ANNALES DE L'INSTITUT FOURIER

ERIC M. FRIEDLANDER
BARRY MAZUR

Correspondence homomorphisms for singular varieties

Annales de l'institut Fourier, tome 44, nº 3 (1994), p. 703-727
<http://www.numdam.org/item?id=AIF_1994__44_3_703_0>

Ann. Inst. Fourier, Grenoble
44, 3 (1994), 703-727

CORRESPONDENCE HOMOMORPHISMS FOR SINGULAR VARIETIES

by E.M. FRIEDLANDER(*†) and B. MAZUR(*)

In this paper, Y and X will be (reduced) projective complex varieties. Homology will be singular homology of underlying topological spaces with $\mathbb{Z}$-coefficients, unless specifically signalled otherwise.

If Y is smooth and connected of dimension n, and if Z is an effective algebraic cycle in $Y \times X$ equidimensional of dimension r over Y, then the fundamental class $[Z] \in H_{2n+2r}(Y \times X)$ determines a homomorphism in homology

$$\phi_Z : H_*(Y) \to H_{*+2r}(X)$$

given by the composition of the Poincaré duality isomorphism $\mathcal{D}$ and slant product with $[Z]$:

$$H_*(Y) \xrightarrow{\mathcal{D}} H^{2n-*}(Y) \xrightarrow{\backslash [Z]} H_{*+2r}(X).$$

We refer to ϕ_Z as the *correspondence homomorphism* in homology attached to Z.

It is not known to us whether one can naturally attach "correspondence homomorphisms" in homology to equidimensional algebraic cycles in general if the smoothness hypothesis on Y is dropped. Nonetheless, we think that a perfectly legitimate requirement for any theory of "geometric (equidimensional) correspondences" between projective varieties is that there be naturally associated correspondence homomorphisms in homology.

By a *Chow correspondence* (of relative dimension $r \geqslant 0$) we mean a continuous algebraic map $f : Y \to \mathcal{C}_r(X)$, a morphism from the seminormalization of Y to the Chow monoid $\mathcal{C}_r(X)$ of effective r-cycles on

---

(*) Partially supported by the N.S.F. and (†) NSA Grant # MDA904-90-H-4006.
Key words : Algebraic variety – Algebraic cycle – Homology – Correspondence.
A.M.S. Classification : 14C – 14E – 55N35.

X (cf. [F1]). A Chow correspondence f has an associated cycle Z_f in $Y \times X$; if Y is normal, then every (effective) algebraic cycle in $Y \times X$ equidimensional of fiber dimension r over Y is the cycle associated to a unique Chow correspondence $f : Y \to \mathcal{C}_r(X)$ (cf. [FL]).

In an earlier paper, we obtained a correspondence homomorphism $\phi_f : \tilde{H}_*(Y;\mathbb{Q}) \to H_{*+2r}(X;\mathbb{Q})$ associated to a Chow correspondence $f : Y \to \mathcal{C}_r(X)$ whose domain of definition is the subspace $\tilde{H}_*(Y;\mathbb{Q}) \subset H_*(X;\mathbb{Q})$ of classes of lowest weight for the Mixed Hodge Structure on the rational homology of a connected variety Y. In this paper, we construct natural correspondence homomorphisms

$$\Phi_f : H_*(Y) \to H_{*+2r}(X)$$

attached to Chow correspondences f for general Y such that $\Phi_f \otimes \mathbb{Q}$ restricts to ϕ_f whenever Y is connected. Furthermore, we show that Φ_f factors through a refinement $\tilde{\Phi}_f : H_*(Y) \to H_{*+2r}(V_f)$, where $V_f \subset X$ denotes the projection to X of the support of the cycle $Z_f \subset Y \times X$ associated to f.

Our construction of Φ_f enables us to extend results of [FM] to possibly singular varieties. Among the examples presented in section 2 are mappings constructed in [FM] only when the domain of the mapping is smooth. Indeed, Theorem 4.3 extends to singular varieties the main result of [FM] concerning filtrations on the homology of projective varieties.

We show that a natural refinement $\langle f \rangle$ of the correspondence homomorphism Φ_f is precisely the total Chern class of a vector bundle generated by global sections in the special case that f is the classifying map for this bundle. This suggests that $\langle f \rangle$ might be viewed as a characteristic class for an equidimensional family of varieties.

Our paper is organized as follows. In section 1, we construct the correspondence homomorphism Φ_f and its refinement $\langle f \rangle$ associated to a Chow correspondence $f : Y \to \mathcal{C}_{r,d}(X)$. We show that this construction is well behaved with respect to compositions and has an evident extension in the relative context. Examples are presented in section 2, including the inverse of the Thom isomorphism for vector bundles and the suspension isomorphism for algebraic suspensions. Section 3 presents the proof that our new construction of the correspondence homomorphism determines the same homomorphism as that considered in [FM] on $\tilde{H}_*(Y;\mathbb{Q})$. Finally, section 4 is devoted to comparing filtrations on homology, thereby extending results of [FM] to singular varieties and refining these results to homology with integral (rather than rational) coefficients.

We anticipate further generalizations of constructions of correspondence homomorphisms (e.g., arising in the context of the "algebraic bivariant cycle complex" of [FG] or possibly in the general framework developed by V. Voevodsky involving his "h-topology" [V]). Is there a theory of "correspondence homomorphisms" in the context of intersection homology?

We gratefully thank Ofer Gabber for useful conversations.

1. Homomorphisms associated to Chow correspondences.

In Appendix B of [FM], we discussed weighted maps $g : T \to S$, $w : T \to \mathbb{N}$ of simplicial sets and the induced trace maps $g^! \equiv (g, w)^! : \mathbb{Z}[S] \to \mathbb{Z}[T]$ which induce trace maps in homology $g^! : H_*(S) \to H_*(T)$. This construction naturally extends to simplicial maps $g : A \to B$ of simplicial complexes equipped with a "weighting" since the associated map of simplicial sets (defined as the map on nerves of the categories of simplices of A and B) is a weighted map of simplicial sets. These trace maps satisfy an evident naturality property with respect to maps $f : S' \to S$ of simplicial sets (and $f : A' \to A$ of simplicial complexes), yielding maps $(f^*g)^! : H_*(S') \to H_*(T')$, where $T' = S' \times_S T$. In a remark in that appendix, we assert that the homotopy invariance property of the trace construction permits one to consider continuous maps $f : A \to |S|$ from a space A homotopy equivalent to a simplicial complex to the geometric realization of a simplicial set. Indeed, the generalization one obtains in this way has an unwanted feature : the resulting trace $(f^*g)^!$ is not realized as the trace of the topological pull-back $A \times_{|S|} |T| \to A$. In the algebro-geometric context of [FM] and of this article, we do indeed require that the transfer is constructed in terms of the geometric fibre-product. To fill this gap in [FM] and for use here, we offer the following proposition.

We introduce the following notation : $SP^d(X)$ denotes the d-fold symmetric product of X with itself and $\Gamma^d(X) \subset SP^d(X) \times X$ denotes the evident incidence correspondence with projection $\gamma : \Gamma^d(X) \to SP^d(X)$.

PROPOSITION 1.1. — *Let $f : Y \to SP^d(X)$ be a morphism of complex, quasi-projective varieties and let*

$$g : \Gamma \equiv Y \times_{SP^d(X)} \Gamma^d(X) \to Y$$

denote the pull-back of γ *via* f. *Then* g *admits the structure of a simplicial map of simplicial complexes equipped with a natural weighting.*

Proof. — We stratify $SP^d(X)$ in the evident way by partitions of the set $\{1,\ldots,d\}$: a point $\sigma \in SP^d(X)$ lies in the $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_k$ stratum provided that σ consists of k distinct points with multiplicities $d_1,\ldots,d_k$. Since this stratification is algebraic, we may apply the triangulation theorem presented in [H] to conclude $SP^d(X)$ admits a semi-algebraic triangulation subordinate to this stratification. Furthermore, this theorem enables us to choose the triangulation so that $f(Y)$ is a subcomplex. We then triangulate Y and $\Gamma^d(X)$ as follows. We first triangulate the pre-images (under f and γ) of the 0-simplices of $SP^d(X)$; these pre-images are complex algebraic varieties and thereby admit a triangulation. Proceeding by induction on k, we triangulate the pre-image of each k-simplex (which are semi-algebraic sets and thereby admit triangulations) compatible with the triangulation given on the pre-image of the boundary of the simplex. Each simplex in the pre-image admits a further triangulation with the property that points $(\sigma, x) \in \Gamma^d(X)$ in an open simplex all have the same multiplicity in $\sigma \in SP^d(X)$.

