r_1)$, i.e.

(2.8)
$$x\,\frac{d\vec{f}_\nu(x)}{dx} = \vec{G}_0(x) + [\ \lambda(x) + A_0\ ]\ \vec{f}_\nu(x) + x^\mu\, \vec{G}(x, \vec{f}_\nu(x)) \quad \text{in} \quad \mathcal{U}_\nu(r_1).$$

We can choose the $\alpha_\nu$ and the $\beta_\nu$ so that, if $j \in \mathcal{R}$ and $\Re\,[\Lambda_j(x)] \leqslant 0$ on

$$\mathcal{W}_\nu(r) = \{x;\ \alpha_\nu \leqslant \arg\ x \leqslant \beta_\nu,\ 0 < |\ x\ | \leqslant r\} \tag{2.9}$$

for sufficiently small $r > 0$, then $\Re\,[\Lambda_j(x, \epsilon; \nu)] < -\delta$ on $\mathcal{W}_\nu(r)$ for some positive number $\delta$ and a sufficiently small $r > 0$ . In particular, we can choose the $\alpha_\nu$ and the $\beta_\nu$ so that

$$\alpha_\nu \neq d_j \qquad \text{and} \qquad \beta_\nu \neq d_j \qquad \text{for} \quad j \in \mathcal{R}_k\ , \quad \nu = 1, 2, \cdots, N.$$

Also we can assume that the directions $\arg x = d_j$ are not in $\mathcal{U}_\nu(r_1) \cap \mathcal{U}_{\nu-1}(r_1)$ for any $\nu$ $(\geqslant 2)$.

We can further assume that $\Re[\Lambda_j(x)] \neq \Re[\Lambda_{j'}(x)]$ on $\mathcal{U}_\nu(r) \cap \mathcal{U}_{\nu'}(r)$ if $\Lambda_j(x) \neq \Lambda_{j'}(x)$ on $\mathcal{U}_\nu(r) \cap \mathcal{U}_{\nu'}(r)$.

As in [11] the main purpose of this paper is to prove the following theorem :

THEOREM 2.1. — *We can modify the $N$ functions $f_1, \cdots, f_N$ by some quantities of $O\left(e^{-\epsilon| x |^{-k}}\right)$, where $\epsilon$ is some positive constant, so that these modified functions also satisfy conditions (a),(b) and (d) of (iv) given above and that moreover they satisfy the following condition (c') : (c') for each $\nu \geqslant 2$, we have*

$$\left|\vec{f}_\nu(x) - \vec{f}_{\nu-1}(x)\right| \leqslant K' e^{-\epsilon_2 | x |^{-k'}} \quad \text{in} \quad \mathcal{U}_\nu(r_2) \cap \mathcal{U}_{\nu-1}(r_2) \tag{2.10}$$

*for some positive numbers $K'$, $r_2$ and $\epsilon_2$.*

## 3. A formal solution by means of a formal normal form.

As in [11] we consider a differential equation :

$$x\frac{d\vec{u}}{dx} = [\lambda(x) + A_0]\vec{u} + x^\mu \vec{F}(x, \vec{u}), \qquad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix}, \tag{3.1}$$

where $\vec{F}$ is a power series in $u_1, \cdots, u_n$ :

$$\vec{F}(x, \vec{u}) = A(x)\vec{u} + \sum_{|\wp|\geqslant 2} \vec{u}^{\wp} \vec{F}_{\wp}(x), \tag{3.2}$$

satisfying the following conditions :

(i) $A(x)$ is an $n \times n$ matrix whose entries are holomorphic and bounded in a sectorial domain :

$$\mathcal{D}(\alpha, \beta, r) = \{ x \,;\, \alpha < \arg x < \beta, \ 0 < |x| < r \}, \tag{3.3}$$

(ii) the $\vec{F}_{\wp}(x)$ are $n$-vectors whose entries are holomorphic and bounded in $\mathcal{D}(\alpha, \beta, r)$,

(iii) the power series $\vec{F}$ is uniformly convergent for

$$x \in \mathcal{D}(\alpha,\beta,r), \qquad |\vec{u}| < 2\rho\,. \tag{3.4}$$

There exists a nonnegative number $L$ such that

$$\left|\vec{F}(x,\vec{u}) - \vec{F}(x,\vec{u}')\right| \leqslant L|\vec{u} - \vec{u}'| \tag{3.5}$$

for

$$x \in \mathcal{D}(\alpha,\beta,r), \qquad |\vec{u}| < \rho\,, \qquad |\vec{u}'| < \rho. \tag{3.6}$$

In particular,

$$\left|\vec{F}(x,\vec{u})\right| \leqslant L|\vec{u}| \tag{3.7}$$

for

$$x \in \mathcal{D}(\alpha,\beta,r), \qquad |\vec{u}| < \rho. \tag{3.8}$$

Let us also assume the following conditions :

(iv) $\beta - \alpha < \dfrac{\pi}{k_1}$;

(v) if, for some $j \in \mathcal{R}$, $\Re[\Lambda_j(x)] \leqslant 0$ in $\mathcal{D}(\alpha,\beta,r)$, then $\Re[\Lambda_j(x)] \leqslant -\delta$ in $\mathcal{D}(\alpha,\beta,r)$ for some positive number $\delta$;

(vi)

$$\mathcal{J}_k = \{j \in \mathcal{R}_k\,;\ \Re[\Lambda_j(x)] < 0 \ \text{ in } \mathcal{D}(\alpha,\beta,r)\} = \{1,\cdots,n_0\}. \tag{3.9}$$

Let us set :

(3.10)

$[\,\lambda(x) + A_0\,]_{n_0}$

$$= \begin{bmatrix} \lambda_1(x)+\mu_1 & \delta_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2(x)+\mu_2 & \delta_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_{n_0-1}(x)+\mu_{n_0-1} & \delta_{n_0-1} \\ 0 & 0 & 0 & \cdots & 0 & \lambda_{n_0}(x)+\mu_{n_0} \end{bmatrix}$$

and

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_{n_0} \end{bmatrix}.$$

Utilizing notations and assumptions given above, we shall state the following lemma which will be proved in §7.

LEMMA 3.1. — *For every $\wp = (p_1, \cdots, p_{n_0})$ where the $p_\ell$ are nonnegative integers such that $|\wp| = p_1 + \cdots + p_{n_0} \geqslant 2$, we can find an $n$-vector $\vec{P}_\wp(x)$ and an $n_0$-vector $\vec{\alpha}_\wp$ :*

$$\vec{P}_\wp(x) = \begin{bmatrix} P_{\wp,1}(x) \\ \cdots \\ P_{\wp,n}(x) \end{bmatrix}, \qquad \vec{\alpha}_\wp = \begin{bmatrix} \alpha_{\wp,1}(x) \\ \cdots \\ \alpha_{\wp,n_0}(x) \end{bmatrix}$$

*together with an $n \times n_0$ matrix $P_0(x)$ in such a way that*

(i) *the matrix $P_0(x)$ is holomorphic in $\mathcal{D}(\alpha, \beta, r)$ and*

$$|x|^{-\mu'} \; |P_0(x) \; - \; C| \qquad\qquad \left(C \; = \; \begin{bmatrix} I_{n_0} \\ O \end{bmatrix}\right)$$

*is bounded in $\mathcal{D}(\alpha, \beta, r)$ for some positive number $\mu'$ such that*

$$\mu' + \Re[\mu_j - \mu_h] > 0 \qquad \text{for} \quad (j, h = 1, \cdots, n),$$

*where $I_{n_0}$ is the $n_0 \times n_0$ identity matrix and $O$ is the $(n - n_0) \times n_0$ zero matrix;*

(ii) *the $\vec{P}_\wp(x)$ and $\vec{\alpha}_\wp(x)$ are holomorphic and bounded in $\mathcal{D}(\alpha, \beta, r)$;*

(iii) *$P_{\wp,j}(x) = 0$ if $\lambda_j(x) = \sum_{1 \leqslant \ell \leqslant n_0} p_\ell \lambda_\ell(x)$;*

(iv) *$\alpha_{\wp,j}(x) = 0$ if $\lambda_j(x) \neq \sum_{1 \leqslant \ell \leqslant n_0} p_\ell \lambda_\ell(x)$;*

(v) *the formal power series :*

$$\vec{P}(x, \vec{w}) \; = \; P_0(x)\vec{w} \; + \; x^\mu \sum_{|\wp| \geqslant 2} \vec{w}^\wp \vec{P}_\wp(x) \tag{3.11}$$

*is a formal solution of differential equation (3.1) if*

$$x\frac{d\vec{w}}{dx} \; = \; [\lambda(x) + A_0]_{n_0}\vec{w} \; + \; x^\mu \sum_{|\wp| \geqslant 2} \vec{w}^\wp \vec{\alpha}_\wp(x). \tag{3.12}$$

Note that power series (3.11) in $\vec{w}$ is a formal solution of differential equation (3.1) in the sense that the following condition is satisfied as power series in $\vec{w}$ (cf. (7.1)) :

$$x\frac{\partial\vec{P}(x,\vec{w})}{\partial x}+\frac{\partial\vec{P}(x,\vec{w})}{\partial\vec{w}}\left([\lambda(x)+A_0]_{n_0}\vec{w}+x^{\mu}\sum_{|\wp|\geqslant 2}\vec{w}^{\wp}\ \vec{\alpha}_{\wp}(x)\right)$$
$$=[\lambda(x)+A_0]\ \vec{P}(x,\vec{w})+x^{\mu}\vec{F}(x,\vec{P}(x,\vec{w})),$$

where $\dfrac{\partial\vec{P}(x,\vec{w})}{\partial\vec{w}}$ is an $n\times n_0$ matrix defined by

$$\frac{\partial\vec{P}(x,\vec{w})}{\partial\vec{w}}=\left[\frac{\partial\vec{P}(x,\vec{w})}{\partial w_1}\quad\cdots\quad\frac{\partial\vec{P}(x,\vec{w})}{\partial w_{n_0}}\right].$$

The left-hand side of this condition comes from a chain-rule by means of differential equation (3.12).

*Observation 3.2.* — In order to find the structure of formal solution (3.11) of Lemma 3.1, let us first look at differential equation (3.12). Since condition (iv) of Lemma 3.1 is satisfied, differential equation (3.12) becomes

$$x\frac{d\vec{v}}{dx}=[A_0]_{n_0}\vec{v}+x^{\mu}\sum_{|\wp|\geqslant 2}\vec{v}^{\wp}\vec{\alpha}_{\wp}(x),\tag{3.13}$$

if we set

$$w_j=e^{\Lambda_j(x)}\,v_j\qquad(j=1,\cdots,n_0),\tag{3.14}$$

where

$$[A_0]_{n_0}=\begin{bmatrix}\mu_1&\delta_1&0&\cdots&0&0\\0&\mu_2&\delta_2&\cdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&0&\cdots&\mu_{n_0-1}&\delta_{n_0-1}\\0&0&0&\cdots&0&\mu_{n_0}\end{bmatrix}.\tag{3.15}$$

Let $\vec{c}=\begin{bmatrix}c_1\\c_2\\\cdots\\c_{n_0}\end{bmatrix}$ be an arbitrary constant $n_0$-vector. If we arrange the $\Lambda_j(x)$ in such a way that

$$\Re\left[\Lambda_1(x)\right]\leqslant\Re\left[\Lambda_2(x)\right]\leqslant\cdots\leqslant\Re\left[\Lambda_{n_0}(x)\right]$$

in a direction $\arg x = \theta$ in $\mathcal{D}(\alpha, \beta, r)$, a general solution of (3.13) is given by

(3.16) $v_j = \psi_j(x; \vec{c}) = c_j x^{\mu_j} + \tilde{\psi}_j(x; c_{j+1}, ..., c_{n_0}) \qquad (j = 1, \cdots, n_0),$

where the $\tilde{\psi}_j$ are holomorphic in $\mathcal{D}(\alpha, \beta, r)$ and

(3.17) $$\tilde{\psi}_j = \sum_{|\wp| \geqslant 1} \vec{c}^{\wp}\, \tilde{\psi}_{j\wp}(x),$$

where

(3.18) $$\tilde{\psi}_{j\wp}(x) = 0 \qquad \text{if} \quad \lambda_j \neq \sum_{\ell=1}^{n_0} p_\ell \lambda_\ell$$

and

(3.19) $$|\tilde{\psi}_{j\wp}(x)| \leqslant \gamma_0 |x|^{\mu_0}$$

in $\mathcal{D}(\alpha, \beta, r)$ for some positive number $\gamma_0$ and some real number $\mu_0$. In fact, the $\psi_j$ are polynomials in $c_1, \cdots, c_{n_0}$.

