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ON LINNIK'S THEOREM ON GOLDBACH NUMBERS IN SHORT INTERVALS AND RELATED PROBLEMS

by A. LANGUASCO and A. PERELLI

1. Introduction.

Define a Goldbach number (G-number) to be an even number representable as a sum of two primes, and write $L = \log N$. The first result concerning the existence of G-numbers in short intervals is due to Linnik [6] who proved, assuming the Riemann Hypothesis (RH), that for any $\varepsilon > 0$ and N sufficiently large, the interval $[N, N + L^{3+\varepsilon}]$ contains a G-number. Linnik's result was improved by Kátai [4] and, independently, by Montgomery-Vaughan [7] who showed that the interval $[N, N + CL^2]$ contains a G-number provided C and N are sufficiently large.

Linnik used the circle method in the proof of his result, while Kátai and Montgomery-Vaughan exploited the connection between G-numbers and primes in short intervals. Indeed, Kátai and Montgomery-Vaughan's result follows easily from the following estimate, due to Selberg [9] under RH,

$$J(N,H) = \int_1^N |\psi(x+H) - \psi(x) - H|^2 \, dx \ll NHL^2. \tag{1}$$

Estimate (1) has been proved by other methods by Saffari-Vaughan [8] and Gallagher [1]. We remark that the slightly weaker estimate, still under RH,

$$J(N,H) \ll NHL^3 \tag{2}$$

may be obtained in a straightforward way using the explicit formula for $\psi(x)$. Estimate (2) corresponds, in a sense, to Linnik's result. The method

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of Saffari-Vaughan for the proof of (1) is based on a preliminary ingenious averaging technique which makes more efficient the use of the explicit formula.

In 1990, Goldston [2] pointed out that the result of Kátai and Montgomery-Vaughan can also be obtained by the circle method and hence, in a way, closer to Linnik's approach. However, Linnik's argument needs no use of (1) as opposed to Goldston who reduces the problem to this estimate, via an application of Gallagher's lemma.

The principal aim of this paper is to show that a variation of Linnik's original approach is capable of proving the result of Kátai and Montgomery-Vaughan without the use of estimate (1). This is obtained first by inserting the Saffari-Vaughan technique into the machinery of the circle method and then by avoiding the use of Parseval's identity in a critical part of the unit interval.

Throughout this paper, we will formulate our arguments in terms of the infinite exponential sum

$$\widetilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/N} e(n\alpha),$$

as Linnik himself did. However, completely analogous results may be obtained by using, instead, the finite exponential sum

$$S(\alpha) = \sum_{n \leq N} \Lambda(n) e(n\alpha).$$

The modifications required in using $S(\alpha)$ in place of $\widetilde{S}(\alpha)$ are based on the explicit formula for $\psi(x)$.

THEOREM 1. — *Assume RH and let* $z = \dfrac{1}{N} - 2\pi i\alpha$. *For N sufficiently large and* $0 \leq \xi \leq \dfrac{1}{2}$ *we have*

$$\int_{-\xi}^{\xi} \left| \widetilde{S}(\alpha)^2 - \frac{1}{z^2} \right| d\alpha \ll N\xi L^2 + N\xi^{1/2} L.$$

We remark that the estimate in Theorem 1 appears to be in contrast with Goldston's statement at the beginning of sect. 4 of [2]. In fact, the loss of a factor L in Linnik's work is due to his treatment of the explicit formula for $\widetilde{S}(\alpha)$, which does not use the smoothing technique of Saffari-Vaughan, and not to the fact that $\widetilde{S}(\alpha)$ does not truncate at N. The above loss corresponds to the loss of a factor L in (2), compared with (1).

