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SEPARATRICES FOR NON SOLVABLE DYNAMICS ON $\mathbb{C}$, 0

by Isao NAKAI

To the memory of Professor Jean Martinet

Introduction.

The topology of germs of holomorphic diffeomorphisms of $\mathbb{C}$ at the origin is regarded from many different view points such as the moduli of differential equations [5], the projective holonomy of singular 1-forms [6], [19], the non isolated singularities of map germs [17], the groups generated by involutions [22] and algebraic correspondences. Recently Il'yashenko and Shcherbakov [10], [20], [21] found that non-solvable groups acting on $\mathbb{C}, 0$ possess special topological properties. Namely Shcherbakov [21] proved the following theorem, which answers affirmatively to a conjecture by Il'yashenko [10] concerning the density theorem of solutions of algebraic differential equations on the projective plane, the so-called Hudai-Verenov theorem [24].

THEOREM 0. — *For non-solvable pseudogroup of holomorphic diffeomorphisms of open neighbourhoods of* $0 \in \mathbb{C}$ *which fix 0, there exists an open dense subset* Ω *of a neighbourhood of* 0 *consisting of finitely many connected components* Ω_i *such that the orbit of each* $z \in \Omega_i$ *is dense in* Ω_i.

The purpose of this paper is to extend and prove the above theorem using the notion of separatrices (Theorem 1) and also to investigate some

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topological properties of orbits of pseudogroups using the holomorphic vector fields associated with pseudogroups (see §3-5). Let Γ be a pseudogroup consisting of diffeomorphisms $f : U_f, 0 \to f(U_f), 0$ of open neighbourhoods U_f of the complex plane $\mathbb{C}$ which fix $0 \in \mathbb{C}$. We call the group Γ_0 of the germs of those $f \in \Gamma$ the *germ* of Γ, and call Γ a *representative* of Γ_0. Γ is *non-solvable* if its germ is non-solvable.

We say that a subset $A \subset \mathbb{C}$ is *invariant under* Γ if $f(A \cap U_f) = A \cap f(U_f)$ for all $f \in \Gamma$. We call a minimal invariant set an *orbit (which is not necessarily closed)*. The orbit containing an x is unique and denoted $\mathcal{O}(x)$, which is the set of those $f(z)$ with $z \in U_f, f \in \Gamma$. Let B_f denote the set of those $z \in U_f$ such that $f^{(n)}(z) \to 0$ as $n \to \infty$, where $f^{(n)}$ stands for the n-iterated $f \circ \cdots \circ f$. If f has the indifferent linear term z (in other words *parabolic* or *flat* at the origin), $B_f \cup B_{f^{(-1)}}$ is an open neighbourhood of 0 (Proposition 2.4). The *basin* B_Γ is the set of points z for which the closure of the orbit contains the origin. Proposition 2.5 asserts that the basin is an open neighbourhood of the origin if an $f \in \Gamma$ is flat.

Assume B_Γ is an open neighbourhood of 0. The *separatrix* $\Sigma(\Gamma)$ for Γ is a closed real semianalytic subset of B_Γ, which possesses the following properties :

(1) $\Sigma(\Gamma)$ is invariant under Γ and smooth off 0.

(2) The germ of $\Sigma(\Gamma)$ at 0 is holomorphically diffeomorphic to a union of 0 and some branches of the real analytic curve $\operatorname{Im} z^k = 0$ for some k.

(3) Any orbit is dense or empty in each connected component of $B_\Gamma - \Sigma(\Gamma)$.

(4) Any orbit is dense or empty in each connected component of $\Sigma(\Gamma) - 0$.

THEOREM 1 (The separatrix theorem). — *If the germ Γ_0 of a pseudogroup Γ is non-solvable, then the basin B_Γ is a neighbourhood of 0 and Γ admits the separatrix $\Sigma(\Gamma)$.*

By definition, the separatrix $\Sigma(\Gamma)$ is unique. From this theorem we obtain

COROLLARY 2. — *If the germ Γ_0 is non-solvable and the subgroup Γ_0^0 of the germs of diffeomorphisms $h \in \Gamma_0$ with the indifferent linear term* *does not admit antiholomorphic involution, then* $\Sigma(\Gamma) - 0$ *and all orbits different from* $0 \in \mathbb{C}$ *are dense or empty on a neighbourhood of 0.*

Example. — Let $f(z) = z/1-z, U_f = \mathbb{C}-\{1 \leq \operatorname{Re} z, \operatorname{Im} z = 0\}, g(z) = \log(1+z) = z - 1/2\ z^2 + 1/3\ z^3 - \cdots$ and let U_g $(\ni 1)$ be a small neighbourhood of 0, on which g restricts to a diffeomorphism onto the image $g(U_g)$. Let Γ be the pseudogroup generated by f and g. Since on the $\tilde{z}$-plane, $\tilde{z} = 1/z$, f induces the translation by -1, the basin B_f is the whole plane $\mathbb{C}$. By Theorem 1.8, the group Γ_0 generated by the germs of f, g at 0 is non solvable. The real line $\mathbb{R}$ is invariant under the group Γ_0 and there is no other invariant curves. If U_g is small enough : $U_g \cap \mathbb{R}$ is contained in the half line $\{-1 < \operatorname{Re} z, \operatorname{Im} z = 0\}$, then g maps the real line $U_g \cap \mathbb{R}$ into $\mathbb{R}$ hence $\mathbb{R}$ is invariant under Γ. Clearly Γ preserves the upper (respectively lower) half plane. Therefore we obtain

$$\Sigma(\Gamma) = \mathbb{R}.$$

Next extend g so that the domain of definition U_g intersects with the half line $\mathbb{R}^-_{-1} = \{\operatorname{Re} z < -1, \operatorname{Im} x = 0\}$. Then g maps the intersection $U_g \cap \mathbb{R}^-_{-1}$ into the complement of the real line, where all orbits are locally dense. The local density holds also at the intersection $U_g \cap \mathbb{R}^-_{-1}$ and propergates to the negative part of the real line $\mathbb{R}^-$. Therefore we obtain

$$\Sigma(\Gamma) = \mathbb{R}^+.$$

This example suggests the following refinement. A pseudo group Γ' is a *restriction* of Γ if for any $g \in \Gamma'$ there exists an $f \in \Gamma$ such that $U_g \subset U_f$ and g is a restriction of f and conversely for any $f \in \Gamma$ there exists a $g \in \Gamma'$ which is a restriction of f. Notice that the restriction Γ' has the same germ as Γ at 0, $B_{\Gamma'} \subset B_\Gamma$ and $\Sigma(\Gamma') \supset \Sigma(\Gamma) \cap B_{\Gamma'}$.

THEOREM 3. — *Assume that the germ* Γ_0 *is non solvable. Then there exist a restriction* Γ' *of* Γ *and a germ of real analytic subset* $\Sigma \subset \mathbb{C}$ *holomorphically diffeomorphic to the germ defined by* $\operatorname{Im} z^{k'} = 0$ *(independent of* Γ'*) such that, for any restriction* Γ'' *of* Γ'*, the separatrix* $\Sigma(\Gamma'')$ *has the germ* Σ *at 0.*

Theorem 1 is proved by a microscopic observation of the orbit structure nearby the origin. More precisely, we observe the local dynamics at a $z \in B_f - 0$ defined by $f^{(-n)}g^{(m)}f^{(n)}$, $m = 0, 1, \ldots$ with a sufficiently large fixed n. When f, g are respectively *i-flat*, *j-flat* $(f(z) = z + az^{i+1} +$

$\cdots, g(z) = z + bz^{j+1} + \cdots)$ and $i < j$, the dynamics is convergent to the identity as $n \to \infty$ but a suitable real scalar multiple $\lambda_n(f^{(-n)}gf^{(n)} - \mathrm{id})$ is convergent to a holomorphic vector field denoted $\chi(f,g)$ defined on $B_f - 0$. By definition the trajectory passing through z is arbitrarily closely approximated by the orbit of type $f^{(-n)}g^{(m)}f^{(n)}(z), m = 0, 1, 2, \ldots$ with a sufficiently large n, so the vector field $\chi(f,g)$ is a time-preserving topological invariant. When the germ Γ_0 is non-solvable, Γ admits many dynamics of this type, which generate dense orbits nearby z. The separatrix theorem is proved by this local density of orbits.

Let Γ' be a pseudogroup and Γ'_0 the germ of Γ'. We say that Γ and Γ' are *topologically equivalent* (respectively *holomorphically equivalent*) if there exists a homeomorphism (resp. holomorphic diffeomorphism) $h : U, 0 \to h(U), 0$ of open neighbourhoods of the origin such that $U_f \subset U, U_g \subset h(U)$ for $f \in \Gamma, g \subset \Gamma'$ and a bijection $\phi : \Gamma \to \Gamma'$, which induces a group isomorphism of Γ_0 to Γ'_0 such that $U_{\phi(f)} = h(U_f)$ and $h \circ f = \varphi(f) \circ h$ hold for $f \in \Gamma$. We call h a *linking homeomorphism* (resp. *linking diffeomorphism*). We say that the germs Γ_0, Γ'_0 are *topologically* (resp. *holomorphically*) *equivalent* if they admit representatives, which are so.

When Γ is topologically equivalent to a Γ', the linking homeomorphism h respects those holomorphic vector fields above defined as well as the orbit structure. By the topological rigidity of generic pairs of holomorphic vector fields (Lemma 5.2), we obtain the following theorem, which is attributed to Shcherbakov [20] (This theorem is restated (Theorem 5.1) and proved in the final section in a generalized form. Similar results were obtained by Cerveau and Sad [5] and Il'yashenko [10]).

THEOREM 4 (Topological rigidity theorem). — *Assume that pseudogroups Γ, Γ' are topologically equivalent and the germs Γ_0, Γ'_0 are non-solvable. Then the restriction of the linking homeomorphism $h : B_\Gamma \to B_{\Gamma'}$ is a holomorphic (respectively anti-holomorphic) diffeomorphism if h is orientation preserving (resp. reversing).*

On the other hand Cerveau and Moussu [5] proved

THEOREM 5. — *Let G, G' be non-commutative subgroups of the group $\mathrm{Diff}^\omega(\mathbb{C}, 0)$ of germs of diffeomorphism of $\mathbb{C}$ which fix $0 \in \mathbb{C}$. Assume that G, G' are non-exceptional and formally equivalent. Then the formal conjugacy is convergent to a germ of diffeomorphism linking G and G'.*

Here a subgroup is *exceptional* if it is formally equivalent to the solvable subgroup $G_{\omega,p}, p \in \mathbb{N}, \omega \in \mathbb{C}$, generated by

$$x \to \omega x \quad \text{and} \quad h_p(x) = x(1 - px^p)^{-1/p} \quad \text{with} \quad \omega^p = -1, \quad (1)^{1/p} = 1.$$

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1. Residue and formal classification of diffeomorphisms and groups.

