ANNALES DE L'INSTITUT FOURIER

NICOLA GAROFALO
ZHONGWEI SHEN

Carleman estimates for a subelliptic operator and unique continuation

Annales de l'institut Fourier, tome 44, nº 1 (1994), p. 129-166
<http://www.numdam.org/item?id=AIF_1994__44_1_129_0>

Ann. Inst. Fourier, Grenoble
44, 1 (1994), 129-166

CARLEMAN ESTIMATES FOR A SUBELLIPTIC OPERATOR AND UNIQUE CONTINUATION

by N. GAROFALO(*) and Z. SHEN(**)

Introduction.

In recent years there has been a large development in the study of unique continuation for second order elliptic equations. We recall that in his celebrated 1939 paper [C], T. Carleman established the strong unique continuation property for the Schrödinger operator $\mathcal{H} = -\Delta + V$ in $\mathbb{R}^2$, under the assumption that $V \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. This result was subsequently extended by several mathematicians to any number of dimensions and to equations with variable coefficients. More recently, the interest of workers in partial differential equations and mathematical physics has been focusing on equations with unbounded lower order terms. See [K] for reference. This development has culminated with Jerison and Kenig's celebrated result [JK] establishing the strong unique continuation property for $\mathcal{H}$ in $\mathbb{R}^n$, $n \geq 3$, when $V \in L^{n/2}_{\text{loc}}(\mathbb{R}^n)$. Their paper has inspired much progress in the subject and nowadays the picture for second order uniformly elliptic equations is almost complete. Not so well understood, instead, is the situation concerning non-elliptic operators.

In this paper we study the unique continuation property for zero-order perturbations of the so-called Grushin operator in $\mathbb{R}^{n+1}$:

$$\mathcal{L} = \Delta_z + |z|^2 \frac{\partial^2}{\partial t^2}. \tag{0.1}$$

(*) Supported in part by the NSF, grant DMS-9104023.
(**) Supported in part by the NSF, grant DMS-9201208.
Key words : Unique continuation – Subelliptic operator – Carleman estimates.
A.M.S. Classification : 35B60 – 35J70.

Here, $z \in \mathbb{R}^n$, $t \in \mathbb{R}$. $\mathcal{L}$ is elliptic for $z \neq 0$ and degenerates on the manifold $\{0\} \times \mathbb{R}$. This operator was studied by Grushin [Gru1], [Gru2], who established its hypoellipticity.

The operator $\mathcal{L}$ in (0.1) possesses a natural family of dilations, namely,

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \qquad \lambda > 0. \tag{0.2}$$

One easily checks that

$$\mathcal{L} \circ \delta_\lambda = \lambda^2 \delta_\lambda \circ \mathcal{L} \tag{0.3}$$

so that $\mathcal{L}$ is homogeneous of degree two with respect to $\{\delta_\lambda\}_{\lambda>0}$. The change of variable formula for Lebesgue measure gives

$$d \circ \delta_\lambda(z,t) = \lambda^Q dz\, dt, \tag{0.4}$$

where

$$Q = n + 2.$$

The number Q plays the role of the Euclidean dimension in the analysis of the Grushin operator. Henceforth, it will be called the homogeneous dimension. A natural problem to consider is : Do couples (p,q) exist such that for some constant $C > 0$ and all $u \in C_0^\infty(\mathbb{R}^{n+1})$, one has

$$\|u\|_{L^p(\mathbb{R}^{n+1})} \leq C \|\mathcal{L}u\|_{L^q(\mathbb{R}^{n+1})} ? \tag{0.5}$$

Using the group $\{\delta_\lambda\}_{\lambda>0}$ one immediately sees from (0.3), (0.4) that a necessary condition for (0.5) to hold is given by

$$\frac{1}{q} - \frac{1}{p} = \frac{2}{Q}. \tag{0.6}$$

It is a nontrivial fact that (0.6) is also sufficient for (0.5). These considerations led in [G] to formulate the following :

Conjecture. — *Suppose that $V \in \mathcal{L}^{Q/2}_{\text{loc}}(\mathbb{R}^{n+1})$. Then, the differential inequality*

$$|\mathcal{L}u| \leq |Vu| \tag{0.7}$$

has the strong unique continuation property at points of the degeneracy manifold $\{(0,t) \in \mathbb{R}^{n+1} | t \in \mathbb{R}\}$.

