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J.-L. JOLY

G. MÉTIVIER

J. RAUCH

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## COHERENT NONLINEAR WAVES AND THE WIENER ALGEBRA

by J.-L. JOLY, G. MÉTIVIER and J. RAUCH<sup>(1)</sup>

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### 1. Introduction.

We study the interaction of high frequency solutions to semilinear systems of the form

$$(1.1) \quad Lu = f(t, x, u, \bar{u})$$

where  $L(t, x, \partial_t, \partial_x)$  is a first order symmetric hyperbolic system of partial differential operators on  $\mathbf{R}^{1+d}$ .

The waves have amplitude  $O(1)$  and wavelength  $\varepsilon$  tending to zero. For the semilinear problems (1.1) this critical size is called weakly nonlinear geometric optics. As epsilon tends to zero, nonlinear effects are negligible for times  $o(1)$  and important for times  $O(1)$ .

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We construct solutions on a fixed time interval  $[0, \underline{t}]$  which have asymptotic description

$$(1.2) \quad u^\varepsilon(t, x) = U(t, x, t/\varepsilon, x/\varepsilon) + o(1)$$

where the profile  $U(t, x, T, X)$  is almost periodic in  $T, X$  and is determined by a system of equations which is easier to analyse or compute numerically than (1.1).

In the introduction, we limit the discussion to constant coefficient operators  $L$  and phases which are linear function of  $t, x$ . Thus,

$$L = \partial_t + \sum A_j \partial / \partial x_j.$$

The general case of variable coefficients with phases satisfying a coherence assumption is presented in §3.

The main advance in this paper compared to earlier works is that it treats multidimensional problems with profiles that are almost periodic in  $T, X$ . Previous work for  $d > 1$  required either quasiperiodicity in  $X$  ([JMR4], [JMR5], [S]), small divisor assumptions on the phases, null conditions on the nonlinearity permitting high order asymptotics ([D], [JMR6]), or an oscillating plane hypothesis which forces the solutions to resemble the case of  $d = 1$ .

The main novelty in the analysis is the space of profiles. We take

$$(1.3) \quad U(t, x, T, X) = \sum_{\tau, \omega \in \mathbf{R}^{1+d}} U_{\tau, \omega}(t, x) e^{i(\tau T + \omega \cdot X)}$$

where

$$(1.4) \quad \sum_{\tau, \omega} \|U_{\tau, \omega}(t, x)\|_{C([0, \underline{t}]; H^s(\mathbf{R}^d))} < \infty.$$

Here  $s > d/2$  so that for  $t, x$  fixed  $U(t, x, T, X)$  is an almost periodic function of  $T, X$  with absolutely convergent Fourier expansion. That is,  $U$  is an element of the Wiener algebra as a function of the fast variables. The possibility of using this algebra to describe profiles was suggested in ([JMR4], [JMR5] §11).

The nonlinear function  $f$  is assumed to be real analytic in its dependence on  $u, \bar{u}$ . This restriction is imposed because the Wiener algebra is invariant under such maps but not under general smooth functions (see [Kat] Th. 8.6).

Solutions of form 1.3 arise as solutions of naturally related oscillatory initial value problems.

$$(1.5) \quad Lu^\varepsilon = f(t, x, u^\varepsilon, \bar{u}^\varepsilon), \quad u^\varepsilon(0, x) = \Gamma(x, x/\varepsilon)$$

where

$$(1.6) \quad \Gamma(x, X) = \sum_{\omega \in \mathbf{R}^d} a_\omega(x) e^{i\omega \cdot X}$$

is an almost periodic function of the fast variables  $X$  such that

$$(1.7) \quad \sum_{\omega} \|a_\omega\|_{H^s(\mathbf{R}^d)} < \infty.$$

Then there is a  $\underline{t} > 0$  so that 1.4 is valid with error  $o(1)$  in  $L^\infty([0, \underline{t}] \times \mathbf{R}^d)$ .

The profile  $U$  is uniquely determined by a system of equations which involve an averaging operator  $\mathbf{E}$  defined on almost periodic functions of  $T, X$  by

$$(1.8) \quad \mathbf{E}(a(t, x) e^{i(\tau T + \omega \cdot X)}) = (\Pi_{\tau, \omega} a(t, x)) e^{i(\tau T + \omega \cdot X)}$$

where  $\Pi_{\tau, \omega}$  is the spectral projection of  $\mathbf{C}^k$  onto  $\ker(L(\tau, \omega))$ . In particular  $\Pi_{\tau, \omega} = 0$  if  $\tau, \omega$  does not belong to the characteristic variety of  $L$ . The system of equations determining  $U$  is then

$$(1.9) \quad \mathbf{E}U = U, \quad U(0, x, 0, X) = \Gamma(x, X),$$

$$(1.10) \quad \mathbf{E}[L(D_{t,x})U + f(t, x, U(t, x, T, X), \bar{U}(t, x, T, X))] = 0.$$

An innovation in this paper is that it is not difficult for us to allow systems with characteristics of variable multiplicity, for example the equations describing conical refraction in crystal optics (see §4). For that system nonlinear effects couple the conical points with others so incoming waves with spectrum far from the optic axis can trigger conical refraction.

The analysis of 1.5 is by decomposition into modes. Interaction generates  $\mathbf{Z}$ -linear combinations of phases and the solution is expressed as sum of terms  $a_\varepsilon e^{i\varphi/\varepsilon}$  where the phase  $\varphi$  belongs to a countable  $\mathbf{Z}$ -module. Decomposing  $f(u, \bar{u})$  into such terms then inverting  $L$  is our approach. The analysis is mode by mode. The key steps are to derive  $\varepsilon$  independent bounds for the inversion of  $L$  and then to analyse the asymptotics relying on linear geometric optics.

In §2 we present some preliminaries on the invariance of almost periodic functions under real analytic maps. The Cauchy problem (1.1) is discussed in §3. In §4 we present three examples. Example 2 is homogeneous oscillations analogous to homogeneous turbulence where the profile equations have an interpretation as an infinite particle dynamical system. Example 3 is the semilinear crystal optics mentioned above.

## 2. Preliminaries on $\mathbf{A}$ .

Let  $\mathbf{A}$  denote the Wiener algebra

$$(2.1) \quad \mathbf{A} \equiv \{u \in \mathcal{S}'(\mathbf{R}^m) : \hat{u} \text{ is a bounded Borel measure on } \mathbf{R}^m\}.$$

Then  $\mathbf{A}$  is contained in  $C(\mathbf{R}^m) \cap L^\infty(\mathbf{R}^m)$ . The norm in  $\mathbf{A}$  is the total variation of  $\hat{u}$ ,

$$(2.2) \quad \|u\|_{\mathbf{A}} \equiv \|\hat{u}\|_{\text{Tot. Var.}} \equiv \int_{\mathbf{R}^m} d|\hat{u}|.$$

Of basic importance in all the analysis to follow is the derivation of sup norm estimates. For that we use elaborations of the elementary estimate

$$(2.3) \quad \|u\|_{L^\infty} \leq (2\pi)^{-m/2} \|u\|_{\mathbf{A}}.$$

**DEFINITION.** — For a positive integer  $m$  and a Banach space  $B$ ,  $\mathbf{A}(B, \mathbf{R}^m)$  is the set of almost periodic  $B$ -valued functions on  $\mathbf{R}^m$  with absolutely summable Fourier coefficients. That is  $a \in \mathbf{A}(B, \mathbf{R}^m)$  if and only if

$$(2.4) \quad a(Y) = \sum a_\alpha e^{i\alpha \cdot Y}$$

where the sum is over  $\alpha \in \mathbf{R}^m$  and the coefficients  $a_\alpha \in B$  satisfy

$$(2.5) \quad \|a\|_{\mathbf{A}(B, \mathbf{R}^m)} \equiv \sum \|a_\alpha\|_B < \infty.$$

The formulas

$$(2.6) \quad a_\alpha(Y) = B - \lim (2R)^{-m} \int_{[-R, R]^m} e^{-i\alpha \cdot Y} a(Y) dY$$

show that the coefficients are uniquely determined.

**DEFINITION.** — For  $a \in \mathbf{A}(B, \mathbf{R}^m)$  the spectrum of  $a$  denoted  $\text{Spec}(a)$  is the (countable) set of  $\alpha \in \mathbf{R}^m$  such that  $a_\alpha \neq 0$ .

**PROPOSITION 2.1.** — Suppose that the Banach space  $B$  is a space of  $\mathbf{C}^k$  valued functions on an open set  $\Omega$  of Euclidean space and that  $B$  is a function algebra in the sense that

$$(2.7) \quad B \subset L^\infty(\Omega : \mathbf{C}^k) \text{ and } \exists c > 0, \forall b \in B, \|b\|_B \geq c \|b\|_{L^\infty(\Omega)}$$

$$(2.8) \quad \exists c > 0, \forall b_1, b_2 \in B, b_1 b_2 \in B \text{ and } \|b_1 b_2\|_B \leq c \|b_1\|_B \|b_2\|_B.$$

Suppose that  $f : \Omega \times \mathbf{C}^k \times \mathbf{C}^k \rightarrow \mathbf{C}^k$  is entire in the sense that

$$(2.9) \quad f(x, u, v) = \sum_{\alpha, \beta \neq (0,0)} f_{\alpha, \beta}(x) u^\alpha v^\beta$$

where for all  $\alpha, \beta$ , multiplication by  $f_{\alpha, \beta}$  maps  $B$  into itself and for all  $r > 0$ , there is a constant  $c(r) > 0$  such that

$$(2.10) \quad \|f_{\alpha, \beta} b\|_B \leq c(r) r^{-|\alpha, \beta|} \|b\|_B \text{ for all } b \in B, \alpha, \beta.$$

(i) Then for  $u \in B$ , the function

$$(2.11) \quad f(u) := f(x, u(x), \bar{u}(x)) = \sum f_{\alpha, \beta}(x) u(x)^\alpha \bar{u}(x)^\beta$$

belongs to  $B$  and the mapping sending  $u$  to  $f(u)$  maps  $B$  to itself and is uniform Lipschitzean on bounded sets in  $B$ .

(ii) If  $a \in \mathbf{A}(B, \mathbf{R}^m)$  then the function  $Y \rightarrow f(a(Y))$  belongs to  $\mathbf{A}(B, \mathbf{R}^m)$  and the map from  $\mathbf{A}(B, \mathbf{R}^m)$  to itself so defined is uniformly Lipschitzean on bounded sets.

*Proof.*

(i) That  $f(u)$  belongs to  $B$  is an immediate consequence of (2.8) and (2.10).

