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HITTING PROBABILITIES AND POTENTIAL THEORY FOR THE BROWNIAN PATH-VALUED PROCESS

by Jean-François LE GALL

0. Introduction.

The object of interest in this paper is the path-valued process studied in [LG1], [LG2], which we call here the Brownian path-valued process because the underlying spatial motion is Brownian motion in $\mathbb{R}^d$. We derive several potential-theoretic results related to this path-valued process, starting from a simple expression for the energy of measures on the path space. We obtain an explicit description of the capacitary measure of certain special subsets of the path space, namely the set of paths that visit a fixed closed set in $\mathbb{R}^d$ and the set of paths that exit a given domain in a certain compact subset of its boundary. We also study the polarity of the previous subsets of the path space. Because of the connections between the Brownian path-valued process and super Brownian motion, these polarity questions are closely related to Dynkin's recent work [Dy2], [Dy3].

Let us describe our main results, and start with a brief presentation of the Brownian path-valued process. This process, denoted by $(W_s)_{s\geq 0}$ takes values in the set $\mathcal{W}_x$ of all stopped paths in $\mathbb{R}^d$ started at a fixed point x. A stopped path can be viewed as a continuous mapping from an interval $[0,\zeta]$ into $\mathbb{R}^d$, the number $\zeta \geq 0$ being called the lifetime of the path. For every $s \geq 0$, W_s can be thought of as a Brownian path in $\mathbb{R}^d$, started at x and stopped at a (random) time ζ_s. The lifetime ζ_s of W_s evolves according to the law of Brownian motion in $\mathbb{R}_+$ killed when it reaches 0

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(or alternatively, Brownian motion reflected at 0, see Section 1). Roughly speaking, when ζ_s decreases, the path W_s is erased (erasing a path means restricting its interval of definition from $[0,\zeta]$ to $[0,\zeta']$, with $\zeta' < \zeta$), and when ζ_s increases, the path W_s is extended, using the law of Brownian motion in $\mathbb{R}^d$ to perform the extension. An important role is played by the "excursion measure" $\mathbb{N}_x$ of (W_s) from the trivial path with lifetime 0 (and initial point x). Under this excursion measure, the lifetime process (ζ_s) is distributed according to Itô measure of positive excursions of linear Brownian motion. Finally, the process (W_s) is symmetric with respect to the reference measure M_x obtained by integrating against dt the law of Brownian motion started at x and stopped at time t. This allows one to use the tools of the potential theory of symmetric Markov processes ([Dy1], [FG]).

Let μ be a finite measure on the path space $\mathcal{W}_x$. For every $t \geq 0$, denote by $\mu_{(t)}$ the restriction of μ to paths whose lifetime is greater than t, viewed as a measure on the σ-field $\mathcal{G}_t$ generated by the coordinates between 0 and t. In order that μ be of finite energy (with respect to the process (W_s)), it is necessary that, for every $t \geq 0$, $\mu_{(t)}$ be absolutely continuous with respect to the Wiener measure P_x restricted to the σ-field $\mathcal{G}_t$, denoted by $P_{x|\mathcal{G}_t}$. Furthermore, the energy of μ is then

$$\mathcal{E}(\mu) = 2\,E_x\left(\int_0^\infty \left(\frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}}\right)^2 dt\right)$$

(see Proposition 1.1 for a more precise statement).

Then, let F be a closed subset of $\mathbb{R}^d$ not containing x, and denote by $H \subset \mathcal{W}_x$ the set of paths that hit F. Assume that H is not M_x-polar (in the sense of [FG]), which in the context of superprocesses means that F is not $\mathcal{R}$-polar in the sense of Dynkin [Dy2]. This is equivalent to saying that the hitting "probability" of H under $\mathbb{N}_x$ is strictly positive. Denote by $u(x)$ this hitting probability, well-defined for $x \in \mathbb{R}^d \backslash F$. As was first observed by Dynkin [Dy2] in terms of super Brownian motion (see [LG2] for a proof via the path-valued process), the function u is the maximal nonnegative solution of the equation $\Delta u = 4u^2$ in $\mathbb{R}^d \backslash F$.

We prove in Section 2 that the set H of paths that hit F is an equilibrium set, in the sense of [FG], and that its capacitary measure is $u(x)$ times the law of the process (x_t) solution of the stochastic differential equation

$$dx_t = dB_t + \frac{\nabla u}{u}(x_t)\,dt$$
$$x_0 = x$$

and stopped at its first hitting time of F. It is a consequence of our argument that this process hits F in finite time. As a consequence of our result, we get that the law of the process (x_t) minimizes the energy $\mathcal{E}(\mu)$ given previously, among all probability measures supported on H.

In Section 3, we turn to the problem of characterizing the sets F such that H is not M_x-polar. For a fixed closed set F, this property does not depend on the choice of $x \in \mathbb{R}^d \backslash F$. This problem was completely solved by Dynkin [Dy2] (in terms of superprocesses, see also Perkins [Pe]). Dynkin's approach relies heavily on analytic results of Baras and Pierre [BP] on removable singularities of the equation $\Delta u = u^q$. We give, in our special situation, a more probabilistic proof of Dynkin's result, in the hope that these arguments will be applicable to other related unsolved problems, such as the one discussed in Section 4. In one direction, which was already treated in [LG2], this is easy : In order to prove that H is not M_x-polar, it suffices to construct a measure μ supported on H and with finite energy. The natural choice is to take for μ the law of an h-process, and this choice gives the right condition. In the reverse direction, things are harder, because, as we already know, the minimizing measure is not the law of an h-process. We use the results of Section 2 to give a probabilistic argument, which still requires an analytic lemma (Lemma 3.3) borrowed from [BP].

Section 4 contains results analogous to those of Sections 2 and 3 in the following different situation. We let D be a Lipschitz domain in $\mathbb{R}^d$, such that $x \in D$, and consider a compact subset K of ∂D. We are now interested in the polarity of the set H of paths that exit D at a point of K. We denote by $u(x)$ the hitting "probability" of H under $\mathbb{N}_x$. In terms of superprocesses, $u(x)$ is related to the probability for super Brownian motion started at δ_x that one of the historical paths will exit D at a point of K. According to Proposition 4.4, the function u is the maximal nonnegative solution of $\Delta u = 4u^2$ in D with zero boundary condition on $\partial D \backslash K$ (no boundary condition is imposed on K). We prove that, in order that H be not M_x-polar (equivalently that $u > 0$ on D), it is sufficient that K supports a nontrivial measure ν such that

$$\int dz\, G(x,z) \left(\int \nu(dy)\, H(z,y) \right)^2 < \infty,$$

where $G(x,z)$ is the Green function of Brownian motion in D, and $H(z,y)$ is the Martin kernel defined on $D \times \partial D$. When D is a C^2 domain, and for instance when $d \geq 4$, this condition reduces to

$$\int\int \nu(dy)\nu(dy')\,|y-y'|^{3-d} < \infty.$$

The previous conditions have been conjectured by Dynkin (personal communication) to be not only sufficient, but also necessary. We have been unable to prove the necessity, but we give a weaker statement involving Hausdorff measures, whose proof is based on the estimates of [AL] for the hitting probabilities of small disks on the boundary. As a special case of the previous results, we obtain that, when K is a singleton, H is M_x-polar if and only if $d \geq 3$. This fact can also be derived from the analytic results in [GV], using the connections with partial differential equations. Finally, assuming that H is not M_x-polar, we obtain an explicit description of its capacitary measure, which is analogous to the results of Section 2. This suggests that it might be possible to adapt the arguments of Section 3 in order to get the necessity of the previous condition. This approach would require an analytic result similar to Lemma 3.3, which however seems harder to obtain.

After the first version of this paper had been completed, we learnt of the recent work of Sheu [Sh], which contains results closely related to Section 4 of the present paper.

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1. Energy of measures on the path space.

1.1. We first recall the basic facts about the path-valued process considered in [LG2], and in [LG1] in a more general situation. We will then establish an important preliminary proposition which allows us to compute the energy, with respect to the path-valued process, of measures on the path space.

The set $\mathcal{W}$ of stopped paths is the set of all pairs $w = (f, \zeta)$, where $\zeta \geq 0$ and f is a continuous mapping from $[0,\infty)$ into $\mathbb{R}^d$, which is constant on $[\zeta, \infty)$. For our purposes, it will be important to make a careful distinction between the stopped path w and the continuous mapping f, so that our notation differs slightly from that in [LG2]. The number ζ is called the lifetime of the path w. We write $(f_{(w)}, \zeta_{(w)})$ instead of (f, ζ) when there is a risk of confusion. The point $f(\zeta)(= f(\infty))$ is usually denoted by $\hat{w}$. The space $\mathcal{W}$ is equipped with the metric

$$d(w, w') = \sup_{t \geq 0} |f_{(w)}(t) - f_{(w')}(t)| + |\zeta_{(w)} - \zeta_{(w')}|.$$

Let $w \in \mathcal{W}$ and $a, b \geq 0$ such that $a \leq b \wedge \zeta_{(w)}$. There is a unique probability measure, denoted by $Q^w_{a,b}(dw')$, on $\mathcal{W}$ such that :

(i) $\zeta_{(w')} = b$, $Q^w_{a,b}(dw')$ a.s.

