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# CIRCLE BUNDLES, ADIABATIC LIMITS OF η-INVARIANTS AND ROKHLIN CONGRUENCES

by Weiping ZHANG

## Introduction.

This work originates from a question of Siye Wu who asked whether the Rokhlin congruence formula [R2] could be proved purely analytically.

Recall that the classical Rokhlin theorem [R1] states that the signature of a compact spin four manifold is divisible by 16. In [AH], Atiyah and Hirzebruch proves the following extension of this result : the $\widehat{A}$ genus of an $8k+4$ dimensional compact spin manifold is an even integer.

Now let $K$ be an oriented compact four manifold not necessarily spin. Let $B$ be an orientable characteristic submanifold of $K$, that is, $B$ is a compact 2 dimensional submanifold of $K$ such that $[B] \in H_2(K, \mathbb{Z}_2)$ is dual to the second Stiefel-Whitney class of $K$. Note $B \cdot B$ the self-intersection of $B$ in $K$.

Rokhlin [R2] established a congruence formula of the type

$$\text{(0.1)} \qquad \frac{\operatorname{sign}(B \cdot B) - \operatorname{sign}(K)}{8} \equiv \phi(B) \pmod{2\mathbb{Z}},$$

where $\phi(B)$ is a spin cobordism invariant associated to $(K, B)$.

The left hand side of (0.1) can also be expressed in terms of $\widehat{A}(K)$ and the Euler number of the normal bundle to $B$ in $K$.

In this paper, we will prove an extension of both (0.1) and the result of Atiyah and Hirzebruch mentioned above. Our results are proved for elliptic genera and hold also in the case where $B$ is non-orientable.

*Key words :* Characteristic classes and characteristic numbers – Index theory and fixed point theory.
*A.M.S. Classification :* 57R – 58G.

The proof of our main result (see Theorem 3.2) is based on a calculation of the adiabatic limit of $\eta$-invariants of Dirac operators on a circle bundle.

More precisely, let $N$ be a tubular neighborhood of $B$, then $\partial N$ is a circle bundle over $B$. An application of the Atiyah-Patodi-Singer index theorem for manifolds with boundary [APS] reduces the problem to a calculation of the $\eta$-invariants of Dirac operators on $\partial N$.

Thanks to the work of Bismut and Cheeger [BC1] and its extension by Dai [D], such a calculation can indeed be carried out explicitly in this case.

The idea of using the adiabatic limit of $\eta$ invariants to study the defects of signatures goes back to Atiyah-Donnelly-Singer and Müller, see Bismut-Cheeger [BC2] and the references there in.

This paper is organized as follows.

In Section 1, we calculate the $\widehat{\eta}$-form of Bismut and Cheeger [BC1] for circle bundles. In Section 2, we calculate the adiabatic limits of $\eta$-invariants of Dirac operators on circle bundles. In Section 3, we prove a congruence formula of Rokhlin type for certain $KO$-characteristic numbers. Section 4 includes some applications of the congruence formula in Section 3. There is also an Appendix in which we try to relate our formula to another extension of (0.1) obtained by Ochanine [O1].

The results of this paper were announced in [Z1]. Finally, we refer to [Z2] for a topological treatment of some results in this paper.

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## 1. The $\widehat{\eta}$-form of circle bundles.

The purpose of this section is to make an explicit calculation for Bismut-Cheeger's $\widehat{\eta}$-form [BC1] of circle bundles over even dimensional manifolds.

This section is organized as follows. In a), we specify what we call an integral power operation on vector bundles. In b), we present our main geometric assumptions and notation. The $\widehat{\eta}$-form is defined in c) and calculated in d).

### a) Power operations on vector bundles.

Let $K$ be a compact manifold. Let $\mathrm{Vect}(K)$ be the ring generated by all real vector bundles over $K$. For any $i \geq 0$ and for any element $E$ in $\mathrm{Vect}(K)$, we use the standard notation $\Lambda^i(E)$ (resp. $S^i(E)$) for the $i^{\text{th}}$ exterior (resp. symmetric) power of $E$. Then each element in the ring $\mathbb{Z}[\Lambda^i, S^i; i = 0, 1, 2, \ldots]$ can be viewed as an operation on $\mathrm{Vect}(K)$. Also it is clear that these operations are well-defined without reference to any base space like $K$.

DEFINITION 1.1. — *An element in $\mathbb{Z}[\Lambda^i, S^i; i = 0, 1, 2, \ldots]$ is called an integral power operation.*

*Notation convention.* — To simplify the notation setting of this paper, we will use the same notation $E$ for a real vector bundle $E$ as well as its complexification. This should not cause any confusion in the context.

### b) Geometric assumptions and notation.

Let $B$ be an even dimensional compact connected spin manifold with a fixed spin structure. Let $g^{TB}$ be a metric on $TB$, let $\nabla^{TB}$ be the associated Levi-Civita connection and $R^{TB}$ the curvature of $\nabla^{TB}$.

Let $N \to B$ be a 2-dimensional oriented vector bundle over $B$. Let $g^N$ be a metric on $N$. Let $\nabla^N$ be a connection on $N$ preserving $g^N$. Note $R^N$ the curvature of $\nabla^N$.

Let $T^H N$ be the horizontal subbundle of $TN$ determined by $\nabla^N$. Then $g^{TB}$ lifts to a metric on $T^H N$.

Let $g^{TN}$ be the metric on $TN$,

$$g^{TN} = g^N \oplus \pi^* g^{TB} \tag{1.1}$$

such that $N$ and $T^H N = \pi^* TB$ are orthogonal to each other with respect to $g^{TN}$.

Set

$$
\begin{aligned}
N_1 &= \{v \in N_x : x \in B, \|v\|_{g^N} \le 1\},\\
M = \partial N_1 &= \{v \in N_x : x \in B, \|v\|_{g^N} = 1\},\\
T^H M &= T^H N|_M.
\end{aligned}
\tag{1.2}
$$

Let $g^{TM}$ be the metric on $TM$,

$$
\begin{aligned}
TM &= TS^1 \oplus \pi^* TB,\\
g^{TM} &= g^{TN}|_M = g^{TS^1} \oplus \pi^* g^{TB}.
\end{aligned}
\tag{1.3}
$$

Then $M$ is a circle bundle over $B$ with structure group $SO(2)$ acting by isometries on the fibres. Furthermore, it carries a canonical induced spin structure induced from the spin structure of $TB$ (cf. [KT]). This in turn determines a spin structure on $TS^1$.