Let $f(z) = z + a_{k+1}z^{k+1} + \cdots, a_{k+1} \neq 0$ be a k-flat germ of diffeomorphism of $\mathbb{C}$ at 0. By a simple calculation we see that f is equivalent to the normal form $z + z^{k+1} + bz^{2k+1} + \cdots$, i.e. there is a germ of diffeomorphism ϕ of $\mathbb{C}, 0$ such that

$$\phi^{-1} \circ f \circ \phi = z + z^{k+1} + bz^{2k+1} + \cdots,$$

and f is formally equivalent to $z + z^{k+1} + bz^{2k+1}$ by a formal diffeomorphism ϕ (see e.g. [2]). The $b \in \mathbb{C}$ is the unique formal invariant for germs of diffeomorhpisms. Define the *residue* of f by

$$\operatorname{res}(f) = -b$$

in other words

$$\operatorname{res}(f) = \frac{1}{2\pi\sqrt{-1}} \oint \frac{1}{f(z) - z} dz$$

and define the *normalized residue* by

$$\operatorname{Res}(f) = \operatorname{res}(f) + \frac{k+1}{2} = -b + \frac{k+1}{2}.$$

These invariants play the role to describe the asymptotic behavior of the f at 0 (see §2). By the definition of the normalized residue and straightforward calculation of $f^{(d)}$ with the above normal form we obtain

PROPOSITION 1.1. — *For integers d, $d \cdot \mathrm{Res}(f^{(d)}) = \mathrm{Res}(f)$.*

It is known ([2]) that a holomorphic vector field χ' is holomorphically equivalent to the following normal form

$$\chi(z) = \frac{z^{k+1}}{1+mz^k}\ \frac{\partial}{\partial z}.$$

The m is called the *residue* of χ' and denoted $\mathrm{res}(\chi')$ ($= \mathrm{res}(\chi)$). By the formula

$$\exp\ \chi(z) = z + z^{k+1} + \left(-m + \frac{k+1}{2}\right) z^{2k+1} + \ \cdots,$$

we obtain the relation $\mathrm{Res}(\exp\ \chi) = \mathrm{res}(\chi) = m$. Let $\tilde{z}' = \phi(z) = z^{-k} + m \log z^{-k}$. Then $d\phi(\chi) = -k\ \partial/\partial\tilde{z}'$ defined at ∞. On the $\tilde{z}'$-plane the trajectory of $-k\ \partial/\partial\tilde{z}'$ along an anti-clockwise cycle $\circlearrowleft$ in the complex time-plane $\mathbb{C}$ with the base point 0 is closed as $\circlearrowleft$ is contractible in the domain of definition for $\exp - tk\partial/\partial\tilde{z}'$, while the trajectory of the induced vector field $\tilde{\chi}$ on the $\tilde{z}$-plane, $\tilde{z} = z^{-k}$, along the $\circlearrowleft$ is not closed as the logarithm has monodromy. Geometrically this phenomenon is interpreted by the functional equation

$$\exp\ \circlearrowleft^k \chi\ (z) = \exp\ (2\pi\sqrt{-1}\ \mathrm{res}(\chi))\chi(z),$$

where $\circlearrowleft^k$ stands for the analytic continuation of the complex time along the k times iteration of $\circlearrowleft$ such that $\exp\ t\chi(z)$ moves around the origin clockwisely from z to $\exp\ 2\pi\sqrt{-1}\ \mathrm{res}(\chi)\chi\ (z)$. Replacing χ with $d\chi$ and the cycle $\circlearrowleft$ in the time plane with a larger cycle homotopic to $\circlearrowleft$ if necessary, we obtain

$$\exp\ \circlearrowleft^k d\chi\ (z) = \exp\ 2\pi\sqrt{-1}\ \frac{1}{d}\mathrm{res}(\chi)\ d\chi(z),$$

from which (or directly by the normal form $\dfrac{z^{k+1}}{1+m/d\ z^k}\partial/\partial z$ of $d\chi$) we obtain

PROPOSITION 1.2. — *For $d \in \mathbb{C}$, $d \cdot\ \mathrm{res}(d\chi) = \mathrm{res}(\chi)$.*

The formal conjugacy class of a germ of a flat diffeomorphism f is determined by the residue $\mathrm{res}(f) = -b$. Therefore there is a formal diffeomorhpism ϕ of $\mathbb{C}, 0$ such that $f = \phi^{(-1)} \circ \exp\ \chi \circ \phi$ with $\mathrm{res}(\chi) = \mathrm{Res}(f)$. The *complex iteration* $f^{(t)}, t \in \mathbb{C}$, is then defined by the formal power series $f^{(t)} = \phi^{(-1)} \circ \exp\ t\chi \circ \phi$.

Proposition 1.1 is generalized as

PROPOSITION 1.3. — *Assume that an* $f(z) = z+az^{k+1}+\cdots$ *commutes with a* $g(z) = z + bz^{k+1} + \cdots$, *and* $a, b \neq 0$. *Then* $a \cdot \mathrm{Res}(f) = b \cdot \mathrm{Res}(g)$ *holds. In general if* $f^{(b)} = g^{(a)}$ *and* $a, b \neq 0$, *then* $a \cdot \mathrm{Res}(f) = b \cdot \mathrm{Res}(g)$.

Proof. — If f, g commute, f, g embed to a complexified formal one parameter family $\exp t\chi$ by Proposition 1.5. Since the residue is a formal invariant, the statement follows from Proposition 1.2 and $\mathrm{Res}(\exp t\chi) = \mathrm{res}(t\chi)$.

The following is a corollary to the above proposition. The proof is left to the readers.

PROPOSITION 1.4. — *Let* $f_1, \ldots, f_i$ *be k-flat diffeomorphisms and all commutative each other. Then*

$$(d^{k+1}f_1(0) + \cdots + d^{k+1}f_i(0))^2 \,\mathrm{Res}(f_1 \circ \cdots \circ f_i) = d^{k+1}f_1(0)^2 \,\mathrm{Res}(f_1) + \cdots + d^{k+1}f_i(0)^2 \,\mathrm{Res}(f_i),$$

where $d^{k+1}f_j(0)$ *denotes the k+1st derivative of* f_j *at* 0 *for* $j = 1, \ldots, i$.

The complexified formal 1-parameter group of $f^{(t)}$ is clearly commutative.

PROPOSITION 1.5. — *Let G be a commutative group consisting of flat germs of diffeomorphisms. Then G embeds to a formal 1-parameter group of* $f^{(t)}, t \in \mathbb{C}$, *of complex iterations of an* $f \in G$. *And also G is formally equivalent to a subgroup of a 1-parameter family of a vector field* $\chi = \dfrac{z^{k+1}}{1+mz^k}\partial/\partial z$.

Proof. — Let k be the smallest order of flatness for diffeomorphisms in G, $f(z) = z + az^{k+1} + \cdots, a \neq 0$ and $f \circ g = g \circ f$. Then the Taylor expansion of g is uniquely determined by its $k+1$-st order term. Clearly the complex iteration $f^{(t)}(z) = z + taz^{k+1} + \cdots$ commutes with f. So $g = f^{(t)}$ with a $t \in \mathbb{C}$ by the uniqueness, and in particular, if g is $k+1$-flat, then $g = \mathrm{id}$. The f is formally equivalent to an $\exp a\chi$, χ being of the normal form, and g is then formally equivalent to $\exp ta\chi$.

Let f be a germ of a flat diffeomorphism and denote by $C^0(f) \subset \mathbb{C}$ the subgroup consisting of those $t \in \mathbb{C}$ for which $f^{(t)}$ is convergent. The following theorem was first stated and partially proved by Baker [1] and later proved by many authors Écalle, Volonin,... (see e.g.[7]).

THEOREM 1.6. — *$C^0(f)$ is either $\mathbb{C}$ or a sequence $c\mathbb{Z}$, c being a real rational number. If $C^0(f) = \mathbb{C}$, there is a germ of holomorphic vector field χ such that $f = \exp \chi$ and $\mathrm{Res}(f) = \mathrm{res}(\chi)$. In other words the centralizer of f in the group of flat germs of diffeomorphisms is holomorphically equivalent to a subgroup of a 1-parameter group $\exp t\chi$.*

Let G be a group consisting of germs of diffeomorphisms of $\mathbb{C}, 0$ and G^0 the subgroup of flat diffeomorphisms. The following proposition is attributed to Il'yashenko [10].

PROPOSITION 1.7. — *G is solvable if and only if G^0 is commutative if and only if G is meta-abelian, that is, $[G, G]$ is commutative. And all diffeomorphisms f in G^0 different from the identity have the same order of flatness k and the projections L, Λ of $G/G^0, G^0$ respectively to the linear and the $(k+1)$-st order terms are injective homomorphisms into $\mathbb{C}^*, \mathbb{C}$.*

Proof. — Assume that a commutator subgroup $[G, G]$ of a group G of germs of diffeomorphisms is commutative and consists of k-flat diffeomorphisms. By Proposition 1.5, the commutator subgroup is equivalent to a subgroup of a one parameter group $f^{(t)}, t \in \mathbb{C}$, of formal k-flat diffeomorphisms. For a k-flat $f \in [G, G]$ and an i-flat $g \in G$, $i \neq 0$, an easy calculation shows that the commutator $[f, g] = f^{(-1)}g^{(-1)}fg \in [G, G]$ is j-flat, $i, k < j$, and formally equivalent to $f^{(0)} = \mathrm{id}$, since the $(k+1)$-st order term is absent. So it follows that $f \circ g = g \circ f$ hence $g = f^{(s)}$ for some $s \in \mathbb{C}$, and if $i \neq k$, g is the identity. This observation tells that if the commutar $[G, G]$ is commutative, then the subgroup G^0 ($\supset [G, G]$) consisting of flat $g \in G$ is commutative, g are k-flat and embeds into the one parameter group of $f^{(t)}$. And the k-jets of those $g \in G$ are determined by their linear terms. Assume G is solvable, and consider the commutator sequence $G \supset G^1 \supset G^2 \supset \cdots \supset G^n = 1, G^{i+1} = [G^i, G^i]$. Since the commutators consist of flat diffeomorphisms, the commutativity of G^{n-1} implies that the flat subgroup G^0 as well as the other commutators.

THEOREM 1.8 (Solvable Groups). — *Assume G is solvable and non commutative group of germs of diffeomorphisms of $\mathbb{C}, 0$.*

(1) *G is formally equivalent to a subgroup of the semidirect product $\mathbb{C}^* \times \mathbb{C}$ acting on $\mathbb{C}$. Here the multiplication in $\mathbb{C}^* \times \mathbb{C}$ is defined by*

$(a,b) * (c,d) = (ac, ad + bc^{k+1})$ and the action of (a,b) on $\mathbb{C}$ is defined by $a \cdot \exp b/a\ \chi(z) = az + bz^{k+1} + \cdots$, where $\chi = z^{k+1}\partial/\partial z$.

(2) *If $G^0 \neq \mathbb{Z}$ then G is holomorphically equivalent to a subgroup of the above $\mathbb{C}^* \times \mathbb{C}$.*

(3) *Assume $G^0 = \mathbb{Z}$. Then $L(G) = \mathbb{Z}_n$, χ is k-flat, and $k = n, 2n, \ldots$, or $n/2, 3n/2, \ldots$. If $k = n, 2n, \cdots$, then G is comutative. If $k = n/2, 3n/2, \ldots$, then G is generated by a k-flat f with $\mathrm{Res}(f) = 0$, a g with a linear term ω, $\omega^{2k} = -1$ and the relation $g^{(-1)}fg = f^{(-1)}$, which is formally equivalent to the exceptional group $G_{\omega,k}$ in Theorem 5.*

Proof of (1). — Assume that G is solvable. Then G^0 is commutative and by Proposition 1.7 G is a central extension of G/G^0 by G^0. By Proposition 1.5 there exists a formal diffeomorphism ϕ and a normal form χ such that G^0 consists of convergent diffeomorphisms $f^{(t)}(z) = \phi^{(-1)} \circ \exp t\chi \circ \phi(z)$ with some $t \in \mathbb{C}$. Since the statement in (1) is formal, we may assume $f = \exp \chi$ and G^0 is a subgroup of $\mathbb{C}$ consisting of the diffeomorphisms $f^{(t)}$ with t in a subgroup $\Lambda \subset \mathbb{C}$. Since $g^{(-1)}f^{(t)}g \in G^0$ and $g^{(-1)}f^{(t)}g(z) = z + cz^{k+1} + \cdots, c = dg(0)^k$ for $g \in G$ by a calculation, the adjoint action μ of G/G^0 on G^0 is presented as

$$(\times) \qquad \mu(g, f^{(t)}) = g^{(-1)}f^{(t)}g = f^{(ct)},\ c = dg(0)^k$$

for $g \in G/G^0$. Since G is non commutative, the action μ is not trivial : $c = dg(0)^k \neq 1$ for a $g \in G$. By the invariance of the normalized residue under coordinate transformations, the relation $(\times)$ implies $\mathrm{res}(f^{(t)}) = \mathrm{res}(f^{(ct)})$. By the multiplicative formula in Proposition 1.3, $\mathrm{res}(f^{(ct)}) = \dfrac{1}{c}\ \mathrm{res}(f^{(t)})$. Therefore $\mathrm{res}(f^{(t)}) = 0$ and $\chi = z^{k+1}\partial/\partial z$. Let g, g' satisfy the relation $(\times)$ and $dg'(0) = dg(0)$. Then $\mu(g,f) = \mu(g',f)$ and $g^{(-1)}g'$ commutes with f, and by Proposition 1.5 $g^{(-1)}g' = f^{(s)}$ hence $g' = gf^{(s)}$ for an s. Clearly g' of this form satisfy $(\times)$. It is easy to see the linear map $g' = az$ satifies $(\times)$. Therefore all formal solutions of $(\times)$ are of the form $af^{(s)}, a \in \mathbb{C}^*, s \in \mathbb{C}$. The correspondence of $af^{(s)}, af^{(s)}(z) = az + as\, z^{k+1} + \cdots$ to (a, as) gives the isomorphism of the group of those diffeomorphisms $af^{(s)}$ onto a semidirect product $\mathbb{C}^* \times \mathbb{C}$. The straightforword calculation

$$(az + bz^{k+1} + \cdots) \circ (cz + dz^{k+1} + \cdots) = acz + (ad + bc^{k+1})z^{k+1} + \cdots$$

tells the multiplication $*$ on $\mathbb{C}^* \times \mathbb{C}$ is defined by $(a,b) * (c,d) = (ac, ad + bc^{k+1})$.

