ANNALES DE L'INSTITUT FOURIER

DONALD I. CARTWRIGHT
WOJCIECH MŁOTKOWSKI
TIM STEGER

Property (T) and $\tilde{A}_2$ groups

Annales de l'institut Fourier, tome 44, nº 1 (1994), p. 213-248
<http://www.numdam.org/item?id=AIF_1994__44_1_213_0>

Ann. Inst. Fourier, Grenoble
44, 1 (1993), 213–248

PROPERTY (T) AND $\widetilde{A}_2$ GROUPS

by D.I. CARTWRIGHT, W. MŁOTKOWSKI (*)
and T. STEGER

1. Introduction and notation.

In two recent papers (see [2], [3]), an infinite family of finitely generated groups Γ was introduced. These groups act simply-transitively on the vertices of certain thick $\widetilde{A}_2$ Tits buildings, and we shall call them $\widetilde{A}_2$ *groups* here. In this paper, for (most) $\widetilde{A}_2$ groups Γ, we :

(i) show that Γ has Kazhdan's property (T), and

(ii) calculate the exact Kazhdan constant $\kappa(\Gamma, S)$ of Γ with respect to its natural set S of generators.

Let us give a definition of property (T) which is convenient for our purposes (see [8] and [9], Chap. 1, Prop. 15). Let Γ be a finitely generated discrete group, and let $S \subset \Gamma$ be a finite generating set. Let π be a unitary representation of Γ with no fixed vector, i.e., with no nonzero vector v in the representation space for which $\pi(g)v = v$ holds for each $g \in \Gamma$. Let $\kappa(\pi, S)$ denote the largest number $\kappa \geq 0$ such that $\max_{s\in S} \|\pi(s)v - v\| \geq \kappa\|v\|$ holds for each vector v in the representation space. Let $\kappa(\Gamma, S)$ denote the infimum of the numbers $\kappa(\pi, S)$ over all unitary representations π of Γ with no fixed vector. We say that Γ has *Kazhdan's property* (T) if the *Kazhdan constant* $\kappa(\Gamma, S)$ is strictly positive. This does not depend on the particular finite generating set S, by Lemma 4 in [8].

Some $\widetilde{A}_2$ groups Γ can be embedded as co-compact lattices in $\mathrm{PGL}(3, F)$ for a suitable non-archimedean local field F. For these groups,

(*) Research carried out while an ARC Research Associate at the University of New South Wales.

Key words : Triangle buildings – Positive definite functions – Kazhdan's property (T).
A.M.S. Classification : 43A35 – 43A90 – 43A65 – 51E24 – 22E50.

property (T) is well known ([9], Chap. 2, Thm 8 and Chap. 3, Thm 4). In [3], § 8, it was seen that many $\widetilde{A}_2$ groups do not embed (in a natural way) as co-compact lattices in any $\mathrm{PGL}(3, F)$. Even when Γ can be embedded in $\mathrm{PGL}(3, F)$, our proof that it has property (T) does not use this fact. These groups therefore provide an answer to Question 2 on page 133 of [9], which sought groups which could be shown to have property (T) without making essential use of the theory of representations of linear groups. These groups are also the first infinite groups for which an exact calculation of $\kappa(\Gamma, S)$ has been possible, and therefore provide an answer to Question 1 on page 133 in [9]. Estimates for some Kazhdan constants have been found by M. Burger [1] for $\mathrm{SL}(3, \mathbb{Z})$. See also [10], p. 230 for estimates of corresponding constants for $\mathrm{SL}(n, \mathbb{R})$.

Let us briefly describe $\widetilde{A}_2$ groups. Suppose that we have a finite projective plane Π, consisting of a set P of points, a set L of lines, and an incidence relation between points and lines. For some integer $q \geq 2$, each point (resp. line) is incident with exactly $q + 1$ lines (resp. points), and q is called the *order* of Π ([11], Thm 3.5) (q is a prime power in all known examples). Also, $|P| = |L| = q^2 + q + 1$. The *Desarguesian plane* $\mathrm{PG}(2, q)$ of order q is formed from a 3-dimensional vector space V over the field $\mathbf{F}_q$ of order q, letting P and L be the sets of 1- and 2-dimensional subspaces of V, respectively, with incidence being inclusion. Given a (not necessarily Desarguesian) plane $\Pi = (P, L)$, let $\lambda : P \to L$ be a bijection, and suppose that we have a set $\mathcal{T}$ (called a *triangle presentation* compatible with λ) of triples (x, y, z), where $x, y, z \in P$, such that :

(A) given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for some $z \in P$ if and only if y and $\lambda(x)$ are incident;

(B) $(x, y, z) \in \mathcal{T}$ implies that $(y, z, x) \in \mathcal{T}$;

(C) given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for at most one $z \in P$.

For any prime power q, triangle presentations are exhibited in [2], § 4, and all possible triangle presentations (up to a natural equivalence) are listed in Appendix B of [3], for the cases $q = 2$ and $q = 3$. An $\widetilde{A}_2$ *group* is a group

$$\Gamma = \big\langle \{a_x\}_{x \in P} \mid a_x a_y a_z = e \quad \text{for all } (x, y, z) \in \mathcal{T} \big\rangle$$

associated with some triangle presentation $\mathcal{T}$. Let S denote the set of generators a_x, $x \in P$, and their inverses, and let $\mu^+ = |P|^{-1} \sum_{x \in P} a_x$ (an element of the group algebra $\mathbb{C}(\Gamma)$ of Γ). It is clear that there is a character $\chi : \Gamma \to \mathbf{T}$ such that $\chi(a_x) = \mathrm{e}^{2\pi i/3}$ for each $x \in P$. Hence $\chi(\mu^+) = \mathrm{e}^{2\pi i/3}$

is in the spectrum $\mathrm{Sp}(\mu^+)$ of μ^+ in the C^*-algebra $C^*(\Gamma)$ of Γ ([8], Prop. 1), as are 1 and $\mathrm{e}^{-2\pi i/3}$. Our main result is the following :

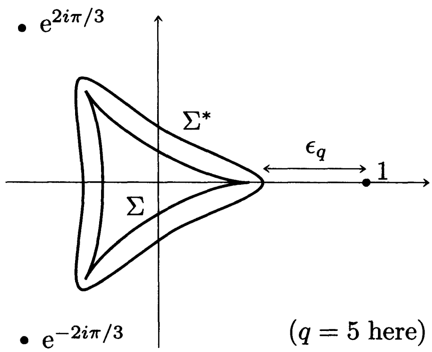

Figure 1

THEOREM. — *Assume that $\Pi = (P, L)$ is isomorphic to the Desarguesian plane of order q. Then the spectrum $\mathrm{Sp}(\mu^+)$ of μ^+ in the C^*-algebra $C^*(\Gamma)$ of Γ is the subset Σ^* of $\mathbb{C}$ consisting of* 1, $\mathrm{e}^{2\pi i/3}$, $\mathrm{e}^{-2\pi i/3}$ *and the region bounded by the curve (see Fig.* 1)

$$(1.1)\qquad \gamma(\theta) = \frac{q}{q^2+q+1}\Big(\big(\sqrt{q}+\sqrt{q^{-1}}\big)\,\mathrm{e}^{i\theta} + e^{-2i\theta}\Big), \qquad 0 \le \theta \le 2\pi.$$

Let $\epsilon_q = 1 - \gamma(0)$. Then Γ has property (T), *and $\kappa(\Gamma, S) = \sqrt{2\epsilon_q}$.*

Remarks

1) The spectrum $\Sigma = \mathrm{Sp}(\lambda(\mu^+))$ of μ^+ in the reduced C^* algebra $C^*_\lambda(\Gamma)$ of Γ was calculated in [4] and [14]. It is the region bounded by the hypocycloid $q(q^2+q+1)^{-1}(2\,\mathrm{e}^{i\theta} + \mathrm{e}^{-2i\theta})$, $0 \le \theta \le 2\pi$, (see Fig. 1).

2) The region Σ^* is the set of $z \in \mathbb{C}$ for which

$$(1.2)\qquad (q+1)^2(z^3+\bar{z}^3) - (q^2+q+1)|z|^4 - (q^2+4q+1)|z|^2 + q \ge 0.$$

This is explained after Corollary 3.4 below. The expression on the left is the denominator in the density of the Plancherel measure found in [4].

3) Notice that $\epsilon_q \to 1$ as $q \to \infty$.

4) We do not know whether the restriction that Π be Desarguesian is necessary. We have no examples of triangle presentations for non-Desarguesian Π's.

While much of the paper is phrased in the language of affine buildings, one can show that Γ has property (T), and obtain the correct lower bound for $\kappa(\Gamma, S)$, using only combinatorial group theory. Let us briefly indicate how this done. If $g \in \Gamma$, there are integers $m(g), n(g) \geq 0$ such that for any word

$$a_{x_1}^{\epsilon_1} \cdots a_{x_k}^{\epsilon_k} \ (x_1, \ldots, x_k \in P, \epsilon_1, \ldots, \epsilon_k = 1 \text{ or } -1)$$

equal to g, with k minimal, then $m(g)$ of the ϵ_i's are $+1$ and $n(g)$ of the ϵ_i's are -1 ([2], Prop. 3.2 and [4], Lemma 6.2). Let $S_{m,n}$ denote the set of $g \in G$ for which $m(g) = m$ and $n(g) = n$; in the notation below, $S_{m,n} = S_{m,n}(e)$, where e is the identity in Γ, thought of as a vertex of the building associated with Γ ([2], Thm 3.4). A function $f : \Gamma \to \mathbb{C}$ is called *biradial* if it is constant on each set $S_{m,n}$. Let $\mu_{m,n}$ denote the function which takes the value $|S_{m,n}|^{-1}$ on $S_{m,n}$ and 0 elsewhere on Γ. Then the finitely supported biradial functions form a commutative convolution algebra $\mathcal{A}$ generated by $\mu^+ = \mu_{1,0}$ and $\mu^- = \mu_{0,1}$, and spanned by the $\mu_{m,n}$. This was proved in [4], Prop. 2.3, (and in [14], Prop. 3.5) using building terminology, but can quite easily be proved using [2], Prop. 3.2 and [4], Lemma 6.2. For each $z, w \in \mathbb{C}$, there is a multiplicative functional $h_{z,w} : \mathcal{A} \to \mathbb{C}$ such that $h_{z,w}(\mu^+) = w$ and $h_{z,w}(\mu^-) = z$. It is given by

$$h_{z,w}(f) = \sum_{x \in \Gamma} f(x) \varphi_{z,w}(x)$$

for a unique biradial function $\varphi_{z,w}$ on Γ ([4], Prop. 3.4, or [14], Prop. 4.5), whose value on $S_{m,n}$ is a certain polynomial $p_{m,n}(z, w)$ in z and w. Now (a special case of) Proposition 4.1 below shows that if $z \in \mathbb{C}$ and if the $2|P| \times 2|P|$ matrix $(\varphi_{z,\bar{z}}(y_i^{-1} y_j))$, where $\{y_1, \ldots, y_{2|P|}\} = \{a_x\}_{x \in P} \cup \{a_x^{-1}\}_{x \in P}$, is positive definite, then $z \in \Sigma^*$. Then using Proposition 4.3 and Lemma 4.4 below, the proofs of Corollary 4.5 and Theorem 4.6 show that $\mathrm{Sp}(\mu^+) \subset \Sigma^*$, and that Γ has property (T), with $\kappa(\Gamma, S) \geq \sqrt{2\epsilon_q}$. A reader wanting only this can avoid almost all of sections 2 and 3.

To obtain the exact value of $\kappa(\Gamma, S)$, we need to define some representations, and it is most natural to do this for the group $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ of «type-rotating» automorphisms of an arbitrary triangle building Δ (i.e., thick building of type $\widetilde{A}_2$). In section 2, we describe a «boundary» Ω for any triangle building Δ. This has been studied in [14], and our results here are generalizations of those in [14]. We use this in section 3 to define the principal and complementary series spherical representations of $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$.

These give us positive definite spherical functions $\varphi_{z,\bar{z}}$ on $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ for each $z \in \Sigma^*$. In fact, in Theorem 3.5 and Proposition 4.1, we show that a certain kernel $k_{z,\bar{z}}(x,y)$ is positive definite on $\mathcal{V}_\Delta$ if and only if $z \in \Sigma^*$. A vertex o having been fixed, $\varphi_{z,\bar{z}}$ and $k_{z,\bar{z}}$ are related by $\varphi_{z,\bar{z}}(g) = k_{z,\bar{z}}(o, go)$. Positive definiteness of the kernel $k_{z,\bar{z}}$ implies positive definiteness of the function $\varphi_{z,\bar{z}}$, of course, these ideas being equivalent if $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ acts transitively on $\mathcal{V}_\Delta$. Finally, in section 4 we prove the above theorem.

Remark (due to the referee). — Thick affine buildings of dimension 2 are the only ones for which one can hope to find an «exotic» (i.e., nonlinear) group with property (T) acting properly with compact quotient. This is clear for affine buildings of dimension 1, which are semihomogeneous trees (and a Kazhdan group acting on a tree has to fix a vertex, see [9, Chap. 6a, Prop. 4]). Now let Γ be a group acting properly and cocompactly on some thick affine building Δ with dimension at least 3. Then : Γ *is a lattice in* $\mathrm{Aut}\,\Delta$, *and* $\mathrm{Aut}\,\Delta$ *is a compact extension of an adjoint simple algebraic group* $G(k)$ *over a non-archimedean local field* k *of* k*-rank* ≥ 2*; in particular,* $\mathrm{Aut}\,\Delta$ *and* Γ *have property* (T).

Indeed, it follows from the main result in [20] that Δ is associated with an adjoint simple algebraic group $G(k)$ over a non-archimedean local field k of k-rank ≥ 2. Now let Δ^∞ be the building at infinity of Δ. Then $\mathrm{Aut}\,\Delta$ embeds in $\mathrm{Aut}\,\Delta^\infty$ (they might actually be equal), and the structure of $\mathrm{Aut}\,\Delta^\infty$ is given in Cor. 5.9 of [19] : it is given as an extension of $(\mathrm{Aut}\,G)(k)$ (itself a finite extension of $G(k)$) by the group $\mathrm{Aut}_G k$ of all automorphisms α of k such that ${}^\alpha G$ is k-isomorphic with G; as k is a nonarchimedean local field, $\mathrm{Aut}_G k$ is a closed subgroup of the compact group $\mathrm{Aut}\,k$.

2. The boundary Ω of a triangle building.

Let Δ be a triangle building. The set of vertices of Δ is denoted $\mathcal{V}_\Delta$. Let $\tau(x) \in \mathbb{Z}/3\mathbb{Z}$ denote the *type* of a vertex x of Δ. Suppose a chamber c in Δ has vertices x, x' and x''. If A is an apartment in Δ containing c, then (see [17], p. 112) the *sector* in A with *(base) vertex* x and *base chamber* c is that part S of A containing c and bounded by the wall w' in A through x and x' and the wall w'' in A through x and x'' (see Fig. 2). If $\tau(x') = \tau(x) + 1 \bmod 3$ and $\tau(x'') = \tau(x) - 1 \bmod 3$, we shall call w'' and w' the *left* and *right* walls of S, respectively.