To prove Lipschitz continuity consider the difference  $u^\alpha \bar{u}^\beta - v^\alpha \bar{v}^\beta$  for  $|\alpha| + |\beta| \neq 0$ . Write

$$u^\alpha \bar{u}^\beta - v^\alpha \bar{v}^\beta = [v + (u - v)]^\alpha [\bar{v} + (\bar{u} - \bar{v})]^\beta - v^\alpha \bar{v}^\beta.$$

The binomial theorem expresses the difference as a sum of terms

$$(u - v)^\gamma (\bar{u} - \bar{v})^\mu (v)^{\alpha - \gamma} (\bar{v})^{\beta - \mu} \binom{\alpha}{\gamma} \binom{\beta}{\mu}$$

with  $|\gamma| + |\mu| \neq 0$ . There is a factor of  $u - v$  or  $\bar{u} - \bar{v}$  in each term. Thus, that there is a constant  $C$  independent of  $\alpha, \beta$  so that

$$\|u^\alpha \bar{u}^\beta - v^\alpha \bar{v}^\beta\|_B \leq \|u - v\|_B C^{|\alpha| + |\beta|} (1 + \|u\|_B + \|v\|_B)^{|\alpha| + |\beta|}.$$

The Lipschitz continuity follows. In the same way one shows that the derivative of  $f$  at  $u$  in the direction  $h$  is equal to  $f_u(u, \bar{u})h + f_{\bar{u}}(u, \bar{u})\bar{h}$ .

(ii) The fact that  $f$  preserves  $\mathbf{A}(B, \mathbf{R}^m)$  and is bounded on bounded sets follows from the fact that  $u \rightarrow \bar{u}$  is an isometry of  $\mathbf{A}(B, \mathbf{R}^m)$  and the map  $a, b \rightarrow ab$  maps  $\mathbf{A}(B, \mathbf{R}^m)$  to itself with

$$(2.12) \quad \|ab\|_{\mathbf{A}(B, \mathbf{R}^m)} \leq c \|a\|_{\mathbf{A}(B, \mathbf{R}^m)} \|b\|_{\mathbf{A}(B, \mathbf{R}^m)}.$$

To prove this consider the Fourier series

$$ab = \sum_{\gamma} \left[ \sum_{\alpha+\beta=\gamma} a_{\alpha} b_{\beta} \right] e^{i\gamma Y}.$$

The triangle inequality and Fubini's inequality yield

$$\|ab\|_{\mathbf{A}(B, \mathbf{R}^m)} \leq \sum_{\gamma} \left[ \sum_{\alpha+\beta=\gamma} c \|a_{\alpha}\|_B \|b_{\beta}\|_B \right] \leq c \sum \|a_{\alpha}\|_B \sum \|b_{\beta}\|_B$$

which is the desired estimate (2.12). □

*Remark.* — The proof of invariance is particularly simple for entire real analytic functions. However, the Weiner-Lévy Theorem shows that it suffices for  $f(x, \zeta, \eta)$  to be holomorphic in  $\zeta, \eta$  on a neighborhood of the values taken by  $u(x), \bar{u}(x)$ . We describe only the case of entire real analytic  $f$  leaving the modifications needed in the more general case to the interested reader.

### 3. Highly oscillatory Cauchy problem.

The goal of this section is to study the oscillatory initial value problem

$$(3.1) \quad L(t, x, D_{t,x})u^{\varepsilon} + f(t, x, u^{\varepsilon}, \bar{u}^{\varepsilon}) = h^{\varepsilon}, \quad u^{\varepsilon}(0, x) = g^{\varepsilon}$$

where

$$(3.2) \quad g^{\varepsilon}(x) \equiv \Gamma(x, \varphi(0, x)/\varepsilon)$$

$$(3.3) \quad h^{\varepsilon}(t, x) \equiv H(t, x, \varphi(t, x)/\varepsilon)$$

with phases  $\varphi = (\varphi_0, \dots, \varphi_m)$  satisfying a restrictive coherence hypothesis.

The function  $f(t, x, u, v)$  with  $f(t, x, 0, 0) = 0$  is assumed to be smooth in  $t, x$  and entire in  $u, v$ . Precisely

$$(3.4) \quad f(t, x, u, v) = \sum_{|\alpha|+|\beta|>0} f_{\alpha,\beta}(t, x) u^{\alpha} v^{\beta}$$

where for all  $r > 0$  and all  $\gamma$  there is a constant  $c = c(\gamma, r)$  such that

$$(3.5) \quad |D_{t,x}^{\gamma} f_{\alpha,\beta}(t, x)| \leq cr^{-|\alpha,\beta|} \text{ for all } |t| \leq r, x \in \mathbf{R}^d, \alpha, \beta.$$

*Symmetric hyperbolicity assumption.*

$$(3.6) \quad L(t, x, D_{t,x}) = A_0 \partial_t + \sum A_j(t, x) \partial_j$$

where the  $A_j$  are smooth  $k \times k$  hermitian symmetric matrix valued functions on a connected open neighborhood  $\mathcal{O}$  of the origin in  $\mathbf{R}^{1+d}$  and  $A_0$  is strictly positive.

*Coherence assumption.*

The phases belong to a real finite dimensional vector space  $\Phi \subset C^\infty(\mathcal{O})$ . The phases are assumed to be *coherent* in the sense that

(i) For each  $\varphi \in \Phi \setminus 0$ ,  $d\varphi$  is nowhere zero on  $\mathcal{O}$ , and,  $\det L(t, x, d\varphi)$  is either everywhere zero or nowhere zero on  $\mathcal{O}$ .

(ii) There is a function  $\varphi_0 \in \Phi \setminus 0$  such that  $\varphi_0|_{t=0} = 0$ .

The reader is referred to [JMR3], [JMR4], [JMR5], [HMR] for a discussion of this hypothesis. The function  $\varphi_0$  is determined uniquely up to a scalar multiple. Thus  $\varphi_0$  is a natural timelike function near  $(0, 0)$ . Making a smooth change of independent variable we may suppose that

$$(3.7) \quad \varphi_0 = t.$$

This done we make a change of dependent variable replacing  $u$  by  $(A_0)^{1/2}u$  which converts the equation

$$(3.8) \quad Lu + f(t, x, u, \bar{u}) = 0$$

to an equation of the same form with

$$(3.9) \quad A_0 = I.$$

The reason for working locally is that a coherent set of phases defined locally need not have a global coherent extension.

*Example.* — The standard example of coherence is when  $L$  has constant coefficients and  $\Phi$  is the  $d+1$  dimensional space of linear functions of  $t, x$ . When  $d > 1$  there are interesting examples which cannot be transformed to such constant coefficient linear phase problems (see [JMR5] §3).

Denote by  $\Phi^0$  the set of restrictions of elements of  $\Phi$  to  $(t = 0)$ .

PROPOSITION 3.1 (Consequences of coherence).

(i)  $\varphi \rightarrow \varphi|_{t=0}$  defines linear map of  $\Phi$  onto  $\Phi^0$  with nullspace equal to  $\mathbf{R}\varphi_0$ .

(ii) For any  $\varphi \in \Phi$ , the eigenvalues of the symmetric matrix  $L(t, x, d\varphi(t, x))$  do not depend on  $t, x$ . In addition their multiplicity is independent of  $t, x$ .

(iii) For  $\chi \in \Phi \setminus 0$ , let  $\lambda_1, \dots, \lambda_{M(\chi)}$  denote the eigenvalues of  $L(d\chi)$ . The functions  $\psi \in \Phi$  such that

$$(3.10) \quad \det(L(t, x, d\psi(t, x))) \equiv 0 \text{ and } \psi|_{t=0} = \chi|_{t=0}$$

are precisely the functions  $\psi_j = \chi - \lambda_j \varphi_0$ . For each  $j$ ,  $\ker(L(t, x, d\psi_j(t, x)))$  is a smoothly varying subspace of  $\mathbf{C}^k$  and one has the orthogonal decomposition

$$(3.11) \quad \mathbf{C}^k = \bigoplus_j \ker(L(t, x, d\psi_j(t, x))).$$

*Proof.*

(i) It suffices to show that if  $\psi \in \Phi$  and  $\psi(0, x) \equiv 0$  then  $\psi \in c\varphi_0$  for some  $c \in \mathbf{R}$ . Fix  $(0, \underline{x}) \in \mathcal{O}$ . Since  $\psi$  vanishes at  $t = 0$ , there is a constant  $c$  such that  $d\psi(0, \underline{x}) = cd\varphi_0(0, \underline{x})$ . Then  $\psi - c\varphi_0 \in \Phi$  and has vanishing differential at one point. Coherence implies that  $\psi - c\varphi_0 \equiv 0$ .

(ii) Coherence implies that the roots  $\lambda$  of the polynomial  $\det(L(t, x, -\lambda d\varphi^0 + d\varphi))$  do not depend on  $t, x$ . However, with the normalizations (3.7) and (3.9), these are precisely the eigenvalues of  $L(d\varphi)$ .

For an eigenvalue  $\lambda$  the multiplicity is equal to

$$\text{trace}[(1/2\pi i) \oint (z - L(d\varphi))^{-1} dz]$$

where the contour is a small circle about  $\lambda$ . This continuous integer valued function must be constant.

(iii) Suppose that  $\psi \in \Phi$  satisfies (3.10). Fix  $(0, \underline{x}) \in \mathcal{O}$ . Then there is a  $\sigma \in \mathbf{R}$  such that  $d\psi(0, \underline{x}) = d\chi(0, \underline{x}) - \sigma d\varphi_0(0, \underline{x})$ . Then  $\psi - \chi + \sigma\varphi_0$  is an element of  $\Phi$  whose differential vanishes at a point. Coherence implies that  $\psi - \chi + \sigma\varphi_0 \equiv 0$ .

In addition at  $t = 0$  one has  $L(d(\chi - \sigma\varphi_0)) = L(d\chi) - \sigma I$ . By (3.10) this is a singular matrix so at  $t = 0$  there is a  $j$  so that  $\sigma = \lambda_j$ . As both sides are constant,  $\sigma = \lambda_j$  throughout  $\mathcal{O}$ , so  $\psi = \chi - \lambda_j \varphi_0 \equiv \psi_j$ .

Finally,  $\ker(L(d\psi_j))$  is the  $\lambda_j$  eigenspace of  $L(d\chi)$  and the smooth orthogonal decomposition follows. □

*Remarks.*

1. Hermitian symmetry implies that the eigenvalues of  $L(t, x, d\varphi(t, x))$  are real and their algebraic and geometric multiplicities are equal. The

proposition shows that the eigenvalues and multiplicities are independent of  $t, x$ .

2. It is important to note that this proposition does not say that the multiplicity of the roots of  $\det(L(t, x, \tau, \xi) = 0$  are independent of  $\tau, \xi$ . For example, consider the case of constant coefficients and linear phases. The proposition then asserts that for  $\tau, \xi$  fixed the multiplicity of the roots of  $L(\tau, \xi)$  is independent of  $t, x$  which is obvious. A striking application in §4 is to the phenomenon of conical refraction which depends exactly on roots of variable multiplicity with respect to  $\tau, \xi$ .