Proof of (2). — In the case (2) the flat subgroup G^0 is holomorphically embedded in a 1-parameter group $\exp t\chi$. Therefore the above formal conjugacy of G to a subgroup of $\mathbb{C}^* \times \mathbb{C}$ is convergent. The convergence is also seen by Theorem 5.

Proof of (3). — The adjoint action μ of $L(G)$ on $\Lambda(G^o) \cong \mathbb{Z}$ is given by $\mu(a,b) = a^k b$. Therefore $a^k = \pm 1$, and $a \in \mathbb{Z}_n \subset \mathbb{Z}_{2k}$. And G is commutative if $a^k = 1$ for all $a \in L(G)$, if and only if $L(G) \cong \mathbb{Z}_n \subset \mathbb{Z}_k$. Let $f \in G^0, g \in G/G^0$ be the generators. If G is non commutative, then f, g generate G with the relation $g^{(-1)} f g = f^{(-1)}$. This relation implies $\mathrm{Res}(f) = 0$.

THEOREM 1.9 (Commutative groups). — *Let G be a commutative group of germs of holomorphic diffeomorphisms of $\mathbb{C}, 0$.*

(1) *If $L(G) \approx \mathbb{Z}_k$, then G is formally equivalent to a subgroup of the Cartesian product $\mathbb{Z}_k \times \mathbb{C}$ acting on $\mathbb{C}$. Here Z_k consists of those $a, a^k = 1$ and the action of (a,b) on $\mathbb{C}$ is defined by $a \cdot \exp b/a\ \chi(z) = az + bz^{k+1} + \cdots$, where $\chi = \dfrac{z^{k+1}}{1+mz^k}\partial/\partial z$.*

(2) *If the linear term $L(G) \subset \mathbb{C}^*$ contains either an $a, |a| \neq 1$ or an $a = \exp 2\pi\sqrt{-1}\ \alpha$ where α is Brjuno number α (see [3] for the definition), then the projection L of G to the linear terms is injective and G is holomorphically equivalent to the $L(G)$ acting linearly on $\mathbb{C}$.*

(3) *If L is not injective and $G^0 \neq \mathbb{Z}$, then G is holomorphically equivalent to a subgroup of $\mathbb{Z}_k \times \mathbb{C}$ in (1).*

Proof. — The proof follows the same argument as the non solvable case. The adjoint action μ is trivial, so the residue m can be arbitrary. The 1-parameter group $\exp t\chi, \chi = \dfrac{z^{k+1}}{1+mz^k}\partial/\partial z$ has the $\mathbb{Z}_k$-symmetry by the linear rotation $\omega z, \omega^k = 1$. Therefore G is formally equivalent to a subgroup of the group consisting of the diffeomorphisms $\omega^i \exp t\chi$, which is isomorphic to $\mathbb{Z}_k \times \mathbb{C}$. In Case(2) the germ of diffeomorphism with the linear term a can be made linear ([3]). Then the other diffeomorphisms are all linear by the commutativity. In Case(3), the formal embedding of G^0 into $\exp t\chi$ is convergent by Theorem 1.6. Then the quotient G/G^0 is generated by the linear rotation $\omega z,\ \omega^k = 1$.

2. The analytic structure of germs of diffeomorphisms and the sectorial normalization theorem.

We begin by introducing the classification method of germs of holomorphic diffeomorphisms of $\mathbb{C}, 0$ due to Écalle, Fatou, Kimura, Malgrange and Voronin [8], [9], [13], [22] (for the various fundamental results on the local holomorphic dynamics, see the book [2]). On the k-sheet covering $\tilde{\mathbb{C}}_k$ of the punctured $\tilde{z}$-plane $\mathbb{C} - 0$, $\tilde{z} = z^{-k}$, a *k-flat* diffeomorphism $f(z) = z + a_{k+1}z^{k+1} + \cdots$ lifts to a germ of diffeomorphism F defined at infinity, which is written in the form with the coordinate $\tilde{z} = z^{-k}$ as

$$F(\tilde{z}) = \tilde{z} - a_{k+1}k + a'\tilde{z}^{-1/k} + a''\tilde{z}^{-2/k} + \cdots,$$

and if f is a *normal form* $f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \cdots$, then

$$F(\tilde{z}) = \tilde{z} - a_{k+1}k + a'\tilde{z}^{-1} + a''z^{-1-1/k} + \cdots,$$

where $\mathrm{res}(f) = -a_{2k+1}/a_{k+1}^2, a' = a_{k+1}^2 k\mathrm{Res}(f)$. From these forms we obtain

$$(*) \qquad \|F(\tilde{z}) - (\tilde{z} - a_{k+1}k)\| \leq c\|\tilde{z}\|^{-1/k}$$

for sufficiently large $\|\tilde{z}\|$ with a positive real constant c. The estimate holds replacing $c\|\tilde{z}\|^{-1/k}$ with $c\|\tilde{z}\|^{-1}$ for f of the normal form.

PROPOSITION 2.1. — *For any small $\epsilon > 0$, there is an $r > 0$ such that, for $n = 0, 1, 2, \ldots$,*

$$F^{(n)}(\tilde{z}) \in S^+(\tilde{z}) = \left\{\tilde{z} + \omega \in \tilde{\mathbb{C}}_k \mid \left\|\arg\left\{\frac{-\omega}{a_{k+1}}\right\}\right\| \leq \sin^{-1}\frac{\epsilon}{\|a_{k+1}\|k}\right\},$$

for $\tilde{z} \in \tilde{S}^+ = \bigcup_{\tilde{z}\in S^+} S^+(\tilde{z}), S^+ = \{r \leq \|\tilde{z}\|, -2\pi/3 \leq \arg\{-\tilde{z}/a_{k+1}\} \leq 2\pi/3\}$ and

$$F^{(-n)}(\tilde{z}) \in S^-(\tilde{z}) = \left\{\tilde{z} + \omega \in \tilde{\mathbb{C}}_k \mid \left\|\arg\left\{\frac{\omega}{a_{k+1}}\right\}\right\| \leq \sin^{-1}\frac{\epsilon}{\|a_{k+1}\|k}\right\},$$

for $\tilde{z} \in \tilde{S}^- = \bigcup_{\tilde{z}\in S^-} S^-(\tilde{z}), S^- = \{r \leq \|\tilde{z}\|, \pi/3 \leq \arg\{-\tilde{z}/a_{k+1}\} \leq 5\pi/3\}$. In particular $F(\tilde{S}^+) \subset \tilde{S}^+$ and $F^{(-1)}(\tilde{S}^-) \subset \tilde{S}^-$.

Proof. — Choose the r large enough so that the estimate

$$\|F(\tilde{z}) - (\tilde{z} - a_{k+1}k)\| \leq c\|\tilde{z}\|^{-1/k} \leq \epsilon$$

holds on $\tilde{S}^+$. Then, for $\tilde{z} \in \tilde{S}^+$, we obtain $F(\tilde{z}) \in S^+(\tilde{z}) \subset \tilde{S}^+$, and by induction, $F^{(n)}(\tilde{z}) \in S^+(\tilde{z})$ for $n = 1, 2, \ldots$. The other statements follow from a similar argument.

LEMMA 2.2. — *Assume $F^{(n)}(\tilde{z}) \to \infty$ as $n \to \infty$. Then*

$$(a)\ \ F^{(n)}(\tilde{z}) = \tilde{z} - na_{k+1}k + \begin{cases} O(\log n) & \text{if } k = 1 \text{ or } f \text{ is a normal form} \\ O(n^{1-\ 1/k}) & \text{if } k \geq 2 \end{cases}$$

for $n = 1, 2, \ldots$.

Proof. — We prove only for a general form f. Since $F^{(n)}(\tilde{z}) \to \infty$, we may assume $\tilde{z} \in S^+$, $F^{(n)}(\tilde{z}) \in S^+(\tilde{z}), n = 0, 1, \ldots$ and also $c\|\tilde{z}\|^{-1/k} \leq \epsilon$ on $\tilde{S}^+$ choosing r large enough. By (*) we obtain

$$\begin{aligned} \|F^{(n+1)}(\tilde{z}) - \tilde{z} + (n+1)a_{k+1}k\| &\leq \|F^{(n+1)}(\tilde{z}) - F^{(n)}(\tilde{z}) + a_{k+1}k\| \\ &\quad + \|F^{(n)}(\tilde{z}) - \tilde{z} + na_{k+1}k\| \\ &\leq \epsilon + \|F^{(n)}(\tilde{z}) - \tilde{z} + na_{k+1}k\|, \end{aligned}$$

from which, by induction, we obtain

$$\|F^{(n)}(\tilde{z}) - \tilde{z} + na_{k+1}k\| < \epsilon n.$$

Choosing ϵ small enough, we obtain

$$(**) \qquad \|F^{(n)}(\tilde{z}) - \tilde{z} + na_{k+1}k\| \leq \epsilon n \leq \frac{1}{2}\ \|\tilde{z} - na_{k+1}k\|$$

for $\tilde{z} \in \tilde{S}^+, n = 1, 2, \ldots$. From this and the estimate

$$(***) \qquad \begin{aligned} \|F^{(n+1)}(\tilde{z}) - \tilde{z} + (n+1)a_{k+1}k\| &\leq \|F^{(n+1)}(\tilde{z}) - F^{(n)}(\tilde{z}) + a_{k+1}k\| \\ &\quad + \|F^{(n)}(\tilde{z}) - \tilde{z} + na_{k+1}k\| \\ &\leq \epsilon\|F^{(n)}(\tilde{z})\|^{-1/k} + \|F^{(n)}(\tilde{z}) - \tilde{z} + na_{k+1}k\|, \end{aligned}$$

it follows

$$\begin{aligned} \|F^{(n)}(\tilde{z}) - \tilde{z} + na_{k+1}k\| &\leq \epsilon(\|\tilde{z}\|^{-1/k} + \|F^{(1)}(\tilde{z})\|^{-1/k} \\ &\quad + \cdots + \|F^{(n-1)}(\tilde{z})\|^{-1/k}) \\ &\leq c \sum_{m=0}^{n-1} (\|\tilde{z} + ma_{k+1}k\|)^{-1/k}. \end{aligned}$$

The statement follows from this and the next estimate (the proof is elementary).

LEMMA 2.3. — *For $\tilde{z} \in S^+$,*

$$\sum_{m=0}^{n} \|\tilde{z} - ma_{k+1}k\|^{-1/k} = \begin{cases} \|\tilde{z}\|^{-1/k} O(n^{1-1/k}) & \text{if } k \geq 2 \\ \|\tilde{z}\|^{-1} O(\log n) & \text{if } k = 1. \end{cases}$$

This turns out to give

PROPOSITION 2.4. — *If $f : U_f \to f(U_f)$ has linear term az at 0 such that $\|a\| \neq 1$ or $a^i = 1$ for an integer i, then $B_f \cup B_{f^{(-1)}}$ is an open neighbourhood of 0. In other words, for any z sufficiently close to 0, the forward or backward orbit of z by f tends to 0.*

Proof. — If $a = 1$, the statement follows from the above argument, and if $a^i = 1$, it reduces to the case $i = 1$, since $B_{f^{(i)}} = B_f$ and $B_{f^{(-i)}} = B_{f^{(-1)}}$. If $\|a\| \neq 1$, the statement follows from Poincaré linearization theorem.