From Theorem 1 we deduce the following

COROLLARY 1. — *Assume RH. There exists a constants $C > 0$ such that, for $N \geq 2$, the interval $[N, N + CL^2]$ contains a G-number.*

Essentially, our method can be used to obtain G-numbers in intervals $[N, N + H]$ for those H for which an estimate of the form

$$\int_{-\frac{1}{H}}^{\frac{1}{H}} \left| \widetilde{S}(\alpha)^2 - \frac{1}{z^2} \right| d\alpha \leq cN$$

holds, where $c > 0$ is a suitable constant. A simple consequence of Theorem 1 is the following

COROLLARY 2. — *Assume RH. For N sufficiently large and $0 \leq \xi \leq \dfrac{1}{2}$ we have*

$$\int_{-\xi}^{\xi} \left| \widetilde{S}(\alpha) \right|^2 d\alpha = \frac{N}{\pi} \arctan 2\pi N\xi + O(N\xi L^2) + O(N\xi^{1/2}L).$$

Corollary 2 should be compared with the result provided by the Parseval identity, i.e.

$$\int_{-1/2}^{1/2} \left| \widetilde{S}(\alpha) \right|^2 d\alpha \sim \frac{NL}{2}. \tag{3}$$

Hence Corollary 2 may be regarded as a conditional truncated version of (3). However, note that taking $\xi = \dfrac{1}{2}$ in Corollary 2 only gives the weaker result

$$\int_{-1/2}^{1/2} \left| \widetilde{S}(\alpha) \right|^2 d\alpha \ll NL^2.$$

A sharper version of Theorem 1, and hence of Corollary 1 and 2, may be obtained by assuming the Montgomery pair correlation conjecture in addition to RH.

In 1959, Lavrik [5] proved that

$$\int_a^b \left| \widetilde{S}(\alpha) \right|^2 d\alpha = \frac{b-a}{2} NL + O(N \log^2 L)$$

if $0 \leq b-a \leq 1$. An unconditional result concerning truncations of Parseval's identity, which improves Lavrik's result, is the following

THEOREM 2. — *Let $0 \leq b-a \leq 1$ and N be sufficiently large. Then*

$$\int_a^b \left|\widetilde{S}(\alpha)\right|^2 d\alpha = \frac{b-a}{2} NL + O(N(L(b-a))^{1/3}) + O(N).$$

We remark that Theorem 2 is essentially best possible, in the sense that one cannot replace the term $O(N)$ by $o(N)$.

An application of Theorem 2 can lead, under suitable circumstances, to a sharpening of results which involve the use of Parseval's identity. For example, the use of Theorem 2 instead of Parseval's identity in Linnik's original arguments allows one to remove the ε in Linnik's result. This should be compared with Goldston's comments on the Fourier polynomial $V(\alpha)$ in sect. 4 of [2]. In the same way, Theorem 2 may replace the partial integration argument in the proof of Corollary 1.

We finally remark that Theorem 2 enables one to deduce the order of magnitude of $\int_{-\xi}^{\xi} \left|\widetilde{S}(\alpha)\right|^2 d\alpha$ in the whole range $0 \leq \xi \leq \frac{1}{2}$. Writing $f(x) \asymp g(x)$ for $g(x) \ll f(x) \ll g(x)$, we have

COROLLARY 3. — *Let N be sufficiently large. Then*

$$\int_{-\xi}^{\xi} \left|\widetilde{S}(\alpha)\right|^2 d\alpha \asymp \begin{cases} N^2\xi & \text{if } 0 \leq \xi \leq \frac{1}{N} \\ N & \text{if } \frac{1}{N} \leq \xi \leq \frac{1}{L} \\ N\xi L & \text{if } \frac{1}{L} \leq \xi \leq \frac{1}{2}. \end{cases}$$

Corollary 3 may be regarded as a truncated version of Parseval's identity.

2. Proof of Theorem 1.

We use the following explicit formula

$$\widetilde{S}(\alpha) = \frac{1}{z} - \sum_{\rho} z^{-\rho}\Gamma(\rho) + O(L^3)$$

where $z = \dfrac{1}{N} - 2\pi i\alpha$ and $\rho = \dfrac{1}{2} + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$, see [6]. Hence

$$\widetilde{S}(\alpha)^2 - \frac{1}{z^2} \ll \left|\sum_\rho z^{-\rho}\Gamma(\rho)\right|^2 + \left|\frac{1}{z}\sum_\rho z^{-\rho}\Gamma(\rho)\right|$$

$$+L^3\left|\sum_\rho z^{-\rho}\Gamma(\rho)\right| + L^3\left|\frac{1}{z}\right| + L^6 = \widetilde{R}_1 + \ldots + \widetilde{R}_5, \tag{4}$$