Let $p^{TS^1}$ be the orthogonal projection on $TS^1$ with respect to $g^{TM}$. Note $\nabla^L$ the Levi-Civita connection of $g^{TM}$. Let $\nabla$ be the connection on $TM$ defined for $U, V \in \Gamma(TS^1)$, $X, Y \in \Gamma(TB)$ as follows (cf. [B]),

$$
\begin{aligned}
\nabla_U V &= p^{TS^1}(\nabla^L_U V),\\
\nabla_X U &= p^{TS^1}(\nabla^L_X U),\\
\nabla_U X &= 0,\\
\nabla_X Y &= \nabla^{TB}_X Y,
\end{aligned}
\tag{1.4}
$$

where we have identified $TB$ with its lift. Such an identification will always be understood in what follows.

Let $S$ be the tensor defined by

$$
S = \nabla^L - \nabla. \tag{1.5}
$$

Let $e \in TS^1$ be the unit vector field determined by $g^{TM}$ and the spin structure on $TS^1$.

LEMMA 1.2. — *The following identity holds,*

$$
S(e)e = 0. \tag{1.6}
$$

*Proof.* — Clearly $e$ generates a one parameter isometry group of $M$. Thus $e$ is a Killing vector field and for any $X \in \Gamma(TM)$, we have

$$
\langle S(e)e, X\rangle = \langle \nabla^L_e e, X - p^{TS^1} X\rangle \tag{1.7}
$$

$$
\begin{aligned}
&= -\left\langle e, \nabla_{X-p^{TS^1}X} e\right\rangle \\
&= -\frac{1}{2}(X - p^{TS^1}X)\langle e, e\rangle \\
&= 0.
\end{aligned}
$$

This proves (1.6). □

Let $T(\cdot,\cdot)$ be the torsion of $\nabla$ defined by

$$T(U,V) = -S(U,V) + S(V,U), \quad U,V \in TB. \tag{1.8}$$

Let $e^* \in T^*S^1$ be the dual of $e$.

LEMMA 1.3. — *The following identity holds,*

$$\langle T(U,V), e\rangle = de^*(U,V), \quad U,V \in TB. \tag{1.9}$$

*Proof.* — Since $\nabla^L$ is torsion free, one verifies easily

$$\langle T(U,V), e\rangle = -\langle [U,V], e\rangle = de^*(U,V). \tag{1.10}$$

□

*Remark* 1.4. — From (1.9), we see that $T(\cdot,\cdot)$ determines a 2-form (still note $T$) in $\Lambda^2(T^*B)$ such that $\frac{T}{2\pi}$ represents the Euler class of $N$.

### c) Dirac operators and $\widehat{\eta}$-forms.

Let $F^{TS^1}$ be the bundle of spinors associated to $(TS^1, g^{TS^1})$ on $M$. The connection $\nabla|_{TS^1}$ lifts to $F^{TS^1}$, which we still note $\nabla$.

Let $\mathcal{R}$ be an integral power operation defined in Definition 1.1. The connection $\nabla$ lifts naturally to $\mathcal{R}(TM)$, which we still note $\nabla$.

For any fibre $S^1$, let $D_{S^1,\mathcal{R}}$ be the Dirac operator acting on $\Gamma(TS^1 \otimes \mathcal{R}(TM|_{S^1}))$ defined as follows,

$$
\begin{aligned}
D_{S^1,\mathcal{R}}(U \otimes V) &= c(e)\nabla_e U \otimes V + c(e) U \otimes \nabla_e V, \\
&\qquad U \in \Gamma(F^{TS^1}), V \in \Gamma(\mathcal{R}(TM)).
\end{aligned} \tag{1.11}
$$

Then $\ker(D_{S^1,\mathcal{R}})$ is of constant rank and forms a vector bundle over $B$. Furthermore, we have

$$\ker(D_{S^1,\mathcal{R}}) = \mathcal{R}(TB \oplus \mathbb{R}). \tag{1.12}$$

Let $b_1, \dots, b_{2n}$ be an orthonormal base of $TB$, $dy^1, \dots, dy^{2n}$ be its dual. Set

$$c(T) = \frac{1}{2} \sum_{\alpha,\beta} dy^\alpha dy^\beta c(T(b_\alpha, b_\beta)). \tag{1.13}$$

Let $\widetilde{\nabla}$ be the natural lifting of $\nabla$ to the infinite dimensional vector bundle $\Gamma(F^{TS^1} \otimes \mathcal{R}(TM|_{S^1}))$ over $B$. By Lemma 1.2, $\widetilde{\nabla}$ is unitary.

Set

$$A_u = \widetilde{\nabla} + \sqrt{u} D_{S^1,\mathcal{R}} - \frac{c(T)}{4\sqrt{u}} . \tag{1.14}$$

This is the Bismut superconnection ([B]) for the family $\{D_{S^1,\mathcal{R}}\}$.

Definition 1.5 ([BC1, 4.39]). — The $\widehat{\eta}$-form is an even form on $B$ defined by

$$\widehat{\eta} = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mathrm{Tr}^{\mathrm{even}} \left[ \left( D_{S^1,\mathcal{R}} + \frac{c(T)}{4u} \right) \exp(-A_u^2) \right] \frac{du}{2\sqrt{u}} . \tag{1.15}$$

The fact that $\widehat{\eta}$ is well-defined follows from the results of Bismut-Cheeger [BC1] and Berline-Getzler-Vergne [BeGeV, Chap. 9].

### d) Calculation of the $\widehat{\eta}$-form for circle bundles.

Recall that $e$ is the unit vector field on $TS^1 \subset TM$ determined by $g^{TM}$ and the spin structure on $TB$.

Let $\nabla$ be the unitary connection defined by (1.4).

Lemma 1.6. — *For any $X \in \Gamma(TM)$, the following identity holds,*

$$\nabla_X e = 0. \tag{1.16}$$

*Proof.* — Clearly,

$$\langle \nabla_X e, e \rangle = 0. \tag{1.17}$$

(1.16) follows from (1.17). □

By Lemma 1.6, we can reduce the calculation of the $\widehat{\eta}$-form to the separated fibres. This is similar to the calculation of Bismut and Cheeger [BC2] for the flat torus bundles.

If $\xi$ is a vector bundle with a connection $\nabla$, we note $L^{\xi,\nabla}$ the curvature of $\nabla$.

THEOREM 1.7. — *The following identity holds,*

$$\widehat{\eta}(D_{S^1,\mathcal{R}}) = \frac{\tanh(\frac{\sqrt{-1}T}{2}) - \frac{\sqrt{-1}T}{2}}{\sqrt{-1}T\tanh(\frac{\sqrt{-1}T}{2})} \cdot \mathrm{Tr}\big[\exp\big(-L^{\mathcal{R}(TB\oplus\mathbb{R}),\nabla}\big)\big]. \tag{1.18}$$

*Proof.* — We split $\Gamma(F^{TS^1})$ through the eigenspaces of $D_{S^1}$,

$$\Gamma(F^{TS^1}) = \bigoplus_k F^{TS^1} \otimes \{e^{ik\theta}\}. \tag{1.19}$$

Let $z$ be an odd auxiliary Grassmann variable.