From this we obtain

PROPOSITION 2.5. — *If a pseudogroup Γ contains a diffeomorphism f which satisfies the condition in Proposition 2.4, then the basin B_Γ is an open neighboourhood of 0.*

Proof. — For a $z \in B_\Gamma$ there is a $g \in \Gamma$ such that $g(z)$ is in the union $B_f \cup B_{f^{(-1)}}$. Then g sends also z' sufficiently close to z to the union, and either the forward or backward orbit of $g(z')$ by f tends to 0. Therefore z' is contained in the basin.

LEMMA 2.6. — *The sequence $dF^{(n)}$ is uniformly convergent (on compact subsets of S^+) to a function $dF^{(\infty)}$ as $n \to \infty$. Furthermore $\log dF^{(\infty)} = O(\|\tilde{z}\|^{-1/k})$ and $\|d^{n+1}F^{(\infty)}\| = O(\|\tilde{z}\|^{-n\ -1/k})$ for $n \geq 1$ on $\tilde{S}^+$, where $d^{n+1}F^{(\infty)} = d^n dF^{(\infty)}$ for $n \geq 0$. If f is a normal form, these estimates hold with $k = 1$.*

Proof. — We prove only for a general form f. We begin with

$$\begin{aligned}(b)\ \log dF^{(n)} &= \log dF(F^{(n-1)}) + \log dF(F^{(n-2)}) + \cdots + \log dF \\ &= \sum_{l=1}^{n-1} \log \left(1 - \frac{A}{k}(F^{(l)})^{-(k+1)/k} - \frac{2A'}{k}(F^{(l)})^{-(k+2)/k} - \cdots\right).\end{aligned}$$

By Lemma 2.2, $\sum_{l=1}^{\infty} \|(F^{(l)})\|^{-(k+1)/k}$ is locally uniformly convergent. Therefore $dF^{(n)}$ is convergent to the $dF^{(\infty)}$ and

$$\begin{aligned}\|\log dF^{(\infty)}\| &\leq \sum_{l=1}^{\infty} \left\|\log\left(1-\frac{A}{k}(F^{(l)})^{-(k+1)/k}-\frac{2A'}{k}(F^{(l)})^{-(k+2)/k}-\cdots\right)\right\| \\ &\leq \sum_{l=1}^{\infty} K \left\|\frac{A}{k}(F^{(l)})^{-(k+1)/k}\right\|\end{aligned}$$

and by Lemma 2.3 and (**) in the proof of Lemma 2.2

$$\|\log dF^{(\infty)}\| \leq \sum_{l=1}^{\infty} K \left\|\frac{A}{2k}(\tilde{z}-na_{k+1}k)\right\|^{-(k+1)/k} \leq K'\|\tilde{z}\|^{-1/k},$$

for $\tilde{z}$ such that $F^{(n)}(\tilde{z}) \to \infty$ as $n \to \infty$ with constants K, K'. The other estimate is obtained inductively by derivating the exponential series of both sides of (b).

From now on in this section we assume that f is of the form

$$f(z) = z - \frac{2\pi\sqrt{-1}}{k} z^{k+1} + cz^{2k+1} + \cdots$$

and

$$F(\tilde{z}) = \tilde{z} + 2\pi\sqrt{-1} + \frac{a'}{\tilde{z}} + \cdots,$$

where $\operatorname{res}(f) = -\dfrac{ck^2}{4\pi^2}$ and $a' = -4\pi^2/k \operatorname{Res}(f)$. By Proposition 2.1 $F(\tilde{S}_i^+) \subset \tilde{S}_i^+$ and $F^{(-1)}(\tilde{S}_i^-) \subset \tilde{S}_i^-$ hold for $i = 1, \ldots, k$, where $\tilde{S}_i^+, \tilde{S}_i^-$ denote the sectors $\tilde{S}^+, \tilde{S}^-$ on the i-th sheet of the covering $\tilde{\mathbb{C}}_k$. Since F is asymptotic to the translation by $2\pi\sqrt{-1}$ at ∞, the quotient space P_i^+ (respectively P_i^-) of $\tilde{S}_i^+$ (resp. $\tilde{S}_i^-$) by F (resp. $F^{(-1)}$) is quasi-conformally homeomorphic hence conformally isomorphic to the punctured 2-sphere $\mathbb{P} - 0 \cup \infty$ endowed with a coordinate t unique up to scalar multiplication (Voronin [2], [22]). We call $P_i^{\pm}$ the *cylinder* for F (or f), and say a fundamental domain $D_i^{\pm}$ in a half plane $\tilde{S}_i^{\pm}$ is *rectangular* if the boundary projects to a real line in $P_i^{\pm}$ joining 0 to ∞. Here 0 (resp. ∞) corresponds to the left (resp. right) end of the fundamental domain $D_i^{\pm}$. The isomorphism from $\mathbb{P} - \{0, \infty\}$ to the quotient space P_i^{ϵ}, $\epsilon = \pm$, defines the isomorphism $\tilde{\phi}_i^{\epsilon}$ of the band $B_\epsilon = \{0 \leq \epsilon \operatorname{Im} z \leq 2\pi\sqrt{-1}\} \subset \mathbb{C}$ to a rectangular fundamental domain in $\tilde{S}_i^{\epsilon}$ for $F^{(\epsilon)}$, which extends to the isomorphism of the upper (if

$\epsilon = +$, and lower if $\epsilon = -$ respectively) half plane into $\tilde{S}_i^\epsilon$ by the relation $\tilde{\phi}_i^\epsilon + 2\pi\sqrt{-1} = \tilde{\phi}_i^\epsilon(F^{(\epsilon)})$. The extension of $\tilde{\phi}_i^\epsilon$ gives the normalization of $F^{(\epsilon)}$ restricted to H_i^ϵ to the translation by $2\pi\sqrt{-1}\epsilon$.

The conjugacy $\tilde{\phi}_i^\epsilon$ is unique up to translation by constant. The inverse is explicitly given by

$$\tilde{\phi}_i^{\epsilon(-1)} = \lim_{n\to\epsilon\infty} F^{(n)} - \left(2\pi\sqrt{-1}n + \frac{a'}{2\pi\sqrt{-1}} \log 2\pi\sqrt{-1}n\right),$$

which is defined on the sector $\tilde{S}_i^\epsilon$ in Proposition 2.1 with a sufficiently large r. The local uniform convergence of $\tilde{\phi}_i^{\epsilon(-1)}$ is seen in [2]. The calculation

$$\begin{aligned}
\tilde{\phi}_i^{\epsilon(-1)}(F) &= \lim_{n\to\epsilon\infty} F^{(n)}(F) - \left(2\pi\sqrt{-1}n + \frac{a'}{2\pi\sqrt{-1}} \log 2\pi\sqrt{-1}n\right) \\
&= \lim_{n\to\epsilon\infty} F^{(n+1)} - \left(2\pi\sqrt{-1}(n+1) + \frac{a'}{2\pi\sqrt{-1}} \log 2\pi\sqrt{-1}(n+1)\right) \\
&\quad + 2\pi\sqrt{-1} + \frac{a'}{2\pi\sqrt{-1}} \log \frac{n+1}{n} \\
&= \tilde{\phi}_i^{\epsilon(-1)} + 2\pi\sqrt{-1}
\end{aligned}$$

tells that $\tilde{\phi}_i^{\epsilon(-1)}$ links the restriction of F to the sector $\tilde{S}_i^\epsilon \cap F^{(-1)}(\tilde{S}_i^\epsilon)$ to the translation by $2\pi\sqrt{-1}$.

The following theorem is attributed to Malgrange and Voronin [22]. Here we recall the idea briefly.

THEOREM 2.7 (Sectorial Normalisation Theorem). — *Let $f(z) = z + z^{k+1} + \cdots$, be a k-flat germ of diffeomorphism and let ϕ be a flat formal conjugacy of f to the normal form $g(z) = \exp \chi(z) = z + z^{k+1} + \cdots$, where $\chi = \dfrac{z^{k+1}}{1+mz^k}\partial/\partial z$, and res$(f) = m$. Then there exist a representative $\tilde{f}$ of f defined on a neighbourhood U of $0 \in \mathbb{C}$, diffeomorphisms $\phi_i^\epsilon : S_i^\epsilon \to T_i^\epsilon, S_i^\epsilon \subset U$, for $i = 1, \ldots, k, \epsilon = \pm$ with the following properties :*

(1) (i) *S_i^+ contains the open sector $\left\{\|z\| < r, \dfrac{2\pi(2i-1)}{2k} - \dfrac{2\pi}{3k} < \arg z < \dfrac{2\pi(2i-1)}{2k} + \dfrac{2\pi}{3k}\right\} \subset \mathbb{C}$, f maps S_i^+ into S_i^+ and $f^{(n)}(z) \to 0$ as $n \to \infty$ uniformly on S_i^+,*

(ii) *S_i^- contains the open sector $\left\{\|z\| < r, \dfrac{2\pi i}{k} - \dfrac{2\pi}{3k} < \arg z < \dfrac{2\pi i}{k} + \dfrac{2\pi}{3k}\right\} \subset \mathbb{C}$, and $f^{(-1)}$ maps S_i^- into S_i^- and $f^{(n)}(z) \to 0$ as $n \to -\infty$ uniformly on S_i^-,*

(2) (i) *T_i^+ contains the open sector* $\left\{\|z\| < r, \frac{2\pi(2i-1)}{2k} - \frac{2\pi}{3k} < \arg z < \frac{2\pi(2i-1)}{2k} + \frac{2\pi}{3k}\right\} \subset \mathbb{C}$, *$g$ maps T_i^+ into T_i^+ and $g^{(n)}(z) \to 0$ as $n \to \infty$ uniformly on T_i^+,*

(ii) *T_i^- contains the open sector* $\left\{\|z\| < r, \frac{2\pi i}{k} - \frac{2\pi}{3k} < \arg z < \frac{2\pi i}{k} + \frac{2\pi}{3k}\right\} \subset \mathbb{C}$, *$g^{(-1)}$ maps T_i^- into T_i^- and $g^{(n)}(z) \to 0$ as $n \to -\infty$ uniformly on T_i^-,*

(3) *$g \circ \phi_i^+ = \phi_i^+ \circ f$ on S_i^+ and $g \circ \phi_i^- = \phi_i^- \circ f$ on S_i^- for $i = 1, \ldots, k$,*

(4) *the Taylor series of ϕ_i^ϵ are asymptotic to ϕ at 0.*

Sketch of the proof. — In order to adjust the various notations to the previous part we assume f is in the form $f(z) = z - \frac{2\pi\sqrt{-1}}{k} z^{k+1} + \cdots$ and $F(\tilde{z}) = \tilde{z} + 2\pi\sqrt{-1} + a'\tilde{z}^{-1} + \cdots$ after linear change of the coordinate z. (Then the sectors $S_i^\epsilon, T_i^\epsilon$ in the theorem are rotated the angle $-\pi/2k$.) Let $\tilde{\psi}_i^{\epsilon(-1)}$ be the conjugacy of the lift G of g to the translation by $2\pi\sqrt{-1}$. Then $\tilde{\psi}_i^{\epsilon(-1)}\tilde{\phi}_i^\epsilon$ links F to G and extends to a diffeomorphism of certain neighbourhoods of the sector $\tilde{S}_i^\epsilon$ on the i-th sheet of $\tilde{\mathbb{C}}_i$ with a sufficiently large r. The diffeomorphism ϕ_i^ϵ in the theorem is defined by $\tilde{\psi}_i^{\epsilon(-1)}\tilde{\phi}_i^\epsilon$ using the coordinate $z = \tilde{z}^{-1/k}$. The other properties can be shown by analyzing the asymptotic property of the Taylor expansion of ϕ_i^ϵ with the estimates in Lemma 2.2, 2.3 and (**).