(ii) $f_{(w')}(t) = f_{(w)}(t)$ for every $t \leq a$, $Q^w_{a,b}(dw')$ a.s.

(iii) the distribution of $(f_{(w')}(a+t), t \geq 0)$ under $Q^w_{a,b}(dw')$ is the law of Brownian motion in $\mathbb{R}^d$ started at $f_{(w)}(a)$ and stopped at time $b - a$.

This definition means that, under $Q^w_{a,b}(dw')$, w' is a stopped path with lifetime b, which coincides with w until time a, but is then independent of w and distributed according to the law of Brownian motion in $\mathbb{R}^d$.

If $f = f_{(w)}$, we may, and will often, write $Q^f_{a,b}(.)$ instead of $Q^w_{a,b}(.)$. Without risk of confusion, we shall also denote by $Q^f_{a,b}(df')$ the law induced by $Q^f_{a,b}(dw')$ on the space $\mathcal{C}$ of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$.

Let us fix $x \in \mathbb{R}^d$ and denote by $\mathcal{W}_x$ the set $\{w \in \mathcal{W};\ f(0) = x\}$. There exists a continuous strong Markov process $(W_s, \mathbb{P}_w)$ with values in $\mathcal{W}_x$ whose law is characterized by the following two properties. Under $\mathbb{P}_w$,

(i) the process $\zeta_s := \zeta_{(W_s)}$ is a reflecting Brownian motion on $\mathbb{R}_+$, which starts at $\zeta_0 = \zeta_{(w)}$;

(ii) conditionally given $(\zeta_s, s \geq 0)$, the process $f_s := f_{(W_s)}$ is a time-inhomogeneous continuous Markov process with values in $\mathcal{C}$, which starts at $f_0 = f_{(w)}$ and whose transition kernel between times r and s is

$$P_{r,s}(f, df') = Q^f_{(\inf_{r \leq u \leq s} \zeta_u), \zeta_s}(df').$$

Property (ii) has the following important consequence, that will used on several occasions in this paper. Outside a $\mathbb{P}_w$-negligible set, for every $r < s$, one has $f_r(t) = f_s(t)$ for every $t \leq \inf_{r \leq u \leq s} \zeta_u$.

We will also need to consider the process (W_s) killed when its lifetime vanishes. The resulting process is denoted by $(W_s, \mathbb{P}^\star_w)$. Its distribution can be described as previously except that reflecting Brownian motion on $\mathbb{R}_+$ is replaced by Brownian motion killed at its first hitting time of 0. The natural state space of $(W_s, \mathbb{P}^\star_w)$ is $\mathcal{W}^\star_x = \{w \in \mathcal{W}_x, \zeta > 0\}$ (augmented with a cemetery point).

We refer to [LG1], [LG2] for the construction and more information about these path-valued Markov processes. The process $(W_s, \mathbb{P}^\star_w)$ is clearly transient. Its potential kernel $U(w, dw')$ is easily computed (cf [LG1], we take this opportunity to point out that a factor 2 is missing in the corresponding formula of [LG1]) :

$$\int U(w, dw')\, \varphi(w') = 2 \int_0^{\zeta(w)} da \int_a^\infty db \int Q^w_{a,b}(dw')\, \varphi(w').$$

Let $P_x(df)$ denote the law on $\mathcal{C}$ of Brownian motion started at x (the Wiener measure with initial point x). For every $a \geq 0$, let P^a_x be the law under $P_x(df)$ of the pair $(f(\cdot \wedge a), a)$ (P^a_x is the law of Brownian motion stopped at time a, viewed as a probability measure on $\mathcal{W}_x$). Both processes $(W_s, \mathbb{P}_w)$, $(W_s, \mathbb{P}^\star_w)$ are symmetric with respect to the reference measure

$$M_x(dw) = \int_0^\infty da\, P^a_x(dw).$$

We can therefore apply the general results of the potential theory of symmetric Markov processes (see [FG] and [Dy1]) to the transient process $(W_s, \mathbb{P}^\star_w)$. Let μ be a finite measure on $\mathcal{W}^\star_x$ such that μU is σ-finite and absolutely continuous with respect to M_x. According to [FG], we can choose a version $U(\mu)$ of the Radon-Nikodym derivative

$$\frac{d\mu U}{dM_x}$$

such that $U(\mu)$ is excessive (with respect to the process $(W_s, \mathbb{P}^\star_w)$). The energy of μ is then defined by

$$\mathcal{E}(\mu) =< \mu, U(\mu) >= \int \mu(dw)\, U(\mu)(w).$$

If μU is not σ-finite, or if the condition $\mu U \ll M_x$ does not hold, one takes $\mathcal{E}(\mu) = \infty$.

1.2. Our first goal is to obtain a more tractable formula for the energy $\mathcal{E}(\mu)$. We denote by $(\mathcal{G}_t)$ the canonical filtration on $\mathcal{C}$ ($\mathcal{G}_t$ is the σ-field generated by the coordinate mappings $f \to f(r)$, $0 \leq r \leq t$).

PROPOSITION 1.1. — *Let μ be a finite measure on $\mathcal{W}_x^\star$. For every $t \geq 0$, let $\mu_{(t)}$ be the finite measure on $(\mathcal{C}, \mathcal{G}_t)$ defined by*

$$\mu_{(t)}(A) = \mu(\zeta > t,\, f \in A).$$

The measure μU is σ-finite and such that $\mu U \ll M_x$, if and only if, for every $t \geq 0$, the measure $\mu_{(t)}$ is absolutely continuous with respect to the measure P_x restricted to $\mathcal{G}_t$. If this condition holds, we can choose for every $t \geq 0$ a version Y_t of the Radon-Nikodym derivative of $\mu_{(t)}$ with respect to $P_{x|\mathcal{G}_t}$, so that the process (Y_t) is both $(\mathcal{G}_t)$-predictable and P_x-a.s. càdlàg. Finally,

$$U(\mu)(w) = 2\int_0^\zeta dt\, Y_t(f) \quad , \; M_x(dw) \text{ a.e.},$$

and

$$\mathcal{E}(\mu) = 2\, E_x\left(\int_0^\infty dt\, Y_t^2\right) = 2\int_0^\infty dt\, E_x\left(\left(\frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}}\right)^2\right).$$

Proof. — Suppose first that μU is σ-finite and such that $\mu U \ll M_x$. Let $t > 0$ and $B \in \mathcal{G}_t$ such that $P_x(B) = 0$. Notice that, for $u \geq t$, the law of $f_{(w)}$ under P_x^u coincides with P_x on the σ-field $\mathcal{G}_t$. Hence,

$$M_x(\zeta > t,\, f \in B) = \int_t^\infty du\, P_x^u(f \in B) = 0.$$

It follows that

$$\begin{aligned} 0 &= \mu U(\zeta > t,\, f \in B) \\ &= 2\int \mu(dw) \int_0^{\zeta_{(w)}} da \int_{a\vee t}^\infty db \int Q_{a,b}^w(dw')\, 1_B(f_{(w')}) \\ &\geq 2\int \mu(dw)\, 1_{(\zeta_{(w)}>t)} \int_t^{\zeta_{(w)}} da \int_a^\infty db\, 1_B(f_{(w)}) \end{aligned}$$

because, when $t \leq a \leq \zeta_{(w)}$, $f_{(w')}$ coincides with $f_{(w)}$ on the time interval $[0, t]$, $Q_{a,b}^w(dw')$ a.s., and we use the fact that B is $\mathcal{G}_t$-measurable. We conclude that

$$\mu_{(t)}(B) = \int \mu(dw)\, 1_{(\zeta>t)}\, 1_B(f) = 0$$

so that $\mu_{(t)} \ll P_{x|\mathcal{G}_t}$.

Next suppose that $\mu_{(t)} \ll P_{x|\mathcal{G}_t}$ for every $t \geq 0$. Let X_t be any version of the Radon-Nikodym derivative of $\mu_{(t)}$ with respect to $P_{x|\mathcal{G}_t}$. It is easy to verify that (X_t) is a $(\mathcal{G}_t, P_x)$ supermartingale : If $u < t$ and $A \in \mathcal{G}_u$,

$$E_x(1_A X_t) = \mu(f_{(w)} \in A, \zeta_{(w)} > t) \leq \mu(f_{(w)} \in A, \zeta_{(w)} > u) = E_x(1_A X_u).$$

Moreover, $E_x(X_t) = \mu(\zeta > t)$ is a right-continuous function of t.