3. Since the eigenvalues of  $L(t, x, \tau, \xi)$  need not have multiplicity independent of  $\tau, \xi$  and need not be smooth functions of  $\tau, \xi$ , the eikonal equation  $\det(L(d\psi)) = 0$  may be singular. We do not know if it is possible for there to be solutions other than those which belong to  $\Phi$ . In case the multiplicities are independent of  $\xi$ , part (iii) describes all solutions of (3.10).

4. The direct sum decomposition in (iii) shows that the solutions in  $\Phi$  suffice to solve the oscillatory initial value problems we encounter.

We work in a compact truncated conical neighborhood

$$\Omega = \Omega_{\beta,r} \equiv \{(t, x) : |x| \leq r - t/\beta, 0 \leq t \leq t_2 \equiv r\beta\}$$

where  $r$  and  $\beta$  are so small that  $\Omega \subset\subset \mathcal{O}$  and the boundaries are all spacelike for  $L$ . The radius  $r$  will be decreased a finite number of times during the proof.

DEFINITION. — For  $\underline{t} \in ]0, t_2[$  and  $s \in \mathbf{N}$ ,  $B(s, \underline{t})$  is the set of restrictions to  $\Omega$  of continuous functions of  $t$  with values in  $H^s(\mathbf{R}^d)$ .  $B(s, \underline{t})$  is a Banach space with norm

$$\|u\|_{s, \underline{t}} = \sup_{0 \leq t \leq \underline{t}} \|u(t, \cdot)\|_{H^s(\{x:(t,x) \in \Omega\})}.$$

$C^\infty(\Omega)$  is dense in  $B(s, \underline{t})$ .

Choose a basis  $\varphi_0 = t, \varphi_1, \varphi_2, \dots, \varphi_m$  of  $\Phi$ . Then  $\dim(\Phi) = m + 1$  and the last  $m$  basis elements restricted to  $t = 0$  form a basis for  $\Phi^0$ .

Denote by  $\Omega_0$  the initial section, that is  $\{|x| \leq r\}$ .

Next we decompose initial oscillations

$$\Gamma_\beta(x) \exp i[(\beta_1 \varphi_1(0, x) + \dots + \beta_m \varphi_m(0, x))/\varepsilon]$$

corresponding to the way these oscillations will be propagated by the system. To understand the recipe, recall the explicit formula for the constant coefficient initial value problem

$$L(D_{t,x})u = 0, \quad u(0, x) = g e^{i\xi \cdot x}, \quad g \in \mathbf{C}^k$$

$$u(t, x) = \sum (\mathbf{E}_{(\tau_j, \xi)} g) \exp(i(\tau_j, \xi) \cdot (t, x))$$

where  $\tau_j$  are the roots of  $\det L(\tau, \xi) = 0$  and  $\mathbf{E}$  denotes the projection on  $\ker(L(\tau_j, \xi))$  along  $\text{Rg}(L(\tau_j, \xi))$ .

In our situation there is an analogous construction. For each  $\xi \in \mathbf{R}^m$ , Proposition 3.1, (iii) shows that there is a finite set of  $\alpha^u(\xi) = (\alpha_0^u, \xi) \in \mathbf{R}^{1+m}$  so that the solutions of the eikonal equation in  $\Phi$  with initial data  $\xi_1 \varphi_1 + \dots + \xi_m \varphi_m$  are equal to  $\alpha^\mu \cdot \varphi$ ,  $\mu = 1, \dots, M(\xi)$ .

For  $\xi$  fixed and  $t, x$  in  $\Omega$ ,  $\mathbf{C}^k$  is a direct sum of the smoothly varying eigenspaces of  $\sum_{j \geq 1} A_j(t, x) \partial(\xi_j \varphi_j) / \partial x_j$  which in turn are equal to the nullspaces,  $\ker(L(\alpha^\mu \cdot d\varphi))$ .

DEFINITION. — For  $\alpha \in \mathbf{Z}^{1+m}$  let  $\mathbf{E}_\alpha(t, x)$  be the spectral projection on  $\ker(L(t, x, \alpha \cdot d\varphi(t, x)))$ .

For  $\Gamma \in \mathbf{A}(H^s(\Omega_0), \mathbf{R}^m)$ , let

$$\Gamma(x, \omega) = \sum g_\xi(x) e^{i\xi \cdot \omega}$$

be the Fourier decomposition of  $\Gamma$ . The above remarks show that  $\text{Id} = \sum_\mu \mathbf{E}_{\alpha^\mu(\alpha')}(t, x)$ , so

$$g_\xi(x) = \sum_\mu \mathbf{E}_{\alpha^\mu(\xi)}(0, x) g_\xi(x),$$

a decomposition of  $g_\xi$  which appears in the next result.

THEOREM 3.2 (Uniform nonlinear existence). — Suppose that  $t_1 \in ]0, t_2[$ ,  $\mathbf{N} \ni s > d/2$ ,  $H \in \mathbf{A}(B(s, t_1), \mathbf{R}^{1+m})$ ,  $\Gamma \in \mathbf{A}(H^s(\Omega_0), \mathbf{R}^m)$ , and  $f$  is as in (3.4). For  $\varepsilon > 0$  let

$$(3.12) \quad h^\varepsilon(t, x) \equiv H(t, x, \varphi_0(t, x)/\varepsilon, \varphi_1(t, x)/\varepsilon, \dots, \varphi_m(t, x)/\varepsilon)$$

$$(3.13) \quad g^\varepsilon(x) \equiv \Gamma(x, \varphi_1(0, x)/\varepsilon, \varphi_2(0, x)/\varepsilon, \dots, \varphi_m(0, x)/\varepsilon).$$

(i) Then, there is a  $\underline{t} \in ]0, t_1]$  so that for all  $0 < \varepsilon \leq 1$ , the initial value problem

$$(3.14) \quad Lu^\varepsilon + f(t, x, u^\varepsilon, \bar{u}^\varepsilon) = h^\varepsilon, \quad u^\varepsilon|_{t=0} = g^\varepsilon$$

has a unique solution in  $C(\Omega \cap \{0 \leq t \leq \underline{t}\})$ .

(ii) The solution  $u^\varepsilon$  is given by  $\mathcal{U}^\varepsilon(t, x, \varphi(x)/\varepsilon)$  where  $\mathcal{U}^\varepsilon(t, x, \theta) \in \mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  satisfies

$$L(t, x, D_{t,x})\mathcal{U}^\varepsilon + \varepsilon^{-1} \sum_{j,k \geq 0} A_j(t, x) (\partial \varphi_k / \partial x_j) \partial \mathcal{U}^\varepsilon / \partial \theta_k + f(\mathcal{U}^\varepsilon) = H(t, x, \theta).$$

$$(3.15) \quad \mathcal{U}^\varepsilon(0, x, \theta_0, \theta_1, \dots, \theta_m) = \sum_{\xi \in \mathbf{R}^m} \sum_{\mu} (\mathbf{E}_{\alpha^\mu(\xi)}(0, x) g_\xi(x)) e^{i\alpha^\mu(\xi) \cdot \theta}$$

with notation as in the paragraph before the theorem. This symmetric hyperbolic initial value problem uniquely determines  $\mathcal{U}^\varepsilon$ , and the family  $\{\mathcal{U}^\varepsilon\}_{\varepsilon \in ]0,1]}$  is bounded in  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}_\theta^{1+m})$ .

*Remark.* — The equations in part (ii) are sufficient but not necessary for  $\mathcal{U}^\varepsilon(t, x, \varphi/\varepsilon)$  to satisfy (3.14). Similarly the initial condition for  $\mathcal{U}$  is sufficient but not necessary for (3.14).

*Proof.* — The proof is by Picard iteration,  $u^\varepsilon = \lim u^{\varepsilon, \nu}$ . The first iterate  $u^{\varepsilon, 1}$  solves the linear problem which one gets by setting  $f \equiv 0$  in (3.14). For  $\nu \geq 2$  one solves the linear problems

$$(3.16) \quad Lu^{\varepsilon, \nu} = -f(t, x, u^{\varepsilon, \nu-1}, \bar{u}^{\varepsilon, \nu-1}) + h^\varepsilon, \quad u^{\varepsilon, \nu}|_{t=0} = g^\varepsilon.$$

The key step is to obtain uniform bounds for  $\|u^{\varepsilon, \nu}\|_{L^\infty}$ . This is done by writing  $u^{\varepsilon, \nu}$  as  $\sum a_\beta(t, x) e^{i\beta \cdot \varphi/\varepsilon}$  and estimating the  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}_\theta^{1+m})$  norm of  $\mathcal{U}^\varepsilon := \sum a_\beta e^{i\beta \cdot \theta}$ .

This in turn is done in two steps. The crucial step is to prove a uniform estimate for high frequency monochromatic linear initial value problems (Proposition 3.3). Superposition then yields  $\mathbf{A}$ -estimates for linear initial value problems (Corollary 3.4). Then the Picard iterated can be controlled.

**PROPOSITION 3.3.** — For each  $s \in \mathbf{N}$  there is a constant  $\gamma > 0$  so that  $\forall \underline{t} \in ]0, t_2[$ ,  $\varphi \in \Phi$ ,  $b \in H^s(\Omega_0)$ ,  $c \in B(s, \underline{t})$  the solution of the linear initial value problem

$$Lu = c(t, x) e^{i\varphi}, \quad u(0, x) = b(x) e^{i\varphi(0, x)}$$

is given by  $u = a(t, x) e^{i\varphi}$  where  $a \in B(s, \underline{t})$  satisfies

$$(3.17) \quad \|a\|_{B(s, \underline{t})} \leq \gamma \{ \|b\|_{H^s(\Omega_0)} + \underline{t} \|c\|_{B(s, \underline{t})} \}.$$

*Proof.* — The case  $\varphi = 0$  is the standard  $L^2$  energy estimate for symmetric hyperbolic systems. To treat  $\varphi \neq 0$ , write the equation for  $a$  as

$$La + iL(t, x, d_{t,x}\varphi)a = c(t, x), \quad a(0, x) = b(x).$$

This is a symmetric hyperbolic initial value problem which determines  $a$  in  $\Omega$ . When  $d\varphi$  is large there is a large variable coefficient lower order term  $iL(d\varphi)a$ .

The first indication that this is ok is that  $iL(d\varphi)$  is antisymmetric so the standard energy method, multiply by  $u$  and integrate by parts in  $\Omega(t)$  yields

$$\|u(t)\|^2 \leq \|u(0)\|^2 + C \int_0^t \|u(\sigma)\| \|c(\sigma)\| d\sigma$$

where  $\|u(\sigma)\|$  is the  $L^2$  norm of  $u$  on the crosssection  $\Omega \cap \{t = \sigma\}$ . Let  $M(t) \equiv \max_{0 \leq \sigma \leq t} \|u(\sigma)\|^2$ . Then the inequality yields

$$M(t) \leq M(0) + CM(t)^{1/2} \int_0^t \|c(\sigma)\| d\sigma \leq M(0) + CtM(t)^{1/2} \|c\|_{B(0,t)}$$

and the case  $s = 0$  of (3.17) follows.