3. Construction of vector fields by commutators.

Let Γ be a pseudogroup of diffeomorphisms $f : U_f, 0 \to f(U_f), 0$ of open neighbourhoods of the origin in $\mathbb{C}$. Assume that the germ Γ_0 of Γ is non-solvable. Then Γ contains diffeomorphisms f, g with Taylor expansions

$$f(z) = z + az^{i+1} + \cdots, \quad g(z) = z + bz^{j+1} + \cdots, \qquad a, b \neq 0,\ i < j$$

and

$$[f, g](z) = z + cz^{k+1} + \cdots, \qquad c \neq 0, j < k.$$

Let $z \in B_f$, in other words, $f^{(n)}(z) \to 0$ as $n \to \infty$. We will show that *the dynamics $f^{(-n)}gf^{(n)}$ is convergent to the identity but a suitable real scalar* *multiple* $\lambda_n(f^{(-n)}gf^{(n)} - \mathrm{id})$ *is convergent to a holomorphic vector field* $\chi = \chi(f,g)$ *on* $B_f - 0$. This vector field may not extend to a neighbourhood of the origin since B_f is not a neighbourhood of the origin in general. On $B_{f^{(-1)}} - 0$ we define $\chi(f^{(-1)}, g)$ replacing f with its inverse. Using g and $[f,g]$, define another dynamics $\zeta = \zeta(f,g) = \chi(g,[f,g])$ on $B_g - 0$. We will show that χ *and* ζ *are* $\mathbb{R}$*-linearly independent at generic points and satisfy the condition of Lemma 5.2* (Lemmas 3.3, 3.4).

Using the coordinate $\tilde{z} = z^{-i}$ on the i-sheet covering $\tilde{\mathbb{C}}_i$ at a $\tilde{z}_0 = z_0^{-i}$, the lift F of f is written as

$$F(\tilde{z}) = f(z^{-1/i})^{-i} = \tilde{z} - ai + A\tilde{z}^{-1/i} + A'\tilde{z}^{-2/i} + \cdots,$$

where $\tilde{z}^{-1/i}$ takes the branch of z_0. On the covering the diffeomorphism g lifts to the slow dynamics

$$G(\tilde{z}) = g(\tilde{z}^{-1/i})^{-i} = \tilde{z} - bi\tilde{z}^{(i-j)/i} + B\tilde{z}^{(i-j-1)/i} + \cdots .$$

Our vector field $\tilde{\chi}$ is defined on the set of those $\tilde{z} \in \tilde{\mathbb{C}}_i$ for which $F^{(n)}(\tilde{z}) \to \infty$ as $n \to \infty$ by

$$\tilde{\chi}(\tilde{z}) = \lim_{n\to\infty} \lambda_n(F^{(-n)}GF^{(n)} - \mathrm{id})\ \partial/\partial\tilde{z}$$

with a suitable sequence of real positive numbers $\lambda_n \to \infty$ as follows. From Lemma 2.6 we obtain

LEMMA 3.1. — *$dF^{(\infty)}$ is holomorphic and $dF^{(\infty)}(\tilde{z}) \to 1$ as $\tilde{z} \to \infty$ in $S^+ = \{r \le \|\tilde{z}\|, -2\pi/3 \le \arg\{-\tilde{z}/a_{k+1}\} \le 2\pi/3\}$.*

Let $\lambda_n = n^{(j-i)/i}$ and define $\tilde{\chi}$ by

$$\begin{aligned}\tilde{\chi} &= \lim_{n\to\infty} \lambda_n\ \{F^{(-n)}GF^{(n)} - \mathrm{id}\}\partial/\partial\tilde{z}\\ &= \lim_{n\to\infty} \lambda_n\ \{dF^{(-n)}\ ((G-\mathrm{id})\circ F^{(n)}\ \partial/\partial\tilde{z})\\ &\qquad + O_n(((G-\mathrm{id})\circ F^{(n)})^2)\ \partial/\partial\tilde{z}\}.\end{aligned}$$

Since the second derivative $d^2F^{(n)}$ is locally uniformly convergent to $d^2F^{(\infty)}$ as $n \to \infty$ (see the proof of Lemma 2.6), the remainder term O_n is independent of n, then

$$\begin{aligned}\tilde{\chi} &= \lim_{n\to\infty} dF^{(-n)}\{-bi\ n^{(j-i)/i} \big/ (F^{(n)})^{(j-i)/i}\ \partial/\partial\tilde{z}\}\\ &= (dF^{(\infty)})^{(-1)}\ (-bi\ (-ai)^{(i-j)/i}\partial/\partial\tilde{z}),\end{aligned}$$

where the branch of $(-ai)^{(j-i)/i}$ depends on the sheet of the covering $\tilde{\mathbb{C}}_i$. Let χ be the holomorphic vector field on $B_f - 0$ induced from $\tilde{\chi}$. Define the vector field ζ on $B_g - 0$ similarly with the vector field defined on the j-sheet covering $\tilde{\mathbb{C}}_j$,

$$\begin{aligned}\tilde{\zeta}' &= \lim_{n\to\infty} n^{(k-j)/j}\{G'^{(-n)}\ [F,G]\ G'^{(n)} - \mathrm{id}\}\partial/\partial\tilde{z} \\ &= (dG'^{(\infty)})^{(-1)}(-cj(-bi)^{(j-k)/j}\ \partial/\partial\tilde{z}),\end{aligned}$$

where $[f,g](z) = z + cz^{k+1} + \cdots$ and G' denotes the diffeomorphism of $\tilde{\mathbb{C}}_j$ induced from g.

LEMMA 3.2. — *The vector field $\tilde{\chi}$ is invariant under dF and induces a linear vector field on each quotient space (cylinder) $P_l^{\pm}, l = 1,\dots,i$, for F as in §2.*

Proof. — We may assume that $F(\tilde{z}) = \tilde{z} + 2\pi\sqrt{-1} + a'/\tilde{z} + \cdots$. By the sectorial normalization theorem (Theorem 2.7), there exists a biholomorphism $\tilde{\phi}$ of S^+ onto an open subset of $\mathbb{C}$ which conjugates F to the translation $t \to t + 2\pi\sqrt{-1} : \tilde{\phi}\circ F(\tilde{z}) = \tilde{\phi}(\tilde{z}) + 2\pi\sqrt{-1}$ for $\tilde{z}\in S^+$, $\|\tilde{z}\|$ being sufficintly large. In the proof of Theorem 2.7 $\tilde{\phi}$ is given by

$$\tilde{\phi} = \lim_{n\to\infty} F^{(n)} - \left(2\pi\sqrt{-1}n + \frac{a'}{2\pi\sqrt{-1}}\ \log 2\pi\sqrt{-1}n\right).$$

From this $d\tilde{\phi} = dF^{(\infty)}$. Then $\tilde{\phi}_*\tilde{\chi} = -bi\ (-ai)^{(i-j)/i}\partial/\partial t$ by the definition of $\tilde{\chi}$.

Similarly, the vector field ζ induces a linear vector field on each cylinder for g. By Lemma 2.2, $\arg g^{(n)}(z), \arg f^{(n)}(z)$ are convergent to constants when $g^{(n)}(z), f^{(n)}(z) \to \infty$ as $n\to\infty$.

LEMMA 3.3. — *Assume $z_0 \in B_g - 0$, $g^{(n)}(z_0) \in B_f$ for $n = 0,1,\dots$ and*

$$\lim_{n\to\infty}\ \arg\,(g^{(n)}(z)/f^{(n)}(z)) \le \pi/2i.$$

Then χ,ζ are nowhere zero holomorphic vector fields and $\mathbb{C}$-linearly independent on a neighbourhood of z_0.

Proof. — Assume that $\zeta = a\chi$ with a real constant a on a neighbourhood of z_0 and let $\tilde{z}_0 = z_0^{-i} \in \tilde{\mathbb{C}}_i$. Since $\tilde{\zeta}$ is invariant under dG, it follows $\tilde{\chi}$ is also invariant and $dG^{(n)}(\tilde{\chi}(\tilde{z}_0)) = \tilde{\chi}(G^{(n)}(\tilde{z}_0)) = (dF^{(\infty)}(G^{(n)}(\tilde{z}_0)))^{(-1)}(-bi(-ai)^{(j-i)/i}\ \partial/\partial\tilde{z})$ is convergent to a non zero constant vector by Lemma 3.1. On the other hand $dG^{(n)}(\tilde{z}_0)$ tends to 0 as n tends to infinity since g lifts on $\tilde{\mathbb{C}}_j$ to the diffeomorphism G' asymptotic to a translation at $\tilde{z} = z^{-j} = \infty$ and $i < j$. This completes the proof.

LEMMA 3.4. — *$[\chi, \zeta]$ is not a constant vector on a neighbourhood of the $\tilde{z}_0$ as in Lemma 3.3.*

Proof. — Assume that $[\tilde{\chi}, \tilde{\zeta}] = a\ \partial/\partial\tilde{z}$ with real constants a. Since $(dF^{(\infty)}(G^{(n)}(\tilde{z}_0)))^{(-1)} \to 1$, $dG^{(n)}(\tilde{z}_0) \to 0$ and $\tilde{\chi}, \tilde{\zeta}$ are respectively invariant under dF, dG , we see that $\tilde{\zeta}(G^{(n)}(\tilde{z}_0)) \to 0$ and $\tilde{\chi}(G^{(n)}(\tilde{z}_0))$ tends to a constant vector hence $[\tilde{\chi}, \tilde{\zeta}](G^{(n)}(\tilde{z}_0)) \to 0$ as n tends to ∞. This implies that $[\chi, \zeta] = 0$ hence χ and ζ are $\mathbb{R}$-linearly dependent on a neighbourhood of $\tilde{z}_0$. This contradicts Lemma 3.3, and completes the proof.

PROPOSITION 3.5. — *Any point* $\exp t\chi(z_0), t \in \mathbb{R}$ *can be approximated by orbits of Γ of type $f^{(-n)}g^{(l_n)}f^{(n)}(z_0)$: if $l_n n^{(i-j)/i} \to t$ as $t \to \infty$, then $f^{(-n)}g^{(l_n)}f^{(n)}(z_0)$ converges to* $\exp t\chi(z_0)$.

Proof. — We prove $F^{(-n)}G^{(l_n)}F^{(n)}(z_0)$ converges to $\exp\ t\tilde{\chi}(\tilde{z}_0)$, $\tilde{z}_0 = z_0^{-k}$. Approximate $\tilde{\chi}$ by $\tilde{\chi}_n(\tilde{z}) = n^{(j-i)/i}(F^{(-n)}GF^{(n)} - \mathrm{id})\partial/\partial\tilde{z}$ with $n = 1, 2, \ldots$. Since $\tilde{\chi}_n$ and $\tilde{\chi}_\infty = \tilde{\chi}$ are holomorphic and $\tilde{\chi}_n$ is locally uniformly convergent to $\tilde{\chi}$ as $n \to \infty$, the real trajectories $\exp t\tilde{\chi}_n(\tilde{z}_0), 0 \le t \le a$ with an a passing through a $\tilde{z}_0$ are arbitrarily closely approximated by sequences $\tilde{z}_{n,l+1} = \tilde{z}_{n,l} + t_l \cdot \tilde{\chi}_n(\tilde{z}_{n,l}), l = 0, 1, \ldots, m-1, \tilde{z}_{n,0} = \tilde{z}_0$ with sufficiently small $0 \le t_l$, $\sum_{l=0}^{m-1} t_l = t$. Then the difference $\|\exp\ s_l\tilde{\chi}_n(\tilde{z}_0) - \tilde{z}_{n,l}\|, s_l = t_1 + \cdots + t_{l-1}$, has a uniform upper bound $C(\delta)$ depending only on $\delta = \max\{t_l\}$ such that $C(\delta) \to 0$ as $\delta \to 0$. Let $t_l = 1/\lambda_n = n^{(i-j)/i}$ and $s_l = l\ n^{(i-j)/i}$ with a sufficiently large n. Then by definition we obtain

$$\begin{aligned}\tilde{z}_{n,l+1} &= \tilde{z}_{n,l} + 1/\lambda_n\ \tilde{\chi}_n(\tilde{z}_{n,l})\\ &= F^{(-n)}GF^{(n)}(\tilde{z}_{n,l})\\ &= F^{(-n)}G^{l+1}F^{(n)}(\tilde{z}_0)\end{aligned}$$

for $l = 0, 1, \ldots$, and the above argument tells

$$\|\exp s_l\ \tilde{\chi}_n(\tilde{z}_0)\ -\ F^{(-n)}G^lF^{(n)}(\tilde{z}_0)\| \le C(1/\lambda_n)$$

for $0 \le s_l \le a$. On the other hand $\exp\ t\tilde{\chi}_n(\tilde{z}_0)$ is uniformly convergent to $\exp t\tilde{\chi}(\tilde{z}_0)$ for $0 \le t \le a$. Any $0 \le t \le a$ is approximated by an $s_{l_n} = l_n/\lambda^n$, for which

$$F^{(-n)}G^{(l_n)}F^{(n)}(\tilde{z}_0) \to \exp s_{l_n}\tilde{\chi}_n(\tilde{z}_0) \to \exp t\tilde{\chi}(\tilde{z}_0)$$

as n tends to infinity. The statement for ζ is proved similarly using the lift $\tilde{\zeta}'$ defined on the j-sheet covering $\mathbb{C}_j$.