say. Since

$$\frac{1}{z} \ll \min\left(N, \frac{1}{\alpha}\right)$$

we have

$$\int_{-\xi}^{\xi} \widetilde{R}_5 d\alpha \ll \xi L^6, \tag{5}$$

$$\int_{-\xi}^{\xi} \widetilde{R}_4 d\alpha \ll N^{1/2}\xi^{1/2}L^3, \tag{6}$$

$$\int_{-\xi}^{\xi} \widetilde{R}_3 d\alpha \ll \xi^{1/2}L^3\left(\int_{-\xi}^{\xi} \widetilde{R}_1 d\alpha\right)^{1/2} \tag{7}$$

and

$$\int_{-\xi}^{\xi} \widetilde{R}_2 d\alpha \ll N^{1/2}\left(\int_{-\xi}^{\xi} \widetilde{R}_1 d\alpha\right)^{1/2}. \tag{8}$$

Since $z^{-\rho} = |z|^{-\rho}\exp(-i\rho\arctan 2\pi N\alpha)$, by Stirling's formula we have that

$$\sum_\rho z^{-\rho}\Gamma(\rho) \ll \sum_\rho |z|^{-1/2}\exp\left(\gamma\arctan 2\pi N\alpha - \frac{\pi}{2}|\gamma|\right).$$

If $\gamma\alpha \le 0$ or $|\alpha| \le \dfrac{1}{N}$ we get

$$\sum_\rho z^{-\rho}\Gamma(\rho) \ll N^{1/2},$$

where, in the first case, ρ runs over the zeros with $\gamma\alpha \le 0$.

Hence

$$\int_{-\xi}^{\xi} \widetilde{R}_1 d\alpha \ll N\xi \tag{9}$$

if $0 \leq \xi \leq \dfrac{1}{N}$, and

$$\int_{-\xi}^{\xi} \widetilde{R}_1 d\alpha \ll \int_{1/N}^{\xi} \Big|\sum_{\gamma>0} z^{-\rho}\Gamma(\rho)\Big|^2 d\alpha + \int_{-\xi}^{-1/N} \Big|\sum_{\gamma<0} z^{-\rho}\Gamma(\rho)\Big|^2 d\alpha + N\xi \tag{10}$$

if $\xi > \dfrac{1}{N}$. We will treat only the first integral on the right hand side of (10), the second being completely similar.

Clearly

$$\int_{1/N}^{\xi} \Big|\sum_{\gamma>0} z^{-\rho}\Gamma(\rho)\Big|^2 d\alpha = \sum_{k=1}^{K} \int_{\eta}^{2\eta} \Big|\sum_{\gamma>0} z^{-\rho}\Gamma(\rho)\Big|^2 d\alpha + O(1) \tag{11}$$

where $\eta = \eta_k = \dfrac{\xi}{2^k}$, $\dfrac{1}{N} \leq \eta \leq \dfrac{\xi}{2}$ and K is a suitable integer satisfying $K = O(L)$. Writing $\arctan 2\pi N\alpha = \dfrac{\pi}{2} - \arctan \dfrac{1}{2\pi N\alpha}$ and using the Saffari-Vaughan technique we have

$$\int_{\eta}^{2\eta} \Big|\sum_{\gamma>0} z^{-\rho}\Gamma(\rho)\Big|^2 d\alpha \leq \int_{1}^{2} \left(\int_{\frac{\delta\eta}{2}}^{2\delta\eta} \Big|\sum_{\gamma>0} z^{-\rho}\Gamma(\rho)\Big|^2 d\alpha \right) d\delta$$

$$= \sum_{\gamma_1>0} \sum_{\gamma_2>0} \Gamma(\rho_1)\overline{\Gamma(\rho_2)}\, e^{\frac{\pi}{2}(\gamma_1+\gamma_2)}\, J \tag{12}$$

where

$$J = J(N,\eta,\gamma_1,\gamma_2) = \int_{1}^{2} \left(\int_{\frac{\delta\eta}{2}}^{2\delta\eta} f_1(\alpha) f_2(\alpha)\, d\alpha \right) d\delta,$$

$$f_1(\alpha) = |z|^{-1-i(\gamma_1-\gamma_2)} \quad \text{and} \quad f_2(\alpha) = \exp\left(-(\gamma_1+\gamma_2) \arctan \frac{1}{2\pi N\alpha} \right).$$