A subcomplex S of Δ is called a *sector in* Δ if it is a sector in some apartment of Δ. Two sectors S and S' in Δ are called *parallel* if their intersection contains a sector ([17], p. 121). Parallelism is an equivalence relation, and we denote by Ω the set of equivalence classes. Thus Ω is the set of chambers of the *building at infinity* Δ^∞ associated to Δ ([17], p. 122). For any $\omega \in \Omega$ and $x \in \mathcal{V}_\Delta$, there is a unique sector $S^x = S^x(\omega)$ in the class ω having base vertex x ([17], Lemma 9.7), and so we can think of Ω as the set of sectors in Δ with base vertex x.

If $x, y \in \mathcal{V}_\Delta$, then y belongs to some sector with base vertex x and we write $y \in S_{m,n}(x)$ if y is at distance m and n, respectively, from the left and right walls of this sector.

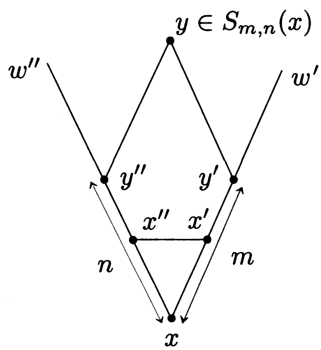

Figure 2

The parallelogram with vertices x, y, y' and y'' (see Fig. 2) is called the *convex hull* of x and y. This depends only on x and y. When $m = 0$ or $n = 0$, the convex hull reduces to a segment, which we sometimes call the *geodesic* between x and y.

The cardinalities $N_{m,n} = |S_{m,n}(x)|$ are independent of x, and given by

$$\begin{aligned} N_{m,n} &= (q^2+q+1)(q^2+q)q^{2(m+n-2)} && \text{if } m,n \geq 1, \\ N_{m,0} &= N_{0,m} = (q^2+q+1)q^{2(m-1)} && \text{if } m \geq 1, \\ N_{0,0} &= 1. \end{aligned}$$

(See [4], Cor. 2.2.) We write $d(x,y) = m+n$ if $y \in S_{m,n}(x)$. A function $f : \mathcal{V}_\Delta \to \mathbb{C}$ is called x-*biradial* if it is constant on each set $S_{m,n}(x)$.

In [4] (see also [14]), an algebra $\mathcal{A}$ of «averaging operators» was studied. It is spanned by the operators $A_{m,n}$, $m, n \in \mathbb{N}$, where

$(A_{m,n}f)(x) = N_{m,n}^{-1}\sum_{y\in S_{m,n}(x)} f(y)$ for functions $f : \mathcal{V}_\Delta \to \mathbb{C}$, is commutative, and is generated by $A^+ = A_{1,0}$ and $A^- = A_{0,1}$. For each pair $(z,w) \in \mathbb{C}^2$, there is a unique multiplicative functional $h_{z,w}$ on $\mathcal{A}$ such that $h_{z,w}(A^+) = z$ and $h_{z,w}(A^-) = w$. These multiplicative functionals can be conveniently indexed by triples $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ such that $s_1s_2s_3 = 1$, related to (z,w) by $z = q(q^2+q+1)^{-1}(s_1+s_2+s_3)$ and $w = q(q^2+q+1)^{-1}(s_1^{-1}+s_2^{-1}+s_3^{-1})$, and we write h_s in place of $h_{z,w}$. Let $p_{m,n}(z,w) = h_{z,w}(A_{m,n})$. This is a polynomial in z and w, calculated explicitly in terms of $s = (s_1,s_2,s_3)$ in [4]. Let $k_{z,w}(x,y)$ or $k_s(x,y)$ be defined to be $p_{m,n}(z,w)$ if $y \in S_{m,n}(x)$. For fixed x, $F : y \mapsto k_{z,w}(x,y)$ is the unique x-biradial function on $\mathcal{V}_\Delta$ satisfying $F(x) = 1$, $A^+F = zF$ and $A^-F = wF$ ([4], Prop. 3.4).

If $\omega \in \Omega$, let $s^x_{m,n}$ or $s^x_{m,n}(\omega)$ denote the unique vertex in $S^x \cap S_{m,n}(x)$ (see Fig. 2).

LEMMA 2.1. — *Let $\omega \in \Omega$, and let $x, y \in \mathcal{V}_\Delta$. Write $s^x_{m,n}$ and $s^y_{m,n}$ for $s^x_{m,n}(\omega)$ and $s^y_{m,n}(\omega)$, respectively. Then there are integers $m(x,y;\omega)$ and $n(x,y;\omega)$ such that*

$$s^x_{i,j} = s^y_{i+m(x,y;\omega),j+n(x,y;\omega)} \quad \text{for } i,j \geq 0 \text{ sufficiently large.} \tag{2.1}$$

Proof. — Now $S^x \cap S^y$ contains a sector. Choose

$$u = s^x_{a,b} = s^y_{c,d} \in S^x \cap S^y.$$

Then $s^x_{a+m,b+n} = s^y_{c+m,d+n}$ for all $m,n \geq 0$. For $T = \{s^x_{a+m,b+n} : m,n \geq 0\}$ and $T' = \{s^y_{c+m,d+n} : m,n \geq 0\}$ are sectors with the same base vertex u and having a sector in common. Thus $T = T'$ by [17], Lemma 9.7, and so $s^x_{a+m,b+n}$ and $s^y_{c+m,d+n}$ are both in $T \cap S_{m,n}(u)$, and thus are equal. It follows that the pair $(c-a, d-b) \in \mathbb{Z}^2$ does not depend on the point chosen in $S^x \cap S^y$, and that (2.1) holds.

Remark. — Note that $m(x,y;\omega)$ and $n(x,y;\omega)$ are two-dimensional analogues of quantities studied for trees by many authors (see [16] for a recent example).

We can easily calculate $m(x,y;\omega)$ and $n(x,y;\omega)$ when $d(x,y) = 1$:

LEMMA 2.2. — *For $\omega \in \Omega$ and $x \in \mathcal{V}_\Delta$, we have (see Fig. 3 next page) :*

$$\big(m(x,y;\omega), n(x,y;\omega)\big) = \begin{cases} (-1,0) & \textit{for one } y \textit{ in } S_{1,0}(x), \\ (1,-1) & \textit{for } q\ y\textit{'s in } S_{1,0}(x), \\ (0,1) & \textit{for } q^2\ y\textit{'s in } S_{1,0}(x), \end{cases}$$

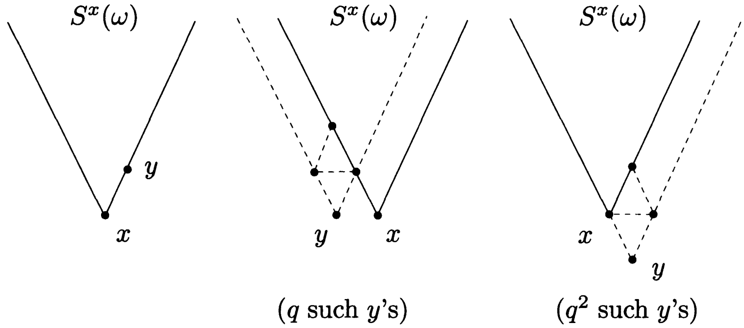

Figure 3

Also,
$$\big(m(x,y;\omega), n(x,y;\omega)\big) = \begin{cases} (0,-1) & \text{for one } y \text{ in } S_{0,1}(x), \\ (-1,1) & \text{for } q \ y\text{'s in } S_{0,1}(x), \\ (1,0) & \text{for } q^2 \ y\text{'s in } S_{0,1}(x). \end{cases}$$

Proof. — This is clear from Fig. 3 (see also the proof of Lemma 2.1 in [4]).

COROLLARY 2.3. — *Let $x, y \in \mathcal{V}_\Delta$ and let $\omega \in \Omega$. Then*
$$s^x_{i,j}(\omega) \in S^x(\omega) \cap S^y(\omega)$$
for $i, j \geq d(x,y)$.

Proof. — Let $y \in S_{m,n}(x)$, where $m \geq 1$, say. Let z be the unique point in $S_{m-1,n}(x) \cap S_{0,1}(y)$ ([4], Lemma 2.1). By induction, we have $s^x_{m+n-1,m+n-1} \in S^z$. Thus $s^x_{m+n,m+n}$ is an interior point of S^z. It follows from the proof of Lemma 2.2 that $s^x_{m+n,m+n} \in S^y$ in all cases.

It is clear that for any $\omega \in \Omega$ and any $x, y, z \in \mathcal{V}_\Delta$ we have the « cocycle relations »

$$\begin{cases} m(x,y;\omega) + m(y,z;\omega) = m(x,z;\omega), \\ n(x,y;\omega) + n(y,z;\omega) = n(x,z;\omega). \end{cases} \tag{2.2}$$

It follows from Lemma 2.2 that $|m(x,y;\omega)|, |n(x,y;\omega)| \leq d(x,y)$. Note also that

$$\begin{cases} m(x,y;\omega) = -m(y,x;\omega), \\ n(x,y;\omega) = -n(y,x;\omega) \end{cases} \tag{2.3}$$

and that $m(x,x;\omega) = n(x,x;\omega) = 0$.

Recall that an automorphism g of Δ is called *type-rotating* if there is an integer c such that $\tau(gx) = \tau(x) + c \bmod 3$ for each $x \in \mathcal{V}_\Delta$. Let $\mathrm{Aut_{tr}}(\Delta)$ denote the group of type-rotating automorphisms of Δ. If g is any automorphism of Δ, and if S and S' are two parallel sectors, then gS and gS' are parallel. If ω is the equivalence class of S we may thus define $g\omega$ to be the class of gS. Suppose that S has base vertex x, and let $s_{m,n}$ be the unique point in $S \cap S_{m,n}(x)$. If $g \in \mathrm{Aut_{tr}}(\Delta)$, then $gs_{m,n}$ is the unique point in $gS \cap S_{m,n}(gx)$. It readily follows that for any $\omega \in \Omega$ and any vertices x, y of Δ

$$\tag{2.4} \begin{cases} m(gx, gy; g\omega) = m(x, y; \omega), \\ n(gx, gy; g\omega) = n(x, y; \omega). \end{cases}$$

If we fix a vertex o in Δ, and set $m(g, \omega) = m(o, go; \omega)$ and $n(g, \omega) = n(o, go; \omega)$, then it is immediate from (2.2) and (2.4) that $m(g, \omega)$ and $n(g, \omega)$ satisfy a «cocycle identity» :

$$\tag{2.5} \begin{cases} m(g_1g_2, \omega) = m(g_1, \omega) + m(g_2, g_1^{-1}\omega), \\ n(g_1g_2, \omega) = n(g_1, \omega) + n(g_2, g_1^{-1}\omega) \end{cases}$$

for $g_1, g_2 \in \mathrm{Aut_{tr}}(\Delta)$ and $\omega \in \Omega$.

There is a natural topology on Ω, making it a totally disconnected compact Hausdorff space. Indeed, if we fix any $x \in \mathcal{V}_\Delta$, there is a natural map of Ω into the product of the finite sets $S_{m,n}(x)$, $m, n \geq 0$ (each endowed with the discrete topology) : one maps ω to $(s^x_{m,n}(\omega))_{m,n\geq 0}$. If $m' \leq m$ and $n' \leq n$, there is a natural map $S_{m,n}(x) \to S_{m',n'}(x)$ (mapping $y \in S_{m,n}(x)$ to the unique vertex in $S_{m',n'}(x)$ and in the convex hull of x and y). The above map is a bijection of Ω onto the inverse limit of this system of maps, which thus induces a compact Hausdorff topology on Ω. For $x, v \in \mathcal{V}_\Delta$, let $\Omega_x(v)$ denote the set of $\omega \in \Omega$ for which the sector $S^x(\omega)$ contains v. Thus $\omega \in \Omega_x(v)$ if and only if $v \in S^x(\omega)$. The sets $\Omega_x(v)$ form a basis of open and closed sets for the topology on Ω.

LEMMA 2.4. — *Let $y \in S_{m,n}(x)$. Suppose that $z \in S_{i,j}(x) \cap S_{k,\ell}(y)$, where $i, j \geq d(x, y) = m + n$. Then $\Omega_x(z) \subset \Omega_y(z)$, and $m(x, y; \omega) = k - i$ and $n(x, y; \omega) = \ell - j$ for all $\omega \in \Omega_x(z)$. Also, $|S_{i,j}(x) \cap S_{k,\ell}(y)|$ does not depend on the particular $y \in S_{m,n}(x)$.*

Proof. — Let $\omega \in \Omega_x(z)$. Then $z = s^x_{i,j}(\omega)$, and thus z is an element of $S^x(\omega) \cap S^y(\omega)$, by Corollary 2.3. Thus $\omega \in \Omega_y(z)$, and $m(x, y; \omega) = k - i$ and $n(x, y; \omega) = \ell - j$ by the proof of Lemma 2.1.

To see the last statement, let $A_{r,s}$, $r, s \geq 0$, denote the averaging operators of [4], described above. Their linear span is an algebra, and we can thus write $A_{k,\ell}A_{j,i} = \sum_{r,s} C_{r,s}A_{r,s}$ for suitable numbers $C_{r,s}$. If we apply both sides to δ_x (where $\delta_x(y) = 1$ if $y = x$, and 0 otherwise) and evaluate at $y \in S_{m,n}(x)$, the right hand side is $C_{n,m}/N_{m,n}$, while the left hand side is $|S_{i,j}(x) \cap S_{k,\ell}(y)|/(N_{i,j}N_{k,\ell})$.

For each vertex x, there is also a natural Borel probability measure ν_x on Ω, which, for each $m, n \geq 0$, assigns equal measure to each of the $N_{m,n}$ disjoint sets $\Omega_x(v)$, $v \in S_{m,n}(x)$. If $g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$, then $\nu_{gx} = g\nu_x$, i.e., $\nu_{gx}(A) = \nu_x(g^{-1}A)$ for any Borel set $A \subset \Omega$, and so ν_x is invariant under each $g \in K_x = \{g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta) : gx = x\}$.