Putting (3.14) and (3.16) together, we can find a general solution of (3.12), i.e.

(3.20)

$$\vec{w} = \vec{\phi}(x; \vec{c}) = \begin{bmatrix} \phi_1(x; \vec{c}) \\ \phi_2(x; \vec{c}) \\ \cdots \\ \phi_{n_0}(x; \vec{c}) \end{bmatrix}, \qquad \phi_j(x; \vec{c}) = e^{\Lambda_j(x)}\, \psi_j(x; \vec{c}) \quad (j = 1, \cdots, n_0).$$

By utilizing (3.18), we can write $\vec{\phi}(x; \vec{c})$ in the following form :

(3.21) $$\vec{\phi}(x; \vec{c}) = x^{[A_0]_{n_0}} e^{[\Lambda(x)]_{n_0}} \vec{c} + \sum_{|\wp| \geqslant 2} \left( e^{[\Lambda(x)]_{n_0}} \vec{c} \right)^{\wp} \vec{\phi}_{\wp}(x),$$

where

$$[\Lambda(x)]_{n_0} = \operatorname{diag}(\Lambda_1(x), \cdots, \Lambda_{n_0}(x))$$

and

(3.22) $$|\vec{\phi}_{\wp}(x)| \leqslant \gamma_0 |x|^{\mu_0}$$

for some positive number $\gamma_0$ and some real number $\mu_0$ in $\mathcal{D}(\alpha, \beta, r)$.

As an analytic justification of Lemma 3.1, we shall prove the following lemma :

LEMMA 3.3. — *For any positive integer $M$ and any positive number $\rho$, differential equation (3.1) admits a solution of the form :*

(3.23)
$$\vec{u}_M(x;\vec{c}) = P_0(x)\vec{\phi}(x;\vec{c}) + x^{\mu} \sum_{2 \leqslant |\wp| \leqslant M-1} \vec{\phi}(x;\vec{c})^{\wp} \vec{P}_{\wp}(x) + \vec{E}_M(x;\vec{c}),$$

*where*

(a) $\vec{E}_M(x;\vec{c})$ *is holomorphic in a domain :*

$$x \in \mathcal{D}(\alpha,\beta,r(M,\rho)) \quad \text{and} \quad |\vec{c}| \leqslant \rho \tag{3.24}$$

*where $r(M,\rho)$ is a suitable positive number depending on $M$ and $\rho$,*

(b) *we have*

$$|\vec{E}_M(x;\vec{c})| \leqslant K_M\ |\vec{c}|^M\ e^{-\delta_0 M|x|^{-k}} \tag{3.25}$$

*in domain (3.24) for some positive numbers $\delta_0$ and $K_M$. Furthermore $\delta_0$ is independent of $M$ and $\rho$.*

*Proof.* — Set

$$\vec{U}_M(x;\vec{c}) = P_0(x)\vec{\phi}(x;\vec{c}) + x^{\mu} \sum_{2 \leqslant |\wp| \leqslant M-1} \vec{\phi}(x;\vec{c})^{\wp} \vec{P}_{\wp}(x)\ .$$

Then, since

$$P_0(x)\vec{\phi}(x;\vec{c}) + x^{\mu} \sum_{|\wp| \geqslant 2} \vec{\phi}(x;\vec{c})^{\wp} \vec{P}_{\wp}(x)$$

is a formal solution of differential equation (3.1), we have

$$\left| x\frac{d\vec{U}_M(x;\vec{c})}{dx} - (\lambda(x) + A_0)\vec{U}_M(x;\vec{c}) - x^{\mu}\vec{F}(x,\vec{U}_M(x;\vec{c})) \right| \leqslant H_M\ |\vec{\phi}(x;\vec{c})|^M$$

for $x \in \mathcal{D}(\alpha,\beta,r_M)$ and $|\vec{\phi}(x;\vec{c})| \leqslant \rho_M$, with sufficiently small positive numbers $r_M$ and $\rho_M$, where $H_M$ is a suitable positive number. Note that

$$|\vec{\phi}(x;\vec{c})| \leqslant \tilde{H}\left(|\vec{c}||x|^{\mu_0}\ e^{-\delta|x|^{-k}}\right)$$

in domain (3.24), where $\delta$ is a suitable positive number and $\mu_0$ is a suitable real number. Therefore, changing (3.1) by the transformation $\vec{u} = \vec{U}_M(x;\vec{c}) + \vec{E}_M$ and utilizing the results of J.-P. Ramis and Y. Sibuya [10] on the differential equation for $\vec{E}_M$, we can prove Lemma 3.3. (See, in particular, §3.3 (pp.78–79) of [10].)

## 4. A normal form of a linear system.

The results in this section will be used in §5 where we shall consider the situation on the intersection of two sectors $\mathcal{D}(\alpha_1,\beta_1,r)$ and $\mathcal{D}(\alpha_2,\beta_2,r)$. Actually Lemmas 3.1 and 3.3 will be used in Step 2 of the proof of Lemma 5.1, whereas Lemma 4.1 will be used in Step 3 of the proof of Lemma 5.1. We shall finish the proof of Lemma 5.1 by utilizing Observations 4.2 and 4.3 in Steps 4 and 5. In this section the intersection of these two sectors is denoted by $\mathcal{D}(\alpha,\beta,r)$. The notation $n_0$ of this section and that of §3 are totally unrelated. Rather, $n_0$ of §3 corresponds to $n_1$ and $n_2$ of this section. Keeping these in mind, we start explaining a normal form of a linear system.

As in [11] let us assume that
$\{1,\cdots,n_1\} \in \mathcal{R}_k$ and $\{n_0+1,\cdots,n_0+n_2\} \in \mathcal{R}_k$ and that

$$\Re\,[\Lambda_j(x)] < 0 \qquad \text{in } \mathcal{D}(\alpha,\beta,r) \tag{4.1}$$

for $j = 1,\cdots,n_1$ and $j = n_0+1,\cdots,n_0+n_2$. Let

$$\vec{\phi}_1(x;\vec{c}_1) = \begin{bmatrix} \phi_1(x;\vec{c}_1) \\ \phi_2(x;\vec{c}_1) \\ \cdots \\ \phi_{n_1}(x;\vec{c}_1) \end{bmatrix}, \qquad \vec{\phi}_2(x;\vec{c}_2) = \begin{bmatrix} \tilde{\phi}_{n_0+1}(x;\vec{c}_2) \\ \tilde{\phi}_{n_0+2}(x;\vec{c}_2) \\ \cdots \\ \tilde{\phi}_{n_0+n_2}(x;\vec{c}_2) \end{bmatrix}$$

be general solutions of the following two differential equations :

$$x\frac{d\vec{w}_1}{dx} = [\ \lambda(x) + A_0\ ]_1\,\vec{w}_1 \ + \ x^{\mu} \sum_{|\wp|\geqslant 2} \vec{w}_1^{\wp}\vec{\alpha}_{1,\wp}(x)$$

and

$$x\frac{d\vec{w}_2}{dx} = [\ \lambda(x) + A_0\ ]_2\,\vec{w}_2 \ + \ x^{\mu} \sum_{|\wp|\geqslant 2} \vec{w}_2^{\wp}\vec{\alpha}_{2,\wp}(x),$$

respectively where

$$[\ \lambda(x)\ +\ A_0\ ]_1$$

$$= \begin{bmatrix} \lambda_1(x)+\mu_1 & \delta_1 & 0 & \cdots & 0 \\ 0 & \lambda_2(x)+\mu_2 & \delta_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \delta_{n_1-1} \\ 0 & 0 & 0 & \cdots & \lambda_{n_1}(x)+\mu_{n_1} \end{bmatrix}$$

and

$$[\ \lambda(x)\ +\ A_0\ ]_2$$

$$= \begin{bmatrix} \lambda_{n_0+1}(x)+\mu_{n_0+1} & \delta_{n_0+1} & 0 & \cdots & 0 & \\ 0 & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \delta_{n_0+n_2-1} & \\ 0 & 0 & 0 & \cdots & \lambda_{n_0+n_2}(x)+\mu_{n_0+n_2} & \end{bmatrix}.$$

Set

$$\vec{\alpha}_{1,\wp}(x) = \begin{bmatrix} \alpha_{1,\wp,1}(x) \\ \alpha_{1,\wp,2}(x) \\ \cdots \\ \alpha_{1,\wp,n_1}(x) \end{bmatrix}, \qquad \vec{\alpha}_{2,\wp}(x) = \begin{bmatrix} \alpha_{2,\wp,n_0+1}(x) \\ \alpha_{2,\wp,n_0+2}(x) \\ \cdots \\ \alpha_{2,\wp,n_0+n_2}(x) \end{bmatrix}.$$

We assume that

$$(4.2)\qquad \begin{cases} \alpha_{1,\wp_1,j}(x) \ = \ 0 & \text{if} \quad \lambda_j \neq \sum\limits_{1\leqslant \ell\leqslant n_1} p_{1,\ell}\lambda_\ell, \\ \alpha_{2,\wp_2,n_0+j}(x) = 0 & \text{if} \quad \lambda_{n_0+j} \neq \sum\limits_{1\leqslant \ell\leqslant n_2} p_{2,n_0+\ell}\lambda_{n_0+\ell} \end{cases}$$

where $\wp_1 = (p_{1,1},\cdots,p_{1,n_1})$ and $\wp_2 = (p_{2,n_0+1},\cdots,p_{2,n_0+n_2})$.

The following lemma will be proved in §8.