Now we proceed to the estimation of J. Integrating twice by parts and denoting by F_1 a primitive of f_1 and by G_1 a primitive of F_1, we get

$$J = \frac{1}{2\eta} \left(G_1(4\eta) f_2(4\eta) - G_1(2\eta) f_2(2\eta) \right)$$

$$-\frac{2}{\eta}\left(G_1(\eta)f_2(\eta)-G_1\Big(\frac{\eta}{2}\Big)f_2\Big(\frac{\eta}{2}\Big)\right)$$

$$-2\int_1^2 G_1(2\delta\eta)f_2'(2\delta\eta)d\delta+2\int_1^2 G_1\Big(\frac{\delta\eta}{2}\Big)f_2'\Big(\frac{\delta\eta}{2}\Big)d\delta$$

$$+\int_1^2\left(\int_{\frac{\delta\eta}{2}}^{2\delta\eta} G_1(\alpha)f_2''(\alpha)d\alpha\right)d\delta. \tag{13}$$

If $\alpha>\frac{1}{N}$ we have

$$f_2'(\alpha)\ll\frac{1}{\alpha}\left(\frac{\gamma_1+\gamma_2}{N\alpha}\right)f_2(\alpha)$$

$$f_2''(\alpha)\ll\frac{1}{\alpha^2}\left\{\left(\frac{\gamma_1+\gamma_2}{N\alpha}\right)+\left(\frac{\gamma_1+\gamma_2}{N\alpha}\right)^2\right\}f_2(\alpha),$$

hence from (13) we get

$$J\ll\frac{1}{\eta}\max_{\alpha\in[\frac{\eta}{2},4\eta]}|G_1(\alpha)|\left\{1+\left(\frac{\gamma_1+\gamma_2}{N\eta}\right)^2\right\}\exp\left(-c\left(\frac{\gamma_1+\gamma_2}{N\eta}\right)\right), \tag{14}$$

where $c>0$ is a suitable constant.

In order to estimate $G_1(\alpha)$ we use the substitution

$$u=u(\alpha)=\left(\frac{1}{N^2}+4\pi^2\alpha^2\right)^{1/2}, \tag{15}$$

thus getting

$$F_1(\alpha)=\frac{1}{2\pi}\int u^{-i(\gamma_1-\gamma_2)}\frac{du}{(u^2-\frac{1}{N^2})^{1/2}}.$$

By partial integration we have

(16)

$$F_1(\alpha)=\frac{1}{2\pi(1-i(\gamma_1-\gamma_2))}\left\{\frac{u^{1-i(\gamma_1-\gamma_2)}}{(u^2-\frac{1}{N^2})^{1/2}}+\int u^{1-i(\gamma_1-\gamma_2)}\frac{u\,du}{(u^2-\frac{1}{N^2})^{3/2}}\right\}.$$

From (15) and (16) we get

$$G_1(\alpha)=\frac{1}{2\pi(1-i(\gamma_1-\gamma_2))}\left\{A(\alpha)+\int B(\alpha)d\alpha\right\}, \tag{13}$$

where

$$A(\alpha) = \frac{1}{2\pi} \int \frac{u^{2-i(\gamma_1-\gamma_2)}}{u^2 - \frac{1}{N^2}}\, du$$

and

$$B(\alpha) = \int \frac{u^{2-i(\gamma_1-\gamma_2)}}{(u^2 - \frac{1}{N^2})^{3/2}}\, du.$$

Again by partial integration we obtain

$$A(\alpha) = \frac{1}{2\pi(3 - i(\gamma_1 - \gamma_2))} \left\{ \frac{u^{3-i(\gamma_1-\gamma_2)}}{u^2 - \frac{1}{N^2}} + 2 \int u^{4-i(\gamma_1-\gamma_2)} \frac{du}{(u^2 - \frac{1}{N^2})^2} \right\}$$

and

$$B(\alpha) = \frac{1}{3 - i(\gamma_1 - \gamma_2)} \left\{ \frac{u^{3-i(\gamma_1-\gamma_2)}}{(u^2 - \frac{1}{N^2})^{3/2}} + 3 \int u^{4-i(\gamma_1-\gamma_2)} \frac{du}{(u^2 - \frac{1}{N^2})^{5/2}} \right\}.$$