LEMMA 2.5. — *The topology on Ω does not depend on the vertex x chosen in the definition above. For any $x, y \in \mathcal{V}_\Delta$, the measures ν_x and ν_y are mutually absolutely continuous, and the Radon Nikodym derivative of ν_y with respect to ν_x is given by*

$$\frac{\mathrm{d}\nu_y}{\mathrm{d}\nu_x}(\omega) = \frac{1}{q^{2(m(x,y;\omega)+n(x,y;\omega))}}. \tag{2.6}$$

Proof. — Let $x, y \in \mathcal{V}_\Delta$, and let $\omega_0 \in \Omega_y(v)$, a basic open set for the topology defined using y, where $v \in S_{r,s}(y)$, say. By Corollary 2.3,

$$s^x_{i,j}(\omega_0) \in S^x(\omega_0) \cap S^y(\omega_0)$$

once $i, j \geq d(x, y)$. Choose i, j so large that also $k = i + m(x, y; \omega_0) \geq d(x, y)$ and $\ell = j + n(x, y; \omega_0) \geq d(x, y)$, and also $k \geq r$ and $\ell \geq s$. Let $z = s^x_{i,j}(\omega_0) = s^y_{k,\ell}(\omega_0)$. Then $z \in S_{i,j}(x) \cap S_{k,\ell}(y)$, and $\Omega_x(z) = \Omega_y(z)$ by Lemma 2.4. Since $v = s^y_{r,s}(\omega_0)$, we have $\omega_0 \in \Omega_x(z) = \Omega_y(z) \subset \Omega_y(v)$. This shows that each «$y$-open» set is «$x$-open». The first statement of the lemma follows. Also, $m(x, y; \omega) = k - i$ and $n(x, y; \omega) = \ell - j$ for all $\omega \in \Omega_x(z) = \Omega_y(z)$, and

$$\nu_y\big(\Omega_x(z)\big) = \frac{1}{N_{k,\ell}} = \frac{1}{q^{2(k-i+\ell-j)}N_{i,j}} = \frac{1}{q^{2(k-i+\ell-j)}}\,\nu_x\big(\Omega_x(z)\big).$$

The remaining statements in the lemma follow easily.

We remark that the action of $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ on Ω is continuous, with the above topology on Ω and the topology of pointwise convergence on $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$.

3. The spherical representations of $\mathbf{Aut_{tr}(\Delta)}$.

Let $\mathcal{C}^\infty(\Omega)$ denote the space of locally constant functions $\Omega \to \mathbb{C}$. For each vertex x, $\mathcal{C}^\infty(\Omega)$ is the linear span of the indicator functions $\mathbf{1}_{\Omega_x(v)}$ of the sets $\Omega_x(v)$. For any vertices x, y of Δ, $m(x,y;\omega)$ and $n(x,y;\omega)$ are locally constant functions of ω. For given $\omega_0 \in \Omega$, let $z = s^x_{i,j}(\omega_0)$, where $i, j \geq d(x,y)$. If $z \in S_{k,\ell}(y)$, Lemma 2.4 shows that $m(x,y;\omega) = k-i$ and $n(x,y;\omega) = \ell - j$ for all ω in the neighbourhood $\Omega_x(z)$ of ω_0.

For $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $s_1 s_2 s_3 = 1$ and for $g \in \mathrm{Aut_{tr}}(\Delta)$, define $\pi_s(g) : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ by

$$(3.1) \qquad \big(\pi_s(g)F\big)(\omega) = \frac{s_1^{m(g,\omega)+n(g,\omega)} s_2^{n(g,\omega)}}{q^{m(g,\omega)+n(g,\omega)}} F(g^{-1}\omega)$$

with the notation defined before (2.5). The cocycle identity (2.5) shows that $\pi_s(g_1 g_2) = \pi_s(g_1) \circ \pi_s(g_2)$ for $g_1, g_2 \in \mathrm{Aut_{tr}}(\Delta)$. Note that the expression on the right in (3.1) can be put into a more symmetric form

$$\frac{s_1^{n_1(g,\omega)} s_2^{n_2(g,\omega)} s_3^{n_3(g,\omega)}}{q^{n_1(g,\omega)-n_3(g,\omega)}} F(g^{-1}\omega)$$

if $n_j(g,\omega)$ are any integers such that $m(g,\omega) = n_1(g,\omega) - n_2(g,\omega)$ and $n(g,\omega) = n_2(g,\omega) - n_3(g,\omega)$.

For the next result, compare Prop. 5.4 in [14].

PROPOSITION 3.1. — *Let $s = (s_1, s_2, s_3)$, where $s_1, s_2, s_3 \in \mathbb{C}$ and $s_1 s_2 s_3 = 1$, and let k_s denote the kernel defined before Lemma 2.1. Then for $x, y \in \mathcal{V}_\Delta$, we have*

$$(3.2) \qquad \int_\Omega \frac{s_1^{m(y,x;\omega)+n(y,x;\omega)} s_2^{n(y,x;\omega)}}{q^{m(x,y;\omega)+n(x,y;\omega)}} \, \mathrm{d}\nu_x(\omega) = k_s(x,y).$$

Proof. — Fix x, and denote the integral in (3.2) by $F(y)$. The integrand is $\alpha^{m(x,y;\omega)}\beta^{n(x,y;\omega)}$, for $\alpha = 1/(s_1 q)$ and $\beta = 1/(s_1 s_2 q) = s_3/q$. We claim that F is the unique x-biradial function on $\mathcal{V}_\Delta$ satisfying $F(x) = 1$, $A^+F = zF$ and $A^-F = wF$ for $z = q(q^2+q+1)^{-1}(s_1+s_2+s_3)$ and $w = q(q^2+q+1)^{-1}(s_1^{-1}+s_2^{-1}+s_3^{-1})$ ([4], Prop. 3.4). Now $F(x) = 1$ because ν_x is a probability measure. To see that F is x-biradial, let

$y \in S_{m,n}(x)$, and let $i, j \geq m+n$. Then as the sets $\Omega_x(z)$, $z \in S_{i,j}(x)$, form a partition of Ω, we have

$$\begin{aligned} F(y) &= \sum_{z \in S_{i,j}(x)} \int_{\Omega_x(z)} \alpha^{m(x,y;\omega)} \beta^{n(x,y;\omega)} \, \mathrm{d}\nu_x(\omega) \\ &= \sum_{k,\ell \geq 0} \sum_{z \in S_{i,j}(x) \cap S_{k,\ell}(y)} \int_{\Omega_x(z)} \alpha^{m(x,y;\omega)} \beta^{n(x,y;\omega)} \, \mathrm{d}\nu_x(\omega). \end{aligned}$$

The last integrand takes the constant value $\alpha^{k-i}\beta^{\ell-j}$ on $\Omega_x(z)$ for any $z \in S_{i,j}(x) \cap S_{k,\ell}(y)$, and $\nu_x(\Omega_x(z)) = 1/N_{i,j}$ for each such z. Thus

$$F(y) = \sum_{k,\ell \geq 0} \left| S_{i,j}(x) \cap S_{k,\ell}(y) \right| \alpha^{k-i} \beta^{\ell-j} / N_{i,j}$$

which, by Lemma 2.4, does not depend on the particular $y \in S_{m,n}(x)$.

Finally, using (2.2), we see that if $z \in \mathcal{V}_\Delta$, then $(A^+F)(z)$ equals

$$\int_\Omega \left\{ \frac{1}{q^2+q+1} \sum_{y \in S_{1,0}(z)} \alpha^{m(z,y;\omega)} \beta^{n(z,y;\omega)} \right\} \alpha^{m(x,z;\omega)} \beta^{n(x,z;\omega)} \, \mathrm{d}\nu_x(\omega).$$

By Lemma 2.2, the expression in braces is

$$\frac{1}{q^2+q+1} \left(\alpha^{-1} + q\alpha\beta^{-1} + q^2\beta \right) = \frac{q}{q^2+q+1} (s_1 + s_2 + s_3)$$

for all $\omega \in \Omega$. Thus $A^+F = zF$, and similarly, $A^-F = wF$. This completes the proof.

Recall that o denotes a fixed vertex of Δ. For $F_1, F_2 \in \mathcal{C}^\infty(\Omega)$, let $\langle F_1, F_2 \rangle = \int_\Omega F_1(\omega) \overline{F_2(\omega)} \, \mathrm{d}\nu_o(\omega)$. This defines an inner product on $\mathcal{C}^\infty(\Omega)$, because any nonzero $F \in \mathcal{C}^\infty(\Omega)$ can be expressed as

$$F = \sum_{z \in S_{m,n}(o)} c_z \mathbf{1}_{\Omega_o(z)}$$

for m, n big enough. Then $\langle F, F \rangle = N_{m,n}^{-1} \sum_{z \in S_{m,n}(o)} |c_z|^2 > 0$.

Corollary 3.2. — *For any $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ such that $s_1 s_2 s_3 = 1$, let $s^* = (\bar{s}_1^{-1}, \bar{s}_2^{-1}, \bar{s}_3^{-1})$. Let $\mathbf{1}$ denote the function on Ω with constant value 1. Then for any $g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ and $F_1, F_2 \in \mathcal{C}^\infty(\Omega)$ we have*

$$\langle \mathbf{1}, \pi_s(g)\mathbf{1} \rangle = \varphi_{s^*}(g) \quad \text{and} \quad \langle \pi_s(g)F_1, F_2 \rangle = \langle F_1, \pi_{s^*}(g^{-1})F_2 \rangle. \tag{3.3}$$

In particular, if $|s_j| = 1$ for $j = 1, 2, 3$, so that $s^* = s$; π_s extends to a continuous unitary representation of $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ on $L^2(\Omega, \nu_o)$, and $\varphi_s(g) = k_s(o, go) = \langle \mathbf{1}, \pi_s(g)\mathbf{1} \rangle$ is a positive definite function on $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$.

Proof. — The first identity in (3.3) is immediate from Proposition 3.1. The second identity follows from Lemma 2.5 and the fact that $\nu_{go} = g\nu_o$.

Remark. — As noted in [4], k_s is determined by $s_1 + s_2 + s_3$ and $s_1^{-1} + s_2^{-1} + s_3^{-1}$, and so equals $k_{s'}$, where $s' = (s_1', s_2', s_3')$, if and only if (s_1', s_2', s_3') is a permutation of (s_1, s_2, s_3). Notice that $s^* = (\bar{s}_1^{-1}, \bar{s}_2^{-1}, \bar{s}_3^{-1})$ is a permutation of (s_1, s_2, s_3) if and only if either (a) $|s_j| = 1$ for each j, or (b) s is a permutation of $(r\,\mathrm{e}^{i\theta}, r^{-1}\,\mathrm{e}^{i\theta}, \mathrm{e}^{-2i\theta})$ for some $r \geq 1$ and $\theta \in \mathbb{R}$.

We now want to describe explicitly the intertwining operators between π_s and $\pi_{s'}$, when $s' = (s_1', s_2', s_3')$ is a permutation of $s = (s_1, s_2, s_3)$.

Recall that $o \in \mathcal{V}_\Delta$ is fixed. Let Ω^ℓ denote the set of left walls w of the sectors with base vertex o. There is a natural map $\Omega \to \Omega^\ell$, and we denote by ν_o^ℓ the probability on Ω^ℓ induced from ν_o by this map (this measure was studied in [14]). For $y \in S_{m,0}(o)$, let

$$\Omega_{w,y} = \big\{\omega \in \Omega : S^o(\omega) \text{ has left wall } w, \text{ and } s^o_{m,0}(\omega) = y\big\}.$$

We write Ω_w in place of $\Omega_{w,o} = \{\omega \in \Omega : S^o(\omega) \text{ has left wall } w\}$. For each $w \in \Omega^\ell$, if $m \geq 1$, then $\Omega_{w,y} \neq \emptyset$ for exactly $(q+1)q^{m-1}$ vertices $y \in S_{m,0}(o)$. Indeed, with the notation of Fig. 4, there are $q+1$ choices for $s_{1,0}$. This, with w, determines the vertices $s_{1,1}$, $s_{1,2}, \ldots$. Then there are q choices for $s_{2,0}$, etc.

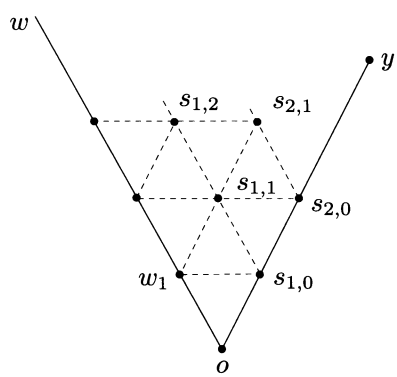

Figure 4

Let $x \in S_{0,n}(o)$ and $y \in S_{m,0}(o)$, where $n \geq m \geq 1$. Assuming that x and y belong to some sector S with base vertex o, a similar counting argument shows that there are q^m vertices $z \in S_{m,n}(o)$ such that x, y and z all belong to such a sector. Hence

$$\begin{aligned}\nu_o\{\omega \in \Omega : S^o(\omega) \text{ contains } x \text{ and } y\} \\ &= q^m/N_{m,n} = 1/((q+1)q^{m-1}N_{0,n}) \\ &= \frac{1}{(q+1)q^{m-1}}\,\nu_o^\ell\{w \in \Omega^\ell : w \text{ contains } x\}.\end{aligned}$$

It follows that if $F \in \mathcal{C}^\infty(\Omega)$, and $F(\omega) = F_{w,y}$ for all $\omega \in \Omega_{w,y}$, then

$$\int_\Omega F(\omega)\,\mathrm{d}\nu_o(\omega) = \frac{1}{(q+1)q^{m-1}} \sum_{y \in S_{m,0}(o)} \int_{\Omega^\ell} F_{w,y}\,\mathrm{d}\nu_o^\ell(w) \tag{3.4}$$

(writing $F_{w,y} = 0$ if $\Omega_{w,y} = \emptyset$).

Suppose that $F \in \mathcal{C}^\infty(\Omega)$. Then for $m, n \geq 0$ large enough, F is constant on each set $\Omega_o(z)$, $z \in S_{m,n}(o)$, and so F is constant on each set $\Omega_{w,y}$, $w \in \Omega^\ell$, $y \in S_{m,0}(o)$. Let $\mathcal{H}^0$ denote the space of all $F \in \mathcal{C}^\infty(\Omega)$ constant on each set Ω_w, and for each $m \geq 1$, let $\mathcal{H}^m$ denote the space of all $F \in \mathcal{C}^\infty(\Omega)$ constant on each set $\Omega_{w,y}$, $y \in S_{m,0}(o)$, and such that for each $w \in \Omega^\ell$ and each $y' \in S_{m-1,0}(o)$ such that $\Omega_{w,y'} \neq \emptyset$, the sum $\sum_y F_{w,y}$ is zero, the sum being over the q y's in $S_{m,0}(o)$ ($q+1$ if $m = 1$) such that $\Omega_{w,y} \neq \emptyset$ and $y \in S_{1,0}(y')$. It follows from (3.4) that $\int_\Omega F_1\overline{F_2}\,\mathrm{d}\nu_o = 0$ if $F_1 \in \mathcal{H}^m$, $F_2 \in \mathcal{H}^n$, and $m \neq n$. To summarize, we have an orthogonal decomposition

$$\mathcal{C}^\infty(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n.$$

Moreover, it is easy to see that any $F \in \mathcal{C}^\infty(\Omega)$ which is constant on each set $\Omega_{w,y}$, $y \in S_{m,0}(o)$, can be written (uniquely) as $F = H^0 + \cdots + H^m$, with $H^j \in \mathcal{H}^j$ for $j = 0, \ldots, m$.