LEMMA 4.1. — *Let an $n\times n$ matrix*

$$(4.3)\qquad A(x,\vec{\phi}_1,\vec{\phi}_2) \ = \sum_{|\wp_1|+|\wp_2|\geqslant 0} \vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2} A_{\wp_1\wp_2}(x)$$

*be given as a power series in $\vec{\phi}_1$ and $\vec{\phi}_2$ , where the $A_{\wp_1\wp_2}(x)$ are holomorphic and bounded in $\mathcal{D}(\alpha,\beta,r)$. Then for every $\wp_1 = (p_{1,1},\cdots,p_{1,n_1})$ and $\wp_2 = (p_{2,n_0+1},\cdots,p_{2,n_0+n_2})$, where $|\wp_1| + |\wp_2| \geqslant 1$, we can find $n\times n$ matrices $\Phi_{\wp_1\wp_2}(x)$ and $B_{\wp_1\ w p_2}(x)$ together with another $n\times n$ matrix $\Phi_0(x)$ in such a way that*

(i) *$\Phi_{\wp_1\wp_2}(x)$ , $B_{\wp_1\wp_2}(x)$ and $\Phi_0(x)$ are holomorphic and bounded in $\mathcal{D}(\alpha,\beta,r)$*

(ii) *if we denote by $\Phi_{\wp_1\wp_2,jj'}(x)$ and $B_{\wp_1\wp_2,jj'}(x)$ the $(j,j')$-th entries of $\Phi_{\wp_1\wp_2}$ and $B_{\wp_1\wp_2}$ respectively, then*

(4.4)
$$\begin{cases} \Phi_{\wp_1\wp_2,jj'}(x) = 0 \quad \textit{if} \quad \lambda_j - \lambda_{j'} = \sum\limits_{1\leqslant\ell\leqslant n_1} p_{1,\ell}\lambda_\ell + \sum\limits_{1\leqslant\ell\leqslant n_2} p_{2,n_0+\ell}\lambda_{n_0+\ell}, \\ \\ B_{\wp_1\wp_2,jj'}(x) = 0 \quad \textit{if} \quad \lambda_j - \lambda_{j'} \neq \sum\limits_{1\leqslant\ell\leqslant n_1} p_{1,\ell}\lambda_\ell + \sum\limits_{1\leqslant\ell\leqslant n_2} p_{2,n_0+\ell}\lambda_{n_0+\ell} \ ; \end{cases}$$

(iii) *there is a positive number $\mu'$ such that $\mu' + \Re[\mu_j - \mu_{j'}] > 0$ for $j,j' = 1,\cdots,n$ and that, if we put*

$$(4.5) \qquad \begin{cases} \Phi(x,\vec{\phi}_1,\vec{\phi}_2) = I + x^{\mu'}\Phi_0 + x^{\mu} \sum\limits_{|\wp_1|+|\wp_2|\geqslant 1} \vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}\Phi_{\wp_1\wp_2}, \\ \\ B(x,\vec{\phi}_1,\vec{\phi}_2) = \sum\limits_{|\wp_1|+|\wp_2|\geqslant 1} \vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2} B_{\wp_1\wp_2}, \end{cases}$$

*we have*

$$(4.6) \qquad x\frac{d\Phi}{dx} = [\ \lambda(x) + A_0 + x^{\mu}B]\,\Phi \ - \ \Phi\,[\ \lambda(x) + A_0 + x^{\mu}A]$$

*as formal power series in $\vec{\phi}_1$ and $\vec{\phi}_2$.*

*Observation 4.2.* — We can construct a fundamental matrix $\Psi(x,\vec{c}_1,\vec{c}_2)$ of the system :

$$(4.7) \qquad x\frac{d\vec{V}}{dx} \ = \ -\vec{V}\,[\ \lambda(x) + A_0 + x^{\mu}B\ ],$$

where $\vec{V}$ is an $n$-dimensional row vector, in such a way that

$$e^{\Lambda(x)}\Psi(x,\vec{c}_1,\vec{c}_2) = \sum_{|\wp_1|+|\wp_2|\geqslant 0} \left(e^{[\Lambda]_1}\vec{c}_1\right)^{\wp_1} \left(e^{[\Lambda]_2}\vec{c}_2\right)^{\wp_2} \Psi_{\wp_1\wp_2}(x)$$

where

$$\begin{cases} \Lambda(x) &= \operatorname{diag}(\Lambda_1(x),\cdots,\Lambda_n(x))\,, \\ [\Lambda(x)]_1 &= \operatorname{diag}(\Lambda_1(x),\cdots,\Lambda_{n_1}(x))\,, \\ [\Lambda(x)]_2 &= \operatorname{diag}(\Lambda_{n_0+1}(x),\cdots,\Lambda_{n_0+n_2}(x)), \end{cases} \tag{4.8}$$

and the $\Psi_{\wp_1\wp_2}(x)$ are holomorphic in $\mathcal{D}(\alpha,\beta,r)$ ; furthermore

$$|\Psi_{\wp_1\wp_2}(x)| \leqslant \gamma_{\wp_1\wp_2}\,|x|^{\mu_{\wp_1\wp_2}} \tag{4.9}$$

in $\mathcal{D}(\alpha,\beta,r)$ for some positive numbers $\gamma_{\wp_1\wp_2}$ and some real numbers $\mu_{\wp_1\wp_2}$.

*Observation 4.3.* — More precisely speaking, if we set

$$\vec{V} = \vec{W}x^{-A_0}e^{-\Lambda(x)}, \tag{4.10}$$

differential equation (4.7) changes to

$$x\frac{d\vec{W}}{dx} = -\vec{W}\,x^{\mu}x^{-A_0}\tilde{B}x^{A_0}, \tag{4.11}$$

where

$$\tilde{B} = e^{-\Lambda(x)}Be^{\Lambda(x)}. \tag{4.12}$$

Since the matrix $B$ satisfies condition (4.4), we have

$$|\tilde{B}| \leqslant \gamma_0\,|x|^{\mu_0} \tag{4.13}$$

in $\mathcal{D}(\alpha,\beta,r)$ for some positive number $\gamma_0$ and some real number $\mu_0$. Furthermore if we assume that

$$\Re[\Lambda_1(x)] \leqslant \Re[\Lambda_2(x)] \leqslant \cdots \leqslant \Re[\Lambda_n(x)], \tag{4.14}$$

and if we denote by $\tilde{B}_{jj'}$ the $(j,j')$-th entry of $\tilde{B}$, we have

$$\tilde{B}_{jj'} = 0 \qquad \text{if} \qquad \Re[\Lambda_j(x)] \geqslant \Re[\Lambda_{j'}(x)]. \tag{4.15}$$

The matrix $\Psi(x,\vec{c}_1,\vec{c}_2)$ has the following form :

(4.16)

$$\Psi(x,\vec{c}_1,\vec{c}_2) = \left[I + \int^x \xi^{\mu}\xi^{-A_0}\tilde{B}(\xi,\vec{c}_1,\vec{c}_2)\xi^{A_0}d\xi + \cdots\right]x^{-A_0}e^{-\Lambda(x)},$$

where $\cdots$ denotes terms of $O(|\vec{c}_1|^2+|\vec{c}_2|^2)$.

## 5. Stokes multipliers.

We shall utilize the same notations as in §2, and consider the following situation : let

$$\mathcal{U}_\nu(r) \;=\; \mathcal{D}(\alpha_\nu, \beta_\nu, r) \qquad (\nu = 1,2)$$

be two subsectors of $\mathcal{D}(a, b, r_0)$ of §2 which satisfy the following conditions :

(i) $0 \;<\; \beta_\nu \;-\; \alpha_\nu \;<\; \dfrac{\pi}{k_1} \qquad (\nu = 1, 2)$;

(ii) $\mathcal{U}_1(r) \cap \mathcal{U}_2(r) \neq \varnothing$;

(iii) there exist a positive number $r_1$ and two functions $\vec{f}_1(x)$ and $\vec{f}_2(x)$ such that

(a) for each $\nu$, the function $\vec{f}_\nu$ is holomorphic in $\mathcal{U}_\nu(r_1)$;

(b) for each $\nu$, we have $\lim\ \vec{f}_\nu(x) \;=\; \vec{0}$ as $x \to 0$ in $\mathcal{U}_\nu(r_1)$;

(c) we have

$$(5.1) \qquad \left|\vec{f}_2(x) \;-\; \vec{f}_1(x)\right| \;\leqslant\; K\, e^{-\epsilon_1 |\, x\, |^{-k}} \qquad \text{in} \quad \mathcal{U}_1(r_1) \cap \mathcal{U}_2(r_1)$$

for some positive numbers $K$ and $\epsilon_1$;

(d) for each $\nu$, $\vec{f}_\nu$ is a solution of differential equation (2.1) in $\mathcal{U}_\nu(r_1)$, i.e.

(5.2)
$$x\, \frac{d\vec{f}_\nu(x)}{dx} \;=\; \vec{G}_0(x) \;+\; [\lambda(x) \;+\; A_0]\ \vec{f}_\nu(x) \;+\; x^\mu\, \vec{G}(x, \vec{f}_\nu(x)) \quad \text{in} \quad \mathcal{U}_\nu(r_1).$$

As in [11] we want to prove the following basic lemma :

LEMMA 5.1. — *We can modify the two functions $f_1$ and $f_2$ by some quantities of $O\left(e^{-\epsilon |\, x\, |^{-k}}\right)$, where $\epsilon$ is some positive constant, so that these modified functions also satisfy conditions (a),(b) and (d) of (iii) given above and that moreover they satisfy the following condition (c') :*

*(c') we have*

$$(5.3) \qquad \left|\vec{f}_1(x) \;-\; \vec{f}_2(x)\right| \;\leqslant\; K'\, e^{-\epsilon_2 |\, x\, |^{-k'}} \qquad \textit{in} \quad \mathcal{U}_1(r_2) \cap \mathcal{U}_2(r_2)$$

*for some positive numbers $K'$, $r_2$ and $\epsilon_2$.*

*Proof.*

Step 1 : For each $\nu = 1, 2$, changing differential equation (2.1) by

$$\vec{y} = \vec{f}_\nu(x) + \vec{u}_\nu, \tag{5.4}$$

we derive another differential equation :

$$x\frac{d\vec{u}_\nu}{dx} = [\lambda(x) + A_0]\,\vec{u}_\nu + x^\mu\,\vec{F}_\nu(x, \vec{u}_\nu) \qquad \text{in} \quad \mathcal{U}_\nu(r_1), \tag{5.5-$\nu$}$$

respectively, where

$$\vec{F}_\nu(x, \vec{u}) = \vec{G}(x, \vec{f}_\nu(x) + \vec{u}) - \vec{G}(x, \vec{f}_\nu(x)) = \sum_{|\wp| \geqslant 1} \vec{u}^{\wp}\,\vec{F}_{\nu,\wp}(x)$$

is a power series which is uniformly convergent in the domain :

$$x \in \mathcal{U}_\nu(r_1), \qquad |\vec{u}| < \rho_0$$

for some positive number $\rho_0$ and the coefficients $\vec{F}_{\nu,\wp}(x)$ are holomorphic and bounded in $\mathcal{U}_\nu(r_1)$.

Step 2 : We shall apply Lemmas 3.1 and 3.3 to each of two differential equations (5.5.$\nu$). To do this, let us assume that

(5.6)

$$\begin{cases} \mathcal{J}_k(1) = \{j \in \mathcal{R}_k;\ \Re[\Lambda_j(x)] < 0 \quad \text{in } \mathcal{U}_1(r_1)\} = \{1, \cdots, n_1\}, \\ \mathcal{J}_k(2) = \{j \in \mathcal{R}_k;\ \Re[\Lambda_j(x)] < 0 \quad \text{in } \mathcal{U}_2(r_1)\} = \{n_0 + 1, \cdots, n_0 + n_2\}. \end{cases}$$

Then we can construct two differential equations :

$$\begin{cases} x\dfrac{d\vec{w}_1}{dx} = [\lambda(x) + A_0]_1\,\vec{w}_1 + x^\mu \displaystyle\sum_{|\wp| \geqslant 2} \vec{w}_1^{\wp}\,\vec{\alpha}_{1,\wp}(x), \\ \\ x\dfrac{d\vec{w}_2}{dx} = [\lambda(x) + A_0]_2\,\vec{w}_2 + x^\mu \displaystyle\sum_{|\wp| \geqslant 2} \vec{w}_2^{\wp}\,\vec{\alpha}_{2,\wp}(x), \end{cases} \tag{5.7}$$

respectively, where

(i) $\vec{w}_1 = \begin{bmatrix} w_{1,1} \\ \cdots \\ w_{1,n_1} \end{bmatrix}$, and $\vec{w}_2 = \begin{bmatrix} w_{2,n_0+1} \\ \cdots \\ w_{2,n_0+n_2} \end{bmatrix}$;

(ii) the two matrices $[\lambda(x) + A_0]_1$ and $[\lambda(x) + A_0]_2$ are the same as in §4;