PROPOSITION 3.6. — *The closure of Γ-orbits are invariant under the flows of the vector fields χ, ζ.*

Proof. — Let $z_i \in \mathcal{O}(w) \cap B_f \cap B_g$ be a sequence convergent to a $z \in B_f \cap B_g$. By Proposition 3.5 $f^{(-n)} g^{(l_n)} f^{(n)}(z)$ is convergent to an $\exp t\chi(z)$ as $n \to \infty$ with a suitable sequence l_n. Then $f^{(-n)} g^{(l_n)} f^{(n)}(z_{i_n}) \in \mathcal{O}(w)$ is convergent to $\exp t\chi(z)$ for a sequence $i_n \to \infty$.

From which we obtain

PROPOSITION 3.7. — *If χ, ζ are $\mathbb{R}$-linearly independent at a $z \in B_f \cap B_g - 0$, any orbit is dense or empty on a neighbourhood of z.*

4. The existence of the separatrix : proof of Theorem 1 and Theorem 3.

Let χ, ζ be the vector fields on $B_f \cap B_g - 0$ constructed in §3. By Proposition 3.7 the property

(*) *Any orbit is dense or empty on a neighbourhood of z,*

holds at $z \in B_f \cap B_g - 0$ if $\chi(z), \zeta(z)$ are $\mathbb{R}$-linearly independent. By Proposition 3.6 this property propagates along the trajectories of χ, ζ. In this section we study the set of those $z \in B_\Gamma$ where this property does not hold, and we prove that this set possesses the various properties to be a separatrix.

Define the holomorphic vector fields $\chi^\epsilon = \chi(f^{(\epsilon)}, g), \zeta^\eta = \zeta(f, g^{(\eta)})$ respectively on $B_{f^{(\epsilon)}}, B_{g^{(\eta)}}$ similarly to χ, ζ replacing f, g by $f^{(\epsilon)}, g^{(\eta)}$ in the previous section for $\epsilon, \eta = \pm 1$. Let

$$B_{f,g} = \{z \in B_g |\ g^{(n)}(z) \in B_f \text{ for } n = 0, 1, \dots \text{ and } \lim_{n \to \infty} \arg g^{(n)}(z)/f^{(n)}(z) \leq \pi/2i\}.$$

By Lemma 2.2 $B_{f,g} - 0$ is open and $B = \bigcup_{\epsilon,\eta=\pm 1} B_{f^{(\epsilon)},g^{(\eta)}}$ is an open neighbourhood of 0, on which those $\chi^\epsilon, \zeta^\eta$ are defined everywhere either for $\epsilon, \eta = 1$ or -1.

First define Σ to be the set of those $z \in B_{f^{(\epsilon)},g^{(\eta)}} - 0$, $\epsilon, \eta = \pm 1$, with the following properties :

(i) $\chi^{\epsilon}, \zeta^{\eta}$ have a common trajectory C passing through z in $B_{f^{(\epsilon)},g^{(\eta)}} - 0$

(ii) $\chi^{-\epsilon}, \zeta^{-\eta}$ are tangent to C as long as they are defined.

4.1. — *Σ is real analytic, smooth and closed in $B - 0$, and Property (*) holds on $B - 0 - \Sigma$.*

Proof. — By Lemma 3.3, $\chi^{\epsilon}, \zeta^{\eta}$ are $\mathbb{C}$-linearly independent on $B_{f^{(\epsilon)},g^{(\eta)}} - 0$, so the Σ is locally a finite union of real analytic curves. Assume $\chi^{\epsilon}, \zeta^{\eta}$ are $\mathbb{R}$-dependent at a $z \in B_{f^{(\epsilon)},g^{(\eta)}} - 0$ and let C, D be respectively their trajectories passing through z in $B_{f^{(\epsilon)},g^{(\eta)}} - 0$. If $C \neq D$, ζ^{η} is $\mathbb{R}$-independent of χ^{ϵ} and Property (*) holds at a generic point on C. Since the closures of Γ-orbits are invariant under χ^{ϵ} by Proposition 3.6, the property propagates to z along C. Next assume $C = D$ and either $\chi^{-\epsilon}$ or $\zeta^{-\eta}$ is not tangent to C at a z'. Then the property holds at z' and propagates to z along C. This completes the proof.

Next let Σ' be the union of connected components C of Σ, on which Property (*) holds nowhere. Then

4.2. — *Σ' is closed in $B - 0$ and Property (*) holds on $B - 0 - \Sigma'$.*

Proof. — The closedness is clear. To show Property (*) on $B - 0 - \Sigma'$ let z be in a connected component C of Σ and assume that the property holds at a z' on C. Since the arc of C joining z' to z is a union of some common trajectories of $\chi^{\epsilon}, \zeta^{\eta}$, $\epsilon, \eta = \pm 1$, and the closure of orbits are invariant under the vector fields (Proposition 3.6), the property at z' propagates along C to z.

Define the separatrix $\Sigma(\Gamma)$ as union of $\{0\}$ with the set of those z in B_{Γ} such that $\mathcal{O}(z) \cap B - 0 \subset \Sigma'$. Then

4.3. — *$\Sigma(\Gamma) \cap B - 0 = \Sigma'$, $\Sigma(\Gamma)$ is closed in B_{Γ}, invariant under Γ and Property (*) holds on $B_{\Gamma} - \Sigma(\Gamma)$ by the Γ-invariance of $\Sigma(\Gamma)$.*

Proof. — The invariance and Property (*) on $B_{\Gamma} - \Sigma(\Gamma)$ follow from the construction. Property (*) holds on $B - 0 - \Sigma'$ and does not hold on Σ' and 0. So by the invariance of the property under Γ we obtain $\Sigma(\Gamma) \cap B - 0 = \Sigma'$. To show the closedness let $z \in \overline{\Sigma(\Gamma)} - \Sigma(\Gamma)$. Then

$f(z) \in B - 0 - \Sigma'$ for an $f \in \Gamma$. By 4.2, Σ' is closed in $B - 0$, so Σ' is empty at $f(z)$, from which it follows $\Sigma(\Gamma)$ is empty at z.

4.4. — *Any Γ-orbit is dense or empty in each connected component of $B_\Gamma - \Sigma(\Gamma)$.*

Proof. — Since Property (*) holds on $B_\Gamma - \Sigma(\Gamma)$, an orbit $\mathcal{O}(z)$ is locally dense at the points in $\mathcal{O}(z)$. Assume $\mathcal{O}(z)$ is not empty in a connected component D of $B_\Gamma - \Sigma(\Gamma)$ and $\overline{\mathcal{O}(z)} \neq \overline{D}$. Let $P = D - \overline{\mathcal{O}(z)}$. Then P is a non empty open subset of D. If $\overline{P} - P \subset \overline{D} - D$, then $P = D$ hence $\overline{D} = \overline{\mathcal{O}(z)}$. So it suffices to show $\overline{P} - P \subset \overline{D} - D$. Let $z' \in (\overline{P} - P) - (\overline{D} - D)$. Then $z' \in \overline{\mathcal{O}(z)}$ and $\mathcal{O}(z)$ is not empty at z'. On the other hand, on a neighbourhood of z', the orbit is dense by Property (*) therefore z' is not in the closure of P. This completes the proof.

4.5. — *Any Γ-orbit is dense or empty in each connected component of $\Sigma(\Gamma) - 0$.*

Proof. — By Proposition 3.6, the closures of Γ-orbits are invariant under the flows of $\chi^\epsilon, \zeta^\eta$ for $\epsilon, \eta = \pm 1$. Since $\Sigma(\Gamma)$ is invariant under Γ and the connected components of $\Sigma' - 0$ in 4.2 are union of trajectories of these vector fields, we see that the following property holds at each point $z \in \Sigma(\Gamma) - 0$:

(**) *Any Γ-orbit is dense or empty on a neighbourhood of z in $\Sigma(\Gamma) - 0$.*

The density we claim follows from this property with a similar argument to the proof of 4.4.

Next we prove

4.6. — *The induced vector field on the cylinder $\mathbb{P} - \{0, \infty\}$ associated to a fundamental domain which has non empty intersection with $\Sigma(\Gamma)$ is a pure imaginary linear flow, and the intersection of $\Sigma(\Gamma) - 0$ with B_f as well as $B_{f^{(-1)}}$ is a preimage of a union of closed cycles by the projection onto the quotient space $P - \{0, \infty\}$.*

Proof. — Let C be a connected component of $\Sigma(\Gamma) - 0$. By definition $\Sigma(\Gamma)$ is invariant under Γ, and the trajectory of the vector field χ passing through a $z_0 \in C \cap B_f - 0$ is contained in C. Let $\tilde{C}, \tilde{\chi}, \tilde{\zeta}$ be the lifts of

C, χ, ζ to $\tilde{\mathbb{C}}_i$ respectively and $\tilde{z}_0 = z_0^{-i} \in \tilde{\mathbb{C}}_i$. The vector field $\tilde{\chi}$ induces a linear vector field on the quotient space (cylinder) $\mathbb{P} - \{0, \infty\}$ in the connected component containing $\tilde{z}_0$ of the set of those $\tilde{z} \in \tilde{\mathbb{C}}_i$, $F^n(\tilde{z}) \to \infty$ (Lemma 3.2). The end of the lift $\tilde{C}$ is invariant under F if and only if $\tilde{C}$ projects to a closed curve in the cylinder. So if the end of $\tilde{C}$ is not invariant under F, the linear coefficient of the induced vector field on the cylinder is not pure imaginary, and the projection of $\tilde{C}$ accumulates at 0 and ∞. This with the invariance of $\Sigma(\Gamma)$ under the flow of χ as well as Γ (Proposition 3.6) tells that the intersections of the iterated images of $\tilde{C}$ under $F, F^{(-1)}$ with a rectangular fundamental domain of F accumulate at both ends corresponding to 0 and ∞. The diffeomorphism F is asymptotic to a translation at infinity, so there is a fundamental domain which is not contained in an arbitrary narrow sector with the vertex 0 in $\tilde{\mathbb{C}}_i$ and does not intersect with any large compact subset of $\tilde{\mathbb{C}}_i$. Choosing appropriately the fundamental domain we may assume that either $\tilde{\zeta}$ or $\tilde{\zeta}^{-1}$ is defined on each of its end. Now recall that the lift of ζ to the j-sheet covering $\tilde{\mathbb{C}}_j$ converges to a constant vector at infinity. So, on the $\tilde{\mathbb{C}}_i$, the ends of those common trajectories of $\tilde{\chi}$ and $\tilde{\zeta}$ have to be contained in arbitrary narrow vertical sectors with vertices at the origin, which contradicts the noncompactness of the intersection of the iterated images of $\tilde{C}$ with the fundamental domain. Therefore the end of $\tilde{C}$ is invariant under F.

Let $\Gamma^0 \subset \Gamma$ be the pseudogroup of flat diffeomorphisms at 0 (the linear term at $0 \in \mathbb{C}$ is z). It follows from 4.6 that

4.7. — *All connected components of $\Sigma(\Gamma) - 0$ containing 0 in their closures are invariant under the sub-pseudogroup Γ^0 of Γ of diffeomorphisms flat at 0 (with the linear term z at 0). And there is no other components of $\Sigma(\Gamma) - 0$ on a neighbourhood of 0.*

Both 4.6 and 4.7 lead to the following picture : given $f(z) = z + az^{k+1} + \cdots$ as before, let $T(f)$ be the union of the real lines consisting of those z' such that z' and az'^{k+1} are $\mathbb{R}$-linearly dependent. Then any connected component of $\Sigma(\Gamma) - 0$ containing 0 in its closure has in fact a tangent line at 0 which is contained in $T(f)$.