The weighting on γ is that defined [FM] App B : a point $(\sigma, x) \in \Gamma^d(X)$ has weight equal to the multiplicity of x in σ. This is readily seen to provide a weighting of γ as a map of simplicial complexes (provided that the triangulations are chosen as above), so that g is equipped with a natural weighting. □

We conclude that a morphism $Y \to SP^d(X)$ of complex projective algebraic varieties induces a "Gysin map"

$$g^! : H_*(Y) \to H^*(Y \times_{SP^d(X)} \Gamma^d(X))$$

and a "correspondence homomorphism"

$$p_* \circ \tilde{f}_* \circ g^! : H_*(Y) \to H_*(X) \tag{1.1.1}$$

where $\tilde{f} : Y \times_{SP^d(X)} \Gamma^d(X) \to \Gamma^d(X)$ is the projection onto the second factor and where $p : \Gamma^d(X) \to X$ is the natural projection.

We can provide a homotopy-theoretic interpretation of this correspondence homomorphism as follows. For any CW complex B, consider the abelian monoid

$$SP(B) \equiv \coprod_{d \geqslant 0} SP^d(B)$$

and let $Z_0(B)$ denote the "naive group completion" of $SP(B)$, defined as the quotient of $SP(B)^2$ modulo the equivalence relation $(\sigma,\mu) \sim (\sigma',\mu')$ whenever $\sigma + \mu' = \mu + \sigma'$ (cf. [DT], where the notation $AG(B)$ is used instead of $Z_0(B)$). One has a natural isomorphism

$$\delta_B : \pi_*(Z_0(B)) \simeq H_*(B)$$

originally provided by Dold and Thom in [DT]. This isomorphism is characterized by the property that it is a natural equivalence of Eilenberg-Steenrod homology theories on the category of CW complexes such that the composition

$$\delta_B \circ i_{B*} : \pi_*(B) \to \pi_*(Z_0(B)) \to H_*(B)$$

equals the Hurewicz homomorphism $\eta_B : \pi_*(B) \to H_*(B)$, where $i_B : B \to Z_0(B)$ is the evident inclusion.

Let $f : A \to SP^d(B)$ be a continuous map of CW complexes and let $f^e : SP^e(A) \to SP^e(SP^d(B)) \to SP^{de}(B)$ denote the induced map for each $e \geqslant 0$. We denote by

$$f^+ : Z_0(A) \to Z_0(B)$$

the group completion of $\coprod_{d\geqslant 0} f^e$. Then, we define

$$\Phi_f \equiv \delta_B \circ f_*^+ \circ \delta_A^{-1} : H_*(A) \to H_*(B). \tag{1.1.2}$$

Since $Z_0(-)$ is a homotopy functor, Φ_f depends only upon the homotopy type of f.

PROPOSITION 1.2. — *Let $f : Y \to SP^d(X)$ be a morphism of complex projective varieties. Then the maps of (1.1.1) and (1.1.2) associated to f,*

$$p_* \circ \tilde{f}_* \circ g^! \ , \ \Phi_f : H_*(Y) \to H_*(X)$$

are equal.

Proof. — In [FM] App.B, it is verified that $\tilde{f}_* \circ g^! = \gamma^! \circ f_*$. We easily reduce to the special case where $A = SP^d(X)$, $B = X$, and $f : A \to B$ is the identity mapping.

Triangulate $\Gamma^d(X) \to SP^d(X)$ as in Proposition 1.1 and consider the associated map of simplicial sets (with respect to which (1.1.1) is defined), $\tau : T \to S'$. This weighted map τ and the weighted map

$\gamma : \Gamma^d(S) \to SP^d(S)$ (where S is the simplicial set associated to the simplicial complex X) have geometric realizations which are related by a homeomorphism which respects weightings. Thus, in our special case, $f = \mathrm{id} : SP^d(X) \to SP^d(X)$, we may take $g^!$ equal to $\gamma^! : H_*(SP^d(S)) \to H_*(\Gamma^d(S))$.

We recall that the Dold-Thom isomorphism for a simplicial set S is merely the identification of the (unnormalized) chain complex of the simplicial abelian group $\mathbb{Z}(S)$ with the chain complex $C_*(S)$, thereby providing a (tautological) isomorphism $\delta_S : \pi_*(\mathbb{Z}(S)) \simeq H_*(S)$. Hence, it suffices to prove that

$$p_* \circ \gamma^! \ = \ \delta_S \circ (\mathrm{id}^+_{SP^d(S)})_* \circ \delta^{-1}_{SP^d(S)}.$$

This equality follows from the explicit identification of $p_* \circ \gamma^!$ given in [FM] App. B, as the map in homology induced by the map of chain complexes $C_*(SP^d(S)) \to C_*(S)$ defined by sending a k-simplex of $SP^d(S)$ (an orbit under the symmetric group of the k-simplices of S^d) to the sum with multiplicities of the underlying k-simplices of S. □

Now let us consider Chow correspondences $f : Y \to C_{r,d}(X)$ whose relative dimension r is not necessarily 0. Let

$$\mathcal{C}_r(X) \equiv \coprod_{d \geqslant 0} C_{r,d}(X)$$

denote the Chow monoid of effective r-cycles on X. The isomorphism class of this algebro-geometric abelian monoid is shown by Barlet [B] to be independent of the projective embedding $X \subset P^n$. We shall view $\mathcal{C}_r(X)$ as an abelian topological monoid whose topology is inherited from the analytic topology of each $C_{r,d}(X)$. Following P. Lima-Filho [LF], we shall consider the abelian topological group $Z_r(X)$, the "naive group completion" of $\mathcal{C}_r(X)$ defined as the quotient space of $\mathcal{C}_r(X)^{\times 2}$ by the equivalence relation $(Z_1, Z_2) \sim (Z_1', Z_2')$ whenever $Z_1 + Z_2' = Z_2 + Z_1'$. Thus, when $r = 0$, $\mathcal{C}_r(X) = SP(X)$ and $Z_0(X)$ is the abelian topological group considered above (with the same name). As can be seen from its step-by-step construction (cf. [LF], [FG]), $Z_r(X)$ admits the structure of an abelian group object in the category of CW complexes.

In previous work, there have been (at least) four approaches to forming the group completion of the topological monoid $\mathcal{C}_r(X)$. Namely, in [F1] the homotopy theoretic group completion $\Omega B\mathcal{C}_r(X)$ was considered; this is an H-space homotopy equivalent to a CW complex (cf. [M]) with component monoid a group such that the "natural" map $\mathcal{C}_r(X) \to \Omega B\mathcal{C}_r(X)$ has the effect in homology of localizing the action of $\pi_0(\mathcal{C}_r(X))$ on $H_*(\mathcal{C}_r(X))$ (cf. [MS]). In [LF], [FG], $\Omega \mathrm{B}\mathcal{C}_r(X)$ was shown to be naturally homotopy equivalent to $Z_r(X)$. In [FM], the simplicial abelian monoid Lim Sing $\mathcal{C}_r(X)$ was considered, defined as the direct limit of copies of the simplical abelian monoid of singular simplices of $\mathcal{C}_r(X)$ indexed by a "base system" associated to $\pi_0(\mathcal{C}_r(X))$. This was seen to be equivalent as a simplicial monoid to the group completion of the simplicial monoid Sing $\mathcal{C}_r(X)$ as well as equivalent to the singular complex of $\Omega B\mathcal{C}_r(X)$. Finally, in [FG], the simplicial abelian group Sing$Z_r(X)$ was replaced by its normalized chain complex $\tilde{Z}_r(X)$.

We define the Lawson homology groups to be the homotopy groups of any of these group completions (or the homology groups of the chain complex $\tilde{Z}_r(X)$). Thus,

$$\begin{aligned} L_rH_{*+2r}(X) &= \pi_*(\Omega B\mathcal{C}_r(X)) \\ &= \pi_*(Z_r(X)) = \pi_*(\mathrm{Lim\ Sing}\,\mathcal{C}_r(X)) = H_*(\tilde{Z}_r(X)). \end{aligned}$$

Various homotopy-theoretic properties of $Z_r(X)$ which we require have been proved for $\Omega \mathrm{B}\mathcal{C}_r(X)$, Lim Sing $\mathcal{C}_r(X)$, and $\tilde{Z}_r(X)$ in [F1], [FM], and [FG]. The equivalences discussed above justify our use of these references.