Choose a norm in  $\Phi$ , whose unit sphere is smooth. For  $\|\varphi\| \leq 1$ , the coefficient  $iL(d\varphi)$  and its derivatives are bounded so a direct energy method attack by differentiating the equation works to prove (3.17) for all  $s$ .

It is for derivative estimates when  $\varphi$  is large that the coherence hypothesis is crucial.

Proposition 3.1 shows that for each  $t, x \in \Omega$  and  $\varphi \in \Phi$ , there is a unitary matrix valued function  $U(t, x, \varphi)$  such that  $UL(t, x, d\gamma(t, x))U^*$  is a real diagonal matrix independent of  $t, x$ .

Next we show that, as in the more general context of ([JMR4] §4), the function  $U$  can be chosen to be homogeneous of degree zero in  $\varphi \neq 0$  and smooth in  $t, x$  near  $(t, x) = (0, 0)$  uniformly in  $\varphi$ . That is, there is an open neighborhood  $\mathcal{N}$  of  $(0, 0)$  so that  $t, x \rightarrow U(t, x, \varphi)$  is a smooth function of  $t, x$  for each  $\varphi$ , and for each  $\gamma$ , there is a constant  $c(\gamma)$  so that

$$\forall \varphi, \|D_{t,x}^\gamma U(\cdot, \cdot, \varphi)\|_{L^\infty(\mathcal{N})} \leq c.$$

Note that no smoothness in  $\varphi$  is asserted.

The columns of  $U^*$  must be a smoothly varying (with respect to  $t, x$ , not  $\varphi$ ) orthogonal eigenbasis. For an eigenvalue of multiplicity  $\mu$  we must choose an orthonormal basis for the eigenspace  $E_\mu(t, x)$  of  $L(t, x, d\varphi(t, x))$ . To do this first fix  $\underline{t}, \underline{x}, \varphi$  and choose an eigenbasis  $v_1, \dots, v_M$  for the eigenspace  $E_\mu(\underline{t}, \underline{x})$  of  $L(d\varphi(\underline{t}, \underline{x}))$ . Let  $\pi(t, x)$  be the orthogonal projector on  $E_\mu(t, x)$ . Then a smooth eigenbasis for  $E_\mu(t, x)$  for  $t, x$  near  $\underline{t}, \underline{x}$  and phases in an open neighborhood  $\omega$  of  $\varphi$  is constructed by applying the Gram-Schmidt algorithm to  $\{\pi(t, x)v_j\}$ . Cover  $\|\varphi\| = 1$  by a finite number of  $\omega_j$  of such neighborhoods. Express  $\|\varphi\| = 1$  as a disjoint union of subsets  $\sigma_j \subset \omega_j$ . The  $t, x$ -smooth eigenbasis is then given for  $\varphi/\|\varphi\|$  in  $\sigma_j$  as the basis constructed above for phases in  $\omega_j$ .

For  $\|\varphi\| \geq 1$ , make the change of dependent variable  $\underline{a} = Ua$ , then the equation for  $\underline{a} = Ua$  is

$$\partial_t \underline{a} + \sum U A_j U^* \partial_j \underline{a} + i \operatorname{diag}(\lambda) \underline{a} + \sum U A_j (\partial_j U^*) \underline{a} = 0.$$

The key observation is that the diagonal matrix, which is the only possibly large coefficient *has constant coefficients*. The other coefficients are bounded together with each of their derivatives uniformly for  $\|\varphi\| \geq 1$ . Thus the equation can be differentiated with respect to  $x$  and the standard energy applied. The large coefficient is no problem since  $\operatorname{Re}(\partial_x^\alpha \underline{a}, i \operatorname{diag}(\lambda) \partial_x^\alpha \underline{a}) = 0$ .

Note that the initial values  $\partial_x^\alpha \underline{a}(0, x) = \partial_x^\alpha (U b)$  are  $L^2$ -bounded for  $|\alpha| \leq s$ . This yields  $L^2$  estimates for  $\partial_x^\alpha \underline{a}(t, \cdot)$  uniformly in  $\varphi$  and  $t$ . These estimates carry over to  $a = U^* \underline{a}$  thanks to the uniform smoothness of  $U(t, x, \varphi)$  with respect to  $t, x$ .  $\square$

*Remark.* — The initial values of the time derivatives of  $a$  are not necessarily bounded independent of  $\varphi$ .

Each of the steps in the Picard iteration involves the solution of a linear initial value problem with source terms which have profiles in a suitable  $B(s, t)$  valued space  $\mathbf{A}$ . Proposition 3.3 allows us to solve such initial value problems by superposition.

**COROLLARY 3.4.** — *Suppose that  $\Gamma \in \mathbf{A}(H^s(\Omega_0), \mathbf{R}^m)$  and  $H \in \mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  and that  $g^\varepsilon, h^\varepsilon$  are defined as in Theorem 3.2. For  $\varepsilon \in ]0, 1]$  let  $u^\varepsilon$  be the solution of the linear initial value problem*

$$Lu^\varepsilon = h^\varepsilon, \quad u^\varepsilon(0, x) = g^\varepsilon.$$

*Then  $u^\varepsilon(t, x) = \mathcal{U}^\varepsilon(t, x, \varphi(t, x)/\varepsilon)$  where  $\mathcal{U}^\varepsilon(t, x, \theta)$  belongs to  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}_\theta^{1+m})$  and is determined by the symmetric hyperbolic initial value problem*

$$(3.18) \quad L(t, x, D_{t,x}) \mathcal{U}^\varepsilon + \varepsilon^{-1} \sum_{j,k \geq 0} A_j(t, x) (\partial \varphi_k / \partial x_j) \partial \mathcal{U}^\varepsilon / d\theta_k = H(t, x, \theta).$$

$$(3.19) \quad \mathcal{U}^\varepsilon(0, x, \theta_0, \theta_1, \dots, \theta_m) = \text{r.h.s. of (3.15)}.$$

*The linear maps  $\mathbf{B}_\varepsilon$  from  $\mathbf{A}(H^s(\Omega_0), \mathbf{R}^m) \times \mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  to  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  defined by  $\mathbf{B}_\varepsilon(\Gamma, H) \equiv \mathcal{U}^\varepsilon$  are uniformly bounded for  $\varepsilon \in ]0, 1]$ , that is there is a  $c > 0$  so that  $\forall \Gamma, H, \varepsilon \in ]0, 1], t \in ]0, \underline{t}]$*

$$\|\mathcal{U}^\varepsilon\|_{\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})} \leq c(\|\Gamma\|_{\mathbf{A}(H^s(\Omega_0), \mathbf{R}^m)} + t\|H\|_{\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})}).$$

*Proof.* — Denote by  $\mathcal{A}$  the set of  $\alpha \in \mathbf{R}^{1+m}$  such that either  $\alpha \in \operatorname{Spec}(h)$ , or,  $\alpha$  is equal to one of the  $\alpha^\mu(\xi)$  corresponding to  $\xi$  in

Spec( $g$ ). Define  $\mathcal{U}^\varepsilon = \sum_{\alpha \in \mathcal{A}} \mathcal{U}_\alpha^\varepsilon(t, x) e^{i\alpha \cdot \theta}$  where the Fourier coefficient with index  $\alpha_0, \alpha_1, \dots, \alpha_m$  is determined by the initial value problem

$$[L + i\varepsilon^{-1}L(t, x, d(\alpha \cdot \varphi))] \mathcal{U}_\alpha^\varepsilon(t, x) = h_\alpha, \quad \mathcal{U}_\alpha^\varepsilon \mu(\xi)(0, x) = \mathbf{E}_{\alpha\mu}(\xi) g_\xi(x), \tag{3.20}$$

$\mathcal{U}_\alpha^\varepsilon(0, x) = 0$  is  $\alpha \in \mathcal{A}$  is not equal to  $\alpha^\mu(\xi)$  for some  $\mu, \xi$ .

Proposition 3.3 shows that  $\mathcal{U}^\varepsilon \in \mathbf{A}(B(s, \underline{t}), \mathbf{R}_\theta^{1+m})$  and that the maps  $\mathbf{B}_\varepsilon$  are uniformly bounded.

The equations (3.20) are equivalent to (3.18)–(3.19). □

Return to the Picard iteration in the proof of Theorem 3.2. Apply the Corollary to analyse (3.16) with right hand side  $H(t, x, \theta) - f(t, x, \mathcal{U}^{\varepsilon, \nu-1}(t, x, \theta), \overline{\mathcal{U}}^{\varepsilon, \nu-1}(t, x, \theta))$  which belongs to  $\mathbf{A}(B(s, t_1), \mathbf{R}^{1+m})$  thanks to Proposition 2.1. We find that  $u^{\varepsilon, \nu} = \mathcal{U}^{\varepsilon, \nu}(t, x, \varphi/\varepsilon)$  with

$$\mathcal{U}^{\varepsilon, \nu} = \sum_{\alpha} a_{\alpha}^{\varepsilon, \nu}(t, x) e^{i\alpha \cdot \theta}$$

where the sum is over the  $\mathbf{Z}$ -module generated by  $\mathcal{A}$  and

$$\|\mathcal{U}^{\varepsilon, \nu}\|_{\mathbf{A}(B(s, t_1), \mathbf{R}^{1+m})} = \sum_{\alpha} \|a_{\alpha}^{\varepsilon, \nu}\|_{B(s, t_1)} < \infty.$$

Let  $R = \|\mathcal{U}^{\varepsilon, 1}\|_{\mathbf{A}}$ . The estimates of Proposition 2.1 and Corollary 3.4 imply that there is a  $\underline{t} \in ]0, t_2]$  so that for all  $\nu > 1$ ,  $\varepsilon \in ]0, 1]$ , and  $\underline{t} \in [0, t_1]$  the Picard iterates in the proof of Theorem 3.1 satisfy

$$\sum_{\alpha} \|a_{\alpha}^{\varepsilon, \nu}\|_{B(s, \underline{t})} < 2R.$$

$$\sum_{\alpha} \|a_{\alpha}^{\varepsilon, \nu} - a_{\alpha}^{\varepsilon, \nu-1}\|_{B(s, \underline{t})} < (C\underline{t})^{\nu-1}.$$

Choose  $\underline{t}$  so that  $C\underline{t} < 1$ . Then as  $\nu$  tends to infinity, the profiles  $\mathcal{U}^{\varepsilon, \nu}$  converge in  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  uniformly in  $\varepsilon$  to a solution  $\mathcal{U}^\varepsilon$  to (3.18)–(3.19). The corresponding function  $u^\varepsilon$  solves our problem.

Uniqueness of the solution  $u^\varepsilon$  is proved by a simple  $L^2$  energy argument. This completes the proof of Theorem 3.2. □

Next consider the high frequency limit  $\varepsilon$  tends to zero. The key here is a linear result which plays a role for asymptotics analogous to the role of Proposition 3.2 for local existence.