Denote by $\bar{\mathcal{G}}_t$ the smallest σ-field that contains $\mathcal{G}_t$ and the P_x-negligible sets of $\mathcal{G}_\infty$. It is well-known that the filtration $(\bar{\mathcal{G}}_t)$ is right-continuous. By a standard regularization theorem, there exists a version $(\bar{X}_t)$ of the process (X_t) (meaning that for every t, $\bar{X}_t = X_t$, P_x-a.s.) which is a càdlàg $(\bar{\mathcal{G}}_t, P_x)$ supermartingale. Since the $(\bar{\mathcal{G}}_t)$ optional and predictable σ-fields coincide, the process $(\bar{X}_t)$ is in particular $(\bar{\mathcal{G}}_t)$-predictable. We then use the following easy fact : For every $(\bar{\mathcal{G}}_t)$-predictable process (H_t), there exists a $(\mathcal{G}_t)$-predictable process (H'_t) such that (H_t) and (H'_t) are indistinguishable (this fact is trivial for elementary processes of the form $H_t = 1_A\, 1_{(u,v]}(t)$, $A \in \bar{\mathcal{G}}_u$, and can then be extended using the monotone class theorem found in [DM], p. 20-22). We let (Y_t) be a $(\mathcal{G}_t)$-predictable process such that $Y_t(f) = \bar{X}_t(f)$ for every t, outside a P_x-negligible set. Obviously (Y_t) is càdlàg outside a P_x-negligible set.

Let us now compute the potential of μ. For any nonnegative measurable function φ,

$$\begin{aligned}
\frac{1}{2}\int \mu U(dw)\,\varphi(w) &= \int \mu(dw) \int_0^\zeta da \int_a^\infty db \int Q^f_{a,b}(dw')\,\varphi(w') \\
&= \int_0^\infty da \int_a^\infty db \int \mu_{(a)}(df) \int Q^f_{a,b}(dw')\,\varphi(w') \\
&= \int_0^\infty da \int_a^\infty db \int P_x(df)\, Y_a(f) \int Q^f_{a,b}(dw')\,\varphi(w') \\
&= \int_0^\infty da \int_a^\infty db\, E^b_x(\varphi(w)\, Y_a(f_{(w)})) \\
&= \int_0^\infty db\, E^b_x\left(\varphi(w) \int_0^{\zeta_{(w)}} da\, Y_a(f_{(w)})\right) \\
&= M_x\left(\varphi \int_0^\zeta da\, Y_a(f)\right).
\end{aligned}$$

For the second equality, notice that $\int Q^f_{a,b}(dw')\,\varphi(w')$ is a $\mathcal{G}_a$-measurable function of f. In the fourth equality, we also use the property that Y_a is $\mathcal{G}_a$-measurable to get $Y_a(f_{(w')}) = Y_a(f)$, $Q^f_{a,b}(dw')$-a.s.

Observe that

$$E_x^b\left(\int_0^{\zeta(w)} da\, Y_a(f_{(w)})\right) = \int_0^b da\, E_x(Y_a) \le b < \mu, 1 > .$$

Therefore, $\int_0^\zeta da\, Y_a(f) < \infty$, $M_x(dw)$ a.e. and the previous calculations imply that μU is σ-finite, $\mu U \ll M_x$ and

$$\frac{d\mu U}{dM_x}(w) = 2\int_0^\zeta da\, Y_a(f)\ ,\quad M_x \text{ a.e.}$$

It remains to compute the energy $\mathcal{E}(\mu)$. Note that the right side of the previous equality may not be an excessive function of w, so that we cannot identify it with $U(\mu)(w)$ for every $w \in \mathcal{W}_x$. To overcome this difficulty, we choose a countable dense subset D of $\mathbb{R}_+$ such that, for every $b \in D$,

$$U(\mu)(w) = 2\int_0^b da\, Y_a(f)\ ,\quad P_x^b \text{ a.s.}$$

We then let h_n be an increasing sequence of measurable functions from $(0,\infty)$ into D such that, for every $t > 0$, $h_n(t) < t$ and

$$t = \lim_{n\to\infty} h_n(t).$$

Let $w \in \mathcal{W}_x^\star$. For $a < \zeta$, set $w_{(a)} = (f_{(w)}(\cdot \wedge a), a) \in \mathcal{W}_x$. We claim that

$$U(\mu)(w) = \lim_{a\uparrow\zeta_{(w)}} \uparrow U(\mu)(w_{(a)}).$$

In fact, if $T_{(a)} = \inf\{s,\, \zeta_s = a\}$ we have $T_{(a)} \downarrow 0$ as $a \uparrow \zeta_{(w)}$, $\mathbb{P}_w^\star$ a.s., and $W_{T_{(a)}} = w_{(a)}$ from the properties of the process (W_s). Therefore, the claimed result follows from the right-continuity of $U(\mu)(W_s)$.

Then,

$$\begin{aligned}
< \mu, U(\mu) > &= \lim_{n\to\infty} \uparrow \int \mu(dw)\, U(\mu)(w_{(h_n(\zeta))}) \\
&= \lim_{n\to\infty} \uparrow 2\int \mu(dw) \int_0^{h_n(\zeta)} da\, Y_a(f(\cdot \wedge h_n(\zeta))) \\
&= \lim_{n\to\infty} \uparrow 2\int \mu(dw) \int_0^{h_n(\zeta)} da\, Y_a(f)
\end{aligned}$$

$$
\begin{aligned}
&= 2\int \mu(dw) \int_0^\zeta da\, Y_a(f) \\
&= 2\int_0^\infty da \int \mu_{(a)}(df)\, Y_a(f) \\
&= 2\int_0^\infty da\, E_x(Y_a^2).
\end{aligned}
$$

For the second equality, note that, for any $b > 0$, the law of $w_{(b)}$ under $\mu(\cdot \cap \{\zeta > b\})$ is absolutely continuous with respect to P_x^b and use the choice of D. The previous calculation completes the proof. □

Remark. — In explicit examples (see below and [LG1]) one can usually verify that the function

$$h(w) = 2\int_0^\zeta dt\, Y_t(f)$$

is excessive, so that the final part of the proof can be simplified.

1.3. Let us now consider the Markov process $(W_s, \mathbb{P}_w)$. The trivial path (f, ζ) such that $\zeta = 0$, $f(0) = x$ is clearly a regular point for this Markov process. We denote by $\mathbb{N}_x$ the excursion measure from this regular point. The measure $\mathbb{N}_x$ is defined on the canonical space $\mathcal{C}(\mathbb{R}_+, \mathcal{W}_x)$ of continuous functions from $\mathbb{R}_+$ into $\mathcal{W}_x$. Without risk of confusion, we also denote by $W_s = (f_s, \zeta_s)$ the canonical process on this space. Under $\mathbb{N}_x$, the lifetime process $\zeta_s = \zeta_{(W_s)}$ is distributed according to the Itô measure of positive excursions of linear Brownian motion, and the conditional distribution of $(f_s)_{s\geq 0}$ knowing $(\zeta_s)_{s\geq 0}$ is the same as previously. We normalize $\mathbb{N}_x$ so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x\left(\sup_{s\geq 0} \zeta_s > \varepsilon\right) = \frac{1}{2\varepsilon},$$

and we denote by σ the duration of the excursion (ζ_s) under $\mathbb{N}_x$ ($\zeta_s = 0$ if and only if $s = 0$ or $s \geq \sigma$).

The range $\mathcal{R}$ of (W_s) is defined under $\mathbb{N}_x$ by

$$\mathcal{R} = \{W_s(t); t \geq 0,\ 0 \leq s < \sigma\} = \{\hat{W}_s; 0 \leq s < \sigma\}$$

(the second equality follows from the special properties of the process (W_s)).

Let F be a closed set in $\mathbb{R}^d$ not containing x and let

$$u(x) = \mathbb{N}_x(\mathcal{R} \cap F \neq \varnothing)$$

be the hitting "probability" of F under $\mathbb{N}_x$. The function u is the maximal nonnegative solution of the equation

$$\Delta u = 4u^2$$

in $\mathbb{R}^d \backslash F$. This result was first proved by Dynkin [Dy2] in terms of superprocesses (see [LG2], Prop. 5.3, for an approach via the path-valued process). If F has a smooth boundary, or more generally, if every boundary point of F is regular for F (with respect to Brownian motion in $\mathbb{R}^d$), then

$$\lim_{x \to y, x \in \mathbb{R}^d \backslash F} u(x) = \infty$$

for every $y \in \partial F$ (see [Dy2] and [LG2], Section 5).

The function u is either strictly positive on $\mathbb{R}^d \backslash F$ or identically zero. In the second case, we say following Dynkin [Dy2] that F is $\mathcal{R}$-polar.

2. Hitting probabilities and capacitary measures.

Let F be a closed subset of $\mathbb{R}^d$, such that $x \notin F$. We assume that F is not $\mathcal{R}$-polar. We introduce the set of all paths that hit F :

$$H = \{w \in \mathcal{W}_x^\star;\ \exists t \in [0, \zeta],\ f(t) \in F\}.$$

In this section, we will investigate the hitting probabilities of H for the killed process $(W_s, \mathbb{P}_w^\star)$. We will prove that H is an equilibrium set, in the sense of [FG], and we will compute its capacitary distribution.