In [FM], the join pairing

$$\# : \mathcal{C}_r(X) \times \mathcal{C}_0(P^1) \to \mathcal{C}_{r+1}(X\#P^1)$$

(sending an irreducible subvariety $Z \subset X \subset P^n$ and point $t \in P^1$ to the cone on Z with vertex t) and the Lawson suspension equivalence $Z_{r+1}(X\#P^1) \to Z_{r-1}(X)$, defined up to homotopy, are combined to provide a pairing

$$s : Z_r(X) \wedge S^2 \to Z_{r-1}(X). \tag{1.2.1}$$

In [FG], the homotopy type of this pairing is shown to be independent of the projective embedding $X \subset P^n$. We shall also denote by s the maps

$$Z_r(X) \to \Omega^2 Z_{r-1}(X) \quad , \quad \pi_*(Z_r(X)) \to \pi_{*+2}(Z_{r-1}(X)) \tag{1.2.2}$$

the first being the adjoint of (1.2.1) and the second being the map in homotopy induced by (1.2.1).

If $f : Y \to C_{\dot{r},d}(X)$ is a Chow correspondence (of relative dimension $r \geqslant 0$), let $f^e : SP^e(Y) \to SP^e(C_{r,d}(X)) \to C_{r,de}(X)$ denote the induced map for each $e \geqslant 0$, and let

$$f^+ : Z_0(Y) \to Z_r(X) \tag{1.2.3}$$

denote the group completion of $\coprod_{e\geqslant 0} f^e$.

Definition 1.3. – *The* **correspondence homomorphism** Φ_f *associated to the Chow correspondence* $f : Y \to C_{r,d}(X)$ *is the following composition*

$$\delta_X \circ s^r \circ (f^+)_* \circ \delta_Y^{-1} : H_*(Y) \to \pi_*(Z_0(Y)) \\ \to \pi_*(Z_r(X)) \to \pi_{*+2r}(Z_0(X)) \to H_{*+2r}(X).$$

If f *restricts to* $f_1 : V \to C_{r,d}(W)$, *then the* relative correspondence homomorphism *is defined to be the composition*

$$\Phi_{f,f_1} \equiv \delta_{X,W} \circ s^r \circ (f^+)_* \circ (\delta_{Y,V})^{-1} : H_*(Y,V) \simeq \pi_*(Z_0(Y)/Z_0(V))$$

$$\to \pi_*(Z_r(X)/Z_r(W)) \to \pi_{*+2r}(Z_0(X)/Z_0(W)) \simeq H_{*+2r}(X,W).$$

If V *is empty, then we denote this composition by*

$$\Phi_f : H_*(Y) \to H_{*+2r}(X,W).$$

Given Chow correspondences $f : Y \to C_{r,d}(X)$ and $g : W \to C_{r,e}(X)$, we obtain the Chow correspondence

$$f + g : Y \times W \to C_{r,d+e}(X)$$

using the additive structure $C_{r,d}(X) \times C_{r,e}(X) \to C_{r,d+e}(X)$. The equality

$$(f+g)^+ = f^+ + g^+ : Z_0(Y \times W) \to Z_r(X)$$

immediately implies the following useful property of the correspondence homomorphism :

$$\Phi_{f+g} = \Phi_g \circ pr_{g*} + \Phi_f \circ pr_{f*}. \tag{1.3.1}$$

We recall the *graph mapping*

$$\Gamma_f : Z_k(Y) \to Z_{r+k}(X)$$

associated to a Chow correspondence $f : Y \to \mathcal{C}_r(X)$ as considered in [F2], defined as the group completion of the composition

$$\mathrm{tr} \circ f_* : \mathcal{C}_k(Y) \to \mathcal{C}_k(\mathcal{C}_r(X)) \to \mathcal{C}_{r+k}(X)$$

where f_* is the map functorially induced by f (cf. [F1], 2.9) and tr is the trace map of [FL], 7.1. In the special case $k = 0$, Γ_f equals f^+ of (1.2.3). Since the graph mapping commutes with the s-operation ([F2], 2.3), we obtain for m, k with $m - 2k \geqslant 0$ the following commutative diagram

$$\text{(1.3.2)} \qquad \begin{array}{ccc} L_k H_m(Y) & \xrightarrow{(\Gamma_f)_*} & L_{r+k} H_{2r+m}(X) \\ \delta_Y \circ s^k \big\downarrow & & \big\downarrow \delta_X \circ s^{r+k} \\ H_m(Y) & \xrightarrow{\Phi_f} & H_{m+2r}(X). \end{array}$$

As defined in [F2], 2.6, the composition product $g \cdot f : Y \to \mathcal{C}_{r+s}(T)$ of Chow correspondences $f : Y \to \mathcal{C}_r(X)$, $g : X \to \mathcal{C}_s(T)$ is defined to be the composition

$$g \cdot f = \mathrm{tr} \circ g_* \circ f : Y \to \mathcal{C}_r(X) \to \mathcal{C}_r(\mathcal{C}_s(T)) \to \mathcal{C}_{r+s}(T).$$

Proposition 1.4. — *Let Y, X, T be projective varieties and consider Chow correspondences*

$$f : Y \to \mathcal{C}_r(X) \quad , \quad g : X \to \mathcal{C}_s(T).$$

Then the correspondence homomorphism associated to the composition product defined above is given as the composition of the correspondence homomorphisms :

$$\Phi_{g \cdot f} = \Phi_g \circ \Phi_f : H_*(Y) \to H_{*+2r+2s}(T).$$

Proof. — We compare the following diagram

$$\begin{array}{ccccc} L_0 H_*(Y) & \xrightarrow{(\Gamma_f)_*} & L_r H_{2r+*}(X) & \xrightarrow{(\Gamma_g)_*} & L_{r+s} H_{*+2r+2s}(T) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H_*(Y) & \xrightarrow{\Phi_f} & H_{*+2r}(X) & \xrightarrow{\Phi_g} & H_{*+2r+2s}(T) \end{array}$$

to the square

$$\begin{array}{ccc} L_0 H_*(Y) & \xrightarrow{(\Gamma_{g \cdot f})_*} & L_{r+s} H_{*+2r+2s}(T) \\ \big\downarrow & & \big\downarrow \\ H_*(Y) & \xrightarrow{\Phi_{g \cdot f}} & H_{*+2r+2s}(T) \end{array}$$

where the vertical maps are the natural homomorphisms from Lawson homology to singular homology. By (1.3.2), these diagrams commute. By [F2], 2.7, the composition of the upper row of the first diagram equals the upper arrow of the square. The proposition now follows, since the left vertical arrow of each diagram is an isomorphism. □

As observed in [F2], the graph mapping $\Gamma_f : Z_k(Y) \to Z_{r+k}(X)$ associated to a Chow correspondence $f : Y \to \mathcal{C}_r(X)$ admits a refinement

$$\tilde{\Gamma}_f : Z_k(Y) \to Z_{r+k}(V_f),$$

where $V_f = \mathrm{pr}_{X*}(|Z_f|)$ is the projection to X of the support of the cycle Z_f on $Y \times X$ associated to f, giving the commutative triangle

$$\begin{tikzcd} H_*(Y) \arrow[d, "\Phi_f"'] \arrow[dr, "\tilde{\Phi}_f"] & \\ H_{*+2r}(X) & H_{*+2r}(V_f) \arrow[l] \end{tikzcd} \tag{1.4.1}$$

Definition 1.5. — *Let $f : Y \to \mathcal{C}_{r,d}(X)$ be a Chow correspondence. The* total characteristic class *$\langle f \rangle$ of f is the homotopy class of the composition*

$$s^r \circ \Gamma_f \circ i_Y : Y \to Z_0(Y) \to Z_r(X) \to \Omega^{2r} Z_0(X)$$

where $i_Y : Y \to Z_0(Y)$ is the natural inclusion $Y = \mathcal{C}_{0,1}(Y) \subset \mathcal{C}_0(Y) \to Z_0(Y)$.