In this paper we prove a Carleman type inequality for the operator $\mathcal{L}$ that implies the strong unique continuation for (0.7), provided $V \in L^r_{\mathrm{loc}}(\mathbb{R}^{n+1})$, where $r > n = Q-2$, when n is even, and $r > 2n^2/(n+1)$, when n is odd. In particular, when $n = 2$, and hence $Q = 4$, we prove that the above conjecture is true, since $(Q/2) = Q-2 = 2$, except that we miss the end-point case $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. It should be emphasized that, in spite of the apparent similarities with the Euclidean Laplacian, the analysis of the Grushin operator presents several subtle novelties that have yet to be fully understood. In this respect, already in the case $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{n+1})$, our result is quite different from its Euclidean predecessor. To explain this point we must bring in the special geometry of the Grushin operator and of its close relative, the sub-Laplacian on the Heisenberg group. Suppose for a moment that $n = 2k$, with $k \in \mathbb{N}$, and for $x, y \in \mathbb{R}^k$ let $z = (x, y) \in \mathbb{R}^n$, $t \in \mathbb{R}$. In the coordinates (z,t) the sub-Laplacian on the Heisenberg group $\mathbb{H}^k$ can be written as follows :

$$\Delta_{\mathbb{H}^k} = \Delta_z + 4|z|^2 \frac{\partial^2}{\partial t^2} + 4\frac{\partial}{\partial t} T, \tag{0.8}$$

where T is the transversal vector field

$$T = \sum_{j=1}^{k} \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right).$$

It is clear from (0.1) and (0.8) that there exists a close connection between the Grushin operator and the sub-Laplacian on the group $\mathbb{H}^k$. In fact, it turns out that, if $\Delta_{\mathbb{H}^k} u = 0$, and moreover $Tu = 0$, then u solves $\mathcal{L}u = 0$. We mention that $Tu = 0$ if and only if u is invariant under the action of the torus $\mathbb{T}$ on $\mathbb{H}^k$ given by

$$\varphi_\theta(z,t) = (e^{i\theta} z, t), \qquad \theta \in \mathbb{T}$$

(here, we have identified $z = (x,y) \in \mathbb{R}^{2k}$ with $z = x + iy \in \mathbb{C}^k$). In spite of this connection between $\mathcal{L}$ and $\Delta_{\mathbb{H}^k}$, for the latter the unique continuation fails, even for $V \in C^\infty$, as a consequence of a result of Bahouri [B]. Recently, one of us [G] has proved the strong unique continuation for (0.7) under suitable size restrictions on V.

For nonsingular potentials the assumption on V in [G] reads

$$|V(z,t)| \le C\psi(z,t), \tag{0.9}$$

where

$$\psi(z,t) = \frac{|z|^2}{(|z|^4 + 4t^2)^{1/2}} \tag{0.10}$$

(here, everything is localized in a neighborhood of the origin). It is clear that (0.9) does not allow for V merely in L^∞_{loc}, but forces vanishing at $z = 0$. The use of the function ψ in the right hand side of (0.9) was suggested by its natural appearance in some representation formulas for the operator $\mathcal{L}$ in (0.1). These are, in turn, related to the polar coordinate decomposition of $\mathcal{L}$, see §1.

Consider the natural pseudo-distance function for $\mathcal{L}$

$$\rho = \rho(z,t) = (|z|^4 + 4t^2)^{1/4}. \tag{0.11}$$

What also makes the operator $\mathcal{L}$ interesting is the fact that it does not map functions of ρ into functions of ρ. In fact, if we let for $f \in C^2(\mathbb{R}_+)$

$$u(z,t) = f(\rho(z,t)),$$

then one has

$$\mathcal{L}u = \psi\Big(f''(\rho) + \frac{Q-1}{\rho} f'(\rho)\Big), \tag{0.12}$$

with ψ being given by (0.10). This feature of the Grushin operator (which is shared by the sub-Laplacian on the Heisenberg group) makes the analysis considerably harder than that of the Euclidean Laplacian.