4.8. — *All branches of the germ of $\Sigma(\Gamma)$ at 0 have distinct tangent directions at the origin.*

Proof. — Assume that two branches C, C' of the germ of $\Sigma(\Gamma) - 0$ have a common tangent direction (which are possibly germs of a common connected component of $\Sigma(\Gamma)-0$). By 4.7, these germs are invariant under f and g. We say that C, C' are *k-separable* if the inequality

$$c\|z\|^{k+1} \leq \operatorname{dist}(C, z) + \operatorname{dist}(C', z) \leq c'\|z\|^{k+1}$$

holds for $z \in C \cup C'$ on a neighbourhood of 0 with positive constants c, c'. To prove the statement it suffices to show that branches C, C' are k-separable if and only if f is k-flat, assuming their lifts $\tilde{C}, \tilde{C}' \subset \tilde{\mathbb{C}}_k$ are contained in the sector $\tilde{S}_i^+ \subset \tilde{\mathbb{C}}_k$ in §2. On a sheet of $\tilde{\mathbb{C}}_k$

$$\frac{1}{2k}\|\tilde{z}\|^{-(k+1)/k}\|\tilde{w}-\tilde{z}\| \leq \|\tilde{w}^{-1/k}-\tilde{z}^{-1/k}\| \leq \frac{2}{k}\|\tilde{z}\|^{-(k+1)/k}\|\tilde{w}-\tilde{z}\|,$$

holds for $\tilde{w}, \tilde{z} \in \tilde{\mathbb{C}}_k$ sufficiently large and at bounded distance each other. By this estimate, it suffices to show the estimate

$$\text{(i)} \qquad 2kc \leq \operatorname{dist}\,(\tilde{C}, \tilde{z}) + \operatorname{dist}\,(\tilde{C}', \tilde{z}) \leq \frac{kc'}{2}$$

for sufficiently large $\tilde{z} \in \tilde{C} \cup \tilde{C}'$. To show this recall that the lifts $\tilde{C}, \tilde{C}'$ are trajectories of the linear vector field $\tilde{\chi}$ and invariant under F by 4.7. Assume $f(z) = z - \dfrac{2\pi\sqrt{-1}}{}kz^{k+1} + cz^{2k+1} + \cdots$, $F(\tilde{z}) = \tilde{z} + 2\pi\sqrt{-1} + \dfrac{a'}{\tilde{z}} + \cdots$ and let $\tilde{\phi}_i^{\epsilon(-1)}$ be the diffeomorphism which normalizes F to the translation by $2\pi\sqrt{-1}$ in §2. By 4.6 we may assume that $\tilde{C}, \tilde{C}'$ are contained in the sector $\tilde{S}_i^+$ on which $\tilde{\phi}_i^{+(-1)}$ is defined. And $\tilde{\phi}_i^{+(-1)}$ maps $\tilde{C}, \tilde{C}'$ respectively to parallel lines L, L', for which the following estimate holds

$$\text{(ii)} \qquad 4kc \leq \operatorname{dist}\,(\tilde{L}, \tilde{z}) + \operatorname{dist}\,(\tilde{L}', \tilde{z}) \leq \frac{kc'}{4}$$

with positive constants c, c'. By definition and Lemma 2.6 we obtain

$$\log d\tilde{\phi}_i^{+(-1)} = \log dF^{(\infty)} = O(\tilde{z}^{-1}),$$

on $\tilde{S}_i^+$, from which

$$d\tilde{\phi}_i^{+(-1)} = 1 + O(\tilde{z}^{-1})$$

on $\tilde{\phi}_i^{+(-1)}(\tilde{S}_i^+)$ and

$$\frac{1}{2}\|\tilde{w}-\tilde{z}\| \leq \|\tilde{\phi}_i^{+(-1)}(\tilde{w}) - \tilde{\phi}_i^{+(-1)}(\tilde{z})\| \leq 2\|\tilde{w}-\tilde{z}\|$$

for $\tilde{w}, \tilde{z}$ at bounded distance from each other. The estimate (i) follows from (ii) with the above estimate. Similar argument shows that C, C' are not l-separable if $l \neq k$.

The following is a corollary to the above results.

4.9. — *Let $T(f)$, $f(z) = z + az^{k+1} + \cdots$, be the union of real lines consisting of those z' for which z' and $a{z'}^{k+1}$ are $\mathbb{R}$-linearly dependent. Then to each half line in $T(f)$ is associated at most one branch of $\Sigma(\Gamma)$.*

4.10. — *The branches of the germ of $\Sigma(\Gamma)$ at 0 are C^∞-smooth at 0.*

Proof. — First we show that a real trajectory $\exp t\chi(z)$ of the vector field $\chi = \dfrac{z^{k+1}}{1+mz^k}\partial/\partial z$ is convergent to 0 as t tends to ∞ or $-\infty$, and the image is C^{k-1}-smooth at 0. On the coordinate plane of $z' = (z^{-k} + m \log z^{-k})^{-1/k}$, the vector field induces $z'^{k+1}\partial/\partial z'$. A trajectory of $z'^{k+1}\partial/\partial z'$ is parametrized as $(c+t)^{-1/k}, c \in \mathbb{C}$ with the real parameter $t \in \mathbb{R}$ and the image is real analytic at 0. Since the $(k-1)$-jet of the transformation $z \to z'$ is asymptotic to the identity at 0, the trajectory of χ is C^{k-1}-smooth at 0. Next let $f \in \Gamma$ be a k-flat diffeomorphism. Assume that $f(z) = z + z^{k+1} + \cdots$ is formally equivalent to $\exp \chi$ and the branch C is contained in the sector S_i^ϵ on a neighbourhood of 0, and let $\phi_i^\epsilon : S_i^\epsilon \to T_i^\epsilon$ the diffeomorphism which normalizes f to an $\exp \chi$ in Theorem 2.7. By 4.6, the image $\phi_i^\epsilon(C)$ is a trajectory of χ and C^{k-1}-smooth at 0. Since the Taylor series of the diffeomorphism ϕ_i^ϵ is asymptotic to a formal series at 0, C is also C^{k-1}-smooth at 0. In a nonsolvable group G, there is an f with an arbitrary large order of flatness k because the commutator sequence of the germ Γ_0 of Γ is infinite and a commutator $[f,g]$ is k-flat, $i,j<k$ if f is i-flat and g is j-flat. Therefore the branch C is C^∞-smooth at 0.

4.11. — *C extends to a real analytic smooth curve at 0.*

Proof. — Let $t \to P(t) = r(t) + \sqrt{-1}s(t) \in \mathbb{C}, t \in \mathbb{R}^+, P(0) = 0, P'(0) \neq 0$, be a C^∞-smooth parametrization of the curve C with real valued functions r, s, and $\hat{P}(t)$ the formal power series of P at 0. Let $\Gamma_0^0 \subset \Gamma_0$ denote the subgroup of flat diffeomorphisms in the germ Γ_0. Since the germ C is invariant under flat diffeomorphisms by 4.7, Γ_0^0 induces a group $P^{-1}\Gamma_0^0$ of germs of C^∞-diffeomorphisms acting on $\mathbb{R}^+$ at 0 with indifferent linear term. The group $\hat{P}^{-1}\Gamma_0^0$ of the formal power series of the germs in $P^{-1}\Gamma_0^0$ is real : all coefficients are real. So it commutes with the complex conjugation I, and $\hat{P}I\hat{P}^{(-1)}$ is an anti-holomorphic formal involution which commutes with Γ_0^0. Let $\bar{\Gamma}_0^0$ be the group of germs of the holomorphic diffeomorphisms $\bar{g} = IgI, g \in \Gamma_0^0$. Then $I\hat{P}I\hat{P}^{(-1)}$ is a formal conjugacy linking Γ_0^0 to $\bar{\Gamma}_0^0$. By Theorem 5, $I\hat{P}I\hat{P}^{(-1)}$ is convergent to a holomorphic conjugacy h linking Γ_0^0 to $\bar{\Gamma}_0^0$: $I\hat{P}I\hat{P}^{(-1)} \circ g = (IgI) \circ I\hat{P}I\hat{P}^{(-1)}$, and then $\hat{P}I\hat{P}^{(-1)} = Ih$ is an anti-holomorphic involution commutative with Γ_0^0. Clearly the fixed point set C' of Ih is invariant under Γ_0^0 and has the same tangent line as the original C at 0. By the uniqueness of the branches for each tangent direction (4.8), $C = C'$ on a neighbourhood of 0.

4.12. — *The germ of $\Sigma(\Gamma)$ at 0 is holomorphically diffeomorphic to a subset of the set* $\operatorname{Im} z^k = 0$ *for a positive integer k.*

Proof. — By 4.11, the branches C_i of the germ of $\Sigma(\Gamma)$ extend to real analytic smooth curves C_i'. Let $\bar{H}$ be the group generated by the anti-holomorphic involutions respecting those C_i', and $H \subset \bar{H}$ the orientation preserving subgroup. Since H commutes with Γ_0^0 by Schwarz reflection principle, if an $h \in H, h \neq \mathrm{id}$, has the indifferent linear term, then Γ_0^0 imbeds to the complexified one parameter family $h^{(t)}, t \in \mathbb{C}$ by Proposition 1.5 and, in particular, Γ_0^0 is commutative hence Γ_0 is solvable by the argument in the begining of §2. By assumption, Γ_0 is nonsolvable. So if $h \in H, h'(0) = 1$, then $h = \mathrm{id}$ and the diffeomorphisms in H are determined by their linear terms. Since H commutes with Γ_0^0 and C_i are invariant under Γ_0^0, the image of those C_i under H is also invariant under Γ_0^0. The argument in 4.6 - 9 applies to any curve invariant under Γ_0^0, and tells the image is a finite union of curves with all distinct tangent directions. Since all germs in H have linear coefficients with absolute value 1, we see that H is holomorphically equivalent to a cyclic group of a finite order k generated by a linear rotation $\omega_k z, \omega_k^k = 1$. Since Γ_0^0 commutes with the rotation, it induces a group $\tilde{\Gamma}_0^0$ of germs of holomorphic diffeomorphisms of the quotient space $\mathbb{C} = \mathbb{C}/\{z \to z^k\}$ at 0, which is semi conjugate to Γ_0^0 via the invariant function z^k. It is easy to see that the smooth curves C_i' project to an irreducible real analytic curve R under z^k and the group $\bar{H}$ induces an anti-holomorphic involution respecting R. Therefore the curve R is smooth and the group $\tilde{\Gamma}_0^0$ commutes with the anti-holomorphic involution.

We may assume that $R \subset \mathbb{C}$ is the real line by a suitable coordinate change. Then we obtain

4.13. — *The flat subgroup Γ_0^0 is induced from a group of germs of real holomorphic diffeomorphisms acting on $\mathbb{C}, 0$ via a finitely branched map at 0.*

Corollary 2 follows from the above 4.12 and 4.13.

4.14. Proof of Theorem 3.

Proof. — Let Σ be the largest germ of real analytic subset of $\mathbb{C}$ at 0 which is invariant under the germ Γ_0 and holomorphically diffeomorphic to the set $\{\mathrm{Im}\ z^{k'} = 0\}$, k' being an integer, and let $\tilde{\Sigma} \subset U$ be a closed real analytic subset of an open neighbourhood U of 0 with the germ Σ at 0. For each $f \in \Gamma$ let f' be a restriction of f to an open neighbourhood $U_{f'} \subset U$ such that $f'(U_{f'}) \subset U$ and $f'(\tilde{\Sigma} \cap U_{f'}) = f'(U_{f'}) \cap \tilde{\Sigma}$ and let Γ' be the pseudogroup generated by those restrictions. Then $\tilde{\Sigma}$ is Γ'-invariant hence $\Sigma(\Gamma') \supset \vec{\Sigma}$. On the other hand the germ of $\Sigma(\Gamma')$ is contained in Σ by the definition of Σ. Therefore $\Sigma(\Gamma') = \Sigma$ at 0. This argument applies to any restriction Γ'' of Γ' and implies that $\Sigma(\Gamma'') = \Sigma$ at 0.