If $\mathrm{Ext}^1(H_{i-1}(X), H_i(X)) = 0$ *for all $i > 0$ so that the identification $Z_0(X) \simeq \prod_i K(H_i(X), i)$ is naturally determined up to homotopy, then we view $\langle f \rangle$ as a* total cohomology class

$$\langle f \rangle \in \prod_i H^i(Y, H_{2r+i}(X)).$$

Observe that $\langle f \rangle$, the homotopy class of $s^r \circ \Gamma_f \circ i_Y$, naturally determines the correspondence homomorphism Φ_f.

2. Examples.

If a subvariety $Z \subset Y \times X$ is flat over Y of relative dimension $r \geqslant 0$, then $Z = Z_f$ for one and only one Chow correspondence $f : Y \to C_{r,d}(X)$. One way to see this is to appeal to Hilbert schemes : the flat "family" $Z \to Y$ is equivalent to a map $Y \to \mathrm{Hilb}_r(X)$ which naturally maps to $\mathcal{C}_r(X)$. Alternatively, any cycle Z on $Y \times X$ each component of which dominates some component of Y determines a generically defined map $\phi_Z : Y - - > C_{r,d}(X)$. The flatness of Z over Y implies that the specializations of the generic fibres of Z at some closed point of Y depend only upon $y \in Y$ and not the "path" of specialization. This is equivalent to the assertion that ϕ extends to a continuous algebraic map. The uniqueness of such an extension is clear.

The following summarizes this first example of a Chow correspondence.

Example 2.1. — A flat map $g : X \to Y$ of relative dimension $r \geqslant 0$ determines a Chow correspondence $f : Y \to \mathcal{C}_r(X)$ which sends a point $y \in Y$ to the Chow point of the cycle $[X_y]$ associated to the scheme-theoretic fibre of g above y.

Example 2.2. — Let E be a rank r (algebraic) vector bundle over Y and let $P(E \oplus 1), P(E)$ denote the projective bundles associated to the bundles $E \oplus 1, E$. Let

$$f_E : Y \to \mathcal{C}_r(P(E \oplus 1))$$

denote the Chow correspondence associated to the flat projection $P(E \oplus 1) \to Y$. Then the associated relative correspondence homomorphism

$$\Phi_E \equiv \Phi_{f_E} : H_*(Y) \to H_{*+2r}(P(E \oplus 1), P(E))$$

is an isomorphism.

Moreover, the inverse of Φ_E is given by cap product with the Thom class $\tau_E \in H^{2r}(P(E \oplus 1), P(E))$.

Proof. — Following Lima-Filho [LF], we let $Z_r(U)$ denote $Z_r(Y)/Z_r(W)$ whenever $i : W \subset Y$ is a closed subvariety of the projective variety Y with complement U. As an object in the derived category, the complex $\tilde{Z}_r(U)$ (defined as the normalized chain complex of the simplicial abelian group

Sing $Z_r(X)$) depends only upon U and not the projective closure $U \subset Y$ [FG], 1.6. Thus, we may write $Z_r(V(E))$ for $Z_r(P(E \oplus 1))/Z_r(P(E))$, where $V(E)$ denotes the quasi-projective variety associated to the symmetric algebra of the dual of E as an O_Y-module. Moreover, $\Gamma_{f_E} : Z_0(Y) \to Z_r(V(E))$ is then identified with flat pull-back of cycles via $\pi : V(E) \to Y$.

Flat pull-back determines a map of distinguished triangles of chain complexes (arising from the localization property of Lawson homology; cf [FG], 1.6)

$$\begin{array}{ccccc} \tilde{Z}_0(W) & \to & \tilde{Z}_0(Y) & \to & \tilde{Z}_0(U) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{Z}_r(V(i^*E)) & \to & \tilde{Z}_r(V(E)) & \to & \tilde{Z}_r(V(E_{|U})). \end{array}$$

Arguing by induction on the dimension of Y, we must show that the relative correspondence homomorphism

$$\Phi_E : H_*(Y, W) \to H_{*+2r}(P(E \oplus 1), P(E) \cup P(i^*(E \oplus 1)))$$

is an isomorphism with inverse given by cap product with τ_E whenever E restricted to U is trivial. A similar argument further reduces the proof to the special case in which E is the trivial rank r bundle on Y.

We are thus reduced to verifying that the composition

$$Z_0(Y) \to Z_r(Y \times P^r) \to \Omega^{2r}(Z_0(Y \times P^r)/Z_0(Y \times P^{r-1}))$$

induces the evident isomorphism in homotopy groups. Using the representation of the s-map given in (1.2.1) and representing S^{2r} as P^r/P^{r-1}, we may interpret this composition as the map sending $y \in Y$ to the map $P^r/P^{r-1} \to Z_0(Y \times P^r)/Z_0(Y \times P^{r-1})$ induced by $P^r \to Z_0(Y \times P^r)$ sending $t \in P^r$ to (y, t). The required isomorphism in homotopy groups is now a special case of the general observation for any simplicial set T that the natural map $\mathbb{Z}(T) \to \Omega\mathbb{Z}(\sigma(T))$ induces the evident isomorphism in homotopy groups, where $\sigma(T)$ is the (topologist's) suspension of T. □

The following example, essentially a special case of our previous example, is a generalization to possibly singular varieties Y of the "suspension isomorphism" of [FM], App. A. Recall that the r-th *algebraic suspension* $\Sigma^r X \subset P^{n+r}$ of $X \subset P^n$ equals the *algebraic join* $X \# P^{r-1}$.

Example 2.3. — Consider the Chow correspondence $\nu_r : X \to C_{r,1}(\Sigma^r X)$ associated to the cycle $\Sigma^r(\Delta/X) \subset X \times \Sigma^r X$ consisting of pairs (x, y) with $y \in x\#P^{r-1} \subset P^{n+r}$. Then the graph mapping

$$\Gamma_{\nu_r} : Z_k(X) \to Z_{k+r}(\Sigma^r X)$$

equals the map which sends a k-cycle to its r-th algebraic suspension. Moreover, the associated correspondence homomorphism

$$\Phi_{\nu_r} : H_*(X) \to H_{*+2r}(X\#P^{r-1})$$

is an isomorphism.

Proof. — Γ_{ν_r} sends a k-dimensional subvariety $Y \subset X$ to the projection via $\mathrm{pr}_2 : Y \times \Sigma^r X \to \Sigma^r X$ of the cycle associated to the restriction of ν_r to Y, $Y \to C_{r,1}(\Sigma^r X)$. This projection is readily seen to be the r-th algebraic suspension of Y, so that Γ_{ν_r} is the map which sends a k-cycle to its r-th algebraic suspension. Since the graph mapping commutes with the s-operation, we factor $s^r \circ \Gamma_{\nu_r}$ as

$$(s \circ \Gamma_\nu)^r : Z_0(X) \to \Omega^2 Z_0(\Sigma X) \to \ldots \to \Omega^{2r} Z_0(\Sigma^r X)$$

where $\nu = \nu_1 : \Sigma^i X \to C_{1,1}(\Sigma(\Sigma^i X))$. Thus, it suffices to consider the case $r = 1$.

We view ΣX as $P(O_X(1) \oplus 1)/P(O_X(1))$. By [FG], 1.6, the projection $P(O_X(1) \oplus 1)/P(O_X(1)) \to (\Sigma X, pt)$ induces a quasi-isomorphism of chain complexes

$$\tilde{Z}_r(P(O_X(1) \oplus 1))/\tilde{Z}_r(P(O_X(1))) \simeq \tilde{Z}_r(\Sigma X)/\tilde{Z}_r(pt).$$

Since $\nu : X \to C_{1,1}(\Sigma X) \subset \mathcal{C}_1(\Sigma X)$ factors through $X \to \mathcal{C}_1(P(O_X(1) \oplus 1))$, the asserted isomorphism now follows from that of Example 2.2. $\square$

We shall have occasion to use the following relative form of the suspension map as first introduced in [FM].