Concerning the approach in this paper, it is based on a suitable Carleman estimate (Theorem 5.1 below), which involves the weight ρ^{-s}, $0 < s < \infty$, as well as positive and negative powers of the function ψ in (0.10).

The structure of the paper is as follows. In §1 we introduce some suitable polar coordinates to obtain a decomposition of $\mathcal{L}$. These coordinates were first introduced by Greiner [Gr] for the Heisenberg group. In §2 we compute the spherical harmonics of the Grushin operator. §§3 and 4 constitute the main technical part of the paper. There, we prove the $L^1 - L^\infty$ and weighted $L^2 - L^2$ estimates for the projection operator onto spherical harmonics of a given degree. The main Carleman estimate (Theorem 5.1) is proved in §5 by using the estimates of the projection operator as a building block. This is the idea of D. Jerison in [J] where a simple proof for the Jerison-Kenig's Carleman-type inequality was discovered. Finally, in

§6 we deduce from the Carleman estimate the strong unique continuation property.

1. Polar coordinates.

In this section we introduce suitable polar coordinates to obtain a decomposition of the operator $\mathcal{L}$ in (0.1).

Let

$$\rho = (|z|^4 + 4t^2)^{1/4}, \qquad z \in \mathbb{R}^n, \quad t \in \mathbb{R} \tag{1.1}$$

and

$$\begin{cases} z_1 = \rho \sin^{1/2}\varphi \sin\theta_1 \dots \sin\theta_{n-2}\sin\theta_{n-1} \\ z_2 = \rho \sin^{1/2}\varphi \sin\theta_1 \dots \sin\theta_{n-2}\cos\theta_{n-1} \\ \vdots \\ z_n = \rho \sin^{1/2}\varphi \cos\theta_1 \\ t = \dfrac{\rho^2}{2}\cos\varphi. \end{cases} \tag{1.2}$$

Here, $0 < \varphi < \pi$, $0 < \theta_i < \pi$, $i = 1, 2, \dots, n-2$ and $0 < \theta_{n-1} < 2\pi$. We plan to compute the Grushin operator in (0.1) in the above coordinates $(\rho, \varphi, \theta_1, \dots, \theta_{n-1})$.

Let $r = |z|$. From (1.2) we obtain

$$r = |z| = \rho \sin^{1/2}\varphi. \tag{1.3}$$

By the usual spherical coordinates in $\mathbb{R}^n$, we have

$$dz = r^{n-1} dr\, d\omega, \tag{1.4}$$

$$\Delta_z = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}. \tag{1.5}$$

Here, $d\omega$ and $\Delta_{S^{n-1}}$, respectively, denote Lebesgue measure and the Laplace-Beltrami operator on S^{n-1}. From (1.2) and (1.3) we have

$$\frac{\partial(r,t)}{\partial(\rho,\varphi)} = \begin{pmatrix} \sin^{1/2}\varphi & \frac{\rho}{2}\sin^{-1/2}\varphi\cos\varphi \\ \rho\cos\varphi & -\frac{\rho^2}{2}\sin\varphi \end{pmatrix}.$$

This gives

$$dr\,dt = \frac{\rho^2}{2}\sin^{-1/2}\varphi d\rho d\varphi. \tag{1.6}$$

Substituting (1.6) in (1.4) yields

$$dz\,dt = \frac{1}{2}\rho^{n+1}(\sin\varphi)^{\frac{n-2}{2}}d\rho d\varphi d\omega. \tag{1.7}$$

We also have

$$\frac{\partial(\rho,\varphi)}{\partial(r,t)} = \begin{pmatrix} \sin^{3/2}\varphi & \rho^{-1}\cos\varphi \\ 2\rho^{-1}\sin^{1/2}\varphi\cos\varphi & -2\rho^{-2}\sin\varphi \end{pmatrix}. \tag{1.8}$$

Note that

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\Delta_{S^{n-1}}. \tag{1.9}$$