PROPOSITION 3.3. — *Let $s = (s_1, s_2, s_3) \in \mathbb{C}^3$, where $s_1s_2s_3 = 1$ and $s_1 \neq qs_2$. Define $J_s : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ by setting $J_sF = j_nF$ for all $F \in \mathcal{H}^n$, where $j_0 = 1$ and $j_n = (s_1/s_2)^{n-1}(qs_1 - s_2)/(qs_2 - s_1)$ for $n \geq 1$. Assume that $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ acts transitively on $\mathcal{V}_\Delta$. Then $J_s \circ \pi_s(g) = \pi_{s'}(g) \circ J_s$ for all $g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$, where $s' = (s_2, s_1, s_3)$.*

Proof. — For each $x \in S_{1,0}(o)$, there is a $g_x \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ such that $g_xo = x$. A simple induction on $d(o, go)$ shows that each $g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ can be expressed as a product of the g_x's and their inverses and a $k \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ such that $ko = o$. Thus all we must do is verify that $J_s(\pi_s(g)F) = \pi_{s'}(g)(J_sF)$ if $F \in \mathcal{H}^n$, for $n \geq 0$ and for each $g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ such that $go = o$ or $go \in S_{1,0}(o)$.

If $go = o$, then $(\pi_s(g)F)(\omega) = F(g^{-1}\omega) = (\pi_{s'}(g)F)(\omega)$. If $\omega \in \Omega_{w,y}$, then $g^{-1}\omega \in \Omega_{g^{-1}w,g^{-1}y}$. So if $F \in \mathcal{H}^n$, then $\pi_s(g)F \in \mathcal{H}^n$, and $J_s(\pi_s(g)F) = j_n\pi_s(g)F = j_n\pi_{s'}(g)F = \pi_{s'}(g)(J_sF)$.

Suppose now that $go = x \in S_{1,0}(o)$, and that $F \in \mathcal{H}^n$. We shall see that $\pi_s(g)F = H^0 + H^1$ if $n = 0$, and $\pi_s(g)F = H^{n-1} + H^n + H^{n+1}$, if $n \geq 1$, where $H^j \in \mathcal{H}^j$ for each j in either case. We shall also use subscripts w and w, y to denote constant values taken by various functions on the sets Ω_w and $\Omega_{w,y}$. Also, we write $\alpha = s_1/q$, $\beta = 1/(s_3q)$, $\alpha' = s_2/q$ and $\beta' = 1/(s_3q) = \beta$. If $w \in \Omega^\ell$ is the left wall of a sector S with base vertex o, let w_1 denote the vertex in $S_{0,1}(o) \cap S$, i.e., the vertex on w at distance 1 from o, and let $g^{-1}w \in \Omega^\ell$ denote the left wall of the sector with base vertex o parallel to $g^{-1}S$.

Now let $F \in \mathcal{H}^0$. Then on $\Omega_{w,y}$, $y \in S_{1,0}(o)$, $\pi_s(g)F$ takes the constant value $(\pi_s(g)F)_{w,y}$, equal to $\alpha^{-1}F_{g^{-1}w}$, $\alpha\beta^{-1}F_{g^{-1}w}$ or $\beta F_{g^{-1}w}$, according as $go = y$, $d(w_1, go) = 1$ but $go \neq y$, or $d(w_1, go) \neq 1$ (see Lemma 2.2). So $\pi_s(g)F = H^0 + H^1$, where H^0_w is the average of the $q+1$ values $(\pi_s(g)F)_{w,y}$, where $y \in S_{1,0}(o)$ and $d(y, w_1) = 1$. Thus

$$H^0_w = \begin{cases} \big((\alpha^{-1} + q\alpha\beta^{-1})/(q+1)\big)F_{g^{-1}w} & \text{if } d(w_1, go) = 1, \\ \beta F_{g^{-1}w} & \text{if } d(w_1, go) = 2. \end{cases}$$

We then find that

$$H^1_{w,y} = \begin{cases} \big(q(\alpha^{-1} - \alpha\beta^{-1})/(q+1)\big)F_{g^{-1}w} & \text{if } d(w_1, go) = 1 \text{ and } go = y, \\ \big(\alpha\beta^{-1} - \alpha^{-1})/(q+1)\big)F_{g^{-1}w} & \text{if } d(w_1, go) = 1 \text{ and } go \neq y, \\ 0 & \text{if } d(w_1, go) = 2. \end{cases}$$

If $\pi_{s'}(g)F = H^{0'} + H^{1'}$ is the corresponding decomposition of $\pi_{s'}(g)F$, then

$$J_s(\pi_s(g)F) = j_0H^0 + j_1H^1 = j_0(H^{0'} + H^{1'}) = \pi_{s'}(g)(J_sF)$$

because $(\alpha^{-1} + q\alpha\beta^{-1})/(q+1) = ({\alpha'}^{-1} + q\alpha'{\beta'}^{-1})/(q+1)$, $\beta' = \beta$ and because $j_1(\alpha^{-1} - \alpha\beta^{-1}) = j_0({\alpha'}^{-1} - \alpha'{\beta'}^{-1})$.

If now $F \in \mathcal{H}^1$, $w \in \Omega^\ell$, $t \in S_{2,0}(o)$ and $\Omega_{w,t} \neq \emptyset$, then, on $\Omega_{w,t}$, $\pi_s(g)F$ takes a constant value $(\pi_s(g)F)_{w,t}$ equal to $\alpha^{-1}F_{g^{-1}w,g^{-1}t}$, $\alpha\beta^{-1}F_{g^{-1}w,g^{-1}w_1}$ or $\beta F_{g^{-1}w,g^{-1}z}$, according as $d(go, w_1) = 1$ and $go = y$

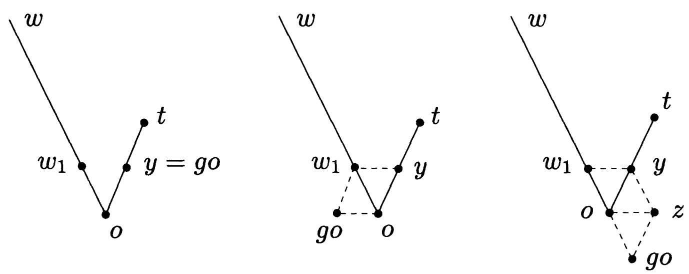

Figure 5

(the point on the geodesic from o to t), $d(go,w_1)=1$ and $go\neq y$, or $d(go,w_1)=2$, where in the last case $z\in S_{0,1}(o)$ is the unique vertex at distance 1 from o, go and y (see Fig. 5). Thus $\pi_s(g)F=H^0+H^1+H^2$, where

(i) if $d(go,w_1)=1$, then
$$H^0_w=\big((q^2\alpha\beta^{-1}-\alpha^{-1})/(q(q+1))\big)F_{g^{-1}w,g^{-1}w_1}$$
and (see Fig. 5)
$$\begin{cases} H^1_{w,y}=-\big((\alpha^{-1}+q\alpha\beta^{-1})/(q+1)\big)F_{g^{-1}w,g^{-1}w_1},\\ H^2_{w,t}=\alpha^{-1}(F_{g^{-1}w,g^{-1}t}+q^{-1}F_{g^{-1}w,g^{-1}w_1})\end{cases}$$
or
$$\begin{cases} H^1_{w,y}=\big((\alpha^{-1}+q\alpha\beta^{-1})/(q(q+1))\big)F_{g^{-1}w,g^{-1}w_1},\\ H^2_{w,t}=0,\end{cases}$$
according as $go=y$ or not.

(ii) if $d(go,w_1)\neq 1$, then $H^0_w=0$, $H^1_{w,y}=\beta F_{g^{-1}w,g^{-1}z}$ and $H^2_{w,t}=0$.

If $\pi_{s'}(g)F=H^{0'}+H^{1'}+H^{2'}$ is the corresponding decomposition of $\pi_{s'}(g)F$, then
$$J_s(\pi_s(g)F)=j_0H^0+j_1H^1+j_2H^2=j_1(H^{0'}+H^{1'}+H^{2'})=\pi_{s'}(g)(J_sF)$$
because $j_2/j_1=\alpha/\alpha'=s_1/s_2$ (which one needs in comparing $j_2H^2_{w,t}$ with $j_1{H^2_{w,t}}'$ in the case $d(go,w_1)=1$ and $go=y$).

Finally, suppose that $F\in\mathcal{H}^n$, where $n\geq 2$, and let $w\in\Omega^\ell$ and $t\in S_{n+1,0}(o)$, where $\Omega_{w,t}\neq\emptyset$. Then, on $\Omega_{w,t}$, $\pi_s(g)F$ takes a constant value $(\pi_s(g)F)_{w,t}$ equal to $\alpha^{-1}F_{g^{-1}w,g^{-1}t}$, $\alpha\beta^{-1}F_{g^{-1}w,g^{-1}v}$ or $\beta F_{g^{-1}w,g^{-1}z}$, according as $d(go,w_1)=1$ and go lies on the geodesic from o to t, $d(go,w_1)=1$ and go lies off this geodesic, or $d(go,w_1)=2$ (see Fig. 6).

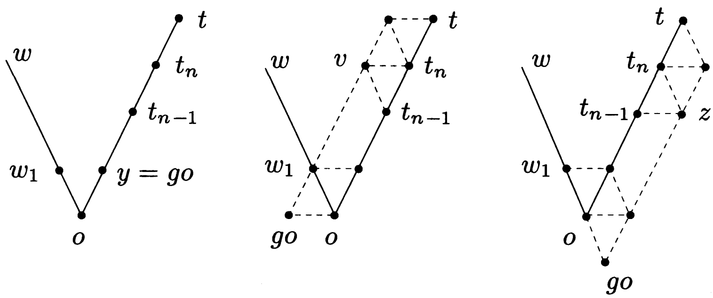

Figure 6

Then $\pi_s(g)F = H^{n-1} + H^n + H^{n+1}$, where if we evaluate at $\omega \in \Omega_{w,t}$, we see that if (a) $d(go, w_1) = 1$ and go lies on the geodesic from o to t, then $H^{n+1}_{w,t} = \alpha^{-1} F_{g^{-1}w, g^{-1}t}$, $H^n_{w,t_n} = H^{n-1}_{w,t_{n-1}} = 0$, if (b) $d(go, w_1) = 1$ and go lies off this geodesic, then $H^{n-1}_{w,t} = \alpha\beta^{-1} F_{g^{-1}w, g^{-1}v}$, $H^{n+1}_{w,t} = H^n_{w,t_n} = 0$, and if (c) $d(go, w_1) \neq 1$, then $H^n_{w,t_n} = \beta F_{g^{-1}w, g^{-1}z}$, $H^{n+1}_{w,t} = H^{n-1}_{w,t_{n-1}} = 0$. If $\pi_{s'}(g)F = H^{n-1\prime} + H^{n\prime} + H^{n+1\prime}$ is the corresponding decomposition of $\pi_{s'}(g)F$, then

$$\begin{aligned} J_s(\pi_s(g)F) &= j_{n-1} H^{n-1} + j_n H^n + j_{n+1} H^{n+1} \\ &= j_n (H^{n-1\prime} + H^{n\prime} + H^{n+1\prime}) \\ &= \pi_{s'}(g)(J_s F) \end{aligned}$$

because $j_{n+1}/j_n = \alpha/\alpha' = s_1/s_2$, $\beta' = \beta$ and $j_{n-1}\alpha\beta^{-1} = j_n \alpha' \beta'^{-1}$ (one needs the first of these in comparing $j_{n+1} H^{n+1}_{w,t}$ with $j_n H^{n+1\prime}_{w,t}$ in the case $d(go, w_1) = 1$ and go lies on the geodesic from o to t, for example). This completes the proof.

If $\mathcal{K}^n$ denotes the subspace of $\mathcal{C}^\infty(\Omega)$ defined like $\mathcal{H}^n$, but with left walls replaced by right walls, and if J_s is replaced by an operator J'_s on $\mathcal{C}^\infty(\Omega)$, where $J'_s F = j'_n F$ for all $F \in \mathcal{K}^n$, $j'_0 = 1$ and $j'_n = (s_2/s_3)^{n-1}(qs_2 - s_3)/(qs_3 - s_2)$ for $n \geq 1$, then, provided $qs_3 \neq s_2$, $J'_s \circ \pi_s(g) = \pi_{s''}(g) \circ J'_s$, where $s'' = (s_1, s_3, s_2)$. Thus, combining the intertwining operators J_s and J'_s, we get intertwining operators between π_s and $\pi_{\tilde{s}}$ for any permutation $\tilde{s}$ of s (provided that $qs_i \neq s_j$ for each $i \neq j$).

Corollary 3.4. — *Let $s = (r\,e^{i\theta}, r^{-1}\,e^{i\theta}, e^{-2i\theta})$, where $\theta \in \mathbb{R}$ and $1/\sqrt{q} < r < \sqrt{q}$. Then, with the notation and assumptions of Proposition 3.3, $\langle F_1, F_2 \rangle_s = \langle F_1, J_s F_2 \rangle$ defines an inner product on $\mathcal{C}^\infty(\Omega)$, and π_s extends to a unitary representation on the corresponding completion* *of $\mathcal{C}^\infty(\Omega)$. Moreover, $\varphi_s(g) = \langle \mathbf{1}, \pi_s(g)\mathbf{1}\rangle_s$, and so φ_s is a positive definite function on* $\mathrm{Aut_{tr}}(\Delta)$. *This last assertion also holds if $1/\sqrt{q} \le r \le \sqrt{q}$.*

Proof. — If $F_1, F_2 \in \mathcal{C}^\infty(\Omega)$, we can write each F_j as a finite sum $\sum_n H_j^n$, where $H_j^n \in \mathcal{H}^n$. Then $\langle F_1, F_2\rangle_s = \sum_n j_n \langle H_1^n, H_2^n\rangle$, and $j_n = r^{2(n-1)}(qr^2 - 1)/(q - r^2) > 0$ for each $n \ge 1$. Since $\langle\,\cdot\,,\,\cdot\,\rangle$ is an inner product on $\mathcal{C}^\infty(\Omega)$, the first statement is proved. Also,

$$\begin{aligned}\langle \pi_s(g)F_1, F_2\rangle_s &= \langle F_1, \pi_{s^*}(g^{-1})J_sF_2\rangle \\ &= \langle F_1, J_s\pi_s(g^{-1})F_2\rangle = \langle F_1, \pi_s(g^{-1})F_2\rangle_s\end{aligned}$$

because of (3.3) and the fact that s^* is, in the present case, obtained from s by interchanging the first two terms of s. Finally, φ_s is still positive definite if $r = \sqrt{q}$ or $1/\sqrt{q}$, being a pointwise limit of a sequence of φ_s's corresponding to $1/\sqrt{q} < r < \sqrt{q}$.