5. Topological rigidity theorem.

Let Γ, Γ' be pseudogroups of diffeomorphisms of open neighbourhoods of $0 \in \mathbb{C}$. Assume that there exists a linking homeomorphism $h : U \to h(U)$ of Γ to Γ' (for the various definitions, see the introduction and §2). The part (1).(i) of the following theorem is attributed to Shcherbakov [20].

THEOREM 5.1 (Topological rigidity theorem). — (1) *The restriction of the linking* $h : B_\Gamma \to B_{\Gamma'}$ *is holomorphic or anti-holomorphic diffeomorphism if one of the following conditions holds :*

(i) *The germs* Γ_0, Γ_0' *are non-solvable, in other words, the commutator subgroups are non-commutative,*

(ii) Γ_0 *is non-commutative but solvable,* $\Lambda(\Gamma_0) \subset \mathbb{C}$ *is dense and the action of the linear term* $L(\Gamma_0) \subset \mathbb{C}^*$ *on* $\Lambda(\Gamma_0)$ *contains a non-real multiplication,*

(iii) Γ_0 *is non-commutative but solvable and the action of* $L(\Gamma_0)$ *on* $\Lambda(\Gamma_0)$ *contains an action of* $\mathbb{Z}_n, n \neq 2, 3, 4$, *or a non-real and non-periodic action.*

(2) *There exists a germ of holomorphic linking diffeomorphism of* Γ_0 *to* Γ_0' *if* Γ_0 *is non-solvable but solvable and the action of* $L(\Gamma_0)$ *on* $\Lambda(\Gamma_0)$ *is not antipodal.*

Proof of 1 (i). — Recall that on the neighbourhood $B = \bigcup_{\epsilon,\eta=\pm 1} B_{f^{(\epsilon)},g^{(\eta)}}$ of the origin the vector fields $\chi^\epsilon, \zeta^\eta$ are defined everywhere either for $\epsilon, \eta = 1$ or -1 (see the begining of §4). Let $f' = h \circ f \circ h^{(-1)}$ and

$g' = h \circ g \circ h^{(-1)}$, and define $\chi'^{\epsilon}, \zeta'^{\eta}$ similarly to $\chi^{\epsilon}, \zeta^{\eta}$ replacing f, g with f', g'. By Lemma 5.3 h sends the flows of $\chi^{\epsilon}, \zeta^{\eta}$ to those of $\chi'^{\epsilon}, \zeta'^{\eta}$ respecting real time, and by Lemmas 3.4 and 5.2, h is holomorphic (anti-holomorphic) on the B respectively if it is orientation preserving (resp. orientation reversing). For $z \in B_{\Gamma}$ there exists an $e \in \Gamma$ such that $e(z) \in B$. Since h sends $z, e(z)$ respectively to $h(z), e'(h(z))$ with $e' = h \circ e \circ h^{(-1)}$, h is also holomorphic or anti-holomorphic at z.

Proof of 1 (ii). — We use the notations in §1. By Theorems 1.6 and 1.8, we may assume the subgroup $\Gamma_0^0 \subset \Gamma_0$ consists of k-flat diffeomorphisms $\exp t\chi$, $t \in \Lambda(\Gamma_0)$ with $\chi(z) = z^{k+1}\, \partial/\partial z$, which lift to the translations by $-kt$ on the k-sheet covering $\tilde{\mathbb{C}}_k$. The order of flatness for flat diffeomorphisms is invariant under topological conjugacy. So we apply a similar argument to Γ'. The lift h' of the topological conjugacy h to $\tilde{\mathbb{C}}_k$ links the translation by $-k\Lambda(\Gamma_0)$ to the translation by $-k\Lambda(\Gamma_0')$ hence it is a real affine isomorphism because $-k\Lambda(\Gamma_0), -k\Lambda(\Gamma_0') \subset \mathbb{C}$ are dense by assumption. The h' induces the isomorphism ϕ of $\Lambda(\Gamma_0)$ to $\Lambda(\Gamma_0')$. Assume $g \in \Gamma, g' = h \circ g \circ h^{(-1)} \in \Gamma'$ have linear terms b, b' respectively, and assume b^k is non-real. The actions of $L(\Gamma_0), L(\Gamma_0')$ on $\Lambda(\Gamma_0), \Lambda(\Gamma_0')$ are also equivalent by the isomorphisms $\psi : L(\Gamma_0) \to L(\Gamma_0')$ induced from h, and in particular, ϕ links the non-real linear multiplication of b^k to that of b'^k hence ϕ is homothety. Therefore h as well as h' is holomorphic or anti-holomorphic diffeomorphisms.

Proof of 1 (iii). — Assume that Γ_0 is non-commutative but solvable. Then Γ_0^0 is isomorphic to $\Lambda(\Gamma_0) \subset \mathbb{C}$, which is invariant under the action of $L(\Gamma_0)$. If the action contains either a periodic action by $\mathbb{Z}_n, n \neq 2, 3, 4$ or a non-real and non-periodic action, then $\Lambda(\Gamma_0)$ is dense and the proof reduces to Case (ii).

Proof of 2. — Assume that Γ_0^0 consists of $\exp t\chi, t \in \Lambda(\Gamma_0) \subset \mathbb{C}, \chi(z) = z^{k+1}\partial/\partial z$. If $\Lambda(\Gamma_0)$ is dense in $\mathbb{C}$, the rigidity holds by 1.(ii). Since the action of $L(\Gamma_0)$ is not antipodal, if $\Lambda(\Gamma_0)$ is not discrete, it is dense. So assume that $\Lambda(\Gamma_0)$ is discrete, that is, a non-degenerate lattice $\lambda\mathbb{Z} + \mu\mathbb{Z}$. By Theorem 1.8, the quotient Γ_0/Γ_0^0 is generated by a $g = \omega_n \exp b\chi$, which is equivalent to $\omega_n z$ by $\exp d\chi$: $(\exp\ -d\chi) \circ (\omega_n \cdot \exp d\chi) = \omega_n \cdot \exp c\chi$, $d = \omega_n^{-k}/(\omega_n^{-k} - 1)$. And then Γ_0^0 remains with the same form. Assume that Γ, Γ' are topologically equivalent. Then Γ'^0_0 is also k-flat and the translations by $-k\Lambda(\Gamma_0), -k\Lambda(\Gamma_0')$ on $\tilde{\mathbb{C}}_k$ are topologically equivalent. So $\Lambda(\Gamma_0')$ is also non-degenerate and Γ'^0_0 embeds to a one parameter family

$\exp t\chi'$. The conjugacy induces also that the linear terms of Γ_0, Γ_0' coincide and their conjugate actions on $\Lambda(\Gamma_0), \Lambda(\Gamma_0')$ are equivalent. Therefore the germs of Γ, Γ' at the origin are holomorphically equivalent.

LEMMA 5.2. — *Let $X_i, Y_i, i = 1, 2$ be holomorphic germs of nonsingular vector fields on $\mathbb{C}, 0$. Assume that there exists a germ of homeomorphism $h : \mathbb{C}, 0 \to \mathbb{C}, h(0)$ such that $h \circ \exp tX_i = \exp tY_i \circ h$ for small $t \in \mathbb{R}$. If $[X_1, X_2], [Y_1, Y_2]$ are not real constant vectors, then h is holomorphic or anti-holomorphic with respect to z.*

Proof. — First we assume that $X_1(0), X_2(0)$ and $Y_1(0), Y_2(0)$ are independent over $\mathbb{R}$. Let $z \in \mathbb{C}$ be a point nearby 0. Since the mappings $A_z(s,t) = \exp sX_1 \circ \exp tX_2\ (z)$ and $A_z'(s,t) = \exp sY_1 \circ \exp tY_2\ (z)$ from $\mathbb{R}^2$ to $\mathbb{C}$ are C^∞-diffeomorphic at $(0,0) \in \mathbb{R}^2$ and $h(A_z(s,t)) = A'_{h(z)}(s,t)$ by assumption, h is a germ of C^∞-diffeomorphism. So we have only to show the analyticity. Define $B_z(s,t) = \exp tX_2 \circ \exp sX_1\ (z)$ and $B_z'(s,t)$ similarly replacing X_1, X_2 with Y_1, Y_2. Since the mappings A_z, B_z have the same non singular 1-jet at $(0,0)$, there exists a function $(s'(s,t,z), t'(s,t,z)) = (s,t) + O(s^2+t^2)$ such that

$$A_z(s,t) = B_z(s',t').$$

Consider the map

$$D_z^{(s,t)}(y) = [\exp(-s'X_1) \circ \exp(-t'X_2) \circ \exp(sX_1) \circ \exp(tX_2)\ ]\ (y).$$

Define $D'^{(s,t)}_z$ similarly with Y_1, Y_2. Since $h(D_z^{(s,t)}(y)) = D_{h(z)}^{(s,t)}(h(y))$ and $D_z^{(s,t)}(z) = z$ (due to the choice of (s',t')), we obtain

$$(*) \qquad dh(z) \circ dD_z^{(s,t)}(z) = dD'^{(s,t)}_{h(z)}(h(z)) \circ dh(z).$$

Here $dD_z^{(s,t)}(z) = \mathrm{id} + st[X_2, X_1](z) + o(s^2+t^2)$ and $dD'^{(s,t)}_{h(z)}(h(z)) = \mathrm{id} + st[Y_2, Y_1](h(z)) + o'(s^2+t^2)$, which are non real homotheties for small s, t, if $[X_2, X_1](z)$, $[Y_2, Y_1](h(z))$ are non real. And then it follows that $dh(z)$ is a homothety from the relation $(*)$.

The set A of those z such that $[X_1, X_2]$ is non-real at z, $[Y_1, Y_2]$ is non-real at $h(z)$, $X_1(z), X_2(z)$ are $\mathbb{R}$-independent and $Y_1(h(z)), Y_2(h(z))$ are $\mathbb{R}$-independent is open dense nearby the origin. On A, h is holomorphic or anti-holomorphic by the above argument. The relation $\exp tY_i \circ h = h \circ \exp tX_i$ implies that if h is holomorphic or anti-holomorphic at a point, h is also holomorphic or anti-holomorphic, uniformly, along its trajectories of X_i. Therefore the $\pm$-holomorphicity of h extends to the set B of those w joined to a $z \in A$ by piecewise-trajectories of X_1, X_2. The complement of the set B is contained in the set C of those z, where X_1, X_2 are $\mathbb{R}$-dependent. The set C is a union of common trajectories of X_1, X_2. The $\pm$-holomorphicity extends to the smooth part of C by Painlevé's theorem, and the extends to the discrete singular point set by Riemann's extension theorem.

LEMMA 5.3. — *The real vector fields χ, ζ defined in §3 are real-time-preservingly invariant under topological equivalence.*

Proof. — We prove the statement only for χ. Let Γ, Γ' be non-solvable pseudogroups and h a topological equivalence from Γ to Γ'. Let $f, g \in \Gamma$ be as in §3 and let $f' = h \circ f \circ h^{(-1)}$ and $g' = h \circ g \circ h^{(-1)}$. Then f' and g' are also i-flat and j-flat respectively. Define the vector field χ' similarly replacing f, g with f', g' and let $\tilde{\chi}'$ be its lift to $\tilde{\mathbb{C}}_i$. By Proposition 3.5 $f^{(-n)} g^{(l_n)} f^{(n)}(z_0) \to \exp t\chi(z_o)$ as $n \to \infty$ if $l_n n^{(i-j)/i} \to t$. Since h sends the orbits $f^{(-n)} g^{(m)} f^{(n)}(z_0)$ of z_0 to those of $h(z_0)$ defined with f', g', $f'^{(-n)} g'^{(l_n)} f'^{(n)}(h(z_0))$ is convergent to $h(\exp t\chi(z_o))$. On the other hand $f'^{(-n)} g'^{(l_n)} f'^{(n)}(h(z_0)) \to \exp t\chi(h(z_o))$ by Proposition 3.5. Therefore $h \circ \exp t\chi = \exp t\chi' \circ h$ for real t.

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