(iii) if we set

$$\vec{\alpha}_{1,\wp}(x) = \begin{bmatrix} \alpha_{1,\wp,1}(x) \\ \alpha_{1,\wp,2}(x) \\ \cdots \\ \alpha_{1,\wp,n_1}(x) \end{bmatrix}, \qquad \vec{\alpha}_{2,\wp}(x) = \begin{bmatrix} \alpha_{2,\wp,n_0+1}(x) \\ \alpha_{2,\wp,n_0+2}(x) \\ \cdots \\ \alpha_{2,\wp,n_0+n_2}(x) \end{bmatrix}.$$

then

$$(5.8) \qquad \begin{cases} \alpha_{1,\wp_1,j}(x) \;=\; 0 & \text{if } \lambda_j \neq \sum\limits_{1\leqslant \ell\leqslant n_1} p_{1,\ell}\lambda_\ell, \\ \alpha_{2,\wp_2,n_0+j}(x) = 0 & \text{if } \lambda_{n_0+j} \neq \sum\limits_{1\leqslant \ell\leqslant n_2} p_{2,n_0+\ell}\lambda_{n_0+\ell}, \end{cases}$$

where $\wp_1 = (p_{1,1}, \cdots, p_{1,n_1})$ and $\wp_2 = (p_{2,n_0+1}, \cdots, p_{2,n_0+n_2})$ . Let

$$\vec{\phi}_1(x;\vec{c}_1) \;=\; \begin{bmatrix} \phi_1(x;\vec{c}_1) \\ \phi_2(x;\vec{c}_1) \\ \cdots \\ \phi_{n_1}(x;\vec{c}_1) \end{bmatrix}, \qquad \vec{\phi}_2(x;\vec{c}_2) \;=\; \begin{bmatrix} \tilde{\phi}_{n_0+1}(x;\vec{c}_2) \\ \tilde{\phi}_{n_0+2}(x;\vec{c}_2) \\ \cdots \\ \tilde{\phi}_{n_0+n_2}(x;\vec{c}_2) \end{bmatrix}$$

be general solutions of differential equations (5.7) respectively as constructed in §3.

The most important meaning of differential equations (5.7) is that there exist two formal power series

$$(5.9) \qquad \vec{P}_\nu(x,\vec{w}_\nu) \;=\; P_{\nu,0}(x)\vec{w}_\nu \;+\; x^\mu \sum_{|\wp|\geqslant 2} \vec{w}_\nu^{\,\wp}\vec{P}_{\nu,\wp}(x)$$

such that

(1) the $\vec{P}_{\nu,\wp}(x)$ are $n$–vectors and the $P_{\nu,0}(x)$ are $n \times n_\nu$ matrices respectively;

(2) the entries of $\vec{P}_{\nu\wp}(x)$ and $P_{\nu,0}(x)$ are holomorphic and bounded in $\mathcal{U}_\nu(r_1)$ respectively;

(3) moreover the quantities

$$|x|^{-\mu'}\;|P_{\nu,0}(x) \;-\; C_\nu|$$

are bounded in $\mathcal{U}_\nu(r_1)$ respectively for some positive number $\mu'$ such that

$$\mu' \;+\; \Re[\mu_j \;-\; \mu_h] \;>\; 0 \qquad \text{for} \quad j,h = 1,2,\cdots,n,$$

where

$$C_1 = \begin{bmatrix} I_{n_1} \\ O \end{bmatrix}, \qquad C_2 = \begin{bmatrix} O \\ I_{n_2} \\ O \end{bmatrix};$$

here $I_{n_\nu}$ is the $n_\nu \times n_\nu$ identity matrix and the $O$ are the zero matrices of suitable sizes;

(4) if we set

$$\vec{P}_{\nu,\wp}(x) = \begin{bmatrix} P_{\nu,\wp,1}(x) \\ \cdots \\ P_{\nu,\wp,n}(x) \end{bmatrix},$$

then we have

$$(5.10) \qquad \begin{cases} P_{1,\wp,j}(x) = 0 & \text{if } \lambda_j = \sum_{\ell=1}^{n_1} p_{1,\ell}\lambda_\ell, \\ P_{2,\wp,n_0+j}(x) = 0 & \text{if } \lambda_{n_0+j} = \sum_{\ell=1}^{n_2} p_{2,n_0+\ell}\lambda_{n_0+\ell}; \end{cases}$$

(5) the formal power series

$$\vec{P}_\nu(x, \vec{\phi}_\nu(x, \vec{c}_\nu)) = P_{\nu,0}(x)\vec{\phi}_\nu(x, \vec{c}_\nu) + x^\mu \sum_{|\wp|\geqslant 2} \vec{\phi}_\nu(x, \vec{c}_\nu)^\wp \vec{P}_{\nu,\wp}(x)$$

are formal solutions of differential equations (5.5.$\nu$) respectively.

Set

$$(5.11) \quad \vec{U}_{M,\nu}(x, \vec{c}_\nu) = P_{\nu,0}(x)\vec{\phi}_\nu(x, \vec{c}_\nu) + x^\mu \sum_{2\leqslant|\wp|\leqslant M-1} \vec{\phi}_\nu(x, \vec{c}_\nu)^\wp \vec{P}_{v,\wp}(x).$$

Then each of differential equations (5.5-$\nu$) admits a solution of the form :

$$\vec{u}_{M,\nu}(x, \vec{c}_\nu) = \vec{U}_{M,\nu}(x, \vec{c}_\nu) + \vec{E}_{M,\nu}(x, \vec{c}_\nu)$$

such that

(a) $\vec{E}_{M,\nu}(x; \vec{c}_\nu)$ is holomorphic in a domain :

$$(5.12\text{-}\nu) \qquad x \in \mathcal{U}_\nu(r_\nu(M, \rho)) \quad \text{and} \quad |\vec{c}| \leqslant \rho$$

where $r_\nu(M, \rho)$ is a suitable positive number depending on $M$ and $\rho$,

(b) we have

$$(5.13)\qquad |\vec{E}_{M,\nu}(x;\vec{c}_\nu)| \leqslant K_M\ |\vec{c}|^M\ e^{-\delta_0 M|x|^{-k}}$$

in domain (5.12-$\nu$) for some positive numbers $\delta_0$ and $K_M$; furthermore $\delta_0$ is independent of $M$ and $\rho$.

Now we modify the two solutions $\vec{f}_1$ and $\vec{f}_2$ by

$$(5.14\text{-}\nu)\qquad \vec{\psi}_{M,\nu}(x,\vec{c}_\nu) = \vec{f}_\nu(x) + \vec{u}_{M,\nu}(x,\vec{c}_\nu),$$

respectively.

Step 3 : If we set

$$(5.15)\qquad \vec{Y}_M = \vec{\psi}_{M,2}(x,\vec{c}_2) - \vec{\psi}_{M,1}(x,\vec{c}_1)$$

in the domain

$$(5.16)\qquad x \in \mathcal{U}_1(r_1(M,\rho)) \cap \mathcal{U}_2(r_2(M,\rho)) \qquad \text{and} \qquad |\vec{c}| \leqslant \rho$$

$\vec{Y}_M$ satisfies a linear homogeneous system :

$$(5.17)\qquad x\frac{d\vec{Y}_M}{dx} = [\lambda(x) + A_0 + x^\mu H_M(x,\vec{c}_1,\vec{c}_2)]\ \vec{Y}_M,$$

where

$$\begin{aligned} H_M(x,\vec{c}_1,\vec{c}_2) &= \int_0^1 \frac{\partial\vec{G}}{\partial\vec{y}}(x, t\vec{\psi}_{M,2} + (1-t)\vec{\psi}_{M,1})\ dt \\ &= \mathcal{G}_0(x) + \sum_{|\wp_1|+|\wp_2|\geqslant 1} \vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}\ \mathcal{G}_{\wp_1,\wp_2}(x) + W_M(x,\vec{c}_1,\vec{c}_2); \end{aligned}$$

here

(i) $\mathcal{G}_0(x)$ is holomorphic and bounded on $\mathcal{U}_1(r_M) \cap \mathcal{U}_2(r_M)$ for some positive number $r_M$;

(ii) the $\mathcal{G}_{\wp_1,\wp_2}(x)$ are holomorphic and bounded on $\mathcal{U}_1(r_M) \cap \mathcal{U}_2(r_M)$ and the series on the right-hand side of (5.17) is uniformly convergent for

$$(5.18)\quad x \in \mathcal{U}_1(r_M) \cap \mathcal{U}_2(r_M), \qquad |\vec{\phi}_1(x;\vec{c}_1)| < \rho_M, \qquad |\vec{\phi}_2(x;\vec{c}_2)| < \rho_M$$

for some $\rho_M > 0$;

(iii) $W_M(x, \vec{c}_1, \vec{c}_2)$ is holomorphic in $(x, \vec{c}_1, \vec{c}_2)$ and

$$|W_M(x, \vec{c}_1, \vec{c}_2)| \leqslant K_M \ |\vec{c}|^M \ e^{-\delta_0 M|x|^{-k}} \tag{5.19}$$

in domain (5.16) for some positive numbers $\delta_0$ and $K_M$ ; furthermore $\delta_0$ is independent of $M$ and $\rho$.

Set

$$\mathcal{G}_1(x, \vec{c}_1, \vec{c}_2) = \sum_{|\wp_1|+|\wp_2| \geqslant 1} \vec{\phi}_1^{\wp_1} \vec{\phi}_2^{\wp_2} \ \mathcal{G}_{\wp_1, \wp_2}(x).$$

Let us apply Lemma 4.1 to the differential equation :

$$x \frac{d\Phi}{dx} = [\lambda(x) + A_0 + x^\mu B]\Phi - \Phi[\lambda(x) + A_0 + x^\mu(\mathcal{G}_0 + \mathcal{G}_1)]. \tag{5.20}$$

Then we can construct two $n \times n$ matrices $\Phi$ and $B$ of the form (4.5) which satisfy conditions (i) and (ii) of Lemma 4.1 (in particular (4.4)) and equation (5.20). Note that the matrix $\Phi$ is a <u>formal</u> power series. However by utilizing an idea similar to the proof of Lemma 3.3, for any given sufficiently large positive integer $M$ and any positive number $\rho$, we can construct an actual solution $\Phi_M(x, \vec{c}_1, \vec{c}_2)$ of the differential equation

$$x \frac{d\Phi_M}{dx} = [\lambda(x) + A_0 + x^\mu B]\Phi_M - \Phi_M[\lambda(x) + A_0 + x^\mu H_M(x, \vec{c}_1, \vec{c}_2)] \tag{5.21}$$

of the form :

$$\Phi_M(x, \vec{c}_1, \vec{c}_2) = I + x^{\mu'}\Phi_0 + x^\mu \sum_{1 \leqslant |\wp_1|+|\wp_2| \leqslant M-1} \vec{\phi}_1^{\wp_1} \vec{\phi}_2^{\wp_2} \Phi_{\wp_1 \wp_2} + V_M(x, \vec{c}_1, \vec{c}_2)$$

such that $V_M(x, \vec{c}_1, \vec{c}_2)$ is holomorphic in $(x, \vec{c}_1, \vec{c}_2)$ and

$$|\vec{V}_M(x, \vec{c}_1, \vec{c}_2)| \leqslant K_M \ |\vec{c}|^M \ e^{-\delta_0 M|x|^{-k}} \tag{5.22}$$

in domain (5.16) for some positive numbers $\delta_0$ and $K_M$ ; furthermore $\delta_0$ is independent of $M$ and $\rho$.