**PROPOSITION 3.5 (Linear asymptotics).** — *Suppose  $s \in \mathbf{N}$ ,  $\varphi \in \Phi$  and  $E(t, x) \in C^\infty(\Omega : \text{Hom}(\mathbf{C}^k))$  is the orthogonal projector on*

$\ker(L(t, x, d\varphi(t, x)))$ . For  $c \in B(s, \underline{t})$  and  $b \in H^s(\Omega_0)$  satisfying  $Eb = b$ , let  $u^\varepsilon$  be the solution of

$$(3.21) \quad Lu^\varepsilon = c(t, x)e^{i\varphi/\varepsilon}, \quad u^\varepsilon(0, x) = b(x)e^{i\varphi(0, x)/\varepsilon}.$$

(i) There is one and only one solution  $a \in B(s, \underline{t})$  of

$$(3.22) \quad Ea = a, \quad E(La - c) = 0, \quad a(0, x) = b(x).$$

There is a constant  $C = C(s)$  independent of  $\varphi$  and  $\underline{t}$  such that

$$(3.23) \quad \|a(\underline{t})\|_{B(s, \underline{t})} \leq C(\|b(x)\|_{H^s(\Omega_0)} + \underline{t}\|c(\sigma)\|_{B(s, \underline{t})}).$$

(ii) There is an  $R^\varepsilon(t, x, \theta) \in \mathbf{A}(B(s, \underline{t}), \mathbf{R}^{m+1})$  such that as  $\varepsilon$  tends to zero,  $\|R^\varepsilon\|_{\mathbf{A}} = o(1)$  and

$$(3.24) \quad u^\varepsilon = ae^{i\varphi(x)/\varepsilon} + R^\varepsilon(t, x, \varphi_0(t, x)/\varepsilon, \dots, \varphi_m(t, x)/\varepsilon).$$

*Proof.* — The analysis depends on whether  $\varphi$  satisfies or does not satisfy the eikonal equation.

If it does not satisfy then  $E = 0$  and  $Ea = a$  implies  $a = 0$  so the existence and uniqueness in part (i) is trivial. In addition  $b = Eb = 0$ . Thanks to Proposition 3.3 it suffices to show that  $u = o(1)$  for  $c(t, x)$  smooth on a neighborhood of  $\Omega$ .

In that case standard elliptic linear geometric optics (a convenient reference is [Hor], p. 272) constructs an asymptotic series

$$v^\varepsilon \sim \varepsilon e^{i\varphi/\varepsilon} \sum_{j=0}^{\infty} a_k(t, x)\varepsilon^k$$

with the property that  $Lv^\varepsilon - ce^{i\varphi/\varepsilon} \sim 0$ .

Denote by  $\psi_1, \dots, \psi_\mu \in \Phi$  the solutions (see Proposition 3.1, (iii)) of the eikonal equation which at  $t = 0$  are equal to  $\varphi(0, x)$ . Hyperbolic geometric optics [L] constructs an asymptotic series

$$w^\varepsilon \sim \varepsilon \sum_j \left[ e^{i\psi_j/\varepsilon} \left( \sum_{k=0}^{\infty} a_{j,k}(t, x)\varepsilon^k \right) \right]$$

with the property that  $Lw^\varepsilon \sim 0$  and  $w^\varepsilon(0, \cdot) - v^\varepsilon(0, \cdot) \sim 0$ .

Fix  $N > s+1$  and let  $v_N^\varepsilon$  and  $w_N^\varepsilon$  be the sums of the terms with  $j \leq N$ . Then  $L(v_N^\varepsilon - w_N^\varepsilon) - ce^{i\varphi/\varepsilon}$  is a sum

$$L(v_N^\varepsilon - w_N^\varepsilon) - ce^{i\varphi/\varepsilon} = o(1) \text{ in } B(s, \underline{t}),$$

$$v_N^\varepsilon(0, \cdot) - w_N^\varepsilon(0, \cdot) - b = o(1) \text{ in } H^s(\Omega_0).$$

Therefore  $u^\varepsilon = v_N^\varepsilon - w_N^\varepsilon + r^\varepsilon$  with  $r^\varepsilon = o(1)$  in  $B(s, \underline{t})$ . This suffices to prove (3.24).

In case  $\varphi$  satisfies the eikonal equation there are three independent ingredients. The first is Proposition 3.3 which shows that  $u^\varepsilon = a^\varepsilon e^{i\varphi/\varepsilon}$  with maps  $b, c \rightarrow a^\varepsilon$  uniformly bounded from  $H^s(\Omega_0) \times B(s, \underline{t})$  to  $B(s, \underline{t})$ . The second ingredient is assertion (i) of the present proposition whose proof is postponed. Given these two things a straightforward approximation argument shows that it suffices to prove assertion (ii) for  $b, c$  smooth on the closed sets  $\Omega_0$  and  $\Omega$  respectively.

Renumbering if necessary we may suppose that  $\varphi = \psi_1$  where the phases  $\psi_j$  are the solutions of the eikonal equation which are equal to  $\varphi(0, \cdot)$  at  $t = 0$ . In that case we follow Lax [L] to construct an asymptotic solution  $v^\varepsilon + w^\varepsilon$  where

$$(3.25) \quad \begin{aligned} v^\varepsilon &\sim e^{i\varphi/\varepsilon} \sum_{k=0}^{\infty} a_k(t, x) \varepsilon^k \\ w^\varepsilon &\sim \varepsilon \sum_j \left[ e^{i\varphi_j/\varepsilon} \left( \sum_{k=0}^{\infty} b_{j,k}(t, x) \varepsilon^k \right) \right]. \end{aligned}$$

Each of the sums on  $k$  satisfies  $Lu \sim 0$ .

The series  $v^\varepsilon$  is determined uniquely by

$$Lv^\varepsilon - ce^{i\varphi/\varepsilon} \sim 0, \quad a_0(0, \cdot) = b, \quad Ea_j(0, \cdot) = 0 \text{ for } j > 0.$$

To derive the equations for the  $a_j$  compute

$$Lv^\varepsilon - ce^{i\varphi/\varepsilon} \sim e^{i\varphi/\varepsilon} [\varepsilon^{-1}L(d\varphi)a_0 + \varepsilon^0(La_0 - c + L(d\varphi)a_1) + \dots].$$

With the goal of forcing  $Lv^\varepsilon - ce^{i\varphi/\varepsilon} \sim 0$ , choose  $a_0$  with  $L(d\varphi)a_0 = 0$  which is equivalent at  $Ea_0 = a_0$ . The  $\varepsilon^0$  term is projected onto its  $E$  and  $I - E$  parts. The  $E$  part yields the equation  $E(La_0 - c) = 0$ . In part (i) we show that this together with initial data for  $a_0$  determine  $a_0$ . Then  $a_1$  is constrained to satisfy  $(1 - E)L(d\varphi)a_1 = (1 - E)(La_0 - c)$  in order that the  $\varepsilon^0$  term vanish. Note that this prohibits one from taking  $a_1(0, \cdot) = 0$ . The closest one can come is to prescribe  $Ea_1 = 0$ . Projecting the  $\varepsilon^1$  term yields  $E(La_1) = 0$ . Part (i) shows that this equation together with the knowledge of  $(I - E)a_1$  and the initial value  $Ea_i(0, x) = 0$  determine  $a_1$ . Continuing in this manner the  $a_j$  satisfying  $Ea_j(0, \cdot) = 0$  are uniquely determined.

Now  $Lv^\varepsilon \sim ce^{i\varphi/\varepsilon}$  and

$$v^\varepsilon(0, x) \sim \varepsilon e^{i\varphi(0, x)/\varepsilon} \sum_{k \geq 1} \varepsilon^{k-1} a_k(0, x).$$

As in the elliptic case, construct  $w^\varepsilon$  as in (3.25) such that  $Lw^\varepsilon \sim 0$  and  $w^\varepsilon(0, \cdot) - v^\varepsilon(0, \cdot) \sim 0$ . Truncating as in the elliptic case yields (3.24).

To complete the proof it remains to prove (i). Here the argument resembles the model provided in [JR5] so we present only a few important features. Let  $G(t, x, D_x) \equiv L - \partial_t$  denote the spatial part of  $L$ . Then, the differential equation for  $a$  is

$$(3.26) \quad \partial_t a + EG(t, x, D_x)a = 0.$$

To derive the  $L^2$  estimate which is (3.23) with  $s = 0$ , use the standard energy method, multiplying the equation by  $a$  and taking real part. The crux is to notice that

$$(3.27) \quad \langle a, EGa \rangle = \langle Ea, Ga \rangle = \langle a, Ga \rangle$$

the first equality because  $E$  is self adjoint thanks to the symmetric hyperbolicity assumption. Thus the derivation of the energy estimate reduces to the same calculations as for the standard hyperbolic operator  $\partial_t + G$ . To prove derivative estimates one does not have  $E\partial a = \partial a$  to make this same trick as simple. However,  $E\partial a = \partial a + (\partial E)a$  and the second term is estimated using the  $L^2$  bounds. In this way one proves estimate (3.23) for smooth solutions of (3.22). Given such a priori estimates it is not difficult to prove the corresponding existence theorem stated in (i).  $\square$

DEFINITION. — *The operator  $\mathbf{E}$  from  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  to itself is defined on monomials by*

$$\mathbf{E}(a(t, x)e^{i\alpha \cdot \theta}) \equiv (\mathbf{E}_\alpha(t, x)a(t, x))e^{i\alpha \cdot \theta}$$

and is then extended by linearity.

COROLLARY 3.6.

(i) *For  $\Gamma, H$  as in Corollary 3.4, there is a unique  $U \in \mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  such that*

$$(3.28) \quad \mathbf{E}U = U, \quad U(0, x, \theta_0, \theta_1, \dots, \theta_m) = \text{r.h.s. of (3.15)}$$

$$(3.29) \quad \mathbf{E}[L(t, x, D_{t,x})U(t, x, \theta) - H(t, x, \theta)] = 0.$$

(ii) *The linear map  $\mathbf{B}_0$  from  $\Gamma, H$  to  $U$  so defined is continuous from  $\mathbf{A}(H^s(\Omega_0), \mathbf{R}^m) \times \mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  to  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$ .*

(iii) *In the strong topology in the space of such mappings,  $s - \lim \mathbf{B}_\varepsilon = \mathbf{B}$  where  $\mathbf{B}_\varepsilon$  is defined in Corollary 3.4.*

*Proof.* — The profile  $U = \sum U_\alpha(t, x)e^{i\alpha \cdot \varphi}$  is constructed mode by mode. If  $\alpha \cdot \varphi$  does not satisfy the eikonal equation the condition  $\mathbf{E}U = U$  implies that  $U_\alpha = 0$ . If  $\alpha \cdot \varphi$  is eikonal, equations (3.28)–(3.29) show that  $U_\alpha$  must satisfy  $\mathbf{E}_\alpha U_\alpha = U_\alpha$  and  $\mathbf{E}_\alpha(LU_\alpha - H_\alpha) = 0$ . In addition with  $\alpha' \equiv (\alpha_1, \dots, \alpha_m)$  one must have

$$\sum U_\alpha(0, x) = g_{\alpha'}(x)$$

where the sum is over all  $\alpha = (\alpha_0^\mu, \alpha')$  such that  $(\alpha_0^\mu, \alpha') \cdot \varphi$  is eikonal. Thus  $U_\alpha(0, x) = \mathbf{E}_\alpha(0, x)g_{\alpha'}(x)$  and there is a constant  $c = c(s)$  independent of  $\alpha'$  and  $\alpha$  such that

$$\|U_\alpha(0, \cdot)\|_{H^s(\Omega_0)} \leq c\|g_{\alpha'}\|_{H^s(\Omega_0)}.$$

The equations for  $U_\alpha$  are uniquely solvable by part (i) of Proposition 3.5. The estimates in that proposition show that  $U \in \mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$ . This proves the existence and boundedness part of parts (i)–(ii) of the corollary.