A straightforward computation based on (1.8) gives

$$\begin{aligned} \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} &= \sin^3\varphi\frac{\partial^2}{\partial\rho^2} + 4\rho^{-1}\sin^2\varphi\cos\varphi\frac{\partial^2}{\partial\rho\partial\varphi} \\ &+ 4\rho^{-2}\sin\varphi\cos^2\varphi\frac{\partial^2}{\partial\varphi^2} \\ &+ (3\rho^{-1}\sin\varphi\cos^2\varphi + (n-1)\rho^{-1}\sin\varphi)\frac{\partial}{\partial\rho} \\ &+ (-8\rho^{-2}\sin^2\varphi\cos\varphi + 2n\rho^{-2}\cos\varphi)\frac{\partial}{\partial\varphi}, \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial t^2} &= \rho^{-2}\cos^2\varphi\frac{\partial^2}{\partial\rho^2} - 4\rho^{-3}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\rho\partial\varphi} \\ &+ 4\rho^{-4}\sin^2\varphi\frac{\partial^2}{\partial\varphi^2} + (2\rho^{-3} - 3\rho^{-3}\cos^2\varphi)\frac{\partial}{\partial\rho} \\ &+ 8\rho^{-4}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}. \end{aligned} \tag{1.11}$$

Substituting (1.10) and (1.11) in (1.9), we obtain

$$\mathcal{L} = \sin\varphi\left\{\frac{\partial^2}{\partial\rho^2} + \frac{n+1}{\rho}\frac{\partial}{\partial\rho} + \frac{4}{\rho^2}\mathcal{L}_\sigma\right\} \tag{1.12}$$

where $\sigma = (\varphi, \omega)$, $\omega \in S^{n-1}$, and

$$\mathcal{L}_\sigma = \frac{\partial^2}{\partial\varphi^2} + \frac{n}{2}\frac{\cos\varphi}{\sin\varphi}\frac{\partial}{\partial\varphi} + \frac{1}{(2\sin\varphi)^2}\Delta_{S^{n-1}}. \tag{1.13}$$

From (1.3) we see that

$$\sin\varphi = \frac{r^2}{\rho^2} = \psi,$$

with ψ defined by (0.10). Recalling the homogeneous dimension $Q = n+2$ introduced in (0.4), we can rewrite (1.12) in the more suggestive way :

$$\mathcal{L} = \psi\left\{\frac{\partial^2}{\partial\rho^2} + \frac{Q-1}{\rho}\frac{\partial}{\partial\rho} + \frac{4}{\rho^2}\mathcal{L}_\sigma\right\}. \tag{1.14}$$

From (1.14) it is clear that if a function u depends solely on the pseudo-distance ρ, i.e., $u(z,t) = f(\rho(z,t))$, then $\mathcal{L}u$ is given by (0.12).

The most interesting feature of formula (1.12) is that the variables ρ and (φ, ω) separate. We mention that for the Heisenberg group in $\mathbb{R}^3$, $\mathbb{H}^1$, the coordinates (1.2) were first introduced by Greiner [Gr]. For the Heisenberg sub-Laplacian, however, the variables ρ and (φ, ω) do not separate.

2. Spherical harmonics for the Grushin operator.

This section is devoted to computing the surface spherical harmonics of the Grushin operator, i.e., the eigenfunctions of $\mathcal{L}_\sigma$ in (1.13).

For $k = 0, 1, \ldots$, we form the function $\rho^k g(\varphi, \omega)$. By (1.12), this is a solution of $\mathcal{L}u = 0$ if and only if

$$\mathcal{L}_\sigma g = -\frac{k(n+k)}{4} g. \tag{2.1}$$

Suppose now that $g(\varphi, \omega) = h(\varphi)Y(\omega)$ where $Y(\omega)$ is a spherical harmonic of degree $\ell \in \{0, 1, \ldots, k\}$. We recall [SW] that

$$\Delta_{S^{n-1}} Y = -\ell(\ell + n - 2)Y. \tag{2.2}$$

Using (1.13) and (2.2), one easily checks that (2.1) holds if and only if

$$\frac{d^2h}{d\varphi^2} + \frac{n}{2}\frac{\cos\varphi}{\sin\varphi}\frac{dh}{d\varphi} + \left[\frac{k(n+k)}{4} - \frac{\ell(\ell+n-2)}{4\sin^2\varphi}\right]h = 0. \tag{2.3}$$