This means that, if we set

$$\vec{Z} = \Phi_M(x, \vec{c}_1, \vec{c}_2)\vec{Y}_M(x, \vec{c}_1, \vec{c}_2),$$

then $\vec{Z}$ satisfies the linear homogeneous system :

$$x \frac{d\vec{Z}}{dx} = [\lambda(x) + A_0 + x^\mu B] \, \vec{Z}.$$

Therefore, if we utilize the fundamental matrix $\Psi(x, \vec{c}_1, \vec{c}_2)$ of system (4.7), i.e.

$$x\frac{d\vec{V}}{dx} = -\vec{V}\left[\lambda(x) + A_0 + x^{\mu}B\right], \tag{4.7}$$

the vector $\vec{\Gamma}_M(\vec{c}_1, \vec{c}_2)$ defined by

(5.23)

$$\vec{\Gamma}_M(\vec{c}_1, \vec{c}_2) = \Psi(x, \vec{c}_1, \vec{c}_2)\vec{Z} = \Psi(x, \vec{c}_1, \vec{c}_2)\Phi_M(x, \vec{c}_1, \vec{c}_2)\vec{Y}_M(x, \vec{c}_1, \vec{c}_2)$$

is independent of $x$. Furthermore

$$e^{\Lambda(x)}\vec{\Gamma}_M(\vec{c}_1, \vec{c}_2) = e^{\Lambda(x)}\Psi(x, \vec{c}_1, \vec{c}_2)\Phi_M(x, \vec{c}_1, \vec{c}_2)\vec{Y}_M(x, \vec{c}_1, \vec{c}_2)$$

can be written in the following form :

(5.24)

$$e^{\Lambda(x)}\vec{\Gamma}_M(\vec{c}_1, \vec{c}_2) = \sum_{|\wp_1|+|\wp_2|\geqslant 0} \left(e^{[\Lambda]_1}\vec{c}_1\right)^{\wp_1}\left(e^{[\Lambda]_2}\vec{c}_2\right)^{\wp_2}\vec{\Gamma}_{\wp_1\wp_2}(x) + \vec{\mathcal{E}}_M(x, \vec{c}_1, \vec{c}_2),$$

where

(i')

$$\begin{cases} \Lambda(x) = \operatorname{diag}\left(\Lambda_1(x), \cdots, \Lambda_n(x)\right), \\ \left[\Lambda(x)\right]_1 = \operatorname{diag}\left(\Lambda_1(x), \cdots, \Lambda_{n_1}(x)\right), \\ \left[\Lambda(x)\right]_2 = \operatorname{diag}\left(\Lambda_{n_0+1}(x), \cdots, \Lambda_{n_0+n_2}(x)\right), \end{cases}$$

(ii') the $\vec{\Gamma}_{\wp_1\wp_2}(x)$ are holomorphic and

$$|\vec{\Gamma}_{\wp_1\wp_2}(x)| \leqslant \gamma_{\wp_1\wp_2}\,|x|^{\mu_{\wp_1\wp_1}}$$

in $\mathcal{U}_2(r_M)\cap\mathcal{U}_1(r_M)$ for some positive numbers $\gamma_{\wp_1\wp_2}$ and some real numbers $\mu_{\wp_1\wp_2}$,

(iii') $\vec{\mathcal{E}}_M(x, \vec{c}_1, \vec{c}_2)$ is holomorphic in $(x, \vec{c}_1, \vec{c}_2)$ and

$$|\vec{\mathcal{E}}_M(x, \vec{c}_1, \vec{c}_2)| \leqslant K_M\,|\vec{c}|^M\,e^{-\delta_0 M|x|^{-k}} \tag{5.25}$$

in domain (5.16) for some positive numbers $\delta_0$ and $K_M$ ; furthermore $\delta_0$ is independent of $M$ and $\rho$.

*Remark.* — If we set $\vec{c}_1 = \vec{0}$ and $\vec{c}_2 = \vec{0}$ in (5.23), we get

$$\vec{\Gamma}_M(\vec{0}, \vec{0}) = x^{-A_0}e^{-\Lambda(x)}\left[I + x^{\mu'}\Phi_0(x)\right](\vec{f}_2(x) - \vec{f}_1(x)). \tag{5.26}$$

Step 4 : Suppose that $M$ is sufficiently large. Then letting $x$ tend to zero in $\mathcal{U}_2(r_M) \cap \mathcal{U}_1(r_M)$ we can compute $\vec{\Gamma}_M(\vec{c}_1, \vec{c}_2)$. To start calculation, let us denote by $\Gamma_j(\vec{c}_1, \vec{c}_2)$ the $j$-th entry of $\vec{\Gamma}_M$, i.e.

$$\vec{\Gamma}_M(\vec{c}_1, \vec{c}_2) = \begin{bmatrix} \Gamma_1(\vec{c}_1, \vec{c}_2) \\ \Gamma_2(\vec{c}_1, \vec{c}_2) \\ \cdots \\ \Gamma_n(\vec{c}_1, \vec{c}_2) \end{bmatrix},$$

and look at (5.24) or

$$\text{(5.24-}j\text{)} \qquad \vec{\Gamma}_j(\vec{c}_1, \vec{c}_2) =$$

$$\sum_{|\wp_1|+|\wp_2|\geqslant 0} e^{-\Lambda_j(x)} \left(e^{[\Lambda]_1}\vec{c}_1\right)^{\wp_1} \left(e^{[\Lambda]_2}\vec{c}_2\right)^{\wp_2} [\vec{\Gamma}_{\wp_1\wp_2}(x)]_j + e^{-\Lambda_j(x)}[\vec{\mathcal{E}}_M(x, \vec{c}_1, \vec{c}_2)]_j$$

where $[\vec{v}]_j$ denotes the $j$-th entry of a vector $\vec{v}$. Note also that we have (5.26). From (5.24-$j$) we can derive immediately

$$\text{(5.27)} \qquad \Gamma_j(\vec{c}_1, \vec{c}_2) = 0 \qquad \text{if} \quad \tau_j < k.$$

If $\tau_j = k$ and $j \notin \mathcal{J}_k(1) \cup \mathcal{J}_k(2)$, then $\Re[\Lambda_j(x)] > 0$ on $\mathcal{U}(r_M) \cap \mathcal{U}_2(r_M)$. Here we use assumption (2) of §2 (cf. figure 1 of §2) and assumption that the directions $\arg x = d_j$ are not in $\mathcal{U}_\nu(r_1) \cap \mathcal{U}_{\nu-1}(r_1)$ for any $\nu$ ($\nu \geqslant 2$). Hence we have

$$\text{(5.28)} \qquad \Gamma_j(\vec{c}_1, \vec{c}_2) = 0 \qquad \text{if} \quad \tau_j = k \quad \text{and} \quad j \notin \mathcal{J}_k(1) \cup \mathcal{J}_k(2).$$

Let us look at $\Gamma_j$ for $j$ such that $\tau_j = k$ and that $j \in \mathcal{J}_k(1) \cup \mathcal{J}_k(2)$. For such $j$, we have

$$\text{(5.29)} \qquad \Gamma_j(\vec{c}_1, \vec{c}_2) = \sum_{|\wp_1|+|\wp_2|\geqslant 0} \vec{c}_1^{\,\wp_1}\vec{c}_2^{\,\wp_2}\Gamma_{j,\wp_1\wp_2}$$

where $\Gamma_{j,\wp_1\wp_2} = 0$ if

$$\text{(5.30)} \qquad \Re\left[\Lambda_j(x) - \sum_{\ell=1}^{n_1} p_{1,\ell}\Lambda_\ell(x) - \sum_{\ell=1}^{n_2} p_{2,n_0+\ell}\Lambda_{n_0+\ell}(x)\right] > 0$$

in $\mathcal{U}(r_M) \cap \mathcal{U}_2(r_M)$. This means that the right-hand side of (5.29) is a polynomial in $\vec{c}_1$ and $\vec{c}_2$ if $M$ is sufficiently large.

Step 5 : More precisely speaking, for $j \in \mathcal{J}_k(1) \cup \mathcal{J}_k(2)$ we can derive
(5.31)

$$\Gamma_j(\vec{c}_1, \vec{c}_2) = \begin{cases} \xi_j + c_{2,j} + \cdots & \text{if } j \in \mathcal{J}_k(2) \text{ and } j \notin \mathcal{J}_k(1), \\ \xi_j + c_{2,j} - c_{1,j} + \cdots & \text{if } j \in \mathcal{J}_k(1) \cap \mathcal{J}_k(2), \\ \xi_j - c_{1,j} + \cdots & \text{if } j \in \mathcal{J}_k(1) \text{ and } j \notin \mathcal{J}_k(2), \end{cases}$$

where $\xi_j = \Gamma_j(\vec{0}, \vec{0})$ and $\cdots$ denotes terms containing only those $c_{\nu,\ell}$ such that $\Re[\Lambda_j(x)] < \Re[\Lambda_\ell(x)]$ in $\mathcal{U}_2(r_M) \cap \mathcal{U}_1(r_M)$. In fact, (5.31) can be derived by a straightforward computation based on the following facts :

(1) $\mu + \Re[\mu_j - \mu_h] > 0$ for $j, h = 1, \cdots, n$,

(2) $\mu' + \Re[\mu_j - \mu_h] > 0$ for $j, h = 1, \cdots, n$,

(3) (5.26),

(4) the quantities $|x|^{-\mu'} \, |P_{\nu,0}(x) - C_\nu|$ are bounded in $\mathcal{U}_\nu(r_M)$ respectively,

(5) (3.21), (4.15) and (4.16).

The main ideas behind this calculation are

(a) we look only for those terms with $|\wp_1| + |\wp_2| = 1$ and

$$\Re\left[\Lambda_j(x) - \sum_{\ell=1}^{n_1} p_{1,\ell}\Lambda_\ell(x) - \sum_{\ell=1}^{n_2} p_{2,n_0+\ell}\Lambda_{n_0+\ell}(x)\right] = 0 \ ,$$

(b) we ignore those quantities that tend to 0 as $x \to 0$ in $\mathcal{U}_2(r_M) \cap \mathcal{U}_1(r_M)$.

For example, if we ignore those terms of $O(|\vec{c}_1|^2 + |\vec{c}_2|^2)$, we have

$$\begin{cases} Y_M(\vec{c}_1, \vec{c}_2) = & \vec{f}_2(x) - \vec{f}_1(x) + [C_2 + O(x^{\mu'})]\vec{\phi}_2(x, c_2) \\ & -[C_1 + O(x^{\mu'})]\vec{\phi}_1(x, c_1) + \cdots \ , \\ \Phi_M(x, \vec{c}_1, \vec{c}_2) = & I + O(x^{\mu'}) \\ & +x^{\mu}[\text{terms linear in } \phi_1 \text{ and } \phi_2 \text{ with bounded coeffi-} \\ & \text{cients}] + \cdots \ , \\ \Psi_M(x, \vec{c}_1, \vec{c}_2) = & \left[I + \int^x \xi^\mu \xi^{-A_0} \tilde{B}(\xi, \vec{c}_1, \vec{c}_2) \xi^{A_0} d\xi\right] x^{-A_0} e^{-\Lambda(x)} + \cdots \ , \\ \phi_\nu(x, \vec{c}_\nu) = & x^{[A_0]_\nu} \, e^{[\Lambda(x)]_\nu} \, \vec{c}_\nu + \cdots \ , \end{cases}$$

where $[A_0]_\nu = [\lambda(x) + A_0]_\nu \,|_{\lambda(x)=0}$ (cf. §4) .

Now if we further ignore those terms which tend to 0 as $x \to 0$ in $\mathcal{U}_2(r_M) \cap \mathcal{U}_1(r_M)$ and those terms corresponding to $\Re[\Lambda_j(x)] < \Re[\Lambda_\ell(x)]$, we conclude that the terms from $C_2\vec{c}_2 - C_1\vec{c}_1$ are the only contribution to the terms linear in $\vec{c}_1$ and $\vec{c}_2$ of $\Gamma_j(\vec{c}_1, \vec{c}_2)$. In this last step we utilize (1), (2), and (3) together with the structure (4.15) of the matrix $\tilde{B}$ . Thus we can derive (5.31).