Hence by (15) we have for $\alpha \in \left[\frac{\eta}{2}, 4\eta\right]$ that

$$A(\alpha) \ll \frac{u}{1 + |\gamma_1 - \gamma_2|} \ll \frac{\alpha}{1 + |\gamma_1 - \gamma_2|}, \tag{18}$$

$$B(\alpha) \ll \frac{1}{1 + |\gamma_1 - \gamma_2|}, \tag{19}$$

where $A(\alpha)$ and $B(\alpha)$ satisfy $A\left(\frac{\eta}{4}\right) = B\left(\frac{\eta}{4}\right) = 0$, and from (17) - (19) we obtain

$$G_1(\alpha) \ll \frac{\alpha}{1 + |\gamma_1 - \gamma_2|^2} \tag{20}$$

for $\alpha \in \left[\frac{\eta}{2}, 4\eta\right]$.

From (14) and (20) we get

$$J \ll \frac{1 + \left(\frac{\gamma_1+\gamma_2}{N\eta}\right)^2}{1 + |\gamma_1 - \gamma_2|^2} \exp\left(-c\left(\frac{\gamma_1 + \gamma_2}{N\eta}\right)\right),$$

hence from (12) and Stirling's formula we have

(21)

$$\int_{\eta}^{2\eta} \Big|\sum_{\gamma>0} z^{-\rho}\Gamma(\rho)\Big|^2 d\alpha \ll \sum_{\gamma_1>0}\sum_{\gamma_2>0} \frac{1 + \left(\frac{\gamma_1+\gamma_2}{N\eta}\right)^2}{1 + |\gamma_1 - \gamma_2|^2} \exp\left(-c\left(\frac{\gamma_1 + \gamma_2}{N\eta}\right)\right).$$

But

$$\left\{1+\left(\frac{\gamma_1+\gamma_2}{N\eta}\right)^2\right\}\exp\left(-c\left(\frac{\gamma_1+\gamma_2}{N\eta}\right)\right) \ll \exp\left(-\frac{c}{2}\frac{\gamma_1}{N\eta}\right),$$

hence (21) becomes

$$\ll \sum_{\gamma_1>0} \exp\left(-\frac{c}{2}\frac{\gamma_1}{N\eta}\right) \sum_{\gamma_2>0} \frac{1}{1+|\gamma_1-\gamma_2|^2} \ll N\eta L^2, \tag{22}$$

since the number of zeros $\rho_2 = \frac{1}{2} + i\gamma_2$ with $n \leq |\gamma_1 - \gamma_2| \leq n+1$ is $O(\log(n+|\gamma_1|))$.

From (9)-(11) and (22) we get

$$\int_{-\xi}^{\xi} \widetilde{R}_1 d\alpha \ll N\xi L^2, \tag{23}$$

and Theorem 1 follows from (4)-(8) and (23).

3. Proof of Corollaries 1 and 2.

Assume that $H, N \in \mathbb{N}$, $H < N$, and define

$$L(\alpha) = \Big|\sum_{m=1}^{H} e(-m\alpha)\Big|^2 = \sum_{m=-H}^{H} a(m)e(-m\alpha),$$

where $a(m) = H - |m|$,

$$R(n) = \sum_{h+k=n} \Lambda(h)\Lambda(k) \quad \text{and} \quad \widetilde{R}(\alpha) = \widetilde{S}(\alpha)^2 - \frac{1}{z^2},$$

where $z = \frac{1}{N} - 2\pi i\alpha$. We have

$$\sum_{n=N-H}^{N+H} a(n-N)e^{-n/N}R(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{S}(\alpha)^2 L(\alpha)e(-N\alpha)d\alpha$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{L(\alpha)}{z^2} e(-N\alpha)d\alpha + \int_{-\frac{1}{2}}^{\frac{1}{2}} \widetilde{R}(\alpha)L(\alpha)e(-N\alpha)d\alpha = I_1 + I_2, \tag{24}$$

say.