Uniqueness is proved by the energy method with multiplier  $U - V$  again following the proof of part (i) of Proposition 3.5.

Part (ii) of Proposition 3.5 shows that  $\mathbf{B}_\varepsilon(\Gamma, H) \rightarrow \mathbf{B}(\Gamma, H)$  for  $\Gamma, H$  in the dense set of finite trigonometric polynomials. The convergence (iii) in the general case follows from uniform boundedness.  $\square$

**THEOREM 3.7 (Nonlinear asymptotics).** — *Suppose that  $u^\varepsilon, \mathcal{U}^\varepsilon, \Gamma, H, g^\varepsilon$  and  $h^\varepsilon$  are as in Theorem 3.2.*

(i) *There is a  $t_1 \in ]0, \underline{t}]$  and a unique  $U \in \mathbf{A}(B(s, t_1), \mathbf{R}^{1+m})$  such that*

$$(3.30) \quad \mathbf{E}U = U, \quad \mathbf{E}[L(D_{t,x})U + f(t, x, U, \bar{U}) - H] = 0,$$

$$(3.31) \quad U(0, x, 0, \theta_1, \dots, \theta_m) = g(x, \theta_1, \dots, \theta_m).$$

(ii) *One has the asymptotic relation as  $\varepsilon$  tends to zero*

$$(3.32) \quad \mathcal{U}^\varepsilon = U + o(1) \text{ in } \mathbf{A}(B(s, t_1), \mathbf{R}^{1+m}).$$

*In particular,*

$$u^\varepsilon(t, x) = U(t, x, \varphi_0(t, x)/\varepsilon, \dots, \varphi_m(t, x)/\varepsilon) + o(1)$$

*with  $o(1)$  measured in  $C(\Omega \cap \{0 \leq t \leq t_1\})$ .*

*Proof.*

(i) The profile  $U$  is constructed as the limit of Picard iterates

$$\begin{aligned} \mathbf{E}U^{\nu+1} &= U^{\nu+1} + \mathbf{E}[L(D_{t,x})U^{\nu+1} + f(t, x, U^\nu, \bar{U}^\nu) - H] = 0, \\ U^{\nu+1}(0, x, 0, \theta_1, \dots, \theta_m) &= g(x, \theta_1, \dots, \theta_m). \end{aligned}$$

For the first iterate,  $\nu = 0$ , the  $f$  term is dropped.

Corollary 3.6 proves the existence of the first iterate  $U^1 = \mathbf{B}(\Gamma, H)$  in  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$ . Given  $U^{\nu-1}$  in  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$ , Proposition 2.2 shows that  $f(t, x, U(t, x, \theta), \bar{U}(t, x, \theta))$  belongs to  $\mathbf{A}(B, s, \underline{t}), \mathbf{R}^{1+m})$  and then  $U^\nu = \mathbf{B}(\Gamma, H - f(t, x, U^{\nu-1}, \bar{U}^{\nu-1}))$  continues the induction.

For convergence note that Proposition 2.1 implies that the map  $W \rightarrow f(t, x, W, \bar{W})$  is locally Lipschitzean from  $\mathbf{A}(B(s, \underline{t}), \mathbf{R}^{1+m})$  to itself and that the map  $H \rightarrow \mathbf{B}(0, H)$  maps the same space to itself with norm  $O(t_1)$ . Thus choosing  $t_1 \leq \underline{t}$  sufficiently small, convergence follows from the contraction mapping principle.

Uniqueness follows from this contraction argument or by a direct  $L^2$  energy estimate multiplying the difference of the equations satisfied by two solutions  $U_1$  and  $U_2$  by  $U_1 - U_2$ .

(ii) The proof is by simultaneous Picard iteration, a technique introduced in [J]. Let  $u^{\varepsilon, \nu}$ ,  $\mathcal{U}^{\varepsilon, \nu}$ , and  $U^\nu$  denote the iterates converging to  $u^\varepsilon$ ,  $\mathcal{U}^\varepsilon$  and  $U$  respectively. Write

$$U - \mathcal{U}^\varepsilon = (U - U^\nu) + (U^\nu - \mathcal{U}^{\varepsilon, \nu}) + (\mathcal{U}^{\varepsilon, \nu} - \mathcal{U}^\varepsilon).$$

In the space  $\mathbf{A}(B(s, t_1), \mathbf{R}^{1+m})$ , we have just shown that the first term tends to zero as  $\nu$  tends to infinity. Similarly, Corollary 3.6 showed that the last term tends to zero as  $\nu$  tends to infinity and the convergence is uniform for  $\varepsilon \in ]0, 1]$ . Finally Corollary 3.6 implies, by induction on  $\nu$ , that for  $\nu$  fixed the middle term tends to zero as  $\varepsilon$  tends to zero.

For any challenge number  $\eta > 0$  choose  $\mu$  so large that the first and last terms are smaller than  $\eta/3$  for  $\nu \geq \mu$  and  $\varepsilon \in ]0, 1]$ . Then choose  $\varepsilon_0$  so that for  $0 < \varepsilon \leq \varepsilon_0$ ,  $\|U^\mu - \mathcal{U}^{\varepsilon, \mu}\| < \eta/3$ . It follows that for  $0 < \varepsilon \leq \varepsilon_0$ ,  $\|U - \mathcal{U}^\varepsilon\| < \eta$  and the proof is complete.  $\square$

Once this framework has been established one can follow [JMR1] to study the lifetime of solutions, the spectrum of solutions, and a sum law for regularity of the profile  $\theta$ . In particular on any interval of existence for the profile  $U$ , the  $u^\varepsilon$  and  $\mathcal{U}^\varepsilon$  exist and are uniformly bounded and (3.32) holds.

One could also use this **A** framework to analyse the example of dense oscillations produced from Cauchy data oscillating with only three phases ([JR4], [JMR4], [JMR5]).

#### 4. Examples.

*Example 1. Constant coefficients and linear phases.*

For constant coefficients and linear phases, the symmetric hyperbolicity assumption is easily relaxed and one can work globally in  $x$ . Suppose that  $L$  has constant coefficients

$$(4.1) \quad L \equiv \partial_t + G(D), \quad G(D) \equiv \sum A_j D_j + B, \quad D_j \equiv (1/i)\partial/\partial x_j.$$

Let  $\Phi$  be the space of linear functions of  $t, x$ .

In this case one can work globally in  $x$  setting

$$\Omega(t) := ]0, t[ \times \mathbf{R}^d.$$

All the results hold under the mild hyperbolicity assumption that  $e^{-tG(D)}$  is a strongly continuous group of bounded operators on  $L^2(\mathbf{R}^d)$ . That is

$$(4.2) \quad \sup\{\|\exp(tG(\xi))\| : \xi \in \mathbf{R}^d\} \leq \gamma e^{\beta|t|} < \infty.$$

There are only two substantial changes that must be made in the analysis to cover this case. The first is to take advantage of the fact that this hyperbolicity assumption has many equivalent aliases. Precisely (4.2) is equivalent to each of the following three conditions (see Kreiss [Kr]).

(i) For all  $\xi \in \mathbf{R}^d \setminus 0$ , the principal symbol

$$(4.3) \quad G_1(\xi) = \sum A_j \xi_j$$

is similar to an imaginary diagonal matrix  $K(\xi)G_1(\xi)K^{-1}(\xi) = \text{imag.diag}$  and the similarity matrix can be chosen so that  $K$  and  $K^{-1}$  are bounded uniformly in  $\xi$ .

(ii)  $G_1$  is uniformly symmetrizable, that is, there is a selfadjoint matrix valued function  $R(\xi)$  such that for all  $\xi \in \mathbf{R}^d$ ,  $0 < cI \leq R(\xi) \leq CI$  and  $R(\xi)G_1(\xi)$  is anti-selfadjoint.

(iii) For all  $\xi \in \mathbf{R}^d \setminus 0$ , the symbol has spectral decomposition

$$(4.4) \quad G_1(\xi) = \sum i\lambda_m(\xi)\pi_m(\xi)$$

where  $\lambda_1 < \lambda_2 < \dots < \lambda_{M(\xi)}$  are distinct and real and valued,  $\sum \pi_m = I_{k \times k}$  and the norms of the eigenprojections  $\pi_m(\xi)$  are bounded independent of  $m$  and  $\xi$ .

The relations useful in passing from one of these to the other are

$$R = \sum (\pi_m)^* \pi_m \quad \text{and} \quad K = (R)^{1/2}.$$

Note that we do not suppose that the matrix functions  $R, K, \pi$  are smooth as functions of  $\xi$ .

Part (iii) is crucial in showing that the averaging operators  $\mathbf{E}$  are bounded. Part (ii) can be used in deriving energy estimates where the natural multiplier is  $R(D)u$ . However, direct estimates using the Fourier transform suffice as in the following analogue Proposition 3.3.

PROPOSITION 4.1. — Consider the linear initial value problem

$$(4.5) \quad Lu = c(t, x)e^{i(\tau t + \omega \cdot x)}, \quad u(0, x) = b(x)e^{i\omega \cdot x}$$

where  $b \in H^s(\mathbf{R}^d)$ , and  $c \in H^s([0, t] \times \mathbf{R}^d)$  for all  $t > 0$ . Then the solution  $u = a(t, x)e^{i(\tau t + \omega \cdot x)}$  satisfies  $\forall t \in \mathbf{R}, |\alpha| \leq s$

$$(4.6) \quad \|D_{t,x}^\alpha a(t, \cdot)\|_{L^2(\mathbf{R}^d)} \leq \gamma [e^{\beta|t|} \|(G, D_x)^\alpha b\|_{L^2(\mathbf{R}^d)} + \int_0^t e^{-\beta|t-\sigma|} \|D_{t,x}^\alpha c(\sigma, \cdot)\|_{L^2(\mathbf{R}_x^d)} d\sigma].$$

*Proof.* — Duhamel's principle reduces the general case to the case  $c \equiv 0$ .