Therefore we can fix arbitrary constants $\vec{c}_1, \vec{c}_2$ in such a way that

$$\Gamma_j(\vec{c}_1, \vec{c}_2) = 0 \qquad \text{for} \quad j \in \mathcal{J}_k(1) \cup \mathcal{J}_k(2).$$

This completes the proof of Lemma 5.1.

## 6. Proof of Theorem 2.1.

We shall utilize the same notations as in §2, and apply Lemma 5.1 to $\{\mathcal{U}_\nu(r_1),\ \mathcal{U}_{\nu-1}(r_1)\}$ and $\{\vec{f}_\nu(x),\ \vec{f}_{\nu-1}(x)\}$ for each $\nu$. To do this, set

$$\text{(6.1)}\quad \mathcal{J}_k(\nu) \;=\; \{j \in \mathcal{R}_k;\ \Re\,[\Lambda_j(x)] < 0 \ \text{ in }\ \mathcal{U}_\nu(r_1)\} \;=\; \{j_{\nu,1},\ \cdots,\ j_{\nu,n_\nu}\},$$

and

$$\text{(6.2)}\qquad \vec{c}_\nu \;=\; \begin{bmatrix} c_{\nu,j_{\nu,1}} \\ c_{\nu,j_{\nu,2}} \\ \cdots \\ c_{\nu,j_{\nu,n_\nu}} \end{bmatrix}.$$

Then for each $\nu$, we have the following system of equations :

(6.3-$\nu$)

$$0 \;=\; \Gamma_{\nu,j}(\vec{c}_{\nu-1}, \vec{c}_\nu)$$

$$= \begin{cases} \xi_j \;+\; c_{\nu,j} \;+\; \cdots & \text{if } \ j \in \mathcal{J}_k(\nu) \ \text{ and } \ j \notin \mathcal{J}_k(\nu-1), \\ \xi_j \;+\; c_{\nu,j} \;-\; c_{\nu-1,j} \;+\; \cdots & \text{if } \ j \in \mathcal{J}_k(\nu) \cap \mathcal{J}_k(\nu-1), \\ \xi_j \;-\; c_{\nu-1,j} \;+\; \cdots & \text{if } \ j \in \mathcal{J}_k(\nu-1) \ \text{ and } \ j \notin \mathcal{J}_k(\nu) \end{cases}$$

where $\xi_j = \Gamma_{\nu,j}(\vec{0}, \vec{0})$ and $\cdots$ denotes terms containing only those $c_{\nu,\ell}$ and $c_{\nu-1,\ell}$ such that $\Re[\Lambda_j(x)] < \Re[\Lambda_\ell(x)]$ in $\mathcal{U}_\nu(r_1) \cap \mathcal{U}_{\nu-1}(r_1)$ (cf. (5.23)).

Let us classify those $j$ in $\mathcal{R}_k$ according to Case A and Case B of figure 1 of §2. Then, since the direction $d$ is not singular on the level $k$, we can choose, for each $\nu$, a point $x_\nu$ in $\mathcal{U}_\nu(r_1) \cap \mathcal{U}_{\nu-1}(r_1)$ in such a way that

$$\text{(6.4)}\qquad \Re[\Lambda_j(x_\nu)] \begin{cases} < \ \Re[\Lambda_j(x_{\nu+1})] & \text{if } j \text{ is } \textit{in Case A}, \\ > \ \Re[\Lambda_j(x_{\nu+1})] & \text{if } j \text{ is } \textit{in Case B} \end{cases}$$

(cf. Lemma 1.4.1 on p. 56 and Lemma 1.7.1 on p. 66 of J.-P. Ramis and Y. Sibuya [10]). Let us order all of equations (6.3-$\nu$) ($\nu = 2, 3, \cdots, N$) by the order of $\{\Re[\Lambda_j(x_\nu)]\ ; j \in \mathcal{J}_k(\nu),\ \ \nu = 2, 3, \cdots, N\ \}$. Then we can solve those equations successively. Thus we can complete the proof of Theorem 2.1.

## 7. Proof of Lemma 3.1.

Let us consider the following partial differential equation :

$$
\begin{aligned}
(7.1) \qquad & x\frac{\partial \vec{P}(x,\vec{w})}{\partial x} + \frac{\partial \vec{P}(x,\vec{w})}{\partial \vec{w}} \left( [\lambda(x) + A_0]_{n_0}\vec{w} + x^\mu \sum_{|\wp|\geqslant 2} \vec{w}^\wp\ \vec{\alpha}_\wp(x) \right) \\
& = [\lambda(x) + A_0]\vec{P}(x,\vec{w}) + x^\mu \vec{F}(x, \vec{P}(x,\vec{w})),
\end{aligned}
$$

where $\dfrac{\partial \vec{P}(x,\vec{w})}{\partial \vec{w}}$ is an $n \times n_0$ matrix defined by

$$
\frac{\partial \vec{P}(x,\vec{w})}{\partial \vec{w}} = \left[ \frac{\partial \vec{P}(x,\vec{w})}{\partial w_1} \quad \cdots \quad \frac{\partial \vec{P}(x,\vec{w})}{\partial w_{n_0}} \right].
$$

Denote by $(P_0)_\ell(x)$ the $\ell$–th column of $P_0(x)$ and set

$$
e_\ell \ = \ (e_{\ell 1}, e_{\ell 2}, \cdots, e_{\ell n_0}), \qquad \text{where} \quad e_{\ell h} = \begin{cases} 1 & \text{if } h = \ell, \\ 0 & \text{if } h \neq \ell. \end{cases}
$$

Then

$$
\frac{\partial \vec{P}(x,\vec{w})}{\partial w_\ell} \ = \ (P_0)_\ell(x) \ + \ x^\mu \sum_{|\wp|\geqslant 2} \ p_\ell\ \vec{w}^{\wp - e_\ell}\ \vec{P}_\wp(x),
$$

and hence

$$
\frac{\partial \vec{P}(x,\vec{w})}{\partial \vec{w}} \ = \ P_0(x) + x^\mu \sum_{|\wp|\geqslant 2} \vec{P}_\wp(x)[p_1\vec{w}^{\wp - e_1} \quad p_2\vec{w}^{\wp - e_2} \quad \cdots \quad p_{n_0}\vec{w}^{\wp - e_{n_0}}].
$$

Also we have

$$
[\lambda(x) + A_0]_{n_0}\vec{w} \ = \ \begin{bmatrix} (\lambda_1(x) + \mu_1)w_1 + \delta_1 w_2 \\ (\lambda_2(x) + \mu_2)w_2 + \delta_2 w_3 \\ \cdots \\ (\lambda_{n_0-1}(x) + \mu_{n_0-1})w_{n_0-1} + \delta_{n_0-1}w_{n_0} \\ (\lambda_{n_0}(x) + \mu_{n_0})w_{n_0} \end{bmatrix}.
$$

Therefore

$$\begin{aligned}
&\frac{\partial \vec{P}(x,\vec{w})}{\partial \vec{w}}\ [\lambda(x)+A_0]_{n_0}\vec{w} \\
&= P_0(x)[\lambda(x)+A_0]_{n_0}\vec{w} \ + \ x^{\mu}\sum_{|\wp|\geqslant 2}\left[\left(\sum_{\ell=1}^{n_0} p_\ell\ (\lambda_\ell(x)+\mu_\ell)\right)\ \vec{w}^{\wp}\right. \\
&\left. + \sum_{\ell=1}^{n_0-1} p_\ell\ \delta_\ell\ \vec{w}^{\wp-e_\ell+e_{\ell+1}}\right]\ \vec{P}_{\wp}(x).
\end{aligned}$$

Thus we derive from (7.1) the following recurrence relations :
(7.2)

$$\begin{cases}
x\dfrac{dP_0(x)}{dx} = [\lambda(x)+A_0+x^{\mu}A(x)]\ P_0(x)\ -\ P_0(x)\ [\lambda(x)+A_0]_{n_0}, \\
x\dfrac{d\vec{\Phi}_{\wp}(x)}{dx} = [\lambda(x)+A_0+x^{\mu}A(x)]\vec{\Phi}_{\wp}(x) \\
\qquad -\ \left(\displaystyle\sum_{\ell=1}^{n_0} p_\ell\ (\lambda_\ell(x)+\mu_\ell)\right)\vec{\Phi}_{\wp}(x) \\
\qquad -\ \displaystyle\sum_{\ell=1}^{n_0-1}(p_\ell+1)\ \delta_\ell\ \vec{\Phi}_{\wp+e_\ell-e_{\ell+1}}(x) \\
\qquad -\ x^{\mu}\Big(\ P_0(x)\ \vec{\alpha}_{\wp}(x)\ +\ \vec{\mathcal{F}}_{\wp}(x)\ \Big),
\end{cases}$$

where

$$\vec{\Phi}_{\wp}(x)\ =\ x^{\mu}\vec{P}_{\wp}(x),$$

and, for each $\wp$, the entries of $\mathcal{F}_{\wp}(x)$ are polynomials in the entries of $\vec{\Phi}_{\wp'}$ and $\vec{\alpha}_{\wp'}$ $(|\wp'|<|\wp|)$ with coefficients holomorphic and bounded in $\mathcal{D}(\alpha,\beta,r)$. Note that $\vec{\Phi}_{\wp+e_\ell-e_{\ell+1}}=\vec{0}$ if some entries of $\wp+e_\ell-e_{\ell+1}$ are negative. We shall determine $P_0$, $\vec{\Phi}_{\wp}$ and $\vec{\alpha}_{\wp}$ by (7.2).

*Step 1* : Set $P_0(x)=C+X$, where $C=\begin{bmatrix} I_{n_0} \\ O\end{bmatrix}$ (cf. condition (i) of Lemma 3.1). Then from (7.2) we derive

$$\begin{aligned}
x\frac{dX}{dx} &= [\lambda(x)+A_0+x^{\mu}A(x)]X - X[\lambda(x)+A_0]_{n_0} \\
&\quad + [\lambda(x)+A_0+x^{\mu}A(x)]C - C[\lambda(x)+A_0]_{n_0} \\
&= [\lambda(x)+A_0+x^{\mu}A(x)]X - X[\lambda(x)+A_0]_{n_0} + x^{\mu}A(x)C.
\end{aligned}\tag{7.3}$$

Choose any positive number $\mu'<\mu$. Then by utilizing the assumption that $\mu+\Re[\mu_j-\mu_h]>0$ $(j,h=1,\cdots,n)$ and the method in §2 of [10], we can construct a solution $X(x)$ of (7.3) such that $x^{-\mu'}X(x)$ is bounded on $\mathcal{D}(\alpha,\beta,r)$. Actually, we can choose $\mu'$ so that

$$\mu' + \Re[\mu_j - \mu_h] > 0 \qquad \text{for} \quad j,h = 1,\cdots,n.$$

Step 2 : Let us denote by $\mathcal{P}(m)$ the set of all $\wp = (p_1,\cdots,p_{n_0})$ such that the $p_\ell$ are nonnegative integers and $|\wp| = m$, i.e.

$$\mathcal{P}(m) \;=\; \{\wp = (p_1,\cdots,p_{n_0})\ ;\ \text{the}\, p_\ell \text{ are nonnegative integers and } |\wp| = m\}.$$

For $\wp$, $\wp' \in \mathcal{P}(m)$, we define an order :

$$\wp = (p_1,\cdots,p_{n_0}) \;>\; \wp' = (p'_1,\cdots,p'_{n_0})$$

if and only if, for some $\ell_0 \in \{1,2,\cdots,n_0\}$, we have

$$p_\ell = p'_\ell \qquad \text{for} \qquad \ell < \ell_0 \qquad \text{and} \qquad p_{\ell_0} \;<\; p'_{\ell_0}.$$