If $1 \le r \le \sqrt{q}$, let

$$\gamma_r(\theta) = q(q^2+q+1)^{-1}\big((r + r^{-1})\,\mathrm{e}^{i\theta} + e^{-2i\theta}\big)$$

for $0 \le \theta \le 2\pi$. Consider the representations π_s defined above or, via the Gelfand, Naimark, Segal construction, from a positive definite φ_s. (One can show that if $qs_j \ne s_i$ for $1 \le i < j \le 3$, then $\mathcal{C}^\infty(\Omega)$ is the linear span of the functions $g \mapsto \pi_s(g)\mathbf{1}$ as g varies over any subgroup of $\mathrm{Aut_{tr}}(\Delta)$ acting transitively on $\mathcal{V}_\Delta$. Thus $\mathbf{1}$ is a cyclic vector for π_s, as defined above.) The γ_r define simple closed curves in the plane. If $z = q(q^2+q+1)^{-1}(s_1+s_2+s_3)$ lies on or inside the hypocycloid γ_1, π_s is called a *principal series spherical representation* of $\mathrm{Aut_{tr}}(\Delta)$. If z lies between γ_1 and $\gamma = \gamma_{\sqrt{q}}$, or on γ, π_s is called a *complementary series spherical representation* of $\mathrm{Aut_{tr}}(\Delta)$.

The z inside and on γ may be described as the $z \in \mathbb{C}$ other than 1, $\mathrm{e}^{2\pi i/3}$ and $\mathrm{e}^{-2\pi i/3}$ satisfying condition (1.2) of § 1. For given $z \in \mathbb{C}$, there exist $s_1, s_2, s_3 \in \mathbb{C}$ such that

$$\begin{gathered} s_1s_2s_3 = 1, \\ q(q^2+q+1)^{-1}(s_1+s_2+s_3) = z, \\ q(q^2+q+1)^{-1}(s_1^{-1}+s_2^{-1}+s_3^{-1}) = \bar{z}, \end{gathered}$$

namely the three roots of

$$X^3 - (q^2+q+1)q^{-1}zX^2 + (q^2+q+1)q^{-1}\bar{z}X - 1 = 0.$$

Then (see the Remark after Corollary 3.2), either (a) $|s_j| = 1$ for each j (in which case z lies within the hypocycloid), or (b) after permuting the s_j if necessary, $(s_1, s_2, s_3) = (r\,e^{i\theta}, r^{-1}e^{i\theta}, e^{-2i\theta})$ for some $r \geq 1$ and $\theta \in \mathbb{R}$. Using the identity

$$\begin{aligned} &q(q^2+q+1)^{-3}\prod_{i\neq j}(qs_i - s_j) \\ &\quad = (q+1)^2(z^3+w^3) - (q^2+q+1)z^2w^2 - (q^2+4q+1)zw + q, \end{aligned}$$

if $s_1s_2s_3 = 1$, $q(q^2+q+1)^{-1}(s_1+s_2+s_3) = z$ and

$$q(q^2+q+1)^{-1}(s_1^{-1}+s_2^{-1}+s_3^{-1}) = w,$$

we see that (1.2) holds in case (b) if and only if $1 \leq r \leq \sqrt{q}$ (so that z lies on γ_1, on γ, or between these curves) or $r = q$ and $e^{3i\theta} = 1$ (so that $z = 1$, $e^{2\pi i/3}$ or $e^{-2\pi i/3}$).

THEOREM 3.5. — *Let Δ be an arbitrary triangle building. Then for any $z \in \Sigma^*$, the kernel $k_{z,\bar{z}}$ is positive definite on $\mathcal{V}_\Delta$. Thus $\varphi_{z,\bar{z}}(g) = k_{z,\bar{z}}(o, go)$ is positive definite on* $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$.

Proof. — If $z = 1$, then $k_{z,\bar{z}}(x,y) = 1$ for any $x, y \in \mathcal{V}_\Delta$. For if we fix $x \in \mathcal{V}_\Delta$, then $F(y) = k_{1,1}(x,y)$ defines an x-biradial function F on $\mathcal{V}_\Delta$ satisfying $A^+F = F$, $A^-F = F$ and $F(x) = 1$. The function taking the constant value 1 on $\mathcal{V}_\Delta$ has the same properties, and so $F(y) = 1$ for all $y \in \mathcal{V}_\Delta$ by the uniqueness mentioned in the proof of Proposition 3.1. Thus $k_{1,1}$ is positive definite.

If $z = \rho$, where $\rho^3 = 1$, then $k_{z,\bar{z}}(x,y) = \rho^{m-n}$ whenever $y \in S_{m,n}(x)$. For then $k_{z,\bar{z}}(x,y) = p_{m,n}(\rho, \rho^{-1}) = \rho^{m-n}p_{m,n}(1,1) = \rho^{m-n}$ by the last paragraph and the property $p_{m,n}(\rho z, \rho^{-1}w) = \rho^{m-n}p_{m,n}(z,w)$ of $p_{m,n}$ mentioned after Proposition 3.1 of [4]. Fix $o \in \mathcal{V}_\Delta$. If $x \in S_{i,j}(o)$, $y \in S_{k,\ell}(o)$ and $y \in S_{m,n}(x)$, then $m - n \equiv (k-\ell) - (i-j) \bmod 3$. This follows from Lemma 2.1 of [4] by induction on $m+n$. It follows that $k_{\rho,\rho^{-1}}(x,y) = \overline{k_{\rho,\rho^{-1}}(o,x)}\,k_{\rho,\rho^{-1}}(o,y)$. So if $x_1, \ldots, x_r \in \mathcal{V}_\Delta$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$, we have

$$\begin{aligned} \sum_{i,j=1}^{r} k_{\rho,\rho^{-1}}(x_i,x_j)\,\alpha_i\,\overline{\alpha_j} &= \sum_{i,j=1}^{r} \overline{k_{\rho,\rho^{-1}}(o,x_i)}\,k_{\rho,\rho^{-1}}(o,x_j)\,\alpha_i\,\overline{\alpha_j} \\ &= \Big|\sum_{j=1}^{r} k_{\rho,\rho^{-1}}(o,x_j)\,\overline{\alpha_j}\Big|^2 \geq 0 \end{aligned}$$

and so $k_{\rho,\rho^{-1}}$ is positive definite. Thus $k_{z,\bar{z}}$ is positive definite if $z \in \{1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{-2\pi i/3}\}$. In fact, the above calculation shows that $\varphi_{z,\bar{z}}$ is a character of $\mathrm{Aut}_{\mathrm{tr}}(\Delta)$ in this case.

Again fix $o \in \mathcal{V}_\Delta$. For any $x \in \mathcal{V}_\Delta$ and for any $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $s_1 s_2 s_3 = 1$, we define $f_s(x) \in \mathcal{C}^\infty(\Omega)$ by

$$f_s(x)(\omega) = \frac{s_1^{m(o,x;\omega)+n(o,x;\omega)} s_2^{n(o,x;\omega)}}{q^{m(o,x;\omega)+n(o,x;\omega)}}. \tag{3.5}$$

Notice that $f_s(x) = \pi_s(g)\mathbf{1}$ if $x = go$, $g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$.

Now suppose that z lies on or inside the hypocycloid γ_1. Then

$$z = q(q^2 + q + 1)^{-1}(s_1 + s_2 + s_3),$$

where $s_1 s_2 s_3 = 1$ and $|s_j| = 1$ for $j = 1, 2, 3$ ([4], Prop. 4.5). It is easy to see using (2.2), (2.3), (2.6) and $\overline{s_j} = s_j^{-1}$ that $\langle f_s(x), f_s(y)\rangle$ equals the integral in Proposition 3.1. (Recall that we write $\langle F_1, F_2\rangle = \int_\Omega F_1(\omega)\overline{F_2(\omega)}\, \mathrm{d}\nu_o(\omega)$ for $F_1, F_2 \in \mathcal{C}^\infty(\Omega)$.) Hence $k_{z,\bar{z}}(x, y) = k_s(x, y) = \langle f_s(x), f_s(y)\rangle$, and so $k_{z,\bar{z}}$ is positive definite ([9], Chap. 5, Exemples 2). This case was proved in [4], Prop. 4.7, by other methods.

Now suppose that z lies between the curves γ_1 and γ. We can write

$$z = q(q^2 + q + 1)^{-1}(s_1 + s_2 + s_3),$$

where $s_1 = r\,\mathrm{e}^{i\theta}$, $s_2 = r^{-1}\,\mathrm{e}^{i\theta}$ and $s_3 = \mathrm{e}^{-2i\theta}$, and where $1 < r < \sqrt{q}$ and $\theta \in \mathbb{R}$. Let $s' = (s_2, s_1, s_3)$. Using $\overline{s_1} = s_2^{-1}$, $\overline{s_2} = s_1^{-1}$, (2.2), (2.3) and (2.6), we now find that $\langle f_s(x), f_{s'}(y)\rangle$ equals the integral in Proposition 3.1. In Lemma 3.6 below, we show that $J_s(f_s(y)) = f_{s'}(y)$. Thus

$$k_{z,\bar{z}}(x, y) = k_s(x, y) = \big\langle f_s(x), J_s(f_s(y))\big\rangle = \big\langle f_s(x), f_s(y)\big\rangle_s$$

and again $k_{z,\bar{z}}$ is positive definite.

Finally, if z lies on the curve γ, then $k_{z,\bar{z}}$ is positive definite, being the pointwise limit of kernels $k_{z_j,\bar{z}_j}$, where $z_j \to z$ and z_j lies between γ_1 and γ.

LEMMA 3.6. — *Let $x \in \mathcal{V}_\Delta$, let $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $s_1 s_2 s_3 = 1$, and define $f_s(x) \in \mathcal{C}^\infty(\Omega)$ as in (3.5). Assume that $q s_2 \neq s_1$. Then $J_s(f_s(x)) = f_{s'}(x)$, where $s' = (s_2, s_1, s_3)$.*

Proof. — We can write

$$f_s(x) = H^0 + H^1 + \cdots,$$

$$f_{s'}(x) = (H')^0 + (H')^1 + \cdots,$$

where $H^\nu, (H')^\nu \in \mathcal{H}^\nu$ for each $\nu \geq 0$. We must show that $j_\nu H^\nu = (H')^\nu$ for each $\nu \geq 0$. This means that for each $w \in \Omega^\ell$ and each $y \in S_{\nu,0}(o)$ such that $\Omega_{w,y} \neq \emptyset$, we must show that $j_\nu H^\nu_{w,y} = (H')^\nu_{w,y}$. We do this by verifying that H^0_w is a symmetric rational function of s_1 and s_2, and that for $\nu \geq 1$, $H^\nu_{w,y}$ has the form $s_1^{-\nu+1}(qs_2 - s_1)f(s_1, s_2)$, where $f(s_1, s_2)$ is a symmetric rational function (depending on w and y).

Suppose that $x \in S_{m,n}(o)$ and that $w \in \Omega^\ell$. For $0 \leq i \leq m$ and $0 \leq j \leq n$, let $x_{i,j}$ denote the vertex of $S_{i,j}(o)$ in the convex hull of o and x. We shall see that there exist $\ell \geq 0$ and $v \in \mathcal{V}_\Delta$ with the following properties :

(i) $v \in S_{\ell,0}(o)$ and $\Omega_{w,v} \neq \emptyset$. We denote by $v_0 = o, \ldots, v_\ell = v$ the vertices on the geodesic from o to v.

(ii) Suppose that $i \geq 0$ and $y \in S_{i,0}(o)$ with $\Omega_{w,y} \neq \emptyset$. Then $m(o, x; \omega)$ and $n(o, x; \omega)$ are constant on $\Omega_{w,y}$, unless $0 \leq i < \ell$ and $y = v_i$.

Using (i) and (ii), it will be easy to calculate any $H^\nu_{w,y}$. For let $y_0 = o, \ldots, y_\nu = y$ denote the vertices on the geodesic from o to y. Then if j is the largest integer $i \leq \nu, \ell$ such that $y_i = v_i$, then $m(o, x; \omega)$ and $n(o, x; \omega)$ are constant on $\Omega_{w,y_{j+1}}$ if $j < \nu$, and on Ω_{w,y_j} if $j = \ell \leq \nu$. Thus $H^\nu_{w,y} = 0$ if $\nu \geq j + 2$, or if $j = \ell$ and $\nu = \ell + 1$. In particular, $H^\nu_{w,y} = 0$ whenever $\nu > \ell$.

Let r denote the largest integer $i \leq m$ such that $\Omega_{w,x_{i,0}} \neq \emptyset$. For $0 \leq i \leq r$ and $j \geq 0$, let $w_{i,j}$ denote the vertex $s^o_{i,j}(\omega)$ (the same for any $\omega \in \Omega_{w,x_{r,0}}$). Thus $w_{i,0} = x_{i,0}$ for $0 \leq i \leq r$. The vertices $w_{i,j}$ lie in a convex planar « strip » S with left wall w and « bottom edge » the segment from o to $x_{r,0}$. When $r < m$, note that $i + j \leq r$ holds for any (i, j) such that $w_{i,j} = x_{i,j}$. For if $w_{i,j} = x_{i,j}$ and $i + j \geq r + 1$, then it is easy to see that $x_{r,1}$ is in the convex hull of $x_{i,j}$ and $x_{r,0}$, and thus $w_{r,1} = x_{r,1}$. But then $d(w_{r,1}, x_{r+1,0}) = 1$, which implies that $\Omega_{w,x_{r+1,0}} \neq \emptyset$, contradicting the definition of r. Now let k denote the largest integer $j \leq r, n$ such that $x_{r-j,j} = w_{r-j,j}$. Notice that $w_{i,k} = x_{i,k}$ for $i = 0, \ldots, r - k$ by convexity of the intersection of S with the convex hull of o and x (see Fig. 7 (a)).

When $r = m$, let h denote the largest integer $j \leq n$ such that $w_{m,j} = x_{m,j}$. Then let k be the largest integer i such that $h \leq i \leq m + h, n$ and $w_{m+h-i,i} = x_{m+h-i,i}$ (see Fig. 7 (b) next page).

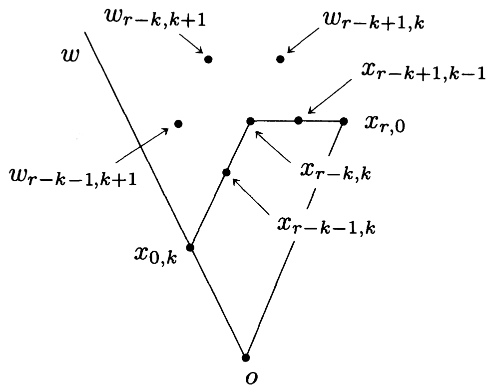

Figure 7 (a)

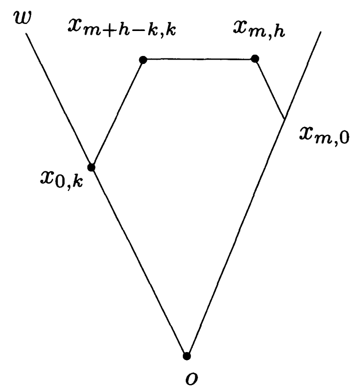

Figure 7 (b)

Let us consider the case when $r < m$ first. We claim that $w_{r-k,k+1}$ and $x_{r-k+1,k}$ are nonadjacent, and (if $k < r, n$) that $w_{r-k-1,k+1}$ and $x_{r-k,k+1}$ are nonadjacent. It will follow that the strip parallel to w having as base the segment $[w_{0,r}, x_{r-k,k}]$ from $w_{0,r}$ to $x_{r-k,k}$ and the parallelogram with corners $x_{r-k,k}$ $x_{r-k,n}$, $x_{m,k}$ and x lie in a single apartment, and in fact in a strip S' of width $\ell = r + n - 2k$ parallel to w with base $[a, x]$ for some vertex a (see Fig. 8).