LEMMA 1.8. — *The following identity holds,*

$$\begin{aligned} -u\Big(\nabla_e + \frac{T}{4u} + z\frac{c(e)}{2\sqrt{u}}\Big)^2 + \frac{1}{2}dy^\alpha dy^\beta \otimes L^{\mathcal{R}(TB\oplus\mathbb{R}),\nabla}(b_\alpha, b_\beta) \\ = A_u^2 - z\Big(\sqrt{u}D_{S^1,\mathcal{R}} + \frac{c(T)}{4\sqrt{u}}\Big). \end{aligned} \tag{1.20}$$

*Proof.* — Clearly, the scalar curvature $k^{S^1} = 0$. (1.20) follows from [BC1, (4.68) - (4.70)], Lemma 1.1 and Lemma 1.6. □

Let $\mathrm{Tr}^z$ be the trace defined by

$$\mathrm{Tr}^z[a + zb] = \mathrm{Tr}[b], \tag{1.21}$$

where $a$ and $b$ do not contain $z$.

From Lemma 1.8, one sees that

$$\begin{aligned} &\mathrm{Tr}^{\mathrm{even}}\Big[\Big(D_{S^1,\mathcal{R}} + \frac{c(T)}{4u}\Big)\exp(-A_u^2)\Big] \\ &\quad = \frac{1}{\sqrt{u}}\mathrm{Tr}^z\Big[\exp\Big(u\Big(\nabla_e + \frac{T}{4u} + z\frac{c(e)}{2\sqrt{u}}\Big)^2\Big)\Big] \\ &\quad\quad \cdot \mathrm{Tr}\Big[\exp\Big(-L^{\mathcal{R}(TB\oplus\mathbb{R}),\nabla}\Big)\Big]. \end{aligned} \tag{1.22}$$

Also, since

$$\mathrm{Tr}[c(e)] = \frac{1}{\sqrt{-1}}, \tag{1.23}$$

one verifies in view of (1.19),

$$(1.24)\quad \frac{1}{\sqrt{u}}\operatorname{Tr}^z\left[\exp\left(u\Big(\nabla_e+\frac{T}{4u}+z\frac{c(e)}{2\sqrt{u}}\Big)^2\right)\right]$$
$$=\frac{1}{\sqrt{u}}\left\{\sum_{n=-\infty}^{+\infty}\left[\exp\left(u\Big(\sqrt{-1}n+\frac{T}{4u}-\frac{z\sqrt{-1}}{2\sqrt{u}}\Big)^2\right)\right]\right\}^z$$
$$=-\sqrt{-1}\sum_{n=-\infty}^{+\infty}\left(\frac{\pi}{u}\right)^{3/2}e^{2\pi n\sqrt{-1}\cdot\frac{T}{4u\sqrt{-1}}}\cdot n\cdot e^{-\frac{n^2\pi^2}{u}},$$

where the last equality follows from a Poisson type summation formula.

Thus,

$$(1.25)\quad \frac{1}{\sqrt{\pi}}\int_0^{+\infty}\frac{1}{\sqrt{u}}\operatorname{Tr}^z\left[\exp\left(u\Big(\nabla_e+\frac{T}{4u}+z\frac{c(e)}{2\sqrt{u}}\Big)^2\right)\right]\frac{du}{2\sqrt{u}}$$
$$=\pi\int_0^{+\infty}\sum_{n=1}^{+\infty}n\cdot\sin\left(\frac{2\pi n}{u}\cdot\frac{T}{4\sqrt{-1}}\right)e^{-\frac{n^2\pi^2}{u}}\frac{du}{u^2}$$
$$=\sum_{n=1}^{+\infty}\frac{\frac{T}{2\sqrt{-1}}}{n^2\pi^2+(\frac{T}{2\sqrt{-1}})^2}$$
$$=\frac{1}{2\sqrt{-1}}\frac{\operatorname{tg}(\frac{T}{2})-\frac{T}{2}}{\frac{T}{2}\operatorname{tg}(\frac{T}{2})}.$$

(1.18) then follows from (1.25), (1.22) and (1.15). □

## 2. Adiabatic limits of $\eta$-invariants of Dirac operators on circle bundles.

In this section, we apply the results of Section 1 to calculate the adiabatic limit of $\eta$-invariants of Dirac operators on circle bundles.

The papers of Bismut-Cheeger [BC1] and its extension by Dai [D] play an essential role.

This section is organized as follows.

In a), we present some notation supplement to those in Section 1, a), b). In b), the adiabatic limit of $\eta$-invariants is calculated. In c), we consider the case where the base manifold $B$ is non-orientable.

### a) Assumptions and notation.

We use the same assumptions and notation as in Section 1.

For any $\varepsilon > 0$, let $g^{TM,\varepsilon}$ be the metric on $TM$ defined by

$$g^{TM,\varepsilon} = g^{TS^1} \oplus \frac{1}{\varepsilon}\pi^* g^{TB}. \tag{2.1}$$

Let $\nabla^{L,\varepsilon}$ be the Levi-Civita connection of $g^{TM,\varepsilon}$. We still use the notation $g^{TM}, \nabla^L$ for $g^{TM,1}, \nabla^{L,1}$.

LEMMA 2.1. — *The following identity holds,*

$$\lim_{\varepsilon \to 0} \nabla^{L,\varepsilon} = \nabla + p^{TS^1} S. \tag{2.2}$$

*Proof.* — Lemma 2.1 is a consequence of [BC1, (4.16)]. □

*Remark* 2.2. — Since $\nabla$ preserves the splitting (1.3), by (2.2), we know that $p^{TS^1} S$ does not contribute to the characteristic forms of $\mathcal{R}(TM)$.

### b) Adiabatic limit of $\eta$-invariants of Dirac operators on circle bundles : the case where the base $B$ is orientable.

For any $\varepsilon > 0$, we will use the same notation $g^{TM,\varepsilon}$, $\nabla^{L,\varepsilon}$ for the canonical metric and connection on a bundle $\mathcal{R}(TM)$ induced from those of $TM$. We will denote by $\mathcal{R}^\varepsilon(TM)$the bundle $\mathcal{R}(TM)$ equipped with the metric (connection) $g^{TM,\varepsilon}(\nabla^{L,\varepsilon})$.