We first recall a few basic results (see [FG]). Let B be a Borel subset of $\mathcal{W}_x^\star$ and let

$$T_B = \inf\{s > 0,\ W_s \in B\}.$$

The function $w \to \mathbb{P}_w^\star(T_B < \infty)$ is excessive (with respect to the process $(W_s, \mathbb{P}_w^\star)$). We say that B is an equilibrium set if this function can be written as the potential of a measure, that is if there exists a σ-finite measure π_B on $\mathcal{W}_x^\star$ such that

$$\pi_B U(dw) = \mathbb{P}_w^\star(T_B < \infty)\, M_x(dw).$$

The measure π_B is unique and is called the capacitary measure of B. The capacity of B is

$$\Gamma(B) =< \pi_B, 1 >$$

(the capacity of B can be defined more generally, see [FG], p. 503). Note that $\Gamma(B) = 0$ if and only if $\mathbb{P}^\star_w(T_B < \infty) = 0$, $M_x(dw)$ a.e., in which case B is called M_x-polar. Our assumption that F is not $\mathcal{R}$-polar is easily seen to be equivalent to saying that H is not M_x-polar (see [LG1]).

LEMMA 2.1. — *For every $w = (f, \zeta) \in \mathcal{W}^\star_x$, set*

$$\tau(w) = \tau(f) = \inf\{t \geq 0,\ f(t) \in F\},$$

where $\inf \varnothing = \infty$. *Then*

$$\mathbb{P}^\star_w(T_H < \infty) = \begin{cases} 1 & , \text{ if } \tau(w) < \zeta\ , \\ 1 - \exp\left(-2\int_0^\zeta u(f(r))\, dr\right) & , \text{ if } \tau(w) \geq \zeta\ , \end{cases}$$

where for every $y \in \mathbb{R}^d \backslash F$,

$$u(y) = \mathbb{N}_y(\mathcal{R} \cap F \neq \varnothing).$$

The set $\{w \in \mathcal{W}^\star_x;\ \tau(w) \leq \zeta_{(w)}, \int_0^{\tau(w)} u(f(r))\, dr < \infty\}$ is M_x-polar.

Proof. — We start by proving the formula for $\mathbb{P}^\star_w(T_H < \infty)$. The case $\tau(w) < \zeta$ is trivial from the behavior of the process (W_s). Suppose now that $\tau(w) \geq \zeta$. According to Proposition 2.5 of [LG2], there exists under $\mathbb{P}^\star_w$ a Poisson measure

$$\sum_i \delta_{\kappa_i}$$

on $\mathcal{C}(\mathbb{R}_+, \mathcal{W})$, with intensity

$$2\int_0^{\zeta_{(w)}} dt\, \mathbb{N}_{f(t)}(\cdot)$$

such that

$$\{W_s(t);\ s > 0,\ 0 \leq t \leq \zeta_s\} = \{f(r);\ 0 \leq r < \zeta\} \cup \left(\bigcup_i \mathcal{R}(\kappa_i)\right)$$

where $\mathcal{R}(\kappa_i)$ denotes the range of κ_i. Since $\tau(w) \geq \zeta$, we have $\{f(r);\, 0 \leq r < \zeta\} \cap F = \varnothing$. Therefore,

$$\begin{aligned}\mathbb{P}^\star_w(T_H < \infty) &= \mathbb{P}^\star_w\left(\left(\bigcup_i \mathcal{R}(\kappa_i)\right) \cap F \neq \varnothing\right)\\ &= 1 - \mathbb{P}^\star_w\left(\bigcap_i \{\mathcal{R}(\kappa_i) \cap F = \varnothing\}\right)\\ &= 1 - \exp\left(-2\int_0^\zeta u(f(r))\,dr\right)\end{aligned}$$

using the Poisson exponential formula and the definition of u.

Then, let H^r denote the set of regular points of H. By the first part of the lemma, the set $H' = \{w;\, \tau(w) = \zeta,\, \int_0^{\tau(w)} u(f(r))\,dr < \infty\}$ is contained in $H \backslash H^r$. This set is therefore semipolar hence M_x-polar ([FG], (2.9)). However, because of the properties recalled in Section 1, the process (W_s) cannot hit the set $\{w;\, \tau(w) \leq \zeta,\, \int_0^{\tau(w)} u(f(r))\,dr < \infty\}$ without also hitting H'. The desired result follows. □

THEOREM 2.2. — *The set H is an equilibrium set. Its capacitary measure μ is characterized by the following property. For every $t \geq 0$, the measure $\mu_{(t)}$ is absolutely continuous with respect to $P_{x|\mathcal{G}_t}$, and*

$$\frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}} = 1_{(t<\tau(f))}\, u(f(t)) \exp\left(-2\int_0^t u(f(s))\,ds\right).$$

Proof. — We first consider the case when F is a closed half-space. Without loss of generality, we may take $F = \{(y^1,\ldots,y^d); y^1 \leq 0\}$. The function u is then easily computed : $u(x^1,\ldots,x^d) = \varphi(x^1)$, where $\varphi'' = 4\,\varphi^2$ and

$$\lim_{x^1 \downarrow 0} \varphi(x^1) = \infty.$$

It follows that $\varphi(x^1) = 3/(2(x^1)^2)$. Observe that

$$\frac{\nabla u}{u}(x^1,\ldots,x^d) = \left(-\frac{2}{x^1}, 0, \ldots, 0\right).$$

Then consider the stochastic differential equation :

$$\begin{aligned}dx_t &= d\beta_t - \left(\frac{2}{x_t^1}, 0, \ldots, 0\right)\, dt\\ x_0 &= x\end{aligned}$$

where (β_t) is a d-dimensional Brownian motion started at x, on a probability space $(\Omega, \mathcal{F}, P)$. The solution is a priori well-defined on the stochastic interval $[0, \bar{\tau}_\varepsilon]$, where $\bar{\tau}_\varepsilon = \inf\{t;\, x_t^1 \le \varepsilon\}$, $\varepsilon > 0$. A simple comparison argument shows that $\bar{\tau} = \lim_{\varepsilon \to 0} \bar{\tau}_\varepsilon < \infty$ a.s. and that $x_{\bar{\tau}} = \lim_{t \uparrow \bar{\tau}} x_t$ exists a.s. We may therefore define a probability measure μ_0 on $\mathcal{W}_x^\star$ as the law of $((x_{t\wedge\bar{\tau}})_{t\ge 0}, \bar{\tau})$. We will check that the capacitary measure of H is $u(x)\mu_0$.

Set $\tau'_\varepsilon = \inf\{t,\, \beta_t^1 \le \varepsilon\}$. By Itô's formula,

$$u(\beta_{t\wedge\tau'_\varepsilon}) \exp\left(-2\int_0^{t\wedge\tau'_\varepsilon} u(\beta_r)\,dr\right)$$

$$= u(x) \exp\left(\int_0^{t\wedge\tau'_\varepsilon} \frac{\nabla u}{u}(\beta_r)\,d\beta_r - \frac{1}{2}\int_0^{t\wedge\tau'_\varepsilon} \left|\frac{\nabla u}{u}(\beta_r)\right|^2 dr\right).$$

By Girsanov's theorem, for every $t \ge 0$, the law of $(x_{r\wedge\bar{\tau}_\varepsilon})_{0\le r\le t}$ coincides with the law of $(\beta_{r\wedge\tau'_\varepsilon})_{0\le r\le t}$ under

$$Q_{(t)} = \frac{u(\beta_{t\wedge\tau'_\varepsilon})}{u(x)} \exp\left(-2\int_0^{t\wedge\tau'_\varepsilon} u(\beta_r)\,dr\right) \cdot P\,.$$

Let $\tau_\varepsilon(f) = \inf\{t,\, f^1(t) \le \varepsilon\}$. It follows that, if $\Phi : \mathcal{C} \to \mathbb{R}_+$ is $\mathcal{G}_t$-measurable,

$$\mu_0(1_{(t<\tau_\varepsilon(f))}\Phi(f)) = E_x\left(\frac{u(f(t))}{u(x)} 1_{(t<\tau_\varepsilon(f))} \exp\left(-2\int_0^t u(f(r))\,dr\right) \Phi(f)\right)$$

and, by letting ε go to 0,

$$\mu_0(1_{(t<\zeta)}\Phi(f)) = E_x\left(\frac{u(f(t))}{u(x)} 1_{(t<\tau(f))} \exp\left(-2\int_0^t u(f(r))\,dr\right) \Phi(f)\right).$$

In the notation of Proposition 1.1, we get

$$Y_t(f) = 1_{(t<\tau(f))} \frac{u(f(t))}{u(x)} \exp\left(-2\int_0^t u(f(r))\,dr\right),$$

P_x-a.s. In particular, M_x-a.e.,

$$2\int_0^\zeta Y_t(f)\,dt = \frac{2}{u(x)} \int_0^{\zeta\wedge\tau(f)} dt\, u(f(t)) \exp\left(-2\int_0^t u(f(r))\,dr\right)$$

$$= \frac{1}{u(x)}\left(1 - \exp -2\int_0^{\zeta\wedge\tau(f)} u(f(r))\,dr\right)$$

which by Lemma 2.1 coincides with the excessive function

$$\frac{1}{u(x)}\mathbb{P}^\star_w(T_H < \infty)$$

outside an M_x-polar set.

We have thus proved that $\mathbb{P}^\star_w(T_H < \infty)$ is the potential of the measure $u(x)\mu_0$. In particular, H is an equilibrium set.