To see that (when $k < r, n$) $w_{r-k-1,k+1}$ and $x_{r-k,k+1}$ are nonadjacent, note that $w_{r-k-1,k+1}$ is already adjacent to $w_{r-k-1,k} = x_{r-k-1,k}$ and to $x_{r-k,k}$ (see Fig. 7 (a)). If it were also adjacent to $x_{r-k,k+1}$, it would have to equal $x_{r-k-1,k+1}$, contradicting the definition of k. Also, $x_{r-k+1,k}$ cannot be adjacent to $w_{r-k,k+1}$. This was noted above when $k = 0$, and if $k \geq 1$, $x_{r-k+1,k}$ is already adjacent to $x_{r-k,k}$ and $x_{r-k+1,k-1}$. If it were also adjacent to $w_{r-k,k+1}$, then it would have to equal $w_{r-k+1,k}$ (see Fig. 7 (a)), which is impossible, as the sum of the subscripts exceeds r.

Let v' denote the vertex on the right wall of the above strip S' which is at distance $\ell + m + k - r$ from x, and thus at distance ℓ from $x_{r-k,n}$. Let $v'_0 = w_{0,r}, \ldots, v'_\ell = v'$ denote the geodesic from $w_{0,r}$ to v'. Consider the part S'' of S' obtained by deleting the vertices «below» this geodesic, i.e., by deleting the convex hull of x, a, $w_{0,r}$ and v'. As $w_{1,r} \neq w_{0,r-1}$ (if $r \geq 1$), we can enlarge S'' to a strip whose left wall is all of w, and whose «lower edge» is a segment from o to a vertex v (see Fig. 9 (a)). The vertices $v_0 = o, \ldots, v_\ell = v$ of this segment form the geodesic from o to v,

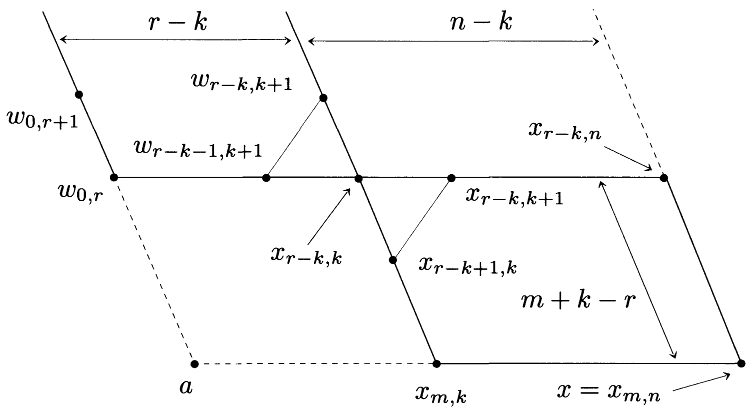

Figure 8

which runs parallel to and at distance r from the geodesic from $w_{0,r}$ to v'. Clearly $v \in S_{\ell,0}(o)$ and $\Omega_{w,v} \neq \emptyset$.

We can now calculate $m(o,x;\omega)$ and $n(o,x;\omega)$ for any $\omega \in \Omega_w$. The calculation involves reducing the problem to that of calculating $m(o,x;\omega')$ and $n(o,x;\omega')$ in the case $d(s^o_{0,1}(\omega'), x_{1,0}) = 2$ and $d(s^o_{1,0}(\omega'), x_{0,1}) = 2$. For then it is easy to see (using Lemma 2.2 and induction, for example) that $(m(o,x;\omega'), n(o,x;\omega')) = (n,m)$.

Let $y \in S_{\nu,0}(o)$ with $\Omega_{w,y} \neq \emptyset$. Let $\omega \in \Omega_{w,y}$. Let j be the largest integer $i \leq \nu, \ell$ such that $y_i = v_i$. Suppose first that $j < \ell, \nu$. Then using the above remark and the fact that $x \in S_{m+k+j-r,n+r-2k-j}(v'_j)$, we have

$$\big(m(v'_j, x;\omega), n(v'_j, x;\omega)\big) = (n+r-2k-j, m+k+j-r)$$

(see Fig. 9 (a) and (b)). Also, $v'_j = s^o_{j,r}(\omega)$, and so

$$\big(m(o, v'_j;\omega), n(o, v'_j;\omega)\big) = (-j, -r).$$

Hence (2.2) implies that

$$\big(m(o,x;\omega), n(o,x;\omega)\big) = (n+r-2k-2j, m+k+j-2r).$$

A similar calculation shows that if $j = \ell$, then

$$\big(m(o,x;\omega), n(o,x;\omega)\big) = (-n-r+2k, m+n-k-r),$$

although the picture is slightly different.

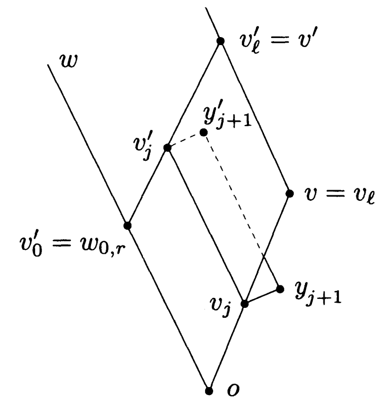

Figure 9 (a)

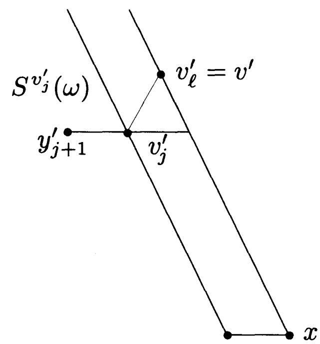

Figure 9 (b)

For $\nu = 1, \ldots, \ell$, consider the distinct vertices $y_\nu^\alpha \in S_{\nu,0}(o)$, $\alpha = 1, \ldots, q$ ($\alpha = 1, \ldots, q+1$ if $\nu = 1$) such that $y_\nu^1 = v_\nu$, and for $\alpha \geq 2$, $v_{\nu-1}$ is the vertex in $S_{\nu-1,0}(o)$ on the geodesic from o to y_ν^α. By the remark after (ii) above, for the given $w \in \Omega^\ell$, the only nonzero $H^\nu_{w,y}$'s are H^0_w and H^ν_{w,y_ν^α}, for $\nu = 1, \ldots, \ell$, and α as above.

Now let $\kappa = -n - r + 2k$ and $\lambda = m + n - k - r$. Thus $\kappa = -\ell \leq 0$, and $(m(o,x;\omega), n(o,x;\omega)) = (\kappa, \lambda)$ for any $\omega \in \Omega_{w,v}$. We first calculate the H^ℓ_{w,y_ℓ^α}. Pick $\omega^\alpha \in \Omega_{w,y_\ell^\alpha}$. Then our calculation above shows that

$$\big(m(o,x;\omega^\alpha), n(o,x;\omega^\alpha)\big) = (\kappa, \lambda)$$

if $\alpha = 1$, and $(\kappa+2, \lambda-1)$ if $\alpha \geq 2$. Thus (assuming $\ell \geq 2$)

$$\left(\frac{s_1}{q}\right)^{\kappa+\lambda} s_2^\lambda = f_s(x)(\omega^1) = H^0_w + \cdots + H^{\ell-1}_{w,v_{\ell-1}} + H^\ell_{w,y_\ell^1},$$

$$\left(\frac{s_1}{q}\right)^{\kappa+\lambda+1} s_2^{\lambda-1} = f_s(x)(\omega^\alpha) = H^0_w + \cdots + H^{\ell-1}_{w,v_{\ell-1}} + H^\ell_{w,y_\ell^\alpha}$$

if $\alpha = 2, \ldots, q$. Summing, and using $\sum\limits_{\alpha=1}^{q} H^\ell_{w,y_\ell^\alpha} = 0$, as required by the definition of $\mathcal{H}^\ell$, we find that

$$H^0_w + \cdots + H^{\ell-1}_{w,v_{\ell-1}} = \frac{s_1^{\kappa+\lambda} s_2^{\lambda-1}}{q^{\kappa+\lambda+2}} \big(qs_2 + (q-1)s_1\big).$$

Substituting this back into the above equations, we get

$$\begin{cases} H^{\ell}_{w,y^1_\ell} = \dfrac{(q-1)(qs_2-s_1)s_1^{\kappa+\lambda}s_2^{\lambda-1}}{q^{\kappa+\lambda+2}}, \\[2ex] H^{\ell}_{w,y^\alpha_\ell} = -\dfrac{(qs_2-s_1)s_1^{\kappa+\lambda}s_2^{\lambda-1}}{q^{\kappa+\lambda+2}} \qquad \text{for } \alpha = 1,\dots,q. \end{cases}$$

Notice that both these expressions are of the form $s_1^{-\ell+1}(qs_2-s_1)$ times a symmetric rational expression in s_1 and s_2. Now a simple (backwards) induction shows that for $i = 2,\dots,\ell$

$$H^i_{w,y^1_i} = \frac{(q-1)(qs_2-s_1)s_1^{\kappa+\lambda}s_2^{i+\kappa+\lambda-1}}{q^{\lambda-i+2}}\left(\frac{s_1^{-\kappa-i+1}-s_2^{-\kappa-i+1}}{s_1-s_2}\right)$$

(the last part of this expression is replaced by its limiting value if $s_1 = s_2$) and that H^i_{w,y^α_i} equals $-H^i_{w,y^1_i}/(q-1)$ for $\alpha = 2,\dots,q$. For once the $H^{i+1}_{w,y^\alpha_{i+1}}$'s have been calculated, choose $\omega^\alpha \in \Omega_{w,y^\alpha_i}$ for $\alpha = 1,\dots,q$, with $\omega^1 \in \Omega_{w,y^\beta_{i+1}}$ for some $\beta \geq 2$, say $\beta = 2$. Then, using the above calculation of $(m(o,x;\omega^\alpha), n(o,x;\omega^\alpha))$, we get equations

$$\left(\frac{s_1}{q}\right)^{\lambda-i} s_2^{\kappa+\lambda+i} = f_s(x)(\omega^1) = H^0_w + \cdots + H^{i-1}_{w,v_{i-1}} + H^i_{w,y^1_i} + H^{i+1}_{w,y^2_{i+1}},$$

$$\left(\frac{s_1}{q}\right)^{\lambda-i+1} s_2^{\kappa+\lambda+i-1} = f_s(x)(\omega^\alpha) = H^0_w + \cdots + H^{i-1}_{w,v_{i-1}} + H^i_{w,y^\alpha_i}$$

if $\alpha = 2,\dots,q$. Adding, and using the known value of $H^{i+1}_{w,y^2_{i+1}}$, we obtain the above formula for the H^i_{w,y^α_i}'s.

Having the above formula for H^2_{w,y^α_2} (or if $\ell = 1$), we find in a similar way that

$$H^1_{w,y^1_1} = \frac{(qs_2-s_1)s_1^{\kappa+\lambda}s_2^{\kappa+\lambda}}{(q+1)q^{\lambda-1}}\left(\frac{s_1^{-\kappa}-s_2^{-\kappa}}{s_1-s_2}\right)$$

and $H^1_{w,y^\alpha_1} = -H^1_{w,y^1_1}/q$ for $\alpha = 2,\dots,q+1$, and

$$H^0_w = \frac{s_1^{\kappa+\lambda}s_2^{\kappa+\lambda}}{(q+1)q^{\lambda-1}}\left(\frac{s_1^{-\kappa+1}-s_2^{-\kappa+1}}{s_1-s_2}\right) - \frac{s_1^{\kappa+\lambda+1}s_2^{\kappa+\lambda+1}}{(q+1)q^{\lambda}}\left(\frac{s_1^{-\kappa-1}-s_2^{-\kappa-1}}{s_1-s_2}\right).$$

This formula is also valid if $\ell = 0$. We see that, as desired, H^0_w is symmetric, and that H^i_{w,y^α_i} has the form $s_1^{-i+1}(qs_2-s_1)f(s_1,s_2)$, where $f(s_1,s_2)$ is symmetric.

In the case $r = m$ illustrated in Fig. 7 (b), an integer $\ell \geq 0$ and a vertex v are found satisfying (i) and (ii) above in almost exactly the same way, with the roles of r and $x_{r-k,k}$ replaced by $m+h$ and $x_{m+h-k,k}$, respectively. This time $\ell = m+n+h-2k$. The formulas for the H^i_{w,y^α_i}'s are exactly as above. In this case, $\kappa = -\ell$ and $\lambda = n-h-k$.

4. Property (T).

Proposition 4.1 below, the converse of Theorem 3.5, is stated in terms of a general triangle building Δ. Recall that if $v \in \mathcal{V}_\Delta$, the set of vertices at distance 1 from v has the structure of a projective plane, which we denote Π_v : the points and lines are the vertices x such that $\tau(x) \equiv \tau(v)+1$ mod 3, respectively $\tau(x) \equiv \tau(v) - 1$ mod 3, with a point x incident with a line y if x, y and v lie in a common chamber.

For the remainder of the paper, we shall be concerned with $\widetilde{A}_2$ groups Γ (see § 1). The Cayley graph of Γ with respect to its natural generators and their inverses is (the 1-skeleton of) a triangle building Δ ([2], Thm 3.4), and Γ acts simply transitively by left multiplication (which is type-rotating) on $\mathcal{V}_\Delta = \Gamma$. The projective plane Π_e of neighbours of e consists of «points» a_y, $y \in P$, and «lines» a_x^{-1}, $x \in P$, and is isomorphic to the plane (P, L) used to define Γ : a point a_y is incident with a line a_x^{-1} if and only if $a_x a_y a_z = e$ for some $z \in P$. The set $S_{m,n}(e)$ equals the set of $g \in \Gamma$ for which, in any minimal word in the generators and their inverses, there are m generators and n inverse generators, as noted in § 1, and the type of $g \in S_{m,n}(e)$ is $(m-n)$ mod 3. Conversely, if Δ is any triangle building admitting a $\Gamma \leq \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ which acts simply transitively on $\mathcal{V}_\Delta$, then Γ is isomorphic to an $\widetilde{A}_2$ group, and Δ is isomorphic to the associated building ([2], Thms 3.1 and 3.5). The algebra $\mathcal{A}$ of averaging operators described in § 2 can be identified with the convolution algebra of e-biradial functions on Γ, as $A^+f = f * \mu^-$ and $A^-f = f * \mu^+$ in the notation of § 1.