Let $F^{TB}$ be the bundle of spinors of $(TB, g^{TB})$. Let $\xi$ be a complex vector bundle on $B$ with a Hermitian metric $g^\xi$ and a connection $\nabla^\xi$ preserving $g^\xi$. Then $\Gamma(F^{TB} \otimes \xi)$ is canonically a Clifford module over $c(TB)$ equipped with canonically induced metric and connection.

DEFINITION 2.3. — The Dirac $D_{B,\xi}$ is the differential operator on $\Gamma(F^{TB} \otimes \xi)$ defined as follows

$$D_{B,\xi}(u \otimes v) = \sum_1^{\dim B} \left( c(b_i)\nabla^L_{b_i} u \otimes v + c(b_i) u \otimes \nabla^\xi_{b_i} v \right), \tag{2.3}$$
$$u \in \Gamma(F^{TB}), v \in \Gamma(\xi).$$

Clearly, $D_{B,\xi}$ is a self-adjoint first order elliptic operator on $\Gamma(F^{TB} \otimes \xi)$.

For any $\varepsilon > 0$, let $F_\varepsilon^{TM}$ (resp. $F_\varepsilon^{TB}$) be the bundle of spinors of $(TM, g^{TM,\varepsilon})$ (resp. $\left(TB, \frac{1}{\varepsilon} g^{TB}\right)$). Then $F_\varepsilon^{TM} = F^{TS^1} \otimes F_\varepsilon^{TB}$.

DEFINITION 2.4. — For any $\varepsilon > 0$, the Dirac operator $D_{M,\mathcal{R}}^\varepsilon$ is the differential operator acting on $\Gamma(F_\varepsilon^{TM} \otimes \mathcal{R}^\varepsilon(TM))$,

$$
\begin{aligned}
(2.4)\quad D_{M,\mathcal{R}}^\varepsilon(u \otimes v) &= c(e)\nabla_e^{L,\varepsilon} u \otimes v + c(e) u \otimes \nabla_e^{L,\varepsilon} v \\
&\quad + \sqrt{\varepsilon} \sum_1^{\dim B} \left( c(b_i) \nabla_{b_i}^{L,\varepsilon} u \otimes v + c(b_i) u \otimes \nabla_{b_i}^{L,\varepsilon} v \right), \\
&\quad u \in \Gamma(F_\varepsilon^{TM}),\ v \in \Gamma(\mathcal{R}^\varepsilon(TM)).
\end{aligned}
$$

Then $D_{M,\mathcal{R}}^\varepsilon$ is a self-adjoint elliptic first order differential operator on $\Gamma(F_\varepsilon^{TM} \otimes \mathcal{R}^\varepsilon(TM))$.

If $D$ is a Dirac operator, denote by $\overline{\eta}(D)$ the reduced $\eta$-invariant of $D$ in the sense of Atiyah, Patodi and Singer [APS].

Let $e$ be the Euler class of $N$ over $B$.

THEOREM 2.5. — *The following identity holds,*

$$
\begin{aligned}
(2.5)\qquad \lim_{\varepsilon \to 0} \overline{\eta}(D_{M,\mathcal{R}}^\varepsilon) &\equiv \frac{1}{2} \dim \ker(D_{B,\mathcal{R}(TB \oplus \mathbb{R})}) \\
&\quad + \left\langle \frac{\widehat{A}(TB) \mathrm{ch}\,(\mathcal{R}(TB \oplus \mathbb{R})) \cdot \tanh(\frac{e}{2}) - \frac{e}{2}}{e \tanh(\frac{e}{2})}, [B] \right\rangle \qquad (\mathrm{mod}\ \mathbb{Z}).
\end{aligned}
$$

*Proof.* — It is clear from [BC1, Proposotion 4.3] that $\lim_{\varepsilon \to 0} \overline{\eta}(D_{M,\mathcal{R}}^\varepsilon)$ exists in $\mathbb{R}/\mathbb{Z}$. Also, by (1.12), $\ker(D_{S^1,\mathcal{R}})$ forms a vector bundle over $B$ which can be identified to $\mathcal{R}(TB \oplus \mathbb{R})$. Proceeding almost identical as in [BC1] and [D], we deduce that

$$
\begin{aligned}
(2.6)\quad \lim_{\varepsilon \to 0} \overline{\eta}(D_{M,\mathcal{R}}^\varepsilon) &\equiv \overline{\eta}(D_{B,\mathcal{R}(TB \oplus \mathbb{R})}) \\
&\quad + \left( \frac{1}{2\pi\sqrt{-1}} \right)^{\frac{\dim B}{2}} \int_B \widehat{A}(\sqrt{-1} R^{TB}) \widehat{\eta} \qquad (\mathrm{mod}\ \mathbb{Z}).
\end{aligned}
$$

A slight difference is that in our case, the connections on the coupled bundles $\mathcal{R}^\varepsilon(TM)$ are allowed to change as $\varepsilon$ varies. However, by using the same arguments as in [BC1] and [D], we obtain (2.6). Compare also with

[BC2, Section 3 f)]. We remark that Lemma 2.1 plays an essential role in such an argument.

Now since $\dim B$ is even, one verifies

$$\overline{\eta}(D_{B,\mathcal{R}(TB\oplus\mathbb{R})}) = \frac{1}{2}\dim\ker(D_{B,\mathcal{R}(TB\oplus\mathbb{R})}). \tag{2.7}$$

(2.5) follows from (2.7), (2.6) and Theorem 1.7. The proof of Theorem 2.5 is completed. □

*Remark* 2.6. — In view of Bismut-Cheeger [BC1, Theorem 4.104], the argument in Dai [D] can be simplified significantly in obtaining (2.6). Alternatively, one can also proceed as in Bismut-Zhang [BZ], where a finite dimensional model is studied. The relationship between the finite and infinite dimensional situations was exploited in Bismut-Cheeger [BC1].

### c) The case where $B$ is non-orientable.

Let now $B$ be non-orientable and carry a fixed pin$^-$-structure (cf. [KT]). We also assume the following equality of Stiefel-Whitney classes,

$$w_1(N) = w_1(TB). \tag{2.8}$$

Then $N$ is still an orientable manifold. We use the same assumptions and notation as before for metrics and connections, etc.

The circle bundle $M$ constructed in (1.2) carries a canonically induced spin-structure not extending to $N_1$ (cf. [KT]).

We still note $D^{\varepsilon}_{M,\mathcal{R}}$ the Dirac operator on $M$ associated to $(F^{TM}_{\varepsilon} \otimes \mathcal{R}^{\varepsilon}(TM), g^{TM,\varepsilon})$.

Let $\widetilde{B}$ be the oriented double covering of $B$. Let $P$ be the covering involution.