Example 2.4. — Consider the Chow correspondence

$$\nu_{r/Y} \equiv \times \circ (1 \times \nu_r) : Y \times X \to Y \times C_{r,1}(\Sigma^r X) \to \mathcal{C}_r(Y \times \Sigma^r X).$$

Then the graph mapping

$$\Gamma_{\nu_r/Y} : Z_k(Y \times X) \to Z_{r+k}(Y \times \Sigma^r X)$$

equals the r-th fibre-wise (over Y) algebraic suspension mapping Σ^r_Y as introduced in [FM], App.A. Moreover, the associated correspondence homomorphism

$$\Sigma^r_{Y*} \equiv \Phi_{\nu_r/Y} : H_*(Y \times X) \to H_{*+2r}(Y \times \Sigma^r X)$$

sends $b \otimes c \in H_i(Y) \otimes H_j(X)$ to $b \otimes \Sigma_*^r(c)$.

Proof. — The identification of $\Gamma_{\nu_r/Y}$ with Σ_Y^r is easily verified using the observation that any point $y \times x \in Y \times X$ is mapped by ν_r/Y to $y \times (x \# P^{r-1})$.

To identify Σ_{Y*}^r on $b \otimes c$ we use the fact proved in [F2], 1.5 that $s^r : \pi_*(Z_r(X)) \otimes \pi_{2r}(Z_0(P^r)) \to \pi_{*+2r}(Z_0(X))$ is induced by $Z_r(X) \times P^r \to Z_r(X \times P^r) \to Z_0(X)$, where the last map is the Gysin map of [FG]. The map sending $b \otimes c$ to $b \otimes \Sigma_*^r(c)$ is determined by the upper row of the following diagram, whereas $\Phi_{\nu_r/Y}$ is determined by the lower row :

$$
\begin{array}{ccccc}
Z_0(Y) \times Z_0(X) \times P^r & \to & Z_0(Y) \times Z_r(\Sigma^r X) \times P^r & & \\
\downarrow & & \downarrow & & \\
Z_0(Y \times X) \times P^r & \to & Z_r(Y \times \Sigma^r X) \times P^r & & \\
& \to & Z_0(Y) \times Z_r(\Sigma^r X \times P^r) & \to & Z_0(Y) \times Z_0(X) \\
& & \downarrow & & \downarrow \\
& \to & Z_r(Y \times \Sigma^r X \times P^r) & \to & Z_0(Y \times \Sigma^r X).
\end{array}
$$

The commutativity of this diagram follows from the naturality of the Gysin map [FG], 3.4.d. □

The following proposition justifies our view of $\langle f \rangle$ as a characteristic class of the Chow correspondence $f : Y \to \mathcal{C}_r(X)$. Because P^n has homology only in even dimensions, we may view $\langle f \rangle$ associated to a Chow correspondence $f : Y \to \mathcal{C}_r(P^n)$ as a cohomology class $\langle f \rangle \in \prod_i H^i(Y, H_{2r+i}(P^n))$.

PROPOSITION 2.5. — *Let $f : Y \to \mathrm{Grass}_{N-r}(P^N) = C_{N-r,1}(P^N)$ be the classifying map associated to the data of a rank r vector bundle E on Y provided with $N+1$ generating global sections. Then*

$$\langle f \rangle \in \prod_i H^i(Y, H_{2N-2r+i}(P^N)) = \prod_{j=0}^{j=r} H^{2j}(Y, \mathbb{Z})$$

equals the total Chern class of E.

Proof. — Clearly, it suffices to take f to be the identity id, corresponding to the universal algebraic vector bundle of rank r over $G = \mathrm{Grass}_{N-r}(P^N)$ generated by $N+1$ global sections.

By Example 2.3, the correspondence homomorphism for the Chow correspondence $\nu_r : P^r \to \mathrm{Grass}_{N-r}(P^N)$ induces an isomorphism

$$\Phi_{\nu_r} : H_*(P^r) \to H_{*+2N-2r}(P^N). \tag{2.5.1}$$

Moreover, using the identification of Φ_E as the inverse of cap product with τ_E and the identification of Φ_{ν_r} in terms of iterates of $\Phi_{O(1)}$, we conclude that (2.5.1) sends the oriented generator of $H_{2j}(P^r)$ to the oriented generator of $H_{2j+2N-2r}(P^N)$.

By Example 2.3, $\Phi_{\nu_r} = \delta_{P^N} \circ s^{N-r} \circ \Sigma^{N-r} \circ (\delta_{P^r})^{-1}$. Using the identification of Φ_{ν_r} on $H_*(P^r)$ achieved above and the usual splitting $Z_0(P^r) \simeq \prod_i K(\mathbb{Z}, 2i)$, we conclude that $\epsilon_r : P^r \to Z_0(P^r)$ is homotopic to the composition $s^{N-r} \circ \Sigma^{N-r} \circ \epsilon_r$. This implies that

$$s^{N-r} \circ \Sigma^{N-r} : Z_0(P^r) \to Z_{N-r}(P^N) \to Z_0(P^r)$$

is homotopic to the identity.

We consider the following diagram

$$\begin{array}{ccccccc}
P^r \to & Z_0(P^r) & = & Z_0(P^r) & \simeq & \prod_{j=0}^r K(\mathbb{Z}, 2j) \\
& {\scriptstyle \Sigma^{N-r}}\big\downarrow & & & & \big\downarrow \\
G \to & Z_{N-r}(P^N) & \overset{s^{N-r}}{\to} & \Omega^{2N-2r} Z_0(P^N) & \simeq & \prod_{i=N-r}^N K(\mathbb{Z}, 2i-2N-2r)
\end{array}$$

whose splittings are chosen in the usual manner and whose right vertical arrow is the evident equivalence. The commutativity of this diagram follows from our verification that $s^{N-r} \circ \Sigma^{N-r}$ is homotopic to the identity. The bottom row of this diagram determines $\langle f \rangle$, whereas the homotopy type of the composition

$$G \to Z_{N-r}(P^N) \overset{(\Sigma^{N-r})^{-1}}{\to} Z_0(P^r) \simeq \prod_{j=0}^r K(\mathbb{Z}, 2j)$$

is shown by Lawson-Michelsohn [LM] to be the total Chern class of the universal bundle over G. □

Recall that the cohomology $H^*(P(E))$ of the projectivization $P(E)$ of a rank r vector bundle E over Y is multiplicatively isomorphic to $H^*(Y) \otimes H^*(P^{r-1})$ if and only if the total Chern class $c(E)$ vanishes in positive degrees.

QUESTION 2.6. — *Let $f : Y \to \mathcal{C}_r(X)$ be a Chow correspondence. What are the implications (if any) for $H^*(|Z_f|)$ of the condition that $\langle f \rangle$ be trivial?*

3. Reformulations.

The purpose of this section is to demonstrate that $\Phi_f \otimes \mathbb{Q}$ restricts to the correspondence homomorphism ϕ_f on $\tilde{H}_*(Y, \mathbb{Q})$, the domain of definition of ϕ_f as constructed in [FM].

For any simplicial set T, there are natural maps

$$SP^e(SP^d(T)) \to SP^{de}(T)$$

which induce a *transfer*

$$\mathrm{tr} : \mathbb{Z}(\mathbb{Z}(T)) \to \mathbb{Z}(T).$$

This leads to a *homology transfer*

$$\tau_* \equiv \delta_T \circ \mathrm{tr}_* \circ (\delta_{\mathbb{Z}(T)})^{-1} : H_*(\mathbb{Z}(T)) \to H_*(T).$$

Similarly, for a CW complex B, we obtain

$$\mathrm{tr} : Z_0(Z_0(B)) \to Z_0(B) \quad , \quad \tau_* : H_*(Z_0(B)) \to H_*(B)$$

where tr is induced by the abelian group structure on $Z_0(B)$.

Recall that the Hurewicz homomorphism $\eta_B : \pi_*(B) \to H_*(B)$ for any C.W. complex B is the composition

$$\eta_B = \delta_B \circ i_* : \pi_*(B) \to \pi_*(Z_0(B)) \to H_*(B)$$

where $i : B \to Z_0(B)$ is the natural inclusion (cf. [DT]). We record the following immediate consequence of the fact that $\mathrm{tr} : Z_0(A) \to A$ induced by the abelian group structure of a topological abelian group A satisfies the condition $\mathrm{tr} \circ i = \mathrm{id}_A$.

LEMMA 3.1. — *Let A be a CW complex with the structure of an abelian topological group. Then*

$$\xi_A \equiv \mathrm{tr}_* \circ (\delta_A)^{-1} : H_*(A) \to \pi_*(A)$$

has the property that $\xi_A \circ \eta_A$ equals the identity of $\pi_*(A)$. □

The following reformulation of the correspondence homomorphism Φ_f involves only maps in homology, so that it lends itself more readily to comparison with the constructions of [FM].