PROPOSITION 4.1. — *Suppose that the kernel $k_{z,\bar{z}}$ defined before Lemma 2.1 is positive definite. Then z belongs to the set Σ^*. In fact, to show that $z \in \Sigma^*$, we need only fix $o \in \mathcal{V}_\Delta$ and assume that $M = (k_{z,\bar{z}}(u,v))_{u,v \in \Pi_o}$ is a positive definite matrix.*

Proof. — Write $p_{m,n}$ in place of $p_{m,n}(z,\bar{z})$, and $S_{m,n}$ for $S_{m,n}(o)$. We calculate the eigenvalues of M. Let $m = \frac{1}{2}|\Pi_o| = q^2+q+1$, and write

$$\lambda I_{2m} - M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are $m \times m$ matrices indexed by $S_{1,0} \times S_{1,0}$, $S_{1,0} \times S_{0,1}$, $S_{0,1} \times S_{1,0}$ and $S_{0,1} \times S_{0,1}$, respectively. Both A and D are $m \times m$ matrices of the form

$$
(4.1) \qquad \begin{pmatrix} \alpha & \beta & \cdots & \beta \\ \beta & \alpha & \cdots & \beta \\ \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \cdots & \alpha \end{pmatrix}
$$

where $\alpha = \lambda - 1$ and $\beta = -p_{1,1}$. Now $C = B^*$, and the (u,v) entry of B is either $-p_{1,0}$ or $-p_{0,2}$ according as $u \in S_{1,0}$ and $v \in S_{0,1}$ are incident in Π_o or not. The values of the $p_{m,n}$ we need can be read off from equations (3.1)–(3.7) in [4] :

$$
\begin{cases} p_{1,0} = z, \\ p_{1,1} = \big((q^2+q+1)|z|^2 - 1\big)/(q^2+q), \\ p_{0,2} = \big((q^2+q+1)\bar{z}^2 - (q+1)z\big)/q^2. \end{cases}
$$

Now for any $m \times m$ matrices A, B, C and D, with A assumed invertible,

$$
\begin{aligned}
\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det\left\{\begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix}\begin{pmatrix} I_m & 0 \\ C & I_m \end{pmatrix}\begin{pmatrix} I_m & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}\right\} \\
&= \det(AD - ACA^{-1}B) \\
&= \det(AD - CB) \quad \text{if } A \text{ and } C \text{ commute.}
\end{aligned}
$$

(Our thanks to R.B. Howlett for pointing this out to us.) The hypothesis that A is invertible can easily seen to be unnecessary for the conclusion to hold.

In the case at hand, A and C commute because C has $q+1$ entries $-\overline{p_{1,0}}$ and q^2 entries $-\overline{p_{0,2}}$ in each row and in each column.

Now AD is an $m \times m$ matrix of the form (4.1), where

$$
\begin{aligned}
\alpha &= (\lambda-1)^2 + (m-1)p_{1,1}^2, \\
\beta &= -2(\lambda-1)p_{1,1} + (m-2)p_{1,1}^2.
\end{aligned}
$$

Also, $CB = B^*B$ has form (4.1). Indeed, if $u, v \in S_{0,1}$ with $u \neq v$, the (u,v) entry $\sum_{w \in S_{1,0}} \overline{B_{w,u}}\, B_{w,v}$ of B^*B equals

$$
|p_{1,0}|^2 + q(\overline{p_{1,0}}\, p_{0,2} + p_{1,0}\, \overline{p_{0,2}}) + (q^2 - q)|p_{0,2}|^2.
$$

For there is a unique $w = w_0 \in S_{1,0}$ incident with both u and v. This gives the $|p_{1,0}|^2$ term. Each of the q w's in $S_{1,0}$ incident with u (resp., with v) but not equal to w_0 gives a $\overline{p_{1,0}}\, p_{0,2}$ (resp., $p_{1,0}\, \overline{p_{0,2}}$) term, and each of the $(q^2 - q)$ w's in $S_{1,0}$ incident with neither u nor v yields a $|p_{0,2}|^2$ term.

Similarly, the diagonal entries of B^*B are all $(q+1)|p_{1,0}|^2 + q^2|p_{0,2}|^2$.

Thus $\det(\lambda I_{2m} - M)$ is the determinant of an $m \times m$ matrix (4.1), where

$$\begin{aligned}
\alpha &= (\lambda - 1)^2 + (q^2 + q)p_{1,1}^2 - \big((q+1)|p_{1,0}|^2 + q^2|p_{0,2}|^2\big), \\
\beta &= -2(\lambda - 1)p_{1,1} + (q^2 + q - 1)p_{1,1}^2 \\
&\qquad\qquad - \big(|p_{1,0}|^2 + q(\overline{p_{1,0}}\, p_{0,2} + p_{1,0}\, \overline{p_{0,2}}) + (q^2 - q)|p_{0,2}|^2\big).
\end{aligned}$$

The determinant of a matrix (4.1) is $(\alpha - \beta)^{m-1}(\alpha + (m-1)\beta)$. Here,

$$\begin{aligned}
\alpha - \beta &= (\lambda - 1 + p_{1,1})^2 - q\,|p_{1,0} - p_{0,2}|^2, \\
\alpha + (m-1)\beta &= \big(\lambda - 1 - (q^2+q)p_{1,1}\big)^2 - \big|(q+1)p_{1,0} + q^2 p_{0,2}\big|^2.
\end{aligned}$$

Setting $\alpha + (m-1)\beta = 0$ and solving for λ, we have :

$$\lambda = 0 \quad \text{or} \quad 2(q^2 + q + 1)|z|^2 \geq 0 \quad \text{for all } z.$$

Setting $\alpha - \beta = 0$ and solving, we get $\lambda = 1 - p_{1,1} \pm \sqrt{q}\,|p_{1,0} - p_{0,2}|$. These values are both nonnegative if and only if $p_{1,1} \leq 1$ and $(1 - p_{1,1})^2 \geq q|p_{1,0} - p_{0,2}|^2$, i.e.,

$$(q+1)^2(z^3 + \bar{z}^3) - (q^2 + q + 1)|z|^4 - (q^2 + 4q + 1)|z|^2 + q \geq 0.$$

That is, if and only if $z \in \Sigma^*$.

Remark. — In fact, to show that $z \in \Sigma^*$, we need only assume that $M = (k_{z,\bar{z}}(u,v))_{u,v \in A}$ is a positive definite matrix, where

$$A = \{o, v_0\} \cup \big\{u \in S_{1,0}(o) \mid u \text{ and } v_0 \text{ are incident in } \Pi_o\big\},$$

where $v_0 \in S_{0,1}(o)$ (see Fig. 10). Note that $|A| = q + 3$. One can calculate $\det(\lambda I_{q+3} - M)$ as in the last proof.

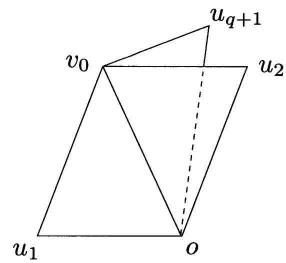

Figure 10

Let Δ be a triangle building. Let $\Gamma \leq \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ act simply transitively on the set of vertices of Δ. Fix a vertex o of Δ, and for $f : \Gamma \to \mathbb{C}$ define $\mathcal{E}f : \Gamma \to \mathbb{C}$ by

$$(\mathcal{E}f)(\gamma) = N_{m,n}^{-1} \sum_{\substack{\gamma' \in \Gamma: \\ \gamma' o \in S_{m,n}(o)}} f(\gamma') \quad \text{if} \quad \gamma o \in S_{m,n}(o).$$

If we identify $\gamma \in \Gamma$ with $\gamma o \in \mathcal{V}_\Delta$, $\mathcal{E}$ is a projection of the space of functions on Γ onto the space of e-biradial functions on Γ (see [4], § 2).

PROPOSITION 4.2. — *Let $K = \{g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta) : go = o\}$. Assume that K acts transitively on each set $S_{m,n}(o)$. With notation as in the preceding paragraph, let $\varphi : \Gamma \to \mathbb{C}$ be positive definite. Then $\mathcal{E}\varphi : \Gamma \to \mathbb{C}$ is positive definite.*

Proof. — Write $G = \mathrm{Aut}_{\mathrm{tr}}(\Delta)$. Since Γ acts simply-transitively, we have $G = \Gamma K$ and $\Gamma \cap K = \{\mathrm{id}\}$. There is a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ and there is a $v_0 \in \mathcal{H}$ such that $\varphi(\gamma) = \langle v_0, \pi(\gamma)v_0 \rangle$ for all $\gamma \in \Gamma$. Let $\pi' = \mathrm{Ind}_\Gamma^G(\pi)$ be the unitary representation of G obtained by inducing π from Γ to G. Thus the representation space $\mathcal{H}'$ of π' is the completion of the space $\mathcal{H}_0'$ of continuous functions $f : G \to \mathcal{H}$ such that $f(g\gamma) = \pi(\gamma^{-1})f(g)$ for all $g \in G$ and $\gamma \in \Gamma$ with respect to the norm given by

$$\|f\|^2 = \int_{G/\Gamma} \|f(g)\|^2 \, \mathrm{d}\mu(g\Gamma).$$

Here μ is the unique G-invariant probability on G/Γ (see, e.g., [13], p. 37; note that G/Γ is compact here). Notice that for $f \in \mathcal{H}_0'$, $\|f(g)\|$ depends only on $g\Gamma$. By [13], p. 38, we have

$$\int_{G/\Gamma} F(g\Gamma) \, \mathrm{d}\mu(g\Gamma) = \int_K F(k\Gamma) \, \mathrm{d}k$$

for continuous functions F on G/Γ, where dk is normalized Haar measure on K.

Let f_0 be the unique element of $\mathcal{H}'_0$ such that $f_0(k) = v_0$ for all $k \in K$; thus $f_0(k\gamma) = \pi(\gamma^{-1})v_0$ for $k \in K$ and $\gamma \in \Gamma$. The statement of the Proposition is immediate from the fact that $(\mathcal{E}\varphi)(\gamma) = \langle f_0, \pi'(\gamma)f_0\rangle$ for $\gamma \in \Gamma$, as we now verify. Indeed,

$$\begin{aligned}\langle f_0, \pi'(\gamma)f_0\rangle &= \int_{G/\Gamma} \langle f_0(g), (\pi'(\gamma)f_0)(g)\rangle \, d\mu(g\Gamma) \\ &= \int_K \langle v_0, f_0(\gamma^{-1}k)\rangle \, dk.\end{aligned}$$

Let φ' be the right K-invariant function on G which agrees with φ on Γ : $\varphi'(\gamma k) = \varphi(\gamma)$ for $\gamma \in \Gamma$ and $k \in K$. Let φ'' be the corresponding function on G/K. If we write $\gamma^{-1}k = k'\gamma'$, where $k, k' \in K$ and $\gamma, \gamma' \in \Gamma$, then

$$\begin{aligned}\langle v_0, f_0(\gamma^{-1}k)\rangle = \langle v_0, f_0(k'\gamma')\rangle &= \langle v_0, \pi({\gamma'}^{-1})v_0\rangle \\ &= \varphi({\gamma'}^{-1}) = \varphi'({\gamma'}^{-1}{k'}^{-1}) \\ &= \varphi'(k^{-1}\gamma) = \varphi''(k^{-1}\gamma K).\end{aligned}$$

Thus

$$\begin{aligned}\int_K \langle v_0, f_0(\gamma^{-1}k)\rangle \, dk &= \int_K \varphi''(k^{-1}\gamma K)\, dk = \int_K \varphi''(k\gamma K)\, dk \\ &= \sum_{\substack{\gamma' \in \Gamma, \\ \gamma' o \in S_{m,n}(o)}} \int_{\{k \in K:\, k\gamma \in \gamma' K\}} \varphi''(\gamma' K)\, dk \qquad (\text{if } \gamma o \in S_{m,n}(o)) \\ &= |S_{m,n}(o)|^{-1} \sum_{\substack{\gamma' \in \Gamma, \\ \gamma' o \in S_{m,n}(o)}} \varphi(\gamma') = (\mathcal{E}\varphi)(\gamma)\end{aligned}$$

because the sets $\{k \in K : k\gamma \in \gamma' K\}$ have equal Haar measure for each $\gamma' \in \Gamma$ such that $\gamma' o \in S_{m,n}(o)$, by our hypothesis on K.

Remarks

1) The hypothesis on K in the Proposition is satisfied if Δ is the building Δ_F associated with $\mathrm{SL}(3, F)$, where F is a local field with residual field of order q (see [17], for example). For then $\mathrm{PGL}(3, F) \leq \mathrm{Aut}_{\mathrm{tr}}(\Delta)$, and $S_{m,n}(o)$ is the K orbit of $g_{m,n}o$, where $g_{m,n}$ is the image in $\mathrm{PGL}(3, F)$ of the diagonal matrix with entries $(1, \varpi^n, \varpi^{m+n})$ (where $\varpi \in F$ has

valuation 1). It may be that any triangle building Δ for which K is transitive on each set $S_{m,n}(o)$ must be a building Δ_F for some F. We do not know whether φ is positive definite implies that $\mathcal{E}\varphi$ is positive definite if we do not assume this transitivity property of K.

2) The method of proof of the last Proposition is applicable also when Γ is a free group, for example, acting simply transitively on a homogeneous tree. The proofs of the corresponding fact for this case appearing in [5], [7] and [12] are not correct. Another proof, due to Haagerup, appears in a related context in [15].

Let Γ be an $\widetilde{A}_2$ group. When the projective plane $\Pi_e \cong (P, L)$ of nearest neighbours of e in the associated building is the usual Desarguesian plane $\mathrm{PG}(2, q)$, we can prove a weak version of Proposition 4.2. All we actually need are certain transitivity properties of the collineation group of Π_e. That these hold for $\mathrm{PG}(2, q)$ follows from the fact that $\mathrm{PGL}(3, \mathbf{F}_q) \leq \mathrm{Aut}(\mathrm{PG}(2, q))$ acts transitively on the set of quadrangles in $\mathrm{PG}(2, q)$ ([11], Thm 2.12) (recall that a *quadrangle* in a projective plane is an ordered set of four distinct points, no three of which are collinear). Actually, these properties characterize $\mathrm{PG}(2, q)$ amongst the finite projective planes of order q ([11], Thm 14.13). As each $g \in \mathrm{Aut}_{\mathrm{tr}}(\Delta)$ which fixes e induces a collineation of Π_e, the hypothesis of these properties is just a weak form of the hypothesis in Proposition 4.2.