We evaluate I_1 using the residue theorem. We have

$$I_1 = \sum_{n=N-H}^{N+H} a(n-N) \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e(-n\alpha)}{z^2} \, d\alpha. \tag{25}$$

If $T \geq \dfrac{1}{2}$ we get

$$\left(\int_{-T}^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{T} \right) \frac{d\alpha}{|z|^2} \ll \int_{\frac{1}{2}}^{T} \frac{d\alpha}{\alpha^2} \ll 1,$$

hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e(-n\alpha)}{z^2} \, d\alpha = \int_{-T}^{T} \frac{e(-n\alpha)}{z^2} \, d\alpha + O(1) \tag{26}$$

uniformly for $T \geq \dfrac{1}{2}$. But

$$\int_{-T}^{T} \frac{e(-n\alpha)}{z^2} \, d\alpha = \frac{e^{-n/N}}{2\pi i} \int_{\frac{1}{N}-2\pi i T}^{\frac{1}{N}+2\pi i T} \frac{\exp(ns)}{s^2} \, ds. \tag{27}$$

Let Γ denote the left half of the circle $\left| s - \dfrac{1}{N} \right| = 2\pi T$. By the residue theorem we get

$$\frac{e^{-n/N}}{2\pi i} \int_{\frac{1}{N}-2\pi i T}^{\frac{1}{N}+2\pi i T} \frac{\exp(ns)}{s^2} \, ds = ne^{-n/N} + \frac{e^{-n/N}}{2\pi i} \int_{\Gamma} \frac{\exp(ns)}{s^2} \, ds$$

$$= ne^{-n/N} + O\left(\frac{1}{T} \right). \tag{28}$$

Letting $T \to \infty$, from (26)-(28) we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e(-n\alpha)}{z^2} \, d\alpha = ne^{-n/N} + O(1) \tag{29}$$

uniformly for $n \leq 2N$. Hence from (25) and (29) we obtain that

$$I_1 = \sum_{n=N-H}^{N+H} a(n-N) ne^{-n/N} + O(H^2)$$

$$= \frac{N}{e} \sum_{n=N-H}^{N+H} a(n-N) + O(H^3) = \frac{H^2 N}{e} + O(H^3). \tag{30}$$

We have

$$I_2 \ll \int_{-\frac{1}{H}}^{\frac{1}{H}} |\widetilde{R}(\alpha)|L(\alpha)d\alpha + \int_{\frac{1}{H}}^{\frac{1}{2}} |\widetilde{R}(\alpha)|L(\alpha)d\alpha. \tag{31}$$

Since

$$L(\alpha) \ll \min\left(H^2, \frac{1}{|\alpha|^2}\right), \tag{32}$$

from Theorem 1 we get

$$\int_{-\frac{1}{H}}^{\frac{1}{H}} |\widetilde{R}(\alpha)|L(\alpha)d\alpha \ll HNL^2 + H^{3/2}NL. \tag{33}$$

From (32) we obtain

$$\int_{\frac{1}{H}}^{\frac{1}{2}} |\widetilde{R}(\alpha)|L(\alpha)d\alpha \ll \int_{\frac{1}{H}}^{\frac{1}{2}} |\widetilde{R}(\alpha)|\frac{d\alpha}{\alpha^2}, \tag{34}$$

and by partial integration and Theorem 1 we get

$$\int_{\frac{1}{H}}^{\frac{1}{2}} |\widetilde{R}(\alpha)|\frac{d\alpha}{\alpha^2} \ll HNL^2 + H^{3/2}NL. \tag{35}$$

Hence from (31) and (33)-(35) we have

$$I_2 \ll HNL^2 + H^{3/2}NL, \tag{36}$$

and from (24), (30) and (36) we finally get

$$\sum_{n=N-H}^{N+H} a(n-N)e^{-n/N}R(n) = \frac{H^2N}{e} + O\left(H^3 + HNL^2 + H^{3/2}NL\right). \tag{37}$$

Choosing $H = CL^2$, $C > 0$ sufficiently large, from (37) we have that

$$\sum_{n=N-H}^{N+H} a(n-N)e^{-n/N}R(n) \gg H^2N$$

and Corollary 1 follows.