Let us now come to the general case. Since F is then contained in a finite union of closed half-spaces not containing x, we can bound $\mathbb{P}^\star_w(T_H < \infty)$ by the potential of a finite measure. According to [FG], (2.7), we know that $\mathbb{P}^\star_w(T_H < \infty)$ is itself the potential of a measure so that H is an equilibrium set. Let μ be the capacitary measure of H. Notice that μ is finite (the previous argument implies that the capacity of H is finite). In the notation of Proposition 1.1, we have, $M_x(dw)$ a.e.,

$$2\int_0^\zeta Y_t(f)\,dt = U(\mu)(w) = \begin{cases} 1 & \text{if } \zeta > \tau(w) \\ 1 - \exp -2\int_0^\zeta u(f(r))\,dr & \text{if not} \end{cases}$$

by Lemma 2.1. Therefore, $P_x(df)$ a.e.,

$$2\int_0^\zeta Y_t(f)\,dt = 1_{(\zeta > \tau(f))} + 1_{(\zeta \le \tau(f))}\left(1 - \exp -2\int_0^\zeta u(f(r))\,dr\right)$$

for all $\zeta \in \mathbb{R}_+$ except possibly on a set of zero Lebesgue measure. Recall that $t \to Y_t(f)$ is P_x a.s. càdlàg. By differentiating the previous formula, it follows that, P_x a.s.,

$$\begin{aligned} Y_t(f) &= 0 &&, \text{ if } t \ge \tau(f), \\ Y_t(f) &= u(f(t))\exp -2\int_0^t u(f(r))\,dr &&, \text{ if } t < \tau(f). \end{aligned}$$

It follows that $\mu_{(t)}$ has the form given in Theorem 2.2. It is then easy to check (for instance by applying the monotone class lemma to sets of the form $\{\zeta > t\}\cap\{f \in A_t\}$, $A_t \in \mathcal{G}_t$) that the collection $(\mu_{(t)})_{t\ge 0}$ characterizes μ. □

Corollary 2.3. — *Let β be a standard d-dimensional Brownian motion started at 0. For every $\varepsilon > 0$, let $(x^\varepsilon_t, t \ge 0)$ be the unique solution to the stochastic integral equation*

$$x^\varepsilon_t = x + \beta_{t\wedge\bar{\tau}^\varepsilon} + \int_0^{t\wedge\bar{\tau}^\varepsilon} \frac{\nabla u}{u}(x^\varepsilon_s)\,ds$$

where $\bar{\tau}^\varepsilon = \inf\{t,\ d(x_t^\varepsilon, F) \le \varepsilon$ or $|x_t^\varepsilon| \ge 1/\varepsilon\}$. *Then, for every sequence* (ε_n) *decreasing to* 0,

$$\bar{\tau} = \lim_{n\to\infty} \uparrow \bar{\tau}^{\varepsilon_n} < \infty \ \text{ a.s.}$$

Furthermore, there exists a (unique) continuous process $(x_t, t \ge 0)$ *such that for every* $\varepsilon > 0$, $x_t^\varepsilon = x_t^\varepsilon$ *for every* $t \in [0, \bar{\tau}^\varepsilon]$, *a.s., and* x_t *is constant over* $[\bar{\tau}, \infty)$.

Finally, the capacitary measure μ *of* H *is* $u(x)$ *times the law in* $\mathcal{W}_x^\star$ *of* $((x_t)_{t\ge 0}, \bar{\tau})$.

Proof. — Since the function $\dfrac{\nabla u}{u}$ is smooth and bounded in $\{y;\ d(y, F) \ge \varepsilon,\ |y| \le 1/\varepsilon\}$, the existence and uniqueness of (x_t^ε) follows from well-known results on stochastic differential equations. It is also clear that for $\varepsilon < \varepsilon'$, x^ε coincides with $x^{\varepsilon'}$ on $[0, \bar{\tau}^{\varepsilon'}]$. We may therefore take any sequence (ε_n) decreasing to 0, set $\bar{\tau} = \lim \uparrow \bar{\tau}^{\varepsilon_n} \in (0, \infty]$ and define on the stochastic interval $[0, \bar{\tau})$ a process (x_t) such that $x_t = x_t^\varepsilon$ for $t \le \bar{\tau}^\varepsilon$, a.s.

For $f \in \mathcal{C}$ set $\tau_\varepsilon(f) = \inf\{t,\ d(f(t), F) \le \varepsilon$ or $|f(t)| \ge 1/\varepsilon\}$, and

$$M_t^\varepsilon(f) = \frac{u(f(t \wedge \tau_\varepsilon(f)))}{u(x)} \exp\left(-2\int_0^{t\wedge\tau_\varepsilon(f)} u(f(r))\,dr\right).$$

Itô's formula ensures that M_t^ε is a bounded $(\mathcal{G}_t, P_x)$-martingale. In particular, M_t^ε converges a.s. as $t \to \infty$. Then there exists a unique probability measure Q^ε on $\mathcal{C}$, defined by

$$Q^\varepsilon(df) = M_\infty^\varepsilon(f)\, P_x(df),$$

such that for every $t \ge 0$, the Radon-Nikodym derivative of Q^ε with respect to P_x on $\mathcal{G}_t$ is $M_t^\varepsilon(f)$.

Since M_t^ε is the exponential martingale associated with

$$\int_0^{t\wedge\tau_\varepsilon(f)} \frac{\nabla u}{u}(f(r)) \cdot df(r)$$

Girsanov's theorem implies that

$$f(t) - \int_0^{t\wedge\tau_\varepsilon(f)} \frac{\nabla u}{u}(f(r))\,dr$$

is a $(Q^\varepsilon, \mathcal{G}_t)$ Brownian motion. This shows that, under Q^ε, the process $(f(t \wedge \tau^\varepsilon(f)), t \ge 0)$ satisfies the same integral equation as $(x_t^\varepsilon, t \ge 0)$.

Therefore these two processes have the same distribution. In particular, for any nonnegative measurable function Φ on $\mathcal{C}$, and any $t \geq 0$,

$$E_x[M_t^\varepsilon \, 1_{(t<\tau^\varepsilon)} \, \Phi(f(r \wedge t), r \geq 0)] = E[1_{(t<\bar{\tau}^\varepsilon)} \, \Phi(x^\varepsilon_{r\wedge t}, r \geq 0)].$$

Using the process (x_t) introduced above, this equality can be rewritten as

$$E_x\left(\frac{u(f(t))}{u(x)} \exp\left(-2\int_0^t u(f(r))\,dr\right) 1_{(t<\tau^\varepsilon)} \, \Phi(f(r \wedge t), r \geq 0)\right)$$
$$= E[1_{(t<\bar{\tau}^\varepsilon)} \, \Phi(x_{r\wedge t}, r \geq 0)].$$

By letting ε go to 0, we obtain

$$E_x\left(\frac{u(f(t))}{u(x)} \exp\left(-2\int_0^t u(f(r))\,dr\right) 1_{(t<\tau)} \, \Phi(f(r \wedge t), r \geq 0)\right)$$
$$= E[1_{(t<\bar{\tau})} \, \Phi(x_{r\wedge t}, r \geq 0)].$$

Let μ be the capacitary measure of H as in Theorem 2.2, and let $\nu = u(x)^{-1}\mu$. By Theorem 2.2, the previous equality can be stated as

$$\int \nu(dw) \, 1_{(t<\zeta)} \, \Phi(f(r \wedge t), t \geq 0) = E[1_{(t<\bar{\tau})} \, \Phi(x_{r\wedge t}, r \geq 0)].$$

If we take $\Phi = 1$ and let t tend to infinity, we obtain $P[\bar{\tau} = \infty] = 0$. Put $x_t^\star = x_t$ if $t < \bar{\tau}$, $x_t^\star = \Delta$ if $t \geq \bar{\tau}$, where Δ is a cemetery point added to $\mathbb{R}^d$. The previous equality implies that the law of $x^\star$ and the law under ν of $f^\star(t) = f(t)\,1_{(t<\zeta)} + \Delta\,1_{(t\geq\zeta)}$ have the same finite-dimensional marginals. It follows that $\lim_{t\uparrow\bar{\tau}} x_t^\star$ exists a.s. We denote this limit by $x_{\bar{\tau}}$ and set $x_t = x_{\bar{\tau}}$ for $t > \bar{\tau}$. Then the previous equality implies that ν is the distribution of $((x_{t\wedge\bar{\tau}}, t \geq 0), \bar{\tau})$. □

Remark. — The previous results show that the capacity of H is

$$\mathrm{cap}(H) =< \mu, 1 >= u(x).$$

On the other hand, we also have $\mathrm{cap}(H) = \mathcal{E}(\mu)$ so that Proposition 1.1 yields the equality

$$u(x) = 2\,E_x\left(\int_0^\tau dt\, u(f(t))^2 \exp -4\int_0^t u(f(r))\,dr\right).$$

Let

$$\mu_0 = \frac{\mu}{< \mu, 1 >}.$$

The probability measure μ_0, which is the law of the process (x_t) of Corollary 2.3, solves the problem

$$\mathcal{E}(\mu_0) = \inf_{\gamma \in \mathcal{P}(H)} \mathcal{E}(\gamma)$$

where $\mathcal{P}(H)$ denotes the set of all probability measures supported on H (see the remarks at the end of [FG]). Therefore, we get

$$\frac{1}{2u(x)} = \inf_{\gamma} E_x \left(\int_0^\infty \left(\frac{d\gamma_{(t)}}{dP_{x|\mathcal{G}_t}}(f) \right)^2 dt \right)$$

where the infimum is taken over all probability measures $\gamma \in \mathcal{P}(H)$ such that $\gamma_{(t)} \ll P_{x|\mathcal{G}_t}$ for every $t \geq 0$. Moreover, the infimum is attained only for $\gamma = \mu_0$.