PROPOSITION 3.2. — *We define* Φ_r *as*

$$\Phi_r \equiv \tau_* \circ (\Sigma^{r+1})^{-1} \circ (\#)_* \circ (1 \otimes [P^r]) :$$

$$H_*(Z_r(X)) \to H_{*+2r}(Z_r(X) \times P^r) \to H_{*+2r}(Z_{r+1}(X\#P^r)) \to H_{*+2r}(Z_0(X)) \to H_{*+2r}(X).$$

Then

$$\Phi_r = \delta_X \circ s^r \circ \xi_{Z_r(X)}.$$

Let $f : Y \to C_{r,d}(X)$ *be a Chow correspondence of relative dimension* $r \geqslant 0$. *Then*

$$\Phi_f = \Phi_r \circ j_* \circ f_* : H_*(Y) \to H_*(C_{r,d}(X)) \to H_*(Z_r(X)) \to H_{*+2r}(X).$$

In particular, if $r = 0$, *then* $\Phi_f = \tau_* \circ j_* \circ f_*$.

Proof. — In [F2], 1.5, the r-th iterate of the s-map is shown to be induced by the pairing

$$(\Sigma^{r+1})^{-1} \circ \# : Z_r(X) \times P^r \to Z_{r+1}(X\#P^r) \to Z_0(X),$$

where Σ^{r+1} is the map which sends a cycle to its $r+1$-st algebraic suspension. Consider the following diagram, whose commutativity follows immediately from the naturality of the Dold-Thom isomorphism and tr_* :

$$\begin{array}{ccccc}
\pi_*(Z_r(X)) \otimes \pi_{2r}(Z_0(P^r)) & \xrightarrow{(\#)_*} & \pi_{*+2r}(Z_{r+1}(X\#P^r)) & \xrightarrow{(\Sigma^{r+1})^{-1}} & \pi_{*+2r}(Z_0(X)) \\
\uparrow {\scriptstyle \xi_{Z_r(X)} \otimes \delta_{P^r}^{-1}} & & \uparrow {\scriptstyle \xi_{Z_{r+1}(X\#P^r)}} & & \uparrow {\scriptstyle \xi_{Z_0(X)}} \\
H_*(Z_r(X)) \times H_{2r}(P^r) & \xrightarrow{(\#)_*} & H_{*+2r}(Z_{r+1}(X\#P^r)) & \xrightarrow{(\Sigma^{r+1})^{-1}} & H_{*+2r}(Z_0(X))
\end{array}$$

in which the map $(\#)_*$ on the tensor product of homotopy groups is the natural map induced from the bilinear mapping of groups $\# : Z_r(X) \times Z_0(P^r) \to Z_{r+1}(X\#P^r)$. Since the lower row when composed with τ_* determines Φ_r and since $\tau_* = \delta_X \circ \xi_{Z_0(X)}$, the commutativity of this diagram implies the equality

$$\Phi_r = \delta_X \circ s^r \circ \xi_{Z_r(X)}.$$

The equality $\Gamma_f = \mathrm{tr} \circ j_* \circ f_* : Z_0(Y) \to Z_r(X)$ implies the commutativity of the square

$$\begin{array}{ccccc} H_*(Y) & \overset{(\delta_Y)^{-1}}{\to} & \pi_*(Z_0(Y)) & \overset{(\Gamma_f)_*}{\to} & \pi_*(Z_r(X)) \\ = \updownarrow & & & & \uparrow \xi_{Z_r(X)} \\ H_*(Y) & \overset{f_*}{\to} & H_*(C_{r,d}(X)) & \overset{j_*}{\to} & H_*(Z_r(X)). \end{array}$$

Since the composition of δ_X and the concatenation of the upper rows of the two squares determine Φ_f, we conclude the asserted equality

$$\Phi_f = \Phi_r \circ j_* \circ f_*. \qquad \square$$

We recall that for e sufficiently large, there exists a map $\psi_e : C_{r+1,d}(X\#P^r) \to C_{0,de}(X)$ with the property that $\Sigma^{r+1} \circ \psi_e$ is algebraically homotopic to multiplication by e (cf. [F], 3.5). Thus, ψ_e represents e times the Lawson inverse of Σ^{r+1}.

Proposition 3.3. — *The correspondence homomorphism $\Phi_f : H_*(Y) \to H_{*+2r}(X)$ sends a class $c \in H_*(Y)$ to*

$$(\Phi_{\psi_{e+1}\#f} - \Phi_{\psi_e\#f})(c \times [P^r]) = \tau_* \circ j_* \circ (\psi_{e+1}\#f - \psi_e\#f)_*(c \times [P^r])$$

where $\psi_e\#f$ denotes the composition

$$Y \times P^r \overset{f\times 1}{\to} C_{r,d}(X) \times P^r \overset{\#}{\to} C_{r+1,d}(X\#P^r) \overset{\psi_e}{\to} C_{0,de}(X)$$

and $j : C_{0,de}(X) \to Z_0(X)$ is the natural inclusion.

Proof. — The equality $\Phi_{\psi_e\#f} = \tau_* \circ j_* \circ (\psi_e\#f)_*$ is given by Proposition 3.2. To compare this with Φ_f, we consider the following diagram :

$$\begin{array}{ccccccccc} Y \times P^r & \to & C_{r,d}(X) \times P^r & \to & C_{r+1,d}(X\#P^r) & \overset{\psi_e}{\to} & & & C_{0,de}(X) \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ Y \times P^r & \to & Z_r(X) \times P^r & \to & Z_{r+1}(X\#P^r) & \overset{(\Sigma^{r+1})^{-1}}{\to} & Z_0(X) & \overset{e}{\to} & Z_0(X) \end{array}$$

whose vertical arrows are the natural inclusions and the composition of whose top row is $\psi_e\#f$. Then all squares but the right-most square commute, whereas the right-most square commutes up to homotopy. Proposition 3.2 implies that the map in homology induced by the bottom row when applied to $c \times [P^r]$ and then composed with τ_* yields $\Phi_f(c)$ for any

$c \in H_*(Y)$. The homotopy commutativity of the diagram implies the asserted identification of $\Phi_f(c)$. □

THEOREM 3.4. — *Let $f : Y \to C_{r,d}(X)$ be a Chow correspondence with Y smooth of dimension n, and let Z be the associated equidimensional geometric correspondence in $Y \times X$. Then*

$$\Phi_f = \phi_Z : H_*(Y) \to H_{*+2r}(X)$$

where ϕ_Z is defined as the composition of the Poincaré duality isomorphism $\mathcal{D}$ and slant product with $[Z]$.

Proof. — For $r = 0$, the equality $\Phi_f = \phi_Z$ follows from Proposition 1.2 and the verification in [FM], 4.5 that $\phi_Z = p_* \circ \gamma_d^! \circ f_*$.

Let $T(e)$ in $Y \times P^r \times X$ denote the cycle associated to $\psi_e \# f$: $Y \times P^r \to C_{0,de}(X)$ for e sufficiently large as in Proposition 3.3. Let W in $(Y \times P^r) \times (X \# P^r)$ denote the cycle associated to the map $f\#1 : Y \times P^r \to C_{r+1,d}(X\#P^r)$. We consider the following diagram

$$\begin{array}{ccccc}
H_*(Y) & \xrightarrow{\times[P^r]} & H_{*+2r}(Y \times P^r) & = & H_{*+2r}(Y \times P^r) \\
\mathcal{D}\Big\downarrow & & \mathcal{D}\Big\downarrow & & \Big\downarrow\mathcal{D} \\
H^{2n-*}(Y) & \xrightarrow{pr_1^*} & H^{2n-*}(Y \times P^r) & = & H^{2n-*}(Y \times P^r) \\
\backslash[Z]\Big\downarrow & & \backslash[W]\Big\downarrow & & \Big\downarrow\backslash[T(e+1)] - [T(e)] \\
H_{*+2r}(X) & \xrightarrow{\Sigma_*^{r+1}} & H_{*+4r+2}(X\#P^r) & \xleftarrow{\Sigma_*^{r+1}} & H_{*+2r}(X).
\end{array}$$

The composition of the maps of the right vertical column is $\phi_{T(e+1)} - \phi_{T(e)}$, which by the special case $r = 0$ equals $\Phi_{\psi_{e+1}\#f} - \Phi_{\psi_e\#f}$. Thus, Proposition 3.3 implies that the composition of the maps of the upper row and right column is Φ_f.