In order to prove Corollary 2 we observe that

$$\left|\widetilde{S}(\alpha)\right|^2 = \frac{1}{|z|^2} + O\left(\widetilde{R}_1 + \ldots + \widetilde{R}_5\right),$$

see (4), and

$$\int_{-\xi}^{\xi} \frac{d\alpha}{|z|^2} = \frac{N}{\pi} \arctan 2\pi N\xi.$$

Corollary 2 follows then from (5)-(8) and (23).

4. Proof of Theorem 2 and Corollary 3.

We recall the properties of Vinogradov's auxiliary function, see [10] p. 196. Let r be any non-negative integer and μ, η, Δ be such that

$$0 < \Delta < \frac{1}{4} \quad \text{and} \quad \Delta \leq \eta - \mu \leq 1 - \Delta. \tag{38}$$

There exists a function $\Psi(\alpha) = \Psi_{\mu,\eta,\Delta}(\alpha)$ periodic of period 1 such that

i) $\Psi(\alpha) = 1$ if $\mu + \frac{\Delta}{2} \leq \alpha \leq \eta - \frac{\Delta}{2}$

ii) $0 < \Psi(\alpha) < 1$ if $\mu - \frac{\Delta}{2} < \alpha < \mu + \frac{\Delta}{2}$, $\eta - \frac{\Delta}{2} < \alpha < \eta + \frac{\Delta}{2}$

iii) $\Psi(\alpha) = 0$ if $\eta + \frac{\Delta}{2} < \alpha < 1 + \mu - \frac{\Delta}{2}$

iv) $\Psi(\alpha) = \eta - \mu + \sum\limits_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} a(m)e(m\alpha)$, where

$$|a(m)| \leq \min\left(\eta - \mu, \frac{1}{\pi m}\left(\frac{r+1}{\pi m \Delta}\right)^r\right). \tag{39}$$

Define

$$I(N, \mu, \eta, \Delta) = \int_0^1 \left|\widetilde{S}(\alpha)\right|^2 \Psi_{\mu,\eta,\Delta}(\alpha) d\alpha.$$

We need the following

LEMMA. — *Let μ, η, Δ satisfy (38). Then*

$$I(N, \mu, \eta, \Delta) = (\eta - \mu)\left(\frac{NL}{2} + O(N)\right) + O\left(N\left(\frac{\eta - \mu}{\Delta}\right)^{1/2}\right).$$

Proof. — By iv) we have

(40)
$$I(N,\mu,\eta,\Delta) = (\eta-\mu)\sum_{n=1}^{\infty}\Lambda^2(n)e^{-2n/N} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} a(m)\widetilde{\Psi}(N,m),$$

where

$$\widetilde{\Psi}(N,m) = \sum_{\substack{h,k=1 \\ h-k=m}}^{\infty} \Lambda(h)\Lambda(k)e^{-(h+k)/N}.$$

From the prime number theorem with remainder term we get

$$\sum_{n=1}^{\infty}\Lambda^2(n)e^{-2n/N} = \frac{NL}{2} + O(N), \tag{41}$$

and from Theorem 3.11 of [3] and partial summation we get

$$\widetilde{\Psi}(N,m) \ll N\mathfrak{S}(|m|) \tag{42}$$

uniformly for $m \neq 0$, where

$$\mathfrak{S}(|m|) = \prod_{p\nmid |m|}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p\nmid |m|}\left(1+\frac{1}{p-1}\right).$$

Hence from (40)-(42) we obtain
(43)
$$I(N,\mu,\eta,\Delta) = (\eta-\mu)\left(\frac{NL}{2}+O(N)\right) + O\left(N\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} |a(m)|\mathfrak{S}(|m|)\right).$$

Now we use the estimate

$$\sum_{n\le x}\mathfrak{S}(n) \ll x, \tag{44}$$

see e.g. [2]. From (39) and the choice $r=1$ we have that

$$a(m) \ll \begin{cases} \eta-\mu & \text{if } |m| \le (\Delta(\eta-\mu))^{-1/2} \\ \dfrac{1}{m^2\Delta} & \text{if } |m| > (\Delta(\eta-\mu))^{-1/2}, \end{cases} \tag{45}$$

hence from (44), (45) and partial summation we obtain

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} |a(m)|\mathfrak{S}(|m|) \ll \left(\frac{\eta-\mu}{\Delta}\right)^{1/2}, \tag{46}$$

and the lemma follows from (43) and (46).