Then $N$ (resp. $M$) lifts to a 2-dimensional vector (resp. circle) bundle $\widetilde{N}$ (resp. $\widetilde{M}$) over $\widetilde{B}$.

The pin$^-$ and spin-structures on $B$ and $M$ lift to compatible spin structures on $\widetilde{B}$ and $\widetilde{M}$ respectively.

We lift the metrics, connections, etc to the covering spaces with notation modified with a "$\widetilde{\ }$". We use the same $P$ to denote the liftings of $P$ on sections of bundles.

Let $\overline{\eta}(D^{\varepsilon}_{\widetilde{M},\mathcal{R}}, P)$ be the equivariant reduced $\eta$-invariant of $D^{\varepsilon}_{\widetilde{M},\mathcal{R}(T\widetilde{M})}$ defined by

$$(2.9)\quad \overline{\eta}(D^{\varepsilon}_{\widetilde{M},\mathcal{R}}, P) = \frac{1}{2}\mathrm{Tr}_{\ker(D^{\varepsilon}_{\widetilde{M},\mathcal{R}})}[P] + \frac{1}{\Gamma(1/2)}\int_0^{+\infty} \mathrm{Tr}\big[PD^{\varepsilon}_{\widetilde{M},\mathcal{R}}\exp\big(-t(D^{\varepsilon}_{\widetilde{M},\mathcal{R}})^2\big)\big]\frac{dt}{\sqrt{t}}.$$

The following identity is easily verified,

$$(2.10)\qquad \frac{1}{2}\big(\overline{\eta}(D^{\varepsilon}_{\widetilde{M},\mathcal{R}}) + \overline{\eta}(D^{\varepsilon}_{\widetilde{M},\mathcal{R}}, P)\big) = \overline{\eta}(D^{\varepsilon}_{M,\mathcal{R}}).$$

Similarly, let $\overline{\eta}(D_{\widetilde{B},\mathcal{R}(T\widetilde{B}\oplus\mathbb{R})}, P)$ be the equivariant reduced $\eta$ - invariant defined by

$$(2.11)\quad \overline{\eta}(D_{\widetilde{B},\mathcal{R}(TB\oplus\mathbb{R})}) = \frac{1}{2}\mathrm{Tr}_{\ker(D_{\widetilde{B},\mathcal{R}})}[P] + \frac{1}{\Gamma(1/2)}\int_0^{+\infty} \mathrm{Tr}\big[PD_{\widetilde{B},\mathcal{R}}\exp(-t(D_{\widetilde{B},\mathcal{R}})^2)\big]\frac{dt}{\sqrt{t}}.$$

Let $\widetilde{e}$ be the Euler class of $\widetilde{N}$ over $\widetilde{B}$.

THEOREM 2.7. — *The following identity holds,*

$$(2.12)\ \lim_{\varepsilon\to 0}\overline{\eta}(D^{\varepsilon}_{M,\mathcal{R}}) \equiv \frac{1}{4}\dim\ker(D_{\widetilde{B},\mathcal{R}(T\widetilde{B}\oplus\mathbb{R})}) + \frac{1}{2}\overline{\eta}(D_{\widetilde{B},\mathcal{R}(T\widetilde{B}\oplus\mathbb{R})}, P) + \frac{1}{2}\left\langle \widehat{A}(T\widetilde{B})\mathrm{ch}\left(\mathcal{R}(T\widetilde{B}\oplus\mathbb{R})\right)\frac{\tanh(\frac{\widetilde{e}}{2}) - \frac{\widetilde{e}}{2}}{\widetilde{e}\tanh(\frac{\widetilde{e}}{2})}, [\widetilde{B}]\right\rangle \pmod{\mathbb{Z}}.$$

*Proof.* — Clearly, $P$ is an isometry for any $\varepsilon > 0$. The proof of (2.12) is the same as the proof of Theorem 2.5, with a trivial modification with respect to the action $P$. This argument proceeds smoothly because i), $P$ commutes with $D^{\varepsilon}_{\widetilde{M},\mathcal{R}}$, so that the large time asymptotics appearing in the works of Bismut-Cheeger [BC1] and Dai [D] hold without change; ii), since $P$ has no fixed points, the small time asymptotics of the trace involving $P$ contribute zero; iii), $\frac{1}{2}(1+P)$ is an orthogonal projection operator, so that all the jumps are in $\mathbb{Z}$.

Details are easy to fill and are left to the reader. □

The operator $\frac{1}{2}(1+P)D_{\widetilde{B},\mathcal{R}(T\widetilde{B}\oplus\mathbb{R})}$ determines a first order self-adjoint elliptic operator, called twisted Dirac operator, on $B$ (cf. [G], [S]). We will denote this operator by $\widetilde{D}_{B,\mathcal{R}(TB\oplus\mathbb{R})}$.

Then the following identity holds,

(2.13)
$$\frac{1}{2}\Big(\frac{1}{2}\dim\ker(D_{\widetilde{B},\mathcal{R}(T\widetilde{B}\oplus\mathbb{R})})+\overline{\eta}(D_{\widetilde{B},\mathcal{R}(T\widetilde{B}\oplus\mathbb{R})},P)\Big)=\overline{\eta}(\widetilde{D}_{B,\mathcal{R}(TB\oplus\mathbb{R})}).$$

Also, $\frac{1}{2}(1+P)\widetilde{T}$ determines an element $T\in\Lambda^2(T^*B)\otimes o(TB)$ such that $\frac{T}{2\pi}$ represents the Euler class of $N$.

In these regards, we can restate Theorem 2.7 as

THEOREM 2.8. — *The following identity holds,*

$$\lim_{\varepsilon\to 0}\overline{\eta}(D^{\varepsilon}_{M,\mathcal{R}})\equiv\overline{\eta}(\widetilde{D}_{B,\mathcal{R}(B\oplus\mathbb{R})})+\Big\langle\widehat{A}(TB)\mathrm{ch}\left(\mathcal{R}(TB\oplus\mathbb{R})\right)\frac{\tanh(\frac{e}{2})-\frac{e}{2}}{e\tanh(\frac{e}{2})},[B]\Big\rangle\pmod{\mathbb{Z}}.\tag{2.14}$$

## 3. Rokhlin congruences for $KO$-characteristic numbers.

In this section we apply the results in Section 2 to prove certain congruence formulas involving $KO$-characteristic numbers of $8k+4$ dimensional oriented manifolds. Formulas of this type were originated by Rokhlin [R2].

This section is organized as follows. In a), we state the main theorem of this section as Theorem 3.2. In b), we prove Theorem 3.2.

### a) Assumptions and notation.

We will use the same assumptions and notation as in Sections 1 and 2.