Since Σ_*^{r+1} is an isomorphism by Example 2.3, to prove the theorem it suffices to prove the commutativity of the diagram. Only the two lower squares require verification. The commutativity of the left lower square is given by the following equalities for any $\alpha \in H^{2n-*}(Y)$:

$$\Sigma_*^{r+1}(\alpha \backslash [Z]) = \alpha \backslash (\Sigma_Y^{r+1})_*([Z]) = \mathrm{pr}_1^*(\alpha) \backslash [W]$$

where Σ_Y denotes the fibrewise suspension over Y. The first of these equalities follows from Example 2.4 and a standard property of slant products (cf [D]). To verify the second equality, observe that

$$(\Sigma_Y^{r+1})_*([Z]) = [\Sigma_Y^{r+1}(Z)]$$

by (1.3.1). Thus, the second equality follows from the projection formula for slant product (cf. [D], 11.7) and the fact that the projection $\mathrm{pr}_{1,3}$: $Y \times P^r \times \Sigma^{r+1}(X) \to Y \times \Sigma^{r+1}(X)$ sends W to $\Sigma_Y^{r+1}(Z)$. (The generic fibre of W over $Y \times P^r$ is a cycle on $\Sigma^{r+1}X$ with Chow point the image under $\Sigma^{r+1} : C_{r,d}(X) \to C_{2r+1,d}(X\#P^r)$ of the Chow point of the generic fibre of Z over Y; since Σ^{r+1} sends the Chow point of a cycle to the Chow point of the $(r+1)$-st suspension of that cycle, we equate $\mathrm{pr}_{1,3}W$ and $\Sigma_Y^{r+1}(Z)$ by comparing generic fibres over Y.)

To prove the commutativity of the right lower square, we demonstrate the equalities

$$\Sigma_*^{r+1}(\beta \setminus [T(e)]) = \beta \setminus \Sigma_*^{r+1}([T(e)]) = \beta \setminus e \cdot [W]$$

for any $\beta \in H^{2n-*}(Y \times P^r)$ and any $e > 0$. The first of these equalities follows as above from Example 2.4. To prove the second, observe that $\Sigma^{r+1}T(e), eW$ are the associated cycles of the Chow correspondences

$$\nu_{r+1} \circ \psi_e \# f \ , \ e \cdot (\# \circ f \times 1) : \ Y \times P^r \to \mathcal{C}_{r+1}(X\#P^r).$$

Since multiplication by e is algebraically homotopic to $\nu_{r+1} \circ \psi_e$, the last equality follows from the following sublemma.

SUBLEMMA. — *Let $F : Y \times A^1 \to C_{r,d}(X)$ be a continuous algebraic map relating $f, g : Y \to C_{r,d}(X)$. Then the associated cycles Z_f, Z_g in $Y \times X$ are rationally equivalent.*

Proof. — Let Z_F denote the cycle in $Y \times A^1 \times X$ associated to F. As shown in [FM], the cycles Z_f, Z_g are given by the intersection-theoretic pull-backs of Z_F via the standard inclusions $i_0, i_1 : Y \times X \to Y \times A^1 \times X$:

$$Z_f = i_0^!(Z_F) \ , \ Z_g = i_1^!(Z_F).$$

On the other hand, since Z_F is flat over A^1, $i_t^!(Z_F)$ equals the fibre associated to the geometric fibre of Z_F above t for any point $t \in A^1$.

For any Chow correspondence $f : Y \to C_{r,d}(X)$, a map

$$\phi_f : \tilde{H}_*(Y; \mathbb{Q}) \to H_{*+2r}(X; \mathbb{Q})$$

is defined in [FM], 4.2, where $\tilde{H}_*(Y;\mathbb{Q}) \subset H_*(Y;\mathbb{Q})$ consists of homology classes of lowest weight with respect to the Mixed Hodge Structure on $H_*(Y;\mathbb{Q})$. The map ϕ_f is defined using a resolution of singularities $q : Y' \to Y$ (i.e., Y' is smooth and q is proper and birational) by the condition that

$$\phi_f(c) = \phi_{Z'}(c')$$

where $c' \in H_*(Y';\mathbb{Q})$ satisfies $p(c') = c$ and where Z' is the cycle associated to $f \circ q : Y' \to C_{r,d}(X)$. □

The following is an immediate corollary of Theorem 3.4 and the naturality of Φ_f.

COROLLARY 3.5. — *For any Chow correspondence $f : Y \to C_{r,d}(X)$, the correspondence homomorphism $\phi_f : \tilde{H}_*(Y;\mathbb{Q}) \to H_{*+2r}(X;\mathbb{Q})$ constructed in [FM] is the restriction to $\tilde{H}_*(Y;\mathbb{Q}) \subset H_*(Y;\mathbb{Q})$ of $\Phi_f \otimes \mathbb{Q}$.*

□

4. Comparison of filtrations.

In this section, we use Φ_f to define and compare filtrations on $H_*(X)$.

DEFINITION 4.1. — *Let r, i be non-negative integers. The r-th* **geometric subgroup** *(whose cohomological formulation was considered by A. Grothendieck in [G])*

$$G_r H_{2r+i}(X) \subset H_{2r+i}(X)$$

is the subgroup generated by elements of $H_{2r+i}(X)$ which lie in the image of maps $f_* : H_{2r+i}(W) \to H_{2r+i}(X)$ as $f : W \to X$ ranges over morphisms with domain W of dimension $\leqslant r+i$. The r-th **correspondence subgroup**

$$C_r H_{2r+i}(X) \subset H_{2r+i}(X)$$

is the subgroup generated by elements of $H_{2r+i}(X)$ which lie in the image of correspondence homomorphisms $\Phi_f : H_i(Y) \to H_{2r+i}(X)$ as f's range over Chow correspondences $f : Y \to C_{r,d}(X)$ with Y projective of dimension $\leqslant i$ and $d \geqslant 0$. The r-th **topological subgroup**

$$T_r H_{2r+i}(X) \subset H_{2r+i}(X)$$

is the image of the composition

$$\pi_i(Z_r(X)) \xrightarrow{s^r} \pi_{i+2r}(Z_0(X)) \overset{\delta_X}{\rightarrow} H_{i+2r}(X).$$

LEMMA 4.2. — *As in Proposition 3.2, let*

$$\Phi_r : H_i(Z_r(X)) \to H_{i+2r}(X)$$

denote the composition $\tau_* \circ (\Sigma^{r+1})^{-1} \circ (\#)_* \circ (1 \otimes [P^r])$. *Then for* $i = 0$, *the image of* Φ_r *is the group generated by the image of* $\pi_0(\mathcal{C}_r(X))$. *Furthermore, for* $i > 0$, *the image of* Φ_r *equals that of its restriction to* $H_i(\mathcal{C}_r(X))$.

Proof. — The assertion for $i = 0$ follows from the observations that $\pi_0(Z_r(X))$ is the group completion of $\pi_0(\mathcal{C}_r(X))$ and that $H_0(Z_r(X))$ is the free abelian group on $\pi_0(Z_r(X))$.

For $i > 0$, we recall from [FM] that

$$H_*(Z_r(X)) = H_*(\mathcal{C}_r(X)) \otimes_{\mathbb{Z}[\pi_0(\mathcal{C}_r(X))]} \mathbb{Z}[\pi_0(Z_r(X))].$$

Thus, to complete the proof, it suffices to show for any effective r-cycle Z on X that "addition of Z" induces $(+Z)_* : H_i(\mathcal{C}_r(X)) \to H_i(\mathcal{C}_r(X))$ which commutes with Φ_r. This readily follows from (1.3.1). □

The following theorem is a generalization of the main result of [FM] to the case of singular varieties. Furthermore, our theorem is a refinement of that of [FM] even for smooth varieties, for it is provided a comparison of filtrations on homology with *integer coefficients.*

THEOREM 4.3. — *Let* r, i *be non-negative integers. Then for any projective variety* X

$$T_r H_{2r+i}(X) \subset C_r H_{2r+i}(X) \subset G_r H_{2r+i}(X).$$

Moreover,

$$T_r H_{2r+i}(X) \otimes \mathbb{Q} = C_r H_{2r+i}(X) \otimes \mathbb{Q}.$$

Proof. — The (elementary) equalities

$$C_r H_{2r}(X) = T_r H_{2r}(X) = G_r H_{2r}(X)$$

are proved in [FM], 7.1; we assume below that $i > 0$.