3. The characterization of polar sets.

In this section, F is a compact subset of $\mathbb{R}^d$ and $x \in \mathbb{R}^d \backslash F$. For reasons that will appear later, we also assume that $d \geq 4$. As we have already observed, the set F is $\mathcal{R}$-polar if and only if the set $H = \{w \in \mathcal{W}_x;\ \exists t \geq 0,\ f(t) \in F\}$ is M_x-polar.

Polar sets have been investigated by Dynkin [Dy2], [Dy3], in fact in a more general situation. Dynkin's work uses analytic results on removable singularities for semilinear partial differential equations. Our goal here is to give a more probabilistic approach to the characterization of polar sets.

Let us briefly recall the arguments used in [LG1]. Suppose we aim to prove that F is not $\mathcal{R}$-polar, or equivalently that H is not M_x-polar. By the general results of [FG] (see also [Dy1]), it is enough to check that H supports a nontrivial measure μ with finite energy (in the sense of Proposition 1.1). We choose a finite measure ν in $\mathbb{R}^d$, supported on F, and we take

$$\mu(dw) = \int \nu(dy)\, G(x,y)\, P_{xy}(dw)$$

where $G(x,y) = |y-x|^{2-d}$ is the Green function of Brownian motion in $\mathbb{R}^d$, and $P_{xy}(dw)$ denotes the law of Brownian motion started at x and conditioned to die at y (P_{xy} can be viewed as a probability measure on $\mathcal{W}_x^\star$). The conditioning here is in the sense of Doob's h-processes, with

$h(\cdot) = G(\cdot, y)$. In the notation of Proposition 1.1,

$$\frac{d(P_{xy})_{(t)}}{dP_{x|\mathcal{G}_t}}(f) = \frac{G(f(t), y)}{G(x, y)}.$$

It follows immediately that

$$Y_t(f) = \frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}}(f) = \int \nu(dy)\, G(f(t), y)$$

and by Proposition 1.1, the energy of μ is

$$\begin{aligned}\mathcal{E}(\mu) &= 2\, E_x \left(\int_0^\infty dt \left(\int \nu(dy)\, G(f(t), y) \right)^2 \right) \\ &= 2 \int dz\, G(x, z) \left(\int \nu(dy)\, G(z, y) \right)^2 .\end{aligned}$$

Therefore, we arrive at the following result (first established by Perkins [Pe] in terms of super Brownian motion).

PROPOSITION 3.1. — *Suppose that F supports a nontrivial measure ν such that*

$$\int dz\, G(x, z) \left(\int \nu(dy)\, G(z, y) \right)^2 < \infty.$$

Then F is not $\mathcal{R}$-polar.

A few lines of calculations show that the condition of Proposition 3.1 is equivalent to

$$\iint \nu(dy)\nu(dz)\, |y - z|^{4-d} < \infty, \text{ if } d > 4,$$

$$\iint \nu(dy)\nu(dz) \log \frac{1}{|y - z|} < \infty, \text{ if } d = 4.$$

We get in particular that straight lines are not $\mathcal{R}$-polar when $d = 4$. By an obvious projection argument, it follows that singletons are not $\mathcal{R}$-polar when $d \leq 3$. Hence, there are no nonempty $\mathcal{R}$-polar sets in dimension $d \leq 3$.

The problem in proving the converse to Proposition 3.1 is that a measure μ supported on F and with finite energy may not be of the type $\mu = \int \nu(dy) P_{xy}$. In fact, we already know that the probability measure that minimizes the energy, namely the equilibrium measure of H, is not of this type. We keep assuming that $d \geq 4$.

PROPOSITION 3.2. — *Suppose that F supports no nontrivial measure ν such that*

$$\int dz\, G(x,z) \left(\int \nu(dy)\, G(z,y) \right)^2 < \infty.$$

Then F is $\mathcal{R}$-polar.

Proof. — Let F be a compact set that satisfies the assumption of Proposition 3.2. We assume that F is not $\mathcal{R}$-polar and will arrive at a contradiction. If F is not $\mathcal{R}$-polar, the function

$$u(x) = \mathbb{N}_x(\mathcal{R} \cap F \neq \varnothing)$$

is a nontrivial nonnegative solution of $\Delta u = 4u^2$ on $\mathbb{R}^d \backslash F$.

We first note that, in the assumption of Proposition 3.2, we may replace the function $G(x,y)$ by the classical Bessel potential $g_2(x-y)$ defined as in Meyers [Me], Section 7. By combining the results of [Me], Theorem 14 and [AP], Theorem A, we obtain that the assumption of Proposition 3.2 is equivalent to the equality $c_{2,2}(K) = 0$, where the capacity $c_{2,2}$ is defined in terms of Sobolev norms, as in [BP] for instance. Then, let O be a bounded open set containing F. By [BP], Lemme 2.1, we know that there exists a sequence $v_n \in C_c^\infty(O)$ such that $0 \leq v_n \leq 1$, $v_n = 1$ on a neighborhood of F and

$$\lim_{n \to \infty} \|v_n\|_{2,2} = 0.$$

Here $\|v_n\|_{2,2} = \|v_n\|_2 + \|\nabla v_n\|_2 + \|\nabla^2 v_n\|_2$ is the usual Sobolev norm.

We will use the sequence (v_n) for the proof of the next lemma which is a special case of [BP], Théorème 2.2. We give the proof for the convenience of the reader.

LEMMA 3.3. — *Under the assumption of Proposition 3.2, any measurable function u on $\mathbb{R}^d$ which solves $\Delta u = 4u^2$ in $\mathbb{R}^d \backslash F$ is locally square integrable.*

Proof. — Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \varphi \leq 1$. Set $\varphi_n = \varphi(1 - v_n)$, so that φ_n converges to φ in $L^2(\mathbb{R}^d)$. Then,

$$4 \int \varphi_n^4\, u^2\, dx = \int \varphi_n^4\, \Delta u\, dx = \int \Delta(\varphi_n^4)\, u\, dx,$$

by Green's identity, using the fact that φ_n vanishes in a neighborhood of K. Then,

$$\begin{aligned}
&\int \varphi_n^4\, u^2\, dx = 3\int \varphi_n^2\, \Delta\varphi_n\, u\, dx + \int \varphi_n^3\, |\nabla\varphi_n|^2\, u\, dx \\
&\leq 3\Big(\int \varphi_n^4\, u^2\, dx\Big)^{1/2}\Big(\int (\Delta\varphi_n)^2\, dx\Big)^{1/2} + \Big(\int \varphi_n^6\, u^2\, dx\Big)^{1/2}\Big(\int |\nabla\varphi_n|^4\, dx\Big)^{1/2} \\
&\leq C\Big(\int \varphi_n^4\, u^2\, dx\Big)^{1/2} \|v_n\|_{2,2},
\end{aligned}$$

using the easy inequality,

$$\Big(\int |\nabla\varphi_n|^4\, dx\Big)^{1/2} \leq C\|\varphi_n\|_\infty\, \|\nabla^2\varphi_n\|_2,$$

which follows (for instance) from an integration by parts. Since the sequence $\|v_n\|_{2,2}$ is bounded, we get

$$\int \varphi_n^4\, u^2\, dx \leq C$$

and Fatou's lemma completes the proof. □

We now complete the proof of Proposition 3.2. Let μ be the equilibrium measure of H as in Theorem 2.2. We know that

$$\mu(dw) \text{ a.e.}, \quad \int_0^\zeta u(f(t))\, dt = \infty,$$

since the set $\{w;\ \hat{w} \in F \text{ and } \int_0^\tau u(f(t))\, dt < \infty\}$ is M_x-polar by Lemma 2.1. Choose $R > 0$ such that

$$\mu(\{w;\ \sup_{t\geq 0} |f(t)| \leq R\}) > 0.$$

It follows that

$$\int \mu(dw) \int_0^\zeta 1_{\{|f(t)|\leq R\}} u(f(t))\, dt = \infty.$$

However, using Theorem 2.2,

$$\begin{aligned}
&\int \mu(dw) \int_0^\zeta 1_{\{|f(t)|\le R\}} u(f(t))\, dt \\
&= \int_0^\infty dt\, E_x\left(1_{\{t<\tau, |f(t)|\le R\}} u^2(f(t)) \exp -2\int_0^t u(f(s))\, ds\right) \\
&\le E_x\left(\int_0^\infty dt\, 1_{\{|f(t)|\le R\}}\, u^2(f(t))\right) \\
&= \int dy\, 1_{\{|y|\le R\}}\, G(x,y)\, u^2(y) \\
&< \infty,
\end{aligned}$$