Let $K$ be a compact connected oriented manifold. By a $KO$-characteristic number of $K$, we will mean a number of the type $\langle\widehat{A}(TK)\,\mathrm{ch}(\mathcal{R}(TK)),[K]\rangle$ where $\mathcal{R}$ as before denotes an integral power operation.

DEFINITION 3.1. — A compact connected submanifold $B$ is said to be a characteristic submanifold of $K$ if $\dim B=\dim K-2$ and

$[B] \in H_{\dim K-2}(K, \mathbb{Z}_2)$ is dual to the Stiefel-Whitney class $w_2(K)$. If $B$ is a characteristic submanifold of $K$, we call $(K, B)$ a characteristic pair.

We now make the assumption that $\dim K = 8k+4, k \in \mathbb{N}$.

Let $(K, B)$ be a characteristic pair. Then $K \setminus B$ is a spin manifold. We fix once and for all a spin structure on $K \setminus B$ . Then $B$ carries a canonically induced pin$^-$-structure (cf. [KT, Lemma 6.2]).

Let $N$ be the normal bundle to $B$ in $K$. Since $K$ is orientable,

$$w_1(N) = w_1(TB). \tag{3.1}$$

Thus we can and we will apply the analysis in Sections 1 and 2 to this pair $(N, B)$. This will always be assumed in what follows.

Recall that $e$ denotes the Euler class of $N$.

The main theorem of this section can be stated as follows.

THEOREM 3.2. — *The following identity holds,*

$$\begin{aligned}(3.2)\quad & \langle \widehat{A}(TK)\mathrm{ch}(\mathcal{R}(TK)), [K]\rangle \equiv \overline{\eta}(\widetilde{D}_{B,\mathcal{R}(TB\oplus\mathbb{R}^2)}) \\ & + \left\langle \widehat{A}(TB)\frac{\mathrm{ch}(\mathcal{R}(TB\oplus N)) - \cosh(\frac{e}{2})\mathrm{ch}(\mathcal{R}(TB\oplus\mathbb{R}^2))}{2\sinh(\frac{e}{2})}, [B]\right\rangle \pmod{2\mathbb{Z}}.\end{aligned}$$

*In particular, if $B$ is orientable, then the reduced $\eta$-invariant in (3.2) can be replaced by $\dfrac{1}{2}\dim\ker(D_{B,\mathcal{R}(TB\oplus\mathbb{R}^2)})$.*

## b) Proof of Theorem 3.2.

Let $N_1$ be the disk bundle defined in (1.7) with fibres $D$ over $B$. Then there are a metric $g^{TD}$ on $TD$ and a series of metrics $g^{TK,\varepsilon}$ $(\varepsilon > 0)$ on $TK$ such that 1), $g^{TK,\varepsilon}$ is product near $M = \partial N_1$; 2), $g^{TK,\varepsilon}|_M = g^{TM,\varepsilon}$ and 3), $g^{TK,\varepsilon}|_{N_1} = g^{TD} \oplus \dfrac{1}{\varepsilon}\pi^* g^{TB}$. Note $R^{K,\varepsilon}$ the curvature of the Levi-Civita connection associated to $g^{TK,\varepsilon}$.

LEMMA 3.3. — *For any $\varepsilon > 0$, the following identity holds,*

$$\begin{aligned}(3.3)\quad & \langle \widehat{A}(TK)\mathrm{ch}(\mathcal{R}(TK)), [K]\rangle \equiv \overline{\eta}(D^{\varepsilon}_{M,\mathcal{R}(TM\oplus\mathbb{R})}) \\ & + \left(\frac{1}{2\pi}\right)^{\frac{\dim K}{2}} \int_{N_1} \widehat{A}(R^{K,\varepsilon})\mathrm{ch}(\mathcal{R}(R^{K,\varepsilon})) \pmod{2\mathbb{Z}}.\end{aligned}$$

*Proof.* — By the index theorem for manifolds with boundary of Atiyah, Patodi and Singer [APS], we have

$$(3.4)\quad \operatorname{ind}(D^{\varepsilon}_{K\setminus N_1,\mathcal{R}(TK)}) = \left(\frac{1}{2\pi}\right)^{\frac{\dim K}{2}} \int_{K\setminus N_1} \widehat{A}(R^{K,\varepsilon})\operatorname{ch}(\mathcal{R}(R^{K,\varepsilon})) - \overline{\eta}(D^{\varepsilon}_{M,\mathcal{R}(TM\oplus\mathbb{R})}),$$

where $D^{\varepsilon}_{K\setminus N_1,\mathcal{R}(TK)}$ is a Dirac operator on $K\setminus N_1$ verifying the Atiyah-Patodi-Singer boundary conditions ([APS]).

Now since an $8k+4$ dimensional spinor space carries a quaternionic structure commuting with the canonical involution operator (cf. [ABS]), one verifies easily that

$$(3.5)\qquad \operatorname{ind}(D^{\varepsilon}_{K\setminus N_1,\mathcal{R}(TK)}) \equiv 0 \qquad (\operatorname{mod}\ 2\mathbb{Z}).$$

(3.3) follows from (3.4) and (3.5). □

Lemma 3.4. — *The following identity holds,*

$$(3.6)\quad \lim_{\varepsilon\to 0}\overline{\eta}(D^{\varepsilon}_{M,\mathcal{R}(TM\oplus\mathbb{R})}) \equiv \overline{\eta}(\widetilde{D}_{B,\mathcal{R}(TB\oplus\mathbb{R}^2)}) + \left\langle \widehat{A}(TB)\operatorname{ch}\left(\mathcal{R}(TB\oplus\mathbb{R}^2)\right)\frac{\tanh(\frac{e}{2})-\frac{e}{2}}{e\tanh(\frac{e}{2})},[B]\right\rangle \quad (\operatorname{mod}\ 2\mathbb{Z}).$$

*Proof.* — Since an $8k+3$ dimensional space carries a quaternionic structure, $\overline{\eta}(D^{\varepsilon}_{M,\mathcal{R}(TM\oplus\mathbb{R})})$ is mod $2\mathbb{Z}$ continuous. Thus in our specific situation, Theorem 2.8 and Theorem 2.5 hold mod $2\mathbb{Z}$. (3.6) follows from these mod $2\mathbb{Z}$ versions of Theorems 2.8 and 2.5. □

Lemma 3.5. — *The following identity holds,*

$$(3.7)\quad \lim_{\varepsilon\to 0}\left(\frac{1}{2\pi}\right)^{\frac{\dim N_1}{2}} \int_{N_1}\widehat{A}(R^{K,\varepsilon})\operatorname{ch}(\mathcal{R}(R^{K,\varepsilon})) = \left\langle \widehat{A}(TB)\frac{1}{e}\left\{\operatorname{ch}\left(\mathcal{R}(TB\oplus N)\right)\frac{\frac{e}{2}}{\sinh(\frac{e}{2})} - \operatorname{ch}\left(\mathcal{R}(TB\oplus\mathbb{R}^2)\right)\right\},[B]\right\rangle.$$

*Proof.* — Since this calculation is local, we can and we will assume that $B$ is orientable.