When  $c \equiv 0$ ,  $a$  is determined by the initial value problem

$$(\partial_t + (G(D + \omega) + i\tau))a = 0, \quad a(0) = b.$$

The proposition follows upon noting that

$$\|e^{tG(D+\omega)}\|_{\text{Hom}(L^2)} = \sup_\xi \|e^{tG(\xi+\omega)}\|_{\text{Hom}(\mathbf{C}^k)} \leq \gamma e^{\beta|t|}. \quad \square$$

Starting with this proposition it is not hard to retrace the steps of the analysis in §3.

*Example 2. Homogeneous oscillations.*

Within the context of constant coefficients and linear phases an interesting case is that of profiles  $U$  which are independent of  $x$ . This is analogous to the theory of homogeneous turbulence. The profiles will

be independent of  $x$  as soon as  $\Gamma$ ,  $H$  and the nonlinear function  $f$  are independent of  $x$ . Then

$$U = U(t, T, X) = \sum U_\alpha(t) e^{i\alpha \cdot (T, x)}$$

is a function of  $t$  with values in the classical Wiener algebra of almost periodic functions corresponding to the choice  $B = \mathbf{C}^k$  in the definition of  $\mathbf{A}(B, \mathbf{R}^{1+m})$ . The function  $f$  has Taylor expansion with infinite radius of convergence

$$f(t, u, v) = \sum_{\mu, \nu \in \mathbf{Z}^k} f_{\mu\nu}(t) u^\mu v^\nu, \quad f_{\mu\nu} \in \mathbf{C}^k.$$

Plugging in  $U$  yields an absolutely convergent expansion

$$f(t, U, \bar{U}) = \sum F_{\beta\gamma\mu\nu}(t) U_\mu^\beta(t) \bar{U}_\nu^\gamma(t) e^{i(\mu|\beta| - \nu|\gamma|) \cdot (T, X)}.$$

The profile equation takes the form of an infinite system of ordinary differential equations

$$(4.7) \quad \mathbf{E}_\alpha U_\alpha = U_\alpha, \quad U(0, 0, X) = \Gamma(X),$$

$$(4.8) \quad dU_\alpha/dt = \sum_{\mu|\beta| - \nu|\gamma| = \alpha} K_{\beta\gamma\mu\nu}^\alpha(t) U_\mu^\beta \bar{U}_\nu^\gamma + \mathbf{E}_\alpha H_\alpha(t)$$

where the interaction matrices  $K$  are defined by

$$(4.9) \quad K_{\beta\gamma\mu\nu}^\alpha \equiv \sum_{\mu|\beta| - \nu|\gamma| = \alpha} \mathbf{E}_\alpha F_{\beta\gamma\mu\nu}(t)$$

and

$$\mathbf{E}_\alpha \equiv \text{spectral projection on } \ker \left( \alpha_0 I + \sum_{j \geq 1} A_j \alpha_j \right).$$

The elusive closure property in the theory of turbulence is supplied here by the fact that  $U(t)$  belongs to the Wiener algebra of almost periodic functions and  $f$  is entire. Thus though the system is infinite it is absolutely convergent. On the other hand, there are no finite closures. The value of  $U_\alpha(t)$  for a single  $\alpha$  and  $t \neq 0$  generically depends on the values of  $U_\beta(0)$  for all  $\beta$ .

It is an interesting question whether the theory of infinite systems of interacting particles has anything to say about the time evolution of  $U$  in special cases.

*Example 3. Conical refraction.*

We have consciously allowed operators with characteristics of variable multiplicity, for example the constant coefficient operator describing

the propagation of electromagnetic radiation in a biaxial crystal. This linear symmetric hyperbolic system describes, among other things the phenomenon of conical refraction. For the linear  $\varphi$  corresponding to propagation along the *optic axis*,  $\ker(L(d\varphi))$  is two dimensional. Thus the amplitude  $a$  in the linear geometric optics (3.22) is a 2-vector valued function and the system (3.22) in this case is a nontrivial  $2 \times 2$  hyperbolic system (see [L], 116–117). This is in sharp contrast to the constant multiplicity case where (3.22) reduces to transport equations, that is, scalar hyperbolic equations. For  $\varphi$  corresponding to the optic axis, the fundamental solution, that is the solution of

$$(4.10) \quad \mathbf{E}A = A, \quad \mathbf{E}LA = 0, \quad A(0, x) = \mathbf{E}\delta(x)$$

has support a set which is the injective linear image of the cone  $z^2 \geq x^2 + y^2$  in  $\mathbf{R}^3$ . Thus it fills a three dimensional cone in  $\mathbf{R}^{1+3}$ . The singular support is equal to the boundary of the cone [Lu]. Thus oscillations initially confined to a small ball about the origin will have leading amplitude nonzero on such a three dimensional cone. This is the phenomenon of *conical refraction*. For strongly localized excitations, the energy is localized near the edge of the cone which corresponds to the thin annulus of light displayed in texts ([BW], 688 bis). The fine structure within the annulus is a more subtle issue (see [MU]). *In the presence of nonlinear lower order terms a new phenomenon is possible. The cone of refracted oscillations can be triggered by resonant interaction of oscillations with spectra far from the optic axis.* We present an example illustrating this possibility.

The strategy is simple. Merely choose a nonlinear interaction such that for a phase  $\alpha \cdot (T, X)$  corresponding to conical refraction, there are  $\beta, \gamma, \mu, \nu$  so that the interaction coefficient  $K_{\beta\gamma\mu\nu}^\alpha$  is nonzero with  $\beta$  and  $\gamma$  not parallel to  $\alpha$ .

Maxwell's equations in a translation invariant medium without free charges are

$$\begin{aligned} \partial_t D &= c \operatorname{curl}(H), \quad \partial_t B = -c \operatorname{curl}(E), \\ D &= \mathcal{E}E, \quad B = \mu H, \quad \operatorname{div}(B) = \operatorname{div}(D) = 0 \end{aligned}$$

where  $\mathcal{E}$  and  $\mu$  are constant positive definite  $3 \times 3$  matrices and  $c$  is the speed of light. The divergence free conditions follow from the others if the divergences vanish at  $t = 0$ . For the case of a biaxial crystal  $\mu$  is a scalar and  $\mathcal{E}$  has three distinct real eigenvalues. We choose units so that  $c = 1$ . The dynamic equations then take the form

$$(\mathcal{E}/\mu)\partial_t E = \operatorname{curl}(B), \quad \partial_t B = -\operatorname{curl}(E).$$

This system is symmetrized by setting

$$u \equiv (\mathcal{E}/\mu)^{1/2} E, \quad v \equiv B.$$

Then with  $A$  defined to be the positive square root  $A \equiv (\mathcal{E}/\mu)^{-1/2}$  and

$$\Omega(\partial) \equiv \text{curl} \equiv \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}$$

$$(4.11) \quad L(\partial_t, \partial_x) \begin{bmatrix} u \\ v \end{bmatrix} \equiv \left\{ \partial_t I - \begin{bmatrix} 0 & \Omega A \\ -\Omega A & 0 \end{bmatrix} \right\} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

This is a symmetric hyperbolic system. An orthogonal change of coordinates in  $\mathbf{C}_u^3$  yields

$$(4.12) \quad A = \text{diag}(\alpha_1, \alpha_2, \alpha_3), \quad \alpha_1 > \alpha_2 > \alpha_3 > 0.$$

For geometric optics we need the characteristic variety and the associated (orthogonal) spectral projections. Equivalently we must find all plane wave solutions

$$(4.13) \quad (u, v) \equiv e^{i(\tau t + x \cdot \xi)}(a, b)$$

with  $(\tau, \xi) \in \mathbf{R}^{1+3}$  and constant vectors  $a, b$ . Then (4.11) holds iff

$$(4.14) \quad \tau a = A(\xi \wedge b) = A\Omega(\xi)b \quad \text{and} \quad \tau b = -\xi \wedge (Aa) = -\Omega(\xi)Aa.$$

For any  $\xi \neq 0$ ,  $\rho = 0$  is a double eigenvalue with eigenspace

$$\ker L(0, \xi) = \text{Span}[(A^{-1}\xi, 0, 0, 0), (0, 0, 0, \xi)].$$

For  $\tau \neq 0$ , use  $a = A\Omega b/\tau$  to eliminate  $a$  to find

$$(4.15) \quad \tau^2 b = -\Omega A^2 \Omega b$$

where the matrix on the right satisfies

$$\Omega A^2 \Omega \equiv \begin{bmatrix} -\alpha_2 \xi_3^2 - \alpha_3 \xi_2^2 & \alpha_3 \xi_1 \xi_2 & \alpha_2 \xi_1 \xi_3 \\ \alpha_2 \xi_1 \xi_2 & -\alpha_1 \xi_3^2 - \alpha_3 \xi_1^2 & \alpha_1 \xi_2 \xi_3 \\ \alpha_2 \xi_1 \xi_3 & -\alpha_1 \xi_2 \xi_3 & -\alpha_1 \xi_2^2 - \alpha_2 \xi_1^2 \end{bmatrix}.$$

$$\text{tr}(-\Omega A^2 \Omega) \equiv \Psi(\xi) \equiv (\alpha_2 + \alpha_1)\xi_3^2 + (\alpha_3 + \alpha_1)\xi_2^2 + (\alpha_3 + \alpha_2)\xi_1^2 \geq 0.$$

$$(4.16) \quad \det(\tau^2 + \Omega A^2 \Omega) = \tau^2(\tau^4 - \Psi(\xi)\tau^2 + |\xi|^2 \Phi(\xi))$$

$$(4.17) \quad \Phi(\xi) \equiv \alpha_1 \alpha_2 \xi_3^2 + \alpha_3 \alpha_1 \xi_2^2 + \alpha_3 \alpha_2 \xi_1^2.$$

Setting (4.16) equal to zero yields a quadratic equation for  $\tau^2 \neq 0$  whose discriminant is given by

$$\text{discriminant} = \Psi^2 - 4|\xi|^2 \Phi = P^2 + Q^2 \geq 0$$

where

$$P \equiv (\alpha_1 - \alpha_2)\xi_3^2 + (\alpha_3 - \alpha_2)\xi_1^2 + (\alpha_3 - \alpha_1)\xi_2^2$$

$$Q \equiv 2\sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}\xi_3\xi_2 \geq 0.$$

There is a double root iff  $\xi$  satisfies  $P = Q = 0$ , that is

$$(4.18) \quad \xi_2 = 0 = (\alpha_1 - \alpha_2)\xi_3^2 - (\alpha_2 - \alpha_3)\xi_1^2, \quad \tau^2 = \Psi(\xi)/2.$$

Our construction turns on resonant interaction of waves all of which satisfy  $\xi_2 = 0$ . When  $\xi_2 = 0$  the roots are  $\tau^2 = (\Psi \pm P)/2$  which yields

$$\tau^2 = \begin{cases} \alpha_1\xi_3^2 + \alpha_3\xi_1^2 & \text{for the } - \text{ sign} \\ \alpha_2(\xi_3^2 + \xi_1^2) & \text{for the } + \text{ sign} \end{cases}.$$

In  $\tau = 1$  this is a circle of radius  $1/\sqrt{\alpha_2}$  and an ellipse with axes  $1/\sqrt{\alpha_1}$  and  $1/\sqrt{\alpha_3}$ . We will use three points. Point  $I$  is defined to be the intersection of the ellipse with  $\xi_3 = 0$  and point  $II$  is the intersection of the circle with  $\xi_1 = 0$ ,

$$\text{point } I = (1, 1/\sqrt{\alpha_3}, 0, 0), \quad \text{point } II = (1, 0, 0, 1/\sqrt{\alpha_2}).$$

The third point is the sum of these two

$$\text{point } III = (2, 1/\sqrt{\alpha_3}, 0, 1/\sqrt{\alpha_2}).$$

Point  $III$  is a double point iff (4.18) holds, that is iff

$$(\alpha_1 - \alpha_2)/\alpha_2 = (\alpha_2 - \alpha_3)/\alpha_3 \quad \text{and} \quad 4 = [(\alpha_1 + \alpha_2)/\alpha_2 + (\alpha_2 + \alpha_3)/\alpha_3]/2.$$

These hold iff

$$(4.19) \quad \alpha_2 = 3\alpha_3 \quad \text{and} \quad \alpha_1 = 9\alpha_3.$$

In this case

$$(4.20) \quad \beta^I + \beta^{II} = \beta^{III}$$

is a resonance relation for  $L$ .