Let

$$\wp_1 < \wp_2 < \cdots < \wp_{M(m)}$$

be all $\wp$'s in $\mathcal{P}(m)$, where

$$\wp_1 = (m,0,\cdots,0) \qquad \text{and} \qquad \wp_{M(m)} = (0,\cdots,0,m).$$

Furthermore

$$\wp \;>\; \wp \;+\; e_\ell \;-\; e_{\ell+1}.$$

We determine $\vec{\Phi}_\wp$ and $\vec{\alpha}_\wp$ starting from $\wp = \wp_1$ successively according with this order. Note that $\vec{\Phi}_{\wp_1+e_\ell-e_{\ell+1}}(x) = 0$ for $\ell = 1,\cdots,n_0-1$. We set

$$\begin{cases} \vec{\Phi}_{\wp,j}(x) \;=\; x^\mu P_{\wp,j}(x) \;=\; \vec{0} & \text{if} \quad \lambda_j(x) = \sum\limits_{\ell=1}^{n_0} p_\ell\lambda_\ell(x), \\ \vec{\alpha}_{\wp,j}(x) \;=\; \vec{0} & \text{if} \quad \lambda_j(x) \neq \sum\limits_{\ell=1}^{n_0} p_\ell\lambda_\ell(x). \end{cases}$$

Then, if $\lambda_j(x) = \sum\limits_{\ell=1}^{n_0} p_\ell\lambda_\ell(x)$, we derive

(7.4)

$$0 = x^\mu[A(x)\vec{\Phi}_\wp(x)]_j - \sum_{\ell=1}^{n_0-1} (p_\ell+1)\ \delta_\ell\ \vec{\Phi}_{\wp+e_\ell-e_{\ell+1},j}(x) - \ x^\mu[P_0(x)\ \vec{\alpha}_\wp(x)]_j$$
$$-\ x^\mu[\vec{\mathcal{F}}_\wp(x)]_j,$$

where $[\vec{Y}]_j$ denotes the $j$–th entry of the vector $\vec{Y}$. It is easy to see that

$$j \in \mathcal{J}_k = \{1, 2, \cdots, n_0\} \qquad \text{if} \qquad \lambda_j(x) = \sum_{\ell=1}^{n_0} p_\ell \lambda_\ell(x)$$

and

$$\begin{aligned}[P_0(x)\ \vec{\alpha}_\wp(x)]_j &= [C\vec{\alpha}_\wp(x)]_j + [(P_0(x) - C)\vec{\alpha}_\wp(x)]_j \\ &= \alpha_{\wp,j}(x) + [(P_0(x) - C)\vec{\alpha}_\wp(x)]_j.\end{aligned}$$

Also we can assume recursively that $\vec{\Phi}_{\wp+e_\ell-e_{\ell+1}}(x) = x^\mu \vec{P}_{\wp+e_\ell-e_{\ell+1}}(x)$, where the entries of $\vec{P}_{\wp+e_\ell-e_{\ell+1}}(x)$ are holomorphic and bounded in $\mathcal{D}(\alpha, \beta, r)$. As a matter of fact, we have

$$\vec{\Phi}_{\wp+e_\ell-e_{\ell+1},j}(x) \ = \ \vec{0} \qquad \text{if} \ \ \lambda_j(x) = \sum_{\ell=1}^{n_0} p_\ell \lambda_\ell(x) \quad \text{and} \quad \delta_\ell \neq 0$$

since $\lambda_\ell = \lambda_{\ell+1}$ if $\delta_\ell \neq 0$. Therefore, (3) determines $\vec{\alpha}_\wp$ in a form :

$$\vec{\alpha}_\wp(x) \ = \ \mathcal{G}_\wp(x)\vec{\Phi}_\wp(x) \ + \ \vec{\mathcal{E}}_\wp(x),$$

where $\mathcal{G}_\wp(x)$ is an $n_0 \times n$ matrix whose entries are holomorphic and bounded in $\mathcal{D}(\alpha, \beta, r)$ and $\vec{\mathcal{E}}_\wp(x)$ is an $n_0$–vector whose entries are holomorphic and bounded in $\mathcal{D}(\alpha, \beta, r)$ (because of recursive assumption).

Now (7.2) yields a system of differential equations :

$$\text{(7.5)} \qquad \begin{aligned} x\frac{d\Phi_{\wp,j}}{dx} \ &= \ \left[\lambda_j(x) \ - \ \sum_{\ell=1}^{n_0} p_\ell \lambda_\ell(x) \ + \ \mu_j \ - \ \sum_{\ell=1}^{n_0} p_\ell \mu_\ell\right] \Phi_{\wp,j} \\ &\quad + \ \delta_j \Phi_{\wp,j+1} \ + \ x^\mu \left[\left(A(x) \ + \ P_0(x)\mathcal{G}_\wp(x)\right) \vec{\Phi}_\wp\right]_j \\ &\quad + \ x^\mu \mathcal{Q}_{\wp,j}(x) \end{aligned}$$

for $j$ such that $\lambda_j(x) \neq \sum_{\ell=1}^{n_0} p_\ell \lambda_\ell(x)$, where the $\mathcal{Q}_{\wp,j}(x)$ are holomorphic and bounded in $\mathcal{D}(\alpha, \beta, r)$.

Finally we can construct $\vec{\Phi}_\wp(x) = x^\mu \vec{P}_\wp(x)$ in such a way that the $\vec{P}_\wp(x)$ are holomorphic and bounded in $\mathcal{D}(\alpha, \beta, r)$ by utilizing the methods and results in J.-P. Ramis and Y. Sibuya [10]. ( In particular, see, §2.1 (pp. 66–70) and Lemma 1.7.2 (p. 66).) Note that, in equations (7.5), we have

$$\lambda_j(x) - \sum_{\ell=1}^{n_0} p_\ell \lambda_\ell(x) \neq 0.$$

## 8. Proof of Lemma 4.1.

Let us determine two formal series $\Phi(x,\vec{\phi}_1,\vec{\phi}_2)$ and $B(x,\vec{\phi}_1,\vec{\phi}_2)$ of (4.5) by differential equation (4.6). Note first that

$$x\frac{d\Phi}{dx}=x\frac{d\left(x^{\mu'}\Phi_0\right)}{dx}+\sum_{|\wp_1|+|\wp_2|\geqslant 1} x\frac{d\left(\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}\right)}{dx}\,x^{\mu}\Phi_{\wp_1\wp_2}$$
$$+\sum_{|\wp_1|+|\wp_2|\geqslant 1}\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}\,x\frac{d\left(x^{\mu}\Phi_{\wp_1\wp_2}\right)}{dx},$$

where

$$x\frac{d\left(\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}\right)}{dx}=\sum_{\ell=1}^{n_1}p_{1,\ell}\,\vec{\phi}_1^{\wp_1-e_{1\ell}}\vec{\phi}_2^{\wp_2}\,x\frac{d\phi_\ell}{dx}+\sum_{\ell=1}^{n_2}p_{2,n_0+\ell}\,\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2-e_{2\ell}}\,x\frac{d\tilde{\phi}_{n_0+\ell}}{dx}.$$

Here

$$e_{1\ell}=(\delta^1_{\ell 1},\cdots,\delta^1_{\ell n_1})\qquad\text{and}\qquad e_{2\ell}=(\delta^2_{\ell 1},\cdots,\delta^2_{\ell n_2}),$$

with

$$\delta^\nu_{\ell j}=\begin{cases}1 & \text{if } j=\ell,\\ 0 & \text{if } j\neq\ell,\end{cases}\qquad \nu=1,2.$$

Note also that

$$\begin{cases} x\dfrac{d\vec{\phi}_1}{dx} & =[\,\lambda(x)+A_0\,]_1\,\vec{\phi}_1\;+\;x^{\mu}\sum_{|\wp|\geqslant 2}\vec{\phi}_1^{\wp}\,\vec{\alpha}_{1,\wp}(x),\\ x\dfrac{d\vec{\phi}_2}{dx} & =[\,\lambda(x)+A_0\,]_2\,\vec{\phi}_2\;+\;x^{\mu}\sum_{|\wp|\geqslant 2}\vec{\phi}_2^{\wp}\,\vec{\alpha}_{2,\wp}(x).\end{cases}$$

Therefore,

$$\vec{\phi}_1^{\wp_1-e_{1\ell}}\vec{\phi}_2^{\wp_2}\,x\frac{d\phi_\ell}{dx}$$
$$=\vec{\phi}_1^{\wp_1-e_{1\ell}}\vec{\phi}_2^{\wp_2}\left((\lambda_\ell(x)+\mu_\ell)\phi_\ell+\delta_\ell\phi_{\ell+1}+x^{\mu}\sum_{|\wp|\geqslant 2}\vec{\phi}_1^{\wp}\,\alpha_{1,\wp,\ell}(x)\right)$$
$$=(\,\lambda_\ell(x)+\mu_\ell\,)\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}+\delta_\ell\vec{\phi}_1^{\wp_1-e_{1\ell}+e_{1\ell+1}}\vec{\phi}_2^{\wp_2}$$
$$+x^{\mu}\vec{\phi}_1^{\wp_1-e_{1\ell}}\vec{\phi}_2^{\wp_2}\sum_{|\wp|\geqslant 2}\vec{\phi}_1^{\wp}\,\alpha_{1,\wp,\ell}(x),$$

and

$$\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2-e_{2\ell}}\, x\frac{d\tilde{\phi}_{n_0+\ell}}{dx}$$
$$= (\ \lambda_{n_0+\ell}(x) + \mu_{n_0+\ell}\ )\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2} + \delta_{n_0+\ell}\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2-e_{2\ell}+e_{2\ell+1}}$$
$$+ x^{\mu}\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2-e_{2\ell}}\sum_{|\wp|\geqslant 2}\vec{\phi}_2^{\wp}\ \alpha_{2,\wp,n_0+\ell}(x),$$

where

$$\vec{\alpha}_{1,\wp}(x) = \begin{bmatrix}\alpha_{1,\wp,1}(x)\\ \cdots\\ \cdots\\ \alpha_{1,\wp,n_1}(x)\end{bmatrix} \qquad \text{and} \qquad \vec{\alpha}_{2,\wp}(x) = \begin{bmatrix}\alpha_{2,\wp,n_0+1}(x)\\ \cdots\\ \cdots\\ \alpha_{2,\wp,n_0+n_2}(x)\end{bmatrix}.$$

Thus we derive

$$x\frac{d\left(\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}\right)}{dx}$$
$$= \left[\sum_{\ell=1}^{n_1} p_{1,\ell}\ (\lambda_\ell(x) + \mu_\ell) + \sum_{\ell=1}^{n_2} p_{2,n_0+\ell}\ (\lambda_{n_0+\ell}(x) + \mu_{n_0+\ell})\right]\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}$$
$$+ \sum_{\ell=1}^{n_1-1}\delta_\ell\ p_{1,\ell}\ \vec{\phi}_1^{\wp_1-e_{1\ell}+e_{1\ell+1}}\vec{\phi}_2^{\wp_2} + \sum_{\ell=1}^{n_2-1}\delta_{n_0+\ell}\ p_{2,n_0+\ell}\ \vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2-e_{2\ell}+e_{2\ell+1}}$$
$$+ \sum_{\ell=1}^{n_1} p_{1,\ell}\ x^{\mu}\vec{\phi}_1^{\wp_1-e_{1\ell}}\vec{\phi}_2^{\wp_2}\sum_{|\wp|\geqslant 2}\vec{\phi}_1^{\wp}\ \alpha_{1,\wp,\ell}(x)$$
$$+ \sum_{\ell=1}^{n_2} p_{2,n_0+\ell}\ x^{\mu}\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2-e_{2\ell}}\sum_{|\wp|\geqslant 2}\vec{\phi}_2^{\wp}\ \alpha_{2,\wp,n_0+\ell}(x).$$