By proceeding as in [BC1], we know that

$$(3.8)\qquad R = \lim_{\varepsilon\to 0} R^{K,\varepsilon}$$

exists. Furthermore, if $R^D, R^B$ are the restrictions of $R$ to $TD$, $\pi^*TB$ respectively, then $R^B = \pi^* R^{TB}$.

We write $R^D$ as $\begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}$. Then by (3.8), we have

$$\lim_{\varepsilon\to 0} \int_{N_1} \widehat{A}(R^{K,\varepsilon})\mathrm{ch}(\mathcal{R}(R^{K,\varepsilon})) = \int_B \widehat{A}(R^{TB}) \int_D \widehat{A}(R^D)\mathrm{ch}\left(\mathcal{R}\left(R^B \oplus \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}\right)\right). \tag{3.9}$$

Now since $g^{TD}$ is product near the boundary, one has $u|_{\partial D} = 0$. Thus, by using the Thom isomorphism theorem, we have the following identity of cohomology classes in $H^\bullet(B)$,

$$\left[\int_D \left(\frac{u}{2\pi}\right)^{2k}\right] = e^{2k-1}\ , \ k = 1, 2, \ldots\ . \tag{3.10}$$

(3.7) follows from (3.9) and (3.10). □

From (3.7), (3.6) and (3.3), (3.2) follows. The proof of Theorem 3.2 is completed. □

## 4. Some application of the congruence theorem.

In this section, we provide some consequences of Theorem 3.2. These include an application to elliptic genera as well as a specialization to the original situation $\dim K = 4$.

This section is organized as follows.

In a), we recall the definition of elliptic genera and the Ochanine genus. In b), we prove a congruence formula involving elliptic genera. In c), we specialize our results to the case $\dim K = 4$.

### a) Elliptic genera and the Ochanine genus.

Let $E$ be a vector bundle over a compact manifold $K$. Let $\Lambda_t(E)$ (resp. $S_t(E)$) be the total exterior (resp. symmetric) power of $E$ defined by

$$\Lambda_t(E) = \Sigma \Lambda^i(E) t^i \tag{4.1}$$

(resp.

$$S_t(E) = \Sigma S^i(E)t^i).$$

Define after Witten [W],

$$\begin{aligned}\Theta_q(E) &= \bigotimes_{n\geq 1} \left(\Lambda_{-q^{2n-1}}(E) \otimes S_{q^{2n}}(E)\right) \\ &= \sum_{n\geq 0} R_n(E)q^n.\end{aligned} \tag{4.2}$$

We will think of each $R_n$ as an integral power operation.

If $F$ is another vector bundle over $K$, then

$$\Theta_q(E \oplus F) = \Theta_q(E)\Theta_q(F). \tag{4.3}$$

We use the notation

$$\widetilde{E} = E - \mathbb{R}^{\dim E}. \tag{4.4}$$

Then $\Theta_q(\widetilde{E})$ is also well-defined. Recall that we use the same notation for a real vector bundle and its complexification.

We will always equip $\mathbb{R}^k$ with the trivial metric and connection.

DEFINITION 4.1 (cf. [L], [O2]). — The Landweber-Stong-Witten class of $E$ is given by

$$\varphi_q(E) = \widehat{A}(E)\mathrm{ch}(\Theta_q(\widetilde{E})). \tag{4.5}$$

If $E$ carries a connection $\nabla$ with curvature $R$, we will use the notation $\varphi_q(R)$ to note the corresponding Chern-Weil representative of $\varphi_q(E)$. Similar notation will also be used for other characteristic forms.

Let now $B$ be an $8k+2$ dimensional spin manifold with a fixed spin structure. Let $g^{TB}$ be a metric on $TB$. Let $\nabla^{TB}$ be the Levi-Civita connection associated to $g^{TB}$. Then $\Theta_q(TB)$ carries canonically induced metric and connection.

Let $D_{B,\Theta_q(\widetilde{TB})}$ be the Dirac operator on $B$ twisted with $\Theta_q(\widetilde{TB})$.

PROPOSITION 4.2. — *The quantity* $\frac{1}{2}\dim\ker(D_{B,\Theta_q(\widetilde{TB})}) \bmod 2\mathbb{Z}[[q]]$ *is a spin cobordism invariant.*

*Proof.* — Proposition 4.2 is an easy consequence of the Atiyah-Patodi-Singer index theorem for manifolds with boundary [APS] and the fact that an $8k+3$ dimensional spinor space carries a quaternionic structure (cf. [ABS]). □

DEFINITION 4.3. — The Ochanine genus $\beta_q(B)$ of a spin manifold $B$ is $\frac{1}{2}\dim\ker(D_{B,\Theta_q(\widetilde{TB})}) \bmod 2\mathbb{Z}[[q]]$.

*Remark* 4.4. — Since $\widetilde{KO}_{8k+2}(pt) = \mathbb{Z}_2$, the above definition is in fact equivalent to the original definition of Ochanine [O2].

## b) A congruence formula for elliptic genera.

Let $K$ be an $8k+4$ dimensional oriented manifold. Let $B$ be a characteristic submanifold of $K$. We assume $B$ is orientable, just to simplify the presentation. We fix a spin structure on $K \setminus B$. Then $B$ carries a canonically induced spin structure (cf. [KT, Lemma 6.2]).

Denote by $e$ the Euler class of the normal bundle $N$ to $B$ in $K$.

THEOREM 4.5. — *The following identity holds,*

$$\begin{aligned}(4.6)\quad &\langle \varphi_q(TK), [K]\rangle \\ &\equiv \beta_q(B) + \left\langle \varphi_q(TB)\frac{\tanh(\frac{e}{2})\varphi_q(e) - \frac{e}{2}}{e\tanh(\frac{e}{2})}, [B]\right\rangle \qquad (\mathrm{mod}\ 2\mathbb{Z}[[q]]).\end{aligned}$$

*Proof.* — From (4.3) we know that

$$\mathrm{ch}(\varphi_q(\widetilde{TB \oplus N})) = \mathrm{ch}(\varphi_q(\widetilde{TB}))\mathrm{ch}(\varphi_q(\widetilde{N})). \tag{4.7}$$

(4.6) follows from Theorem 3.2, (4.7) and the definitions of the elliptic genera $\varphi_q$ and $\beta_q(B)$. □

*Remark* 4.6. — If $B$ is non-orientable, then (4.6) still holds if we define

$$\beta_q(B) \equiv \overline{\eta}(\widetilde{D}_{B,\Theta_q(\widetilde{TB})}) \qquad (\mathrm{mod}\ 2\mathbb{Z}[[q]]). \tag{4.8}$$

In this case, $\beta_q(B)$ is a pin$^-$-cobordism invariant of $B$.