by Lemma 3.3. We arrive at a contradiction, which completes the proof.□

4. Hitting probabilities for boundary sets.

In this section, we consider a bounded domain D in $\mathbb{R}^d$. We assume that D is a Lipschitz domain, meaning that the boundary of D can be locally represented as the graph of a Lipschitz function (see [HW] for a precise definition). We also assume that $x \in D$. For $w \in \mathcal{W}_x$, we set $\tau(w) = \inf\{t;\ f(t) \notin D\} \le \infty$. The range $\mathcal{R}^D$ is defined by

$$\mathcal{R}^D = \{W_s(t);\ s \ge 0,\ 0 \le t \le \zeta_s \wedge \tau(W_s)\}.$$

Obviously, $\mathcal{R}^D \subset \bar{D}$, $\mathbb{N}_x$ a.e., where $\bar{D}$ denotes the closure of D. A subset K of ∂D is called ∂-polar if

$$\mathbb{N}_x(\mathcal{R}^D \cap K \neq \varnothing) = 0.$$

As was the case for $\mathbb{N}_x(\mathcal{R} \cap K \neq \varnothing)$, it is easy to check that this condition does not depend on the choice of $x \in D$ (use Proposition 2.5 of [LG2]). Our goal is to investigate the class of ∂-polar subsets of ∂D. Notice that K is ∂-polar if and only if the set

$$H = \{w \in \mathcal{W}_x^\star;\ \tau(w) = \zeta_{(w)},\ \hat{w} \in K\}$$

is M_x-polar.

We fix a reference point $x_0 \in D$ and we denote by $H(x,y)$, $x \in D$, $y \in \partial D$ the corresponding Martin kernel :

$$H(x,y) = \lim_{x' \to y, x' \in D} \frac{G_D(x,x')}{G_D(x_0,x')}$$

where G_D is Green function of Brownian motion in D. This definition is correct because the Martin boundary of D can be identified with its Euclidean boundary (see [HW]). For any fixed $y \in \partial D$, the function $H(\cdot, y)$ is positive harmonic in D, and for any fixed $x \in D$, the function $H(x, \cdot)$ is continuous on ∂D.

Our first result is analogous to Proposition 3.1.

PROPOSITION 4.1. — *Suppose that K supports a nontrivial measure ν such that*

$$\int_D dz\, G_D(x,z) \Big(\int \nu(dy)\, H(z,y) \Big)^2 < \infty.$$

Then K is not ∂-polar.

Proof. — We check that H is not M_x-polar by constructing a measure μ supported on H with finite energy. We take

$$\mu(dw) = \int \nu(dy)\, H(x,y)\, P^D_{xy}(dw),$$

where P^D_{xy} denotes the law (in $\mathcal{W}_x$) of Brownian motion started at x, conditioned to exit D at y, and stopped at that exit time. More precisely, we consider the h-process of Brownian motion started at x associated with the harmonic function $h(\cdot) = H(\cdot, y)$. By adapting the arguments of Doob [Do], p. 692, it is easy to verify that this h-process converges to y at its lifetime. Therefore, we can interpret its law as a probability measure P^D_{xy} on $\mathcal{W}^\star_x$, which is supported on

$$\{w;\ \tau(w) = \zeta_{(w)},\ \hat{w} = y\}.$$

By the definition of an h-process,

$$\frac{d(P^D_{xy})_{(t)}}{dP_{x|\mathcal{G}_t}} = 1_{(t<\tau)}\, \frac{H(f(t),y)}{H(x,y)}.$$

It follows that

$$\frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}} = 1_{(t<\tau)} \int \nu(dy)\, H(f(t),y)$$

and, by Proposition 1.2, the energy of μ is

$$\begin{aligned}\mathcal{E}(\mu) &= E_x\left(\int_0^{\mathcal{T}} dt\left(\int \nu(dy)\, H(f(t),y)\right)^2\right)\\ &= \int_D dz\, G_D(x,z)\left(\int \nu(dy)\, H(z,y)\right)^2\end{aligned}$$

which is finite by assumption. □

COROLLARY 4.2. — *Suppose that D is a C^2-domain. If $d=2$, any nonempty compact subset of ∂D is not ∂-polar. If $d\geq 3$, if K supports a nontrivial measure ν such that*

$$\begin{aligned}&\iint \nu(dy)\nu(dz)\,\log\frac{1}{|z-y|} < \infty \quad \text{if } d=3,\\ &\iint \nu(dy)\nu(dz)\,|z-y|^{3-d} < \infty \quad \text{if } d\geq 4,\end{aligned}$$

then K is not ∂-polar.

Proof. — Note that

$$\begin{aligned}&\int_D dz\, G_D(x,z)\left(\int \nu(dy)\, H(z,y)\right)^2\\ &\qquad = \iint \nu(dy)\nu(dy')\int dz\, G_D(x,z)\, H(z,y)\, H(z,y').\end{aligned}$$

When D is a C^2 domain, it is known that G_D and H satisfy the following estimates. For $|z-x|\geq\varepsilon>0$,

$$\begin{aligned}&G_D(x,z)\leq C(x,\varepsilon)\,\rho(z)\\ &H(z,y)\leq C\,\rho(z)\,|z-y|^{1-d}\end{aligned}$$

where $\rho(z)=d(z,\partial D)$ (these estimates can be easily derived by comparing G_D to the Green function of suitable domains such as an interior tangent sphere or the complement of an exterior tangent sphere). It follows that

$$\int dz\, G_D(x,z)\, H(z,y)\, H(z,y') \leq \begin{cases} C(x) & \text{if } d=2,\\ C(x)\,(1+\log_+(\frac{1}{|y-y'|})) & \text{if } d=3,\\ C(x)\,|y-y'|^{3-d} & \text{if } d\geq 4,\end{cases}$$

which completes the proof. □

In view of Proposition 3.2, one may expect that the converse to Corollary 4.2 (or to Proposition 4.1) holds. We now present a partial converse, which relies on the results in Abraham and Le Gall [AL]. If h is a suitable function from $\mathbb{R}_+$ into $\mathbb{R}_+$, we denote by $h - m$ the associated Hausdorff measure.

PROPOSITION 4.3. — *Suppose that $d \geq 3$ and that D is a C^2 domain. Set $h_3(r) = |\log r|^{-1}$ and $h_d(r) = r^{d-3}$ if $d \geq 4$. The condition $h_d - m(K) = 0$ implies that K is ∂-polar.*

Proof. — It is proved in [AL] that, for $x \in D$, $y \in \partial D$ and $\varepsilon \in (0, 1/2)$,

$$\mathbb{N}_x(\mathcal{R}^D \cap B_{\partial D}(y, \varepsilon) \neq \varnothing) \leq C(x)\, h_d(\varepsilon),$$

where $B_{\partial D}(y, \varepsilon) = \{y' \in \partial D;\ |y' - y| < \varepsilon\}$ and $C(x)$ is a constant depending on x. Let $\delta > 0$. By assumption, there exists a covering of K by balls $B_{\partial D}(y_i, \varepsilon_i)$, with $y_i \in \partial D$, $\varepsilon_i \in (0, 1/2)$, such that

$$\sum_i h_d(\varepsilon_i) < \delta.$$

It follows that

$$\mathbb{N}_x(\mathcal{R}^D \cap K \neq \varnothing) \leq \sum_i \mathbb{N}_x(\mathcal{R}^D \cap B_{\partial D}(y_i, \varepsilon_i) \neq \varnothing) \leq C(x)\, \delta,$$

which gives the desired result since δ was arbitrary. □

In particular, points are ∂-polar as soon as $d \geq 3$. The latter fact can also be derived from the results of Gmira and Véron [GV] and the connection with partial differential equations described in the next proposition.

PROPOSITION 4.4. — *Let K be a compact subset of ∂D. The function*

$$u(x) = \mathbb{N}_x(\mathcal{R}^D \cap K \neq \varnothing)$$

solves $\Delta u = 4u^2$ in D, with boundary condition

$$\lim_{x \to y, x \in D} u(x) = 0$$

for every $y \in \partial D \backslash K$. Moreover, u is the maximal nonnegative solution of this problem.

Proof. — Let O be a subset of ∂D, which is open for the relative topology of ∂D. Denote by X^D the exit measure of D (see [LG2]). As a consequence of [LG2], Proposition 5.5, which can be applied here because D is a Lipschitz domain, we know that the topological support of X^D coincides $\mathbb{N}_x$ a.e. with the set $\{W_s(\tau(W_s)); s \geq 0, \tau(W_s) < \infty\}$. Hence,

$$\mathbb{N}_x(\mathcal{R}^D \cap O \neq \varnothing) = \mathbb{N}_x(X^D(O) > 0).$$

However,

$$\mathbb{N}_x(X^D(O) > 0) = \lim_{\lambda \to \infty} \mathbb{N}_x(1 - \exp(-\lambda X^D(O))),$$

and the function $v_\lambda(x) = \mathbb{N}_x(1 - \exp(-\lambda X^D(O)))$ solves $\Delta u_\lambda = 4u_\lambda^2$ in D ([LG2], Section 4). Using the associated integral equations, it is easy to obtain that $v(x) = \lim_{\lambda \to \infty} v_\lambda(x)$ solves the same equation (see [LG2], Section 5 for similar arguments).