Suppose first that

$$0<\Delta<\frac{1}{4} \quad \text{and} \quad 0\le b-a\le 1-2\Delta, \tag{47}$$

and choose $\mu = a-\frac{\Delta}{2}$ and $\eta = b+\frac{\Delta}{2}$. Hence (38) is satisfied and we have

$$\begin{aligned}\int_a^b \left|\widetilde{S}(\alpha)\right|^2 d\alpha &= I(N,\mu,\eta,\Delta) \\ &\quad - \int_{a-\Delta}^{a} \left|\widetilde{S}(\alpha)\right|^2 \Psi_{\mu,\eta,\Delta}(\alpha)d\alpha - \int_{b}^{b+\Delta} \left|\widetilde{S}(\alpha)\right|^2 \Psi_{\mu,\eta,\Delta}(\alpha)d\alpha \\ &= I(N,\mu,\eta,\Delta) - I_1 - I_2, \end{aligned} \tag{48}$$

say. By i) and ii) we have

$$I_1 \le I(N,\mu',\eta',\Delta) \tag{49}$$

where $\mu' = a-\frac{3}{2}\Delta$ and $\eta' = a+\frac{\Delta}{2}$. It is easy to verify that μ',η',Δ satisfy (38). An analogous upper bound holds for I_2 too.

Choosing

$$\Delta = \left(\frac{b-a}{L^2}\right)^{1/3}, \tag{50}$$

from (48), (49) and the lemma we obtain

$$\int_a^b \left|\widetilde{S}(\alpha)\right|^2 d\alpha = \frac{b-a}{2}NL + O(N(L(b-a))^{1/3}) + O(N), \tag{51}$$

provided that (47) holds.

If

$$1-2\Delta < b-a \le 1, \tag{52}$$

then

$$\int_a^b \left|\widetilde{S}(\alpha)\right|^2 d\alpha = \int_0^1 \left|\widetilde{S}(\alpha)\right|^2 d\alpha - \int_I \left|\widetilde{S}(\alpha)\right|^2 d\alpha \tag{53}$$

with $|I| < 2\Delta$. Hence we may treat $\int_I \left|\widetilde{S}(\alpha)\right|^2 d\alpha$ in a way similar to the treatment of I_1, thus getting

$$\int_I \left|\widetilde{S}(\alpha)\right|^2 d\alpha \ll \Delta NL + N. \tag{54}$$

Since $b - a = 1 + O(\Delta)$ we have

$$\int_0^1 \left|\widetilde{S}(\alpha)\right|^2 d\alpha = \frac{b-a}{2} NL + O(\Delta NL) + O(N), \tag{55}$$

hence from (50) and (53)-(55) we obtain (51) under condition (52), and Theorem 2 follows.

The proof of Corollary 3 runs as follows. If $0 \le \xi \le \frac{1}{N}$, by Stirling's formula and the zero-free region of $\zeta(s)$ we have

$$\sum_\rho z^{-\rho}\Gamma(\rho) \ll \sum_\rho |z|^{-\beta}|\gamma|^{\beta-\frac{1}{2}} \exp\left(\gamma \arctan 2\pi N\alpha - \frac{\pi}{2}|\gamma|\right) = o(N),$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. Hence, arguing as in the proof of Corollary 2, we get

$$\int_{-\xi}^{\xi} \left|\widetilde{S}(\alpha)\right|^2 d\alpha \asymp N^2\xi.$$

Since $\int_{-\xi}^{\xi} \left|\widetilde{S}(\alpha)\right|^2 d\alpha$ is an increasing function of ξ, from the previous result we have that

$$\int_{-\xi}^{\xi} \left|\widetilde{S}(\alpha)\right|^2 d\alpha \gg N$$

for $\frac{1}{N} \le \xi \le \frac{1}{L}$. The corresponding upper bound follows from Theorem 2. Corollary 3 follows then arguing in a similar way in the range $\frac{1}{L} \le \xi \le \frac{1}{2}$.

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