PROPOSITION 4.3. — *Assume that the projective plane Π_e of nearest neighbours of e is the usual Desarguesian plane* $\mathrm{PG}(2, q)$. *Let $\varphi : \Gamma \to \mathbb{C}$ be positive definite. Then $((\mathcal{E}\varphi)(x^{-1}y))_{x,y\in\Pi_e}$ is a positive definite matrix.*

Proof. — Let $n = |\Pi_e| = 2(q^2 + q + 1)$, and let M be the $n \times n$ matrix $(\varphi(x^{-1}y))_{x,y\in\Pi_e}$. Let $A = \mathrm{Aut}(\Pi_e)$ be the group of collineations π of Π_e, i.e., bijections of Π_e mapping points to points, lines to lines and preserving incidence. For each $\pi \in A$, let P^π be the corresponding $n \times n$ permutation matrix : $(P^\pi)_{x,y} = 1$ if $x = \pi(y)$ and 0 otherwise. Now M is a positive definite matrix, and so

$$M' = \frac{1}{|A|} \sum_{\pi\in A} (P^\pi)^{-1} M P^\pi$$

is positive definite, and the (x, y) entry of M' is

$$\frac{1}{|A|} \sum_{\pi\in A} \varphi\big(\pi(x)^{-1}\pi(y)\big).$$

But this is $(\mathcal{E}\varphi)(x^{-1}y)$. For let $x \in \Pi_e$ be a point (i.e., $x \in S_{1,0} = S_{1,0}(e)$) and let $y = z^{-1}$ be a line in Π_e (i.e., $y \in S_{0,1} = S_{0,1}(e)$) incident with x. As A is transitive on the set of quadrangles in Π_e because of the hypothesis $\Pi_e \cong \mathrm{PG}(2,q)$, it is also transitive on the set of incident point-line pairs. As there are $(q+1)(q^2+q+1)$ such pairs, the above entry is

$$\frac{1}{(q+1)(q^2+q+1)} \sum_{\substack{x' \in S_{1,0} \\ \text{incident with} \\ z'^{-1} \in S_{0,1}}} \varphi(x'^{-1}z'^{-1}) = \frac{1}{(q^2+q+1)} \sum_{x'' \in S_{1,0}} \varphi(x'')$$

the last equality because each point $x'' \in S_{1,0}$ may be expressed $x'' = x'^{-1}y'^{-1}$, where $x', z' \in S_{1,0}$, in precisely $q+1$ ways. So the entry is $(\mathcal{E}\varphi)(x^{-1}y)$, because $x^{-1}y = x^{-1}z^{-1} \in S_{1,0}$. Similarly, using transitivity of A on nonincident point-line pairs, and on pairs of distinct points, and on pairs of distinct lines, we find that $(\mathcal{E}\varphi)(x^{-1}y) = M'_{x,y}$ for all $x, y \in \Pi_e$.

LEMMA 4.4. — *Let Γ be a countable discrete group. Let $h : \Gamma \to \mathbb{C}$ have finite support and satisfy $h * h^* = h^* * h$, and let $z \in \mathbb{C}$. Then the following assertions are equivalent :*

(i) $z \in \mathrm{Sp}(h)$, *the spectrum of h in the full C^* algebra $C^*(\Gamma)$ of Γ;*

(ii) $z \in \mathrm{Sp}(\pi(h))$ *for some unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ of Γ;*

(iii) z *is an eigenvalue of $\pi(h)$ for some unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ of Γ;*

(iv) $h * \varphi = z\varphi$, *or, equivalently, $\varphi * h = z\varphi$, for some nonzero positive definite function φ on Γ.*

Proof. — If $\varphi : \Gamma \to \mathbb{C}$ is positive definite, with $\varphi(e) = 1$, there is a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ of Γ and a cyclic unit vector $v \in \mathcal{H}_\pi$ such that $\varphi(x) = \langle v, \pi(x)v \rangle$ for all $x \in \Gamma$. Then

$$\big\langle zv - \pi(h)v, \pi(x)v \big\rangle = z\varphi(x) - (h * \varphi)(x) \tag{4.2}$$

for all $x \in \Gamma$. Hence $h * \varphi = z\varphi$ if and only if $\pi(h)v = zv$, as v is cyclic. Equivalently, $\pi(h^*)v = \bar{z}v$, because $\pi(h)$ is normal (and because $\|T^*v\| = \|Tv\|$ for normal operators T), so that $h^* * \varphi = \bar{z}\varphi$, and $\varphi * h = (h^* * \varphi)^* = (\bar{z}\varphi)^* = z\varphi$, as $\varphi^* = \varphi$. This shows that (iii) and the two forms of (iv) are equivalent.

We next show that (ii) implies (iv). If $z \in \mathrm{Sp}(\pi(h))$, then as $\pi(h)$ is normal, there is for each $\epsilon > 0$ a unit vector $v_\epsilon \in \mathcal{H}_\pi$ such that

$\|zv_\epsilon - \pi(h)v_\epsilon\| < \epsilon$. Then for each $x \in \Gamma$, (4.2) implies that

$$\big|z\varphi_\epsilon(x) - (h * \varphi_\epsilon)(x)\big| = \big|\langle zv_\epsilon - \pi(h)v_\epsilon, \pi(x)v_\epsilon\rangle\big| < \epsilon$$

where $\varphi_\epsilon(y) = \langle v_\epsilon, \pi(y)v_\epsilon\rangle$ for $y \in \Gamma$. We have $|\varphi_\epsilon(y)| \leq 1$ for all $y \in \Gamma$ and $\epsilon > 0$, and so for a suitable sequence $\epsilon_1 > \epsilon_2 > \cdots > 0$, $\varphi(y) = \lim_{j\to\infty} \varphi_{\epsilon_j}(y)$ exists for each $y \in \Gamma$. Then φ is positive definite, and $h * \varphi = z\varphi$ because h has finite support. Thus (iv) holds.

As for the remaining implications, that (i) implies (ii) is immediate, because $\mathrm{Sp}(h) = \mathrm{Sp}(\pi_{\mathrm{un}}(h))$, where π_{un} is the universal representation of Γ. Obviously, (iii) implies (ii), and it is easy to see that (ii) implies (i).

Remark. — The referee pointed out to us that the equivalence of (ii) and (iii) in the last lemma is valid in a more general context (see Thm V.1.4 in [18]).

COROLLARY 4.5. — *Let Γ be an $\widetilde{A}_2$ group. Assume that $\Pi_e \cong \mathrm{PG}(2, q)$. Then $\mathrm{Sp}(\mu^+) = \Sigma^*$.*

Proof. — Let $z \in \mathrm{Sp}(\mu^+)$. Then as μ^+ and $\mu^{+*} = \mu^-$ commute, Lemma 4.4 shows that $\mu^+ * \varphi = \varphi * \mu^+ = z\varphi$ for some positive definite function φ satisfying $\varphi(e) = 1$. Thus $\mu^- * \varphi = \varphi * \mu^- = \bar{z}\varphi$ too. Now $\mathcal{E}$ commutes with any operator $f \mapsto f * g$, where g is biradial ([4], § 2), and so

$$\begin{cases} z\mathcal{E}\varphi = \mathcal{E}(\varphi * \mu^+) = (\mathcal{E}\varphi) * \mu^+, \\ \bar{z}\mathcal{E}\varphi = \mathcal{E}(\varphi * \mu^-) = (\mathcal{E}\varphi) * \mu^-. \end{cases}$$

As $\mathcal{E}\varphi$ is e-biradial, this implies that $\mathcal{E}\varphi$ is the spherical function $\varphi_{\bar{z},z}$ ([4], Prop. 3.4). But Proposition 4.3 shows that $((\mathcal{E}\varphi)(x^{-1}y))_{x,y\in\Pi_e}$ is positive definite, so that, by Proposition 4.1, $\bar{z} \in \Sigma^*$, and therefore $z \in \Sigma^*$.

Conversely, if $z \in \Sigma^*$ then $\varphi = \varphi_{\bar{z},z}$ is positive definite by Theorem 3.5, and $\varphi(e) = 1$ and $\varphi * \mu^+ = z\varphi$. Thus $z \in \mathrm{Sp}(\mu^+)$ by Lemma 4.4. Note that we don't really need to use Theorem 3.5 here. For if $z \in \Sigma^* \setminus \{1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{-2\pi i/3}\}$, then $\varphi_{\bar{z},z}$ is positive definite by Corollary 3.4, while if $z \in \{1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{-2\pi i/3}\}$, then $\varphi_{\bar{z},z}$ is a character of Γ (see the beginning of the proof of Theorem 3.5), and hence positive definite.

THEOREM 4.6. — *Let Γ be an $\widetilde{A}_2$ group. Assume that $\Pi_e \cong \mathrm{PG}(2,q)$. Let ϵ_q be as in the introduction. Then Γ has property* (T), *and if S is the set of natural generators of Γ and their inverses, then*

$$\kappa(\Gamma, S) = \sqrt{2\epsilon_q}.$$

Proof. — Let π be a unitary representation of Γ without fixed vectors. Then if $\mu = \frac{1}{2}(\mu^+ + \mu^-)$, then $\mathrm{Sp}(\pi(\mu)) \subset \{\Re z : z \in \Sigma^*\} \subset [-1, 1-\epsilon_q] \cup \{1\}$. Now $1 \notin \mathrm{Sp}(\pi(\mu))$, for if it were, then 1 would be eigenvalue of $\pi(\mu)$. But $\pi(\mu)v = v$ implies that $\pi(x)v = v$ for each $x \in S$, and thus for each $x \in \Gamma$, by strict convexity of $\mathcal{H}_\pi$ (or [8], Lemma 3), and this would contradict our hypothesis. Thus $\kappa(\pi, S) \geq \sqrt{2\epsilon_q}$ ([8], Proposition I(6)). Thus Γ has property (T), and $\kappa(\Gamma, S) \geq \sqrt{2\epsilon_q}$.

To see that $\kappa(\Gamma, S) \leq \sqrt{2\epsilon_q}$ holds, let $z \in \Sigma^*$, so that $\varphi_{z,\bar{z}}$ is positive definite (Corollary 3.4). Let $\pi_z : \Gamma \to \mathcal{H}_z$ be a unitary representation of Γ with cyclic unit vector v satisfying $\langle v, \pi_z(x)v \rangle = \varphi_{z,\bar{z}}(x)$ for $x \in \Gamma$. Now π_z has no nonzero fixed vector unless $z = 1$. For if it did, then for some $\epsilon > 0$, $\varphi : x \mapsto \varphi_{z,\bar{z}}(x) - \epsilon$ would be positive definite on Γ. Thus

$$0 \leq \sum_{x \in \Gamma} \varphi(x)(f^* * f)(x^{-1}) = f^* * (f * \varphi)(e)$$

would hold for every finitely supported $f : \Gamma \to \mathbb{C}$. Applying this to $f = \mu^- - z\delta_e$, and using $\mu^- * \varphi_{z,\bar{z}} = z\varphi_{z,\bar{z}}$ and $\mu^+ * \varphi_{z,\bar{z}} = \bar{z}\varphi_{z,\bar{z}}$, we obtain $0 \leq -\epsilon|1-z|^2$, which can only happen if $z = 1$.

Now let $z = 1 - \epsilon_q$. Then

$$\left\|\pi_z(x)v - v\right\|^2 = 2\big(1 - \mathrm{Re}\,\langle v, \pi_z(x)v \rangle\big) = 2(1-z) = 2\,\epsilon_q$$

for each $x \in S$. Hence $\kappa(\Gamma, S) \leq \kappa(\pi_z, S) \leq \sqrt{2\epsilon_q}$.

Remark. — The first half of the last proof can be slightly simplified if we appeal to the following generalization of Proposition I (6) in [8] (which has exactly the same proof) : if $\pi(h)$ is normal and $\mathrm{Sp}\,\pi(h) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z \leq 1 - \epsilon\}$, then $\kappa(\pi, S) \geq \sqrt{2\epsilon}$.

BIBLIOGRAPHY

[1] M. BURGER, Kazhdan constants for SL(3, $\mathbb{Z}$), J. reine angew. Math., 413 (1991), 36–67.

[2] D.I. CARTWRIGHT, A.M. MANTERO, T. STEGER, A. ZAPPA, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$ I, Geom. Ded., 47 (1993), 143–166.

[3] D.I. CARTWRIGHT, A.M. MANTERO, T. STEGER, A. ZAPPA, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$ II : the cases $q = 2$ and $q = 3$, Geom. Ded., 47 (1993), 167–223.

[4] D.I. CARTWRIGHT, W. MŁOTKOWSKI, Harmonic analysis for groups acting on triangle buildings, to appear, J. Aust. Math. Soc.

[5] J.M. COHEN, L. DE MICHELE, The radial Fourier–Stieltjes algebra of free groups, Operator Algebras and Theory Contemporary Mathematics, 10, Am. Math. Soc., Providence (1982), 33–40.

[6] M. COWLING and T. STEGER, The irreducibility of restrictions of unitary representations to lattices, J. reine angew. Math., 420 (1991), 85–98.

[7] A. FIGÀ–TALAMANCA and M.A. PICARDELLO, Harmonic Analysis on Free Groups, Lect. Notes Pure Appl. Math., 87 (1983).

[8] P. DE LA HARPE, A.G. ROBERTSON and A. VALETTE, On the spectrum of the sum of generators for a finitely generated group, Israel J. Math., 81 (1993), 65–96.

[9] P. DE LA HARPE and A. VALETTE, La propriété (T) de Kazhdan pour les groupes localment compacts, Astérisque, Soc. Math. France, 175 (1989).

[10] R. HOWE, ENG CHYE TAN, Non–Abelian Harmonic Analysis, Applications of SL(2, $\mathbb{R}$), Universitext, Springer–Verlag, New York (1992).

[11] D.R. HUGHES, F.C. PIPER, Projective Planes, Graduate Texts in Mathematics, 6 (1973).

[12] A. IOZZI, M.A. PICARDELLO, Spherical functions on symmetric graphs, p. 344–386 in Harmonic Analysis, Lecture Notes in Math. 992, Springer Verlag, Berlin Heidelberg New York (1983).

[13] S. LANG, $SL_2(\mathbb{R})$, Graduate Texts in Mathematics 105, Springer Verlag, New York Berlin Tokyo (1985).

[14] A.M. MANTERO and A. ZAPPA, Spherical functions and spectrum of the Laplacian operators on buildings of rank 2, to appear, Boll. Un. Mat. Ital.

[15] W. MŁOTKOWSKI, Positive Definite Radial Functions on Free Product of Groups, Bollettino Un. Mat. Ital. (7), 2-B (1988), 53–66.

[16] I. PAYS and A. VALETTE, Sous–groupes libres dans les groupes d'automorphismes d'arbres, L'Enseignement Mathématique, 37 (1991), 151–174.

[17] M. RONAN, Lectures on Buildings, Perspectives in Math., vol. 7., Academic Press, (1989).

[18] H.H. SCHAEFER, Banach lattices and positive operators, Grundlehren der Math. Wiss., Springer–Verlag, Berlin, (1974).

[19] J. TITS, Buildings of spherical type and finite BN–pairs, Lecture Notes in Math., 386 (1974).

[20] J. TITS, Immeubles de type affine in Buildings and the Geometry of Diagrams, Proc. CIME Como 1984 (L.A. Rosati, ed), Lecture Notes in Math., 1181, Springer–Verlag, Berlin (1986), 159–190.

Manuscrit reçu le 1er avril 1993,
révisé le 23 juillet 1993.

D.I. CARTWRIGHT,
School of Mathematics and Statistics
The University of Sydney
NSW 2006 Sydney (Australia)
&
W. MŁOTKOWSKI,
Institute of Mathematics
The University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław (Poland)
&
T. STEGER
Department of Mathematics
Boyd Graduate Study Research Center
University of Georgia
Athens GA 30602-7403 (USA).