Let's still assume $B$ is orientable.

COROLLARY 4.7. — *The following identity holds,*

(4.9)
$$\left\langle \widehat{A}(TK),[K]\right\rangle \equiv \frac{1}{2}\dim\ker(D_B)-\left\langle \widehat{A}(TB)\cdot\frac{1}{2}\tanh\left(\frac{e}{4}\right),[B]\right\rangle \pmod{2\mathbb{Z}}.$$

*Proof.* Proof (4.9) follows from (4.6) by setting $q=0$. □

*Remark* 4.8. — If $K$ is spin, then $B=\varnothing$, (4.9) reduces to a result of Atiyah and Hirzebruch [AH]. Also a special case of Corollary 4.7 for complex manifolds has been proved by Esnault, Seade and Viehweg [ESV].

### c) The case $\dim K = 4$.

We now make the special assumption $\dim K=4$. By setting $q=0$ in (4.6) and (4.8), we get

THEOREM 4.9. — *Let $B\cdot B$ be the self-intersection of $B$ in $K$, then*

$$\frac{\operatorname{sign}(B\cdot B)-\operatorname{sign}(K)}{8}\equiv\overline{\eta}(\widetilde{D}_B) \pmod{2\mathbb{Z}}. \tag{4.10}$$

Now let

$$\beta:\Omega_2^{\mathrm{pin}^-}\to\mathbb{Z}/8\mathbb{Z} \tag{4.11}$$

be the Brown invariant ([Br], [KT]) of the two dimensional pin$^-$-cobordism group $\Omega_2^{\mathrm{pin}^-}$.

Comparing (4.10) with the extended Rokhlin congruences of Guillou-Marin [GM] and Kirby-Taylor [KT], we deduce that

$$4\overline{\eta}(\widetilde{D}_B)\equiv\beta(B) \pmod{8\mathbb{Z}}. \tag{4.12}$$

Thus $\overline{\eta}(\widetilde{D}_B)$ gives a natural analytic interpretation of the Brown invariant.

*Remark* 4.10. — In fact, (4.12) can be checked directly, as both $\overline{\eta}(\widetilde{D}_B)$ and $\beta(B)$ are pin$^-$-cobordism invariants. Since $\Omega_2^{\mathrm{pin}^-}$ is generated by one of $\mathbb{R}P^2_{\pm}$ (cf. [KT]), we need only to check it for one of them. That they are indeed equal follows from an argument of Gilkey [G] and

Stolz [S]. As for orientable $B$, the identification of the Atiyah invariant $\frac{1}{2}\dim\ker(D_B)\ (\mathrm{mod}\ 2\mathbb{Z})$ with the Arf invariant [R2] of $H_1(B,\mathbb{Z}/2\mathbb{Z})$, which corresponds to the Brown invariant in this case, was carried out by Johnson [J].

## Appendix. On the Ochanine-Rokhlin congruences.

In this appendix, we try to relate our higher dimensional Rokhlin congruences to those of Ochanine [O1].

The first observation is that while our theorem extends the Atiyah-Hirzebruch's result on the divisibility of $\widehat{A}$-genus, the congruence of Ochanine extends his following theorem on the divisibility of the signature.

THEOREM A1 (Ochanine [O1]). — *The signature of an $8k+4$ dimensional compact spin manifold is divisible by 16.*

To see the relevance, we prove a congruence formula for the signature in dimension 12, which is equivalent to that of Ochanine in this dimension.

Let $(K,B)$ be a characteristic pair such that $\dim K = 12$. For simplicity we assume $B$ is orientable.

THEOREM A.2. — *The following identity holds,*

$$\text{(A.1)}\qquad \frac{\mathrm{sign}(B\cdot B)-\mathrm{sign}(K)}{8} \equiv \frac{\dim\ker(D_{B,TB})}{2} \qquad (\mathrm{mod}\ 2\mathbb{Z}).$$

*Proof.* — Let $L(x)$ be the Hirzebruch function

$$\text{(A.2)}\qquad L(x) = \frac{x/2}{\tanh x/2}.$$

The corresponding characteristic forms will be understood in standard way.

By the so called "miraculous cancellation" of Alvarez-Gaumé and Witten [AGW], we have

$$\text{(A.3)}\qquad \frac{1}{8}(2^6L(R^{K,\varepsilon}))_{(12)} = (\widehat{A}(R^{K,\varepsilon})\mathrm{ch}(R^{K,\varepsilon}))_{(12)} - 4(\widehat{A}(R^{K,\varepsilon}))_{(12)}$$

where we use the notation in Section 3 b).

From (A.3), the Atiyah-Patodi-Singer index theorem for manifolds with boundary ([APS]) and the fact that the 12 dimensional spinor space carries a quaternionic structure, we deduce that

$$\text{(A.4)} \qquad \frac{\operatorname{sign}(K)}{8} = \overline{\eta}(D^{\varepsilon}_{M,TM}) - 3\overline{\eta}(D^{\varepsilon}_{M}) + 2^3 \int_{N_1} L(R^{K,\varepsilon}) \pmod{2\mathbb{Z}}.$$

(A.1) then follows from (A.4) by passing to the adiabatic limit $\varepsilon \to 0$. Details are left to the reader. □

It is clear that the "miraculous cancellation" of Alvarez-Gaumé and Witten plays an essential role in above proof of Theorem A.2.

Now we make the observation that this cancellation can be written as

$$\text{(A.5)} \qquad (2^6 L)_{(12)} = (-8)^3(\widehat{A})_{(12)} + (-8)(-1)(72\widehat{A} + \widehat{A}\operatorname{ch}(\widetilde{TK}))_{(12)}.$$

This is obviously related to Landweber's proof of Theorem A.1 ([L]).

Thus the "miraculous cancellation" (A.5) should be a special case of a general identity in elliptic genera expressing the Hirzebruch $L$-forms by $KO$-characteristics forms.

While the existence of such an identity is clear, the problem is to what extent these involved $KO$-characteristic forms can be expressed explicitly. This would lead to a better understanding of the congruences of Ochanine, as well as their extension by Finashin [F].

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