The vector  $b$  in (4.13) is determined from  $(\tau^2 I + \Omega A \Omega)b = 0$ . One finds

$$\begin{aligned} \text{point } I : \quad & \tau^2 I + \Omega A^2 \Omega = \text{diag}(1, 0, -2), \quad b^I = [0, 1, 0] \\ \text{point } II : \quad & \tau^2 I + \Omega A^2 \Omega = \text{diag}(0, -8, 1), \quad b^{II} = [1, 0, 0]. \end{aligned}$$

Equation (4.14) shows that  $a = A(\xi \wedge b)/\tau$  so

$$a^I = [0, 0, 1], \quad a^{II} = [0, 1/\sqrt{3}, 0].$$

For the third point,  $\ker(\tau^2 + \Omega A^2 \Omega)$  has dimension equal to two. Denote by  $e_1 \equiv (\sqrt{3}, 0, 1)/2 = \sqrt{3}\xi^{III}/2$ , the unit vector in the direction of  $\xi^{III}$ . Then

$$\tau^2 I + \Omega A^2 \Omega = \begin{bmatrix} 3 & 0 & \sqrt{3} \\ 0 & 0 & 0 \\ \sqrt{3} & 0 & 1 \end{bmatrix} = 4 \times (\text{orth. proj. on } \mathbf{C}e_1).$$

The kernel is the set of vectors  $b$  orthogonal to  $e_1$ . An oriented orthonormal basis for  $\mathbf{R}^3$  is

$$e_1, e_2, e_3 \equiv (\sqrt{3}, 0, 1)/2, \quad (0, 1, 0), \quad (-1, 0, \sqrt{3})/2.$$

The last two are a basis for the set of  $b$ 's. The vector  $a$  corresponding to  $b = e_3$  is given by

$$A(\xi^{III} \wedge e_3)/\tau = A(2/\sqrt{3})e_1 \wedge e_3/2 = A(-e_2/\sqrt{3}) = (0, -1/\sqrt{3}, 0).$$

For  $b = e_2$  one finds

$$A((2/\sqrt{3})e_1 \wedge e_2)/2 = A(e_3/\sqrt{3}) = A(-1/\sqrt{3}, 0, 1)/2 = (-3\sqrt{3}, 0, 1)/2.$$

Thus the two dimensional space of plane waves associated to point  $III$  is given by

$$(u, v) = e^{i(2t+x_1/\sqrt{\alpha_3}+x_3/\sqrt{\alpha_2})}(a, b), \\ (a, b) \in \text{Span}[(-3\sqrt{3}/2, 0, 1/2, 0, 1, 0), (0, -1/\sqrt{3}, 0, -1/2, 0, \sqrt{3}/2)].$$

This span together with  $(a^I, b^I)$ ,  $(a^{II}, b^{II})$  generate a four dimensional subspace. Therefore it contains a vector  $W$  which is orthogonal to  $(a^I, b^I)$  and  $(a^{II}, b^{II})$ . Introduce the semilinear equation

$$(4.21) \quad L(\partial_{t,x})(u, v) = u_2 u_3 W.$$

First consider homogeneous oscillations with profile

$$(U(t, T, X), V((t, T, X))) = \sum (U_\beta(t), V_\beta(t)) e^{i\beta \cdot (T, X)}.$$

Let  $\beta(j)$ ,  $j = I, II, III$  be the  $(\tau, \xi)$  corresponding to points  $I$ ,  $II$  and  $III$  respectively. The initial oscillations have phases corresponding to the points  $I$  and  $II$ , that is  $U_\beta(0), V_\beta(0) = 0$  unless  $\beta = \beta(j)$  for  $j = I$  or  $II$ . Precisely,

$$(4.22) \quad (U_\beta(0), V_\beta(0)) = \begin{cases} (a^j, b^j) & \text{for } \beta = \beta(j), \quad j = I, II \\ 0 & \text{otherwise} \end{cases}.$$

Since  $E_{\beta(I)}W = 0 = E_{\beta(II)}W$ , it follows that  $E_{\beta(III)}(u_2 u_3 W) = 0$  and therefore that  $d(U_{\beta(j)}, V_{\beta(j)})/dt = 0$  for  $j = I, II$ . Thus

$$(4.23) \quad (U_{\beta(j)}(t), V_{\beta(j)}(t)) = (a^j, b^j) \text{ for } j = I, II \text{ and all } t.$$

It is easy to see, for example by considering the Picard iterates  $U^\nu$  converging to  $U$ , that the spectrum of  $U$  is contained in  $(n\beta^I + m\beta^{II} \in \text{char}(L) : (n, m) \in \mathbf{N}^2 \setminus \{0\})$ . There are only nonnegative  $n, m$  because the nonlinearity has no complex conjugate terms.

LEMME 4.2. — *If  $n$  and  $m$  are real numbers and  $n\beta^I + m\beta^{II}$  belongs to the characteristic variety of  $L$ , then either  $n = 0$ ,  $m = 0$ , or  $n = \pm m$ .*

*Proof.* —  $\text{Det } L(n\beta^I + m\beta^{II}) = -4mn(n + m)^2(n - m)^2.$   $\square$

The only relation  $\beta(III) = \alpha + \gamma$  with  $\alpha, \gamma \in \text{Spec}(U, V)$  is (4.20). This yields the equation  $d(U_{\beta(III)}, V_{\beta(III)})/dt = a_2^I a_3^{II} W$  with exact solution

$$(U_{\beta(III)}(t), V_{\beta(III)}(t)) = tW/\sqrt{3}.$$

Thus oscillations along the optic axis are triggered in mode III.

For homogeneous oscillations, the oscillations fill space time so that this example does not show the spread of oscillations typical of conical refraction. Consider next the nonhomogeneous case with initial data which are narrow ray bundles with phases corresponding to  $\beta(I)$  and  $\beta(II)$ .

Since  $E_{\beta(j)}W = 0$  for  $j = I, II$ , the equations for the  $\beta(I)$  and  $\beta(II)$  components of the profile are the linear systems

$$\begin{aligned} \mathbf{E}_{\beta(j)}(U_{\beta(j)}(t, x), V_{\beta(j)}(t, x)) &= (U_{\beta(j)}(t, x), V_{\beta(j)}(t, x)) \\ \mathbf{E}_{\beta(j)}L(\partial_{t,x})(U_{\beta(j)}(t, x), V_{\beta(j)}(t, x)) &= 0. \end{aligned}$$

Standard geometric optics analysis shows that the general solution is of the form

$$(4.24) \quad (U_{\beta(j)}(t, x), V_{\beta(j)}(t, x)) = (a^j, b^j)\chi(x - s_j t).$$

The group velocities  $s^j$  are computed as follows. Near the point  $\beta$  represent the characteristic variety as  $\tau = \tau(\xi)$  so  $\tau(\xi)$  is homogeneous of degree one. Then  $s = \nabla_{\xi}\tau$ . For example at point  $I$  symmetry shows that the derivatives of  $\tau$  with respect to  $\xi_{2,3}$  vanish. Homogeneity shows  $\tau(\xi_1, 0, 0) = \xi_1\sqrt{\alpha_3}$ . Thus

$$(4.25) \quad s^I = (1, 0, 0).$$

Similarly,

$$(4.26) \quad s^{II} = (0, 0, \sqrt{3}).$$

Consider the initial data

$$(4.27) \quad (U_{\beta}(0, x), V_{\beta}(0, x)) = \begin{cases} (a^j, b^j)\chi(x) & \text{for } \beta = \beta(j), \quad j = I, II \\ 0 & \text{otherwise} \end{cases}$$

where  $\chi \in C_0^\infty(\mathbf{R}^3)$  has support near the origin. Then (4.24) holds for all  $t, x$  and all other modes vanish at  $t = 0$ .

Again thanks to the lemma the profile equation is explicitly solvable with

$$(4.28) \quad (U_{\beta}, V_{\beta}) = 0 \text{ for } \beta \notin \{\beta(I), \beta(II), \mathbf{N}\beta(III)\}$$

and for  $\beta = \beta(III)$

$$(4.29) \quad \mathbf{E}_\beta(U_\beta, V_\beta) = (U_\beta, V_\beta), \quad (U_\beta(0, x), V_\beta(0, x)) = 0,$$

$$(4.30) \quad \mathbf{E}_\beta L(\partial_{t,x}(U_\beta, V_\beta) = \chi(x - s^I t)\chi(x - s^{II} t)W/\sqrt{3}.$$

The solution is given by  $A * (\chi(x - s^I t)\chi(x - s^{II} t)W/\sqrt{3})$  where the fundamental solution  $A$  is defined in (4.10). Typically, the resulting  $\beta(III)$  Fourier coefficients will have support filling a solid three dimensional convex cone in  $\mathbf{R}^{1+4}$ . Thus the interaction of off-axis oscillations has triggered a cone of nonlinear conical refraction.

*Critique.* — The semilinear equation (4.21) has no relation as far as we know to any realistic physical model. However it is our belief that the phenomenon described here is robust, and is likely to be present in models of conical refraction which embrace nonlinear phenomena.

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J.-L. JOLY,  
Université de Bordeaux  
Département de Mathématiques  
351, cours de la libération  
F-33405 Talence Cedex

G. MÉTIVIER,  
Université de Rennes I  
IRMAR  
Campus de Beaulieu  
F-35042 Rennes Cedex

J. RAUCH,  
Department of Mathematics  
University of Michigan  
Ann Arbor MI 48109 (USA).