We may then find a decreasing sequence of relative open sets O_n such that $K = \cap O_n$. Denote by $v_{(n)}$ the corresponding functions. Obviously, $u(x) = \lim \downarrow v_{(n)}(x)$ and the same arguments as previously show that $\Delta u = 4u^2$ in D.

Let us now fix $y \in \partial D \backslash K$, and choose $a \in \left(0, \frac{1}{2} d(y, K)\right)$. Then, for $\delta > 0$, $\varepsilon > 0$, and $|x - y| < a$,

$$\begin{aligned}\mathbb{N}_x(\mathcal{R}^D \cap K \neq \varnothing) \leq\ & \mathbb{N}_x(\exists s \in [0, \delta] \cup [(\sigma - \delta)_+, \sigma],\ |\hat{W}_s - x| > a) \\ & + \mathbb{N}_x(\sigma \geq 2\delta,\ \inf\{\zeta_s,\ s \in [\delta, \sigma - \delta]\} \leq \varepsilon) \\ & + \mathbb{N}_x(\zeta_\delta \geq \varepsilon,\ \{W_\delta(t), 0 \leq t \leq \varepsilon\} \subset D),\end{aligned}$$

(recall that σ denotes the duration of the excursion (ζ_s) under $\mathbb{N}_x$). To get this inequality, one argues as follows. The event

$$(\{\sigma < 2\delta\} \cup \{\exists s \in [0, \delta] \cup [(\sigma - \delta)_+, \sigma] \text{ and } t \geq 0 \text{ s.t. } W_s(t) \in K\}) \cap \{\mathcal{R} \cap K \neq \varnothing\}$$

is contained in

$$\{\exists s \in [0, \delta] \cup [(\sigma - \delta)_+, \sigma],\ |\hat{W}_s - x| > a\}.$$

Suppose then that $\sigma \geq 2\delta$ and that the paths W_s, $s \in [0, \delta] \cup [(\sigma - \delta)_+, \sigma]$ do not intersect K. By the properties of the path-valued process, the paths W_s, $s \in [\delta, \sigma - \delta]$ all coincide with W_δ on the interval $[0, h]$, where

$$h = \inf\{\zeta_s,\ s \in [\delta, \sigma - \delta]\}.$$

In particular, if $\{W_\delta(t), 0 \leq t \leq h\}$ exits D (automatically at a point of $\partial D \backslash K$ by our assumption), then none of the paths W_s, $s \in [\delta, \sigma - \delta]$ can exit D at a point of K.

We then observe that

$$\lim_{\delta\downarrow 0} \mathbb{N}_x(\exists s \in [0,\delta] \cup [(\sigma-\delta)_+,\sigma],\ |\hat{W}_s - x| > a) = 0$$

from the continuity of the mapping $s \to \hat{W}_s$. Also, for every fixed $\delta > 0$,

$$\lim_{\varepsilon\downarrow 0} \mathbb{N}_x(\sigma \geq 2\delta,\ \inf\{\zeta_s,\ s \in [\delta, \sigma-\delta]\} \leq \varepsilon) = 0.$$

Finally, if $\varepsilon > 0$ is fixed, the fact that W_δ is a Brownian path started at x, together with our assumption on D implies that

$$\lim_{x\to y} \mathbb{N}_x(\zeta_\delta \geq \varepsilon,\ \{W_\delta(t), 0 \leq t \leq \varepsilon\} \subset D) = 0.$$

It remains to verify that u is the maximal solution of the given problem. To this end, consider a decreasing sequence K_n of closed neighborhoods of K such that

$$K_n \subset \{y \in \mathbb{R}^d;\ d(y,K) \leq 2^{-n}\}.$$

We may assume that every point of ∂K_n is regular for K_n. We then take n large enough so that $x_0 \notin K_n$, and we let D_n be the connected component of $D\backslash K_n$ that contains x_0. We set $U_n = \partial D_n \backslash \partial D \subset \partial K_n$ and

$$u_n(x) = \mathbb{N}_x(\mathcal{R}^{D_n} \cap U_n \neq \varnothing)$$

for $x \in D_n$. It is easy to verify that $u_n \geq u$ and more precisely that $u(x) = \lim \downarrow u_n(x)$.

Note that U_n is open in ∂D_n and that $u_n(x) \geq \mathbb{N}_x(X^{D_n}(U_n) > 0) \geq \mathbb{N}_x(1 - \exp(-\lambda X^{D_n}(U_n)))$, for every $\lambda > 0$. Using Corollary 4.3 of [LG2], we easily get

$$\lim_{x\to y, x\in D_n} u_n(x) = \infty,$$

for every $y \in U_n$. The maximum principle (see e.g. Dynkin [Dy2], Appendix) then implies that any other nonnegative solution v of the problem of Proposition 4.4 satisfies $v \leq u_n$ on D_n. This completes the proof, since D_n increases to D as $n \to \infty$. □

Remark. — The result of Proposition 4.4 holds more generally under the assumption that every point of D is regular for $\mathbb{R}^d\backslash D$ (the first part of the proof has to be modified).

In view of Proposition 4.4, we see that K is not ∂-polar if and only if the problem of Proposition 4.4 has a nontrivial nonnegative solution. In particular, the condition of Proposition 4.1 ensures the existence of such a solution.

The next theorem presents results analogous to Lemma 2.1 and Theorem 2.2.

THEOREM 4.5. — *Suppose that K is not ∂-polar. Let*

$$T_H = \inf\{s \geq 0,\, W_s \in H\}.$$

Then, for every $w \in \mathcal{W}_x^\star$,

$$\mathbb{P}_w^\star(T_H < \infty) = \begin{cases} 1 & \text{, if } \tau(w) < \zeta \text{ and } \hat{w} \in K, \\ 1 - \exp(-2\int_0^{\zeta\wedge\tau(w)} u(f(r))\,dr) & \text{, if not.} \end{cases}$$

In particular, the set $\{w \in \mathcal{W}_x^\star;\ \tau(w) \leq \zeta_{(w)},\ f(\tau(w)) \in K,\ \int_0^{\tau(w)} u(f(r))\,dr < \infty\}$ is M_x-polar.

The set H is an equilibrium set. Its capacitary measure μ is such that, for every $t \geq 0$, the measure $\mu_{(t)}$ is absolutely continuous with respect to $P_{x|\mathcal{G}_t}$, and

$$\frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}} = 1_{(t<\tau(f))}\, u(f(t)) \exp\left(-2\int_0^t u(f(s))\,ds\right).$$

Proof. — The proof of the first part of Theorem 4.5 is exactly similar to the proof of Lemma 2.5, using again Proposition 2.5 of [LG2]. The fact that H is an equilibrium set is immediate from a domination argument. We can then argue as in the proof of Theorem 2.2. If $(X_t(f))$ is a $(\mathcal{G}_t)$-predictable, P_x-a.s. càdlàg version of the Radon-Nikodym derivative

$$\frac{d\mu_{(t)}}{dP_{x|\mathcal{G}_t}}$$

we know by Proposition 1.2 that

$$2\int_0^\zeta X_t(f)\,dt = U(\mu)(w) = \mathbb{P}_w^\star(T_H < \infty),$$

$M_x(dw)$ a.e. The desired result follows by differentiation. □

A result analogous to Corollary 2.3 also holds in the present setting. If (x_t) denotes the solution of the stochastic differential equation

$$\begin{aligned} dx_t &= d\beta_t + \frac{\nabla u}{u}(x_t)\, dt, \\ x_0 &= x, \end{aligned}$$

stopped at its hitting time τ of ∂D, the process $(x_t, 0 \leq t < \tau)$ can be continuously extended to the time interval $[0, \tau]$, and $x_\tau \in K$. Furthermore, the law of $(x_{t\wedge\tau},\, t \geq 0)$ in $\mathcal{W}_x$ is $u(x)^{-1}$ times the equilibrium measure of H. Finally, the law of $(x_{t\wedge\tau},\, t \geq 0)$ solves a variational problem analogous to the one stated at the end of Section 2. Namely,

$$\frac{1}{2u(x)} = \inf_{\gamma}\ E_x \left(\int_0^\infty \left(\frac{d\gamma_{(t)}}{dP_{x|\mathcal{G}_t}}(f) \right)^2 dt \right)$$

where γ runs over all probability measures supported on H, and the infimum is attained only when γ is the law of $(x_{t\wedge\tau},\, t \geq 0)$.

Remark. — One might think of using the method of proof of Proposition 3.2 to get the converse to Corollary 4.2 (which seems easier than the converse to Proposition 4.1). The problem however is to get the analogue of Lemma 3.3. More precisely, assuming that K supports no nontrivial measure ν satisfying the assumption of Corollary 4.2, one has to show that any nonnegative solution of the problem of Proposition 4.4 satisfies

$$\int_D \rho(x)\, u(x)^2\, dx < \infty.$$

The same argument as in Section 3 would then imply that K is ∂-polar.

Note added in proof : We are now able to prove that the converse of Corollary 4.2 holds provided that the boundary of D is sufficiently smooth. The method of proof is that described in the final remark of the paper.

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