Set $\Phi = I + \tilde{\Phi}$, $\quad A = A_{00}(x) + \tilde{A}$. Then

$$\begin{aligned}
&[\lambda(x) + A_0 + x^{\mu}B]\Phi - \Phi[\lambda(x) + A_0 + x^{\mu}A]\\
&\quad =[\lambda(x) + A_0 + x^{\mu}B]\tilde{\Phi} - \tilde{\Phi}[\lambda(x) + A_0 + x^{\mu}(A_{00}(x) + \tilde{A})]\\
&\qquad + x^{\mu}(B - A_{00}(x) - \tilde{A})\\
&\quad =[\lambda(x) + A_0]\tilde{\Phi} - \tilde{\Phi}[\lambda(x) + A_0 + x^{\mu}A_{00}(x)]\\
&\qquad + x^{\mu}B\tilde{\Phi} - x^{\mu}\tilde{\Phi}\tilde{A} + x^{\mu}(B - A_{00}(x) - \tilde{A}).
\end{aligned}$$

If we write $[\lambda(x)+A_0+x^\mu B]\Phi-\Phi[\lambda(x)+A_0+x^\mu A]$ in the form :

$$\begin{aligned}&[\lambda(x)+A_0+x^\mu B]\Phi-\Phi[\lambda(x)+A_0+x^\mu A]\\ &=W_0(x)+\sum_{|\wp_1|+|\wp_2|\geqslant 1}\vec{\phi}_1^{\wp_1}\vec{\phi}_2^{\wp_2}W_{\wp_1\wp_2}(x),\end{aligned}$$

we have

$$\begin{cases} W_0(x)= & [\lambda(x)+A_0](x^{\mu'}\Phi_0)-(x^{\mu'}\Phi_0)[\lambda(x)+A_0+x^\mu A_{00}(x)]\\ & -x^\mu A_{00}(x),\\ W_{\wp_1\wp_2}(x)= & [\lambda(x)+A_0](x^\mu\Phi_{\wp_1\wp_2})-(x^\mu\Phi_{\wp_1\wp_2})[\lambda(x)+A_0+x^\mu A_{00}(x)]\\ & +x^\mu B_{\wp_1\wp_2}(x)[I+x^{\mu'}\Phi_0]+x^\mu\Psi_{\wp_1\wp_2}(x),\end{cases}$$

where $\Psi_{\wp_1\wp_2}(x)$ is a polynomial in $B_{\wp_1'\wp_2'}(x)$ and $\Phi_{\wp_1'\wp_2'}(x)$ $(|\wp_1'|+|\wp_2'|<|\wp_1|+|\wp_2|)$ with coefficient holomorphic and bounded in $\mathcal{D}(\alpha,\beta,r)$.

Thus we derive the following differential equations :

$$\begin{cases} & x\dfrac{d(x^{\mu'}\Phi_0)}{dx}\\ = & [\lambda(x)+A_0](x^{\mu'}\Phi_0)-(x^{\mu'}\Phi_0)[\lambda(x)+A_0+x^\mu A_{00}(x)]-x^\mu A_{00}(x),\\ \\ & x\dfrac{d(x^\mu\Phi_{\wp_1\wp_2})}{dx}\\ = & [\lambda(x)+A_0](x^\mu\Phi_{\wp_1\wp_2})-(x^\mu\Phi_{\wp_1\wp_2})[\lambda(x)+A_0+x^\mu A_{00}(x)]\\ & -[\ \sum_{\ell=1}^{n_1}\ p_{1,\ell}\ (\lambda_\ell(x)+\mu_\ell)\\ & +\sum_{\ell=1}^{n_2}\ p_{2,n_0+\ell}\ (\lambda_{n_0+\ell}(x)+\mu_{n_0+\ell})\ ]\,(x^\mu\Phi_{\wp_1\wp_2})\\ & -\sum_{\ell=1}^{n_1-1}\ \delta_\ell\ (p_{1,\ell}+1)\ (x^\mu\Phi_{\wp_1+e_{1\ell}-e_{1\ell+1},\wp_2})\\ & -\sum_{\ell=1}^{n_2-1}\ \delta_{n_0+\ell}\ (p_{2,n_0+\ell}+1)\ (x^\mu\Phi_{\wp_1,\wp_2+e_{2\ell}-e_{2\ell+1}})\\ & +x^\mu B_{\wp_1\wp_2}(x)[I+x^{\mu'}\Phi_0]+x^\mu\tilde{\Psi}_{\wp_1\wp_2}(x),\end{cases}$$

where $\tilde{\Psi}_{\wp_1\wp_2}(x)$ is a polynomial in $B_{\wp_1'\wp_2'}(x)$ and $\Phi_{\wp_1'\wp_2'}(x)$ $(|\wp_1'|+|\wp_2'|<|\wp_1|+|\wp_2|)$ with coefficient holomorphic and bounded in $\mathcal{D}(\alpha,\beta,r)$.

Writing the equations on $x^\mu\Phi_{\wp_1\wp_2}$ componentwise and utilizing an argument similar to the proof of Lemma 3.1, we can complete the proof of Lemma 4.1.

## 9. Multisummability of formal power series solutions of non linear meromorphic differential equations.

We will prove multisummability of formal power series solutions of differential equation (2.1) according to the Malgrange-Ramis definition of multisummability [7]. This definition is slightly different from those of [1], [2], [3], [8] : one replaces the notion of multisummability in one direction by the notion of multisummability on a family $\{I_1, \ldots, I_n\}$ of nested closed arcs of the unit circle $S$ (or more generally of its universal covering $\tilde{S}$). In this paragraph we will use the definitions and notations of [7].

DEFINITION 9.1. — *Let $I_1 \subset \ldots \subset I_p$ be a set of nested closed intervals of the circle $S$ with $|I_j| \geqslant \dfrac{\pi}{k_j}$ $(j = 1, \ldots p)$. We set $I_{p+1} = S$. Let $\hat{\phi} \in C[[x]]$ be a formal power series expansion. We will say that $\hat{\phi}$ is $(k_1, \ldots, k_p)$-summable on $(I_1, \ldots I_p)$, with sum $\phi_1$, if :*

(i) $\hat{\phi} \in C[[x]]_{\frac{1}{k_p}}$,

(ii) *there exists a sequence $(\phi_1, \ldots, \phi_p, \phi_{p+1})$ where :*

*a) $\phi_{p+1} \in \Gamma(S; \mathcal{A}/\mathcal{A}^{\leqslant -k_p})$ and $\phi_{p+1}$ corresponds to $\hat{\phi}$ by the natural isomorphism*

$$\Gamma(S; \mathcal{A}/\mathcal{A}^{\leqslant -k_p}) \to C[[x]]_{\frac{1}{k_p}},$$

*b) $\phi_j \in \Gamma(I_j; \mathcal{A}/\mathcal{A}^{\leqslant -k_{j-1}})$ $(j = 1, \ldots, p+1)$, and $\phi_j = \phi_{j+1|I_j}$ modulo $\mathcal{A}^{\leqslant -k_j}$, for $j = 1, \ldots, p$.*

We denote by $d'_j$ the bisecting line of $I_j$ $(j = 1, \ldots, p)$. If $d'_1 = \ldots = d'_p = d'$, we will say that $\hat{\phi}$ is $(k_1, \ldots, k_p)$-summable in the direction $d'$. (This definition is equivalent with the definitions of [1], [2], [3], [8].)

If $\vec{f} \in (C[[x]])^n$ is a formal solution of the differential equation

$$(2.1) \qquad x\,\frac{d\vec{y}}{dx} = \vec{G}_0(x) + [\,\lambda(x) + A_0\,]\,\vec{y} + x^{\mu}\,\vec{G}(x, \vec{y}),$$

then it is proved in [12] (Theorem A.2.4.2, p. 209) that $\vec{f} \in (C[[x]]_{\frac{1}{k_p}})^n$.

*Observation 9.2.* — We can choose $q = 1$ in (1.9) and set $k = k_1$, $k' = k_0 = +\infty$. Then we can replace the condition (c') by $\vec{f}_\nu(x) = \vec{f}_{\nu-1}(x)$ in $\mathcal{U}_\nu(r_2) \cap \mathcal{U}_{\nu-1}(r_2)$. Then the theorem 2.1 remains true.

THEOREM 9.3. — *Let $\vec{\phi} \in (C[[x]])^n$ be a formal solution of the differential equation*

$$x\,\frac{d\vec{y}}{dx} = \vec{G}_0(x) + [\lambda(x) + A_0]\;\vec{y} + x^{\mu}\,\vec{G}(x,\vec{y}).$$

*Let $I_1 \subset \ldots \subset I_p$ be a set of nested closed intervals of the circle $S$ with $|I_j| \geqslant \frac{\pi}{k_j}$ $(j = 1,\ldots p)$. Let us assume that the bisecting line $\arg x = d'_j$ of $I_j$ is not a singular line of the level $k_j$ (that is $\Re[\Lambda_j(x)]$ does not change its sign across the direction $\arg x = d'_j$ for $j = 1,\ldots,p$). Then $\vec{\phi}$ is $(k_1,\ldots,k_p)$-summable on $(I_1,\ldots I_p)$.*

We will build a sequence $(\vec{\phi}_1,\ldots,\vec{\phi}_p,\vec{\phi}_{p+1})$ with

a) $\vec{\phi}_j \in (\Gamma(S;\mathcal{A}/\mathcal{A}^{\leqslant -k_{j-1}}))^n$ by a descending recursion on $j$.

First we get $\vec{\phi}_{p+1}$ from $\vec{\phi}$ using the natural isomorphism

$$(C[[x]]_{\frac{1}{k_p}})^n \to (\Gamma(S;\mathcal{A}/\mathcal{A}^{\leqslant -k_p}))^n.$$

Now we suppose that we know $(\vec{\phi}_r,\ldots,\vec{\phi}_p,\vec{\phi}_{p+1})$ for $3 \leqslant r < p+1$, such that $\vec{\phi}_j \in (\Gamma(I_j;\mathcal{A}/\mathcal{A}^{\leqslant -k_{j-1}}))^n$ $(j = r,\ldots,p+1)$, and $\vec{\phi}_j = \vec{\phi}_{j+1|I_j}$ modulo $(\mathcal{A}^{\leqslant -k_j})^n$ for $j = r,\ldots,p$.

Then we set $k_{r-1} = k$ and $k_{r-2} = k'$, $I = I_{r-1}$, $d = d'_{r-1}$.

We can represent $\vec{\phi}_{r|I_{r-1}}$ by a 0-cochain $\vec{f} = \{\vec{f}_1,\ldots,\vec{f}_N\}$ associated to a covering $\{\mathcal{U}_\nu(r)\}$ of a closed sector $\mathcal{W}_0(r) = \{x \in C; |\arg x - d| \leqslant \frac{\pi}{2k}, 0 < |x| < r\}$ (corresponding to the closed arc $I$ on $S$) as in §2. The coboundary of $\vec{f}$ takes its values in $(\mathcal{A}^{\leqslant -k})^n$. Then applying Theorem 2.1 we get a 0-cochain $\vec{g} = \{\vec{g}_1,\ldots,\vec{g}_N\}$ representing also $\vec{\phi}_{r|I_{r-1}}$ but such that its coboundary takes its values in $(\mathcal{A}^{\leqslant -k'})^n$. Therefore $\vec{g}$ defines an element $\hat{\phi}_{r-1} \in (\Gamma(I_{r-1};\mathcal{A}/\mathcal{A}^{\leqslant -k_{r-2}}))^n$ such that $\vec{\phi}_{r-1} = \vec{\phi}_{r|I_{r-1}}$ modulo $(\mathcal{A}^{\leqslant -k_{r-1}})^n$.

Utilizing the Observation 9.2 we can do the same construction with $r = 2$. Finally we get a sequence $(\vec{\phi}_1,\ldots,\vec{\phi}_p,\vec{\phi}_{p+1})$ satisfying our conditions.

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