We recall from Proposition 3.2 the definition

$$\Phi_r \equiv \tau_* \circ (\Sigma^{r+1})^{-1} \circ (\#)_* \circ (1 \otimes [P^r])$$

and we define

$$\Phi_{r,d} \equiv \Phi_r \circ j_{r,d*} : H_*(C_{r,d}(X)) \to H_{*+2r}(X)$$

where $j_{r,d} : C_{r,d}(X) \to Z_r(X)$ is the natural inclusion. Since any correspondence homomorphism Φ_f factors through some $\Phi_{r,d}$ by Proposition 3.2,

$$C_r H_{2r+i}(X) \subset \Phi_r(H_i(Z_r(X))).$$

For any $d > 0$, we intersect $C_{r,d}(X)$ with hypersurfaces $H_1, ..., H_{t(d)}$ ($\dim C_{r,d}(X) = t(d) + i$), such that H_j contains the singular locus and all irreducible components of dimension $< t(d) + i - (j-1)$ of $C_{r,d}(X) \cap H_1 \cap \cdots \cap H_{j-1}$ and furthermore meets properly each irreducible component of dimension $t(d) + i - (j-1)$ of this intersection. Let

$$f_d : Y_d = C_{r,d}(X) \cap H_1 \cap ... \cap H_{t(d)} \to C_{r,d}(X)$$

denote the inclusion; so defined, Y_d has dimension i. The Lefschetz hyperplane theorem for singular varieties (cf. [AF]) applied to these successive intersections implies that

$$\Phi_{f_d}(H_i(Y_d)) = \Phi_{r,d}(H_i(C_{r,d}(X))).$$

We conclude that

$$C_r H_{2r+i}(X) = \Phi_r(H_i(Z_r(X))).$$

In the above discussion, we are applying the Andreotti-Frankel result [AF] to ambient varieties (the varieties $C_{r,d}(X) \cap H_1 \cap \cdots H_{j-1}$) which are not necessarily irreducible. Since the statement of the theorem in [AF] requires the ambient variety be irreducible, let us note that their proof does not, in fact, need irreducibility : the essential key to their proof is that the complement of the hypersurface in the ambient variety is a Stein variety. But since (even in the general case of a not necessarily irreducible ambient variety) the hypersurface to be removed contains the singular locus of the ambient variety, the complement of that hypersurface is a disjoint union of Stein varieties and thus itself Stein. Hence, their theorem holds without the assumption of irreducibility.

The equality $\Phi_r = \delta_X \circ s^r \circ \xi_{Z_r(X)}$ of Proposition 3.2 and the equality $\xi_{Z_r(X)} \circ \eta_{Z_r(X)} = 1$ imply the equality

$$\delta_X \circ s^r = \Phi_r \circ \eta_{Z_r(X)}, \tag{4.3.1}$$

which immediately implies the inclusion

$$T_r H_{2r+i}(X) \subset C_r H_{2r+i}(X).$$

The Milnor-Moore theorem [MM] implies that

$$\eta_{Z_r(X)} \otimes \mathbb{Q} : \pi_*(Z_r(X)) \otimes \mathbb{Q} \to H_*(Z_r(X), \mathbb{Q})$$

is an isomorphism onto the primitive elements of the Hopf algebra $H_*(Z_r(X), \mathbb{Q})$. Consequently, by (4.3.1), to prove the asserted equality

$$T_r H_{2r+i}(X) \otimes \mathbb{Q} = C_r H_{2r+i}(X) \otimes \mathbb{Q},$$

it suffices to prove that Φ_r vanishes on decomposable elements. By Lemma 4.2, a decomposable element of $H_*(Z_r(X))$ can be represented as $\Phi_r(u \otimes v)$, for some $u \in H_k(\mathcal{C}_r(X)), v \in H_\ell(\mathcal{C}_r(X))$ with $k, \ell > 0$. In other words, such a decomposable element can be represented as $\Phi_{f+g}(u_Y \otimes u_W)$ for some $f : Y \to \mathcal{C}_r(X)$, $g : W \to \mathcal{C}_r(X)$ with $u_Y \in H_k(Y)$, $u_W \in H_\ell(W)$, $k, \ell > 0$. Since $\mathrm{pr}_{g*}(u_Y \otimes u_W) = 0 = \mathrm{pr}_{f*}(u_Y \otimes u_W)$, the vanishing of $\Phi_{f+g}(u_Y \otimes u_W)$ follows from (1.3.1).

To prove the inclusion $C_r H_{2r+i}(X) \subset G_r H_{2r+i}(X)$, we consider $f : Y \to \mathcal{C}_{r,d}(X)$ with Y of dimension $\leqslant i$. Then (1.4.1) implies that Φ_f has image contained in the image of $H_{i+2r}(V_f))$, where $V_f = \mathrm{pr}_{X*}(|Z_f|)$. Since the dimension of $|Z_f|$ is $\leqslant i + r$, we conclude that $\mathrm{im}(\Phi_f) \subset G_r H_{2r+i}(X)$. On the other hand, $C_r H_{2r+i}(X)$ is by definition the subgroup generated by such $\mathrm{im}(\Phi_f)$. □

BIBLIOGRAPHY

[AF] A. ANDREOTTI and T. FRANKEL, The Lefschetz theorem on hyperplane sections, Ann. of Math., (2) 69 (1959), 713-717.

[B] D. BARLET, Notes on the "Joint Theorem", preprint, Institut Elie Cartan, Université Nancy, 1991.

[D] A. DOLD, Lectures on Algebraic Topology, Springer-Verlag, 1972.

[DT] A. DOLD and R. THOM, Quasifaserungen und unendliche symmetrische produkte, Ann. of Math., (2) 67 (1958), 239-281.

[F1] E. FRIEDLANDER, Algebraic cycles, Chow varieties, and Lawson homology, Compositio Mathematica, 77 (1991), 55-93.

[F2] E. FRIEDLANDER, Filtrations on algebraic cycles and homology, to appear in Ann. Ec. Norm. Sup.

[FG] E. FRIEDLANDER and O. GABBER, Cycle spaces and intesection theory, in Topological Methods in Modern Mathematics, Publish or Perish 1993, 325-370.

[FL] E. FRIEDLANDER and H. B. LAWSON, A theory of algebraic cocycles, Annals of Math., 136 (1992), 361-428.

[FM] E. FRIEDLANDER and B. MAZUR, Filtrations on the homology of algebraic varieties, to appear as a Memoir of the A.M.S.

[G] A. GROTHENDIECK, Hodge's general conjecture is false for trivial reasons, Topology, 8 (1969), 299-303.

[H] H. HIRONAKA, Triangulation of algebraic sets, pp. 165-185, in Algebraic Geometry, Arcata 1974 : Proceedings of a Symposium in Pure Mathematics 29, A.M.S. Providence, 1975.

[L] H. B. LAWSON, Algebraic cycles and homotopy theory, Annals of Math., 129 (1989), 253-291.

[LM] H. B. LAWSON and M. L. MICHELSOHN, Algebraic cycles, Bott periodicity, and the Chern characteristic map, The Mathematical Heritage of Herman Weyl, AMS 1988, 241-264.

[LF] P. LIMA-FILHO, Completion and fibrations for topological monoids and excision for Lawson homology, Trans. A.M.S., 340, n° 1 (1993), 127-147.

[MS] D. McDUFF and G. SEGAL, Homology fibrations and the "group completion" theorem, Inventiones Math., 31 (1967), 279-284.

[M] J. MILNOR, On spaces having the homotopy type of a CW complex, Trans. A.M.S., 90 (1959), 272-280.

[MM] J. MILNOR and J. MOORE, On the structure of Hopf algebras, Annals of Math., (2) 81 (1965), 211-264.

[V] V. VOEVODSKY, Harvard Ph. D Thesis, 1992.

Manuscrit reçu le 5 juillet 1993,
révisé le 9 mai 1994.

E. M. FRIEDLANDER,
Department of Mathematics
Northwestern University
Evanston, IL 60208 (USA)
and
B. MAZUR,
Department of Mathematics
Harvard University
Cambridge, MA 02318 (USA).