We let $\tau = \cos\varphi$, $u(\tau) = h(\varphi)$ in (2.3). By this change of variable, the latter equation transforms into

$$\frac{d^2u}{d\tau^2} - \left(\frac{n}{2}+1\right)\tau\frac{du}{d\tau} + \left[\frac{k(n+k)}{4} - \frac{\ell(\ell+n-2)}{4(1-\tau^2)}\right]u = 0. \tag{2.4}$$

Setting $v(\tau) = (1-\tau^2)^{-\ell/4}u(\tau)$ one verifies that v satisfies

$$(1-\tau^2)\frac{d^2v}{d\tau^2} - \left(\frac{n}{2}+\ell+1\right)\tau\frac{dv}{d\tau} + \left(\frac{k-\ell}{2}\right)\left(\frac{k-\ell}{2}+\ell+\frac{n}{2}\right)v = 0. \tag{2.5}$$

This is a Jacobi differential equation, provided $\ell \equiv k(\text{mod } 2)$ (see [E], vol. 2, p. 169). One polynomial solution of (2.5) is given by the ultraspherical (or Gegenbauer) polynomial

$$v(\tau) = C_{\frac{k-\ell}{2}}^{\frac{\ell}{2}+\frac{n}{4}}(\tau)$$

(see [E], vol. 2, p. 174).

To summarize, we have proved

LEMMA 2.6. — *Let k be a nonnegative integer and $\ell \equiv k(\text{mod}\,2)$, with $0 \le \ell \le k$. Suppose that Y_ℓ is a spherical harmonic of degree ℓ. Then*

$$g(\varphi,\omega) = \sin^{\frac{\ell}{2}}\varphi\, C_{\frac{k-\ell}{2}}^{\frac{\ell}{2}+\frac{n}{4}}(\cos\varphi)Y_\ell(\omega)$$

satisfies (2.1).

Fix now an integer $\ell \ge 0$ and denote by $\{Y_{\ell,j}\}_{j=1,2,\dots,d_\ell}$ an orthonormal basis for the space of spherical harmonics of degree ℓ on S^{n-1}. Recall [SW] that

$$d_\ell = \frac{(n+2\ell-2)\Gamma(n+\ell-2)}{\Gamma(\ell+1)\Gamma(n-1)}. \tag{2.7}$$

We define,

$$\mathcal{H}_k = \operatorname{span}\left\{ \sin^{\frac{\ell}{2}} \varphi \mathrm{C}^{\frac{\ell}{2}+\frac{n}{4}}_{\frac{k-\ell}{2}}(\cos\varphi)\mathrm{Y}_{\ell,\mathrm{j}}(\omega) \right| \tag{2.8}$$
$$j = 1, 2, \ldots, d_\ell, \qquad 0 \le \ell \le k, \qquad \ell \equiv k (\mathrm{mod}\ 2) \Big\}.$$

Consider the measure on

$$\Omega = \left\{ (z,t) \in \mathbb{R}^{n+1} \middle| \rho = (|z|^4 + 4t^2)^{\frac{1}{4}} = 1 \right\} \tag{2.9}$$

given by

$$d\Omega = \sin^{\frac{n}{2}} \varphi \, d\varphi \, d\omega. \tag{2.10}$$

Here, we have parametrized Ω (see (1.2)) by

$$z_1 = \sin^{\frac{1}{2}} \varphi \sin\theta_1 \ldots \sin\theta_{n-2} \sin\theta_{n-1},$$
$$z_2 = \sin^{\frac{1}{2}} \varphi \sin\theta_1 \ldots \sin\theta_{n-2} \cos\theta_{n-1}, \ldots,$$
$$z_n = \sin^{\frac{1}{2}} \varphi \cos\theta_1, \qquad t = \frac{1}{2}\cos\varphi.$$

We have

LEMMA 2.11. — *The following direct sum decomposition holds :*

$$L^2(\Omega, d\Omega) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

Proof. — We begin by observing that the spaces $\mathcal{H}_k$ are mutually orthogonal in $L^2(\Omega, d\Omega)$. This follows from the orthogonality properties of spherical harmonics together with the fact (see [Sz], p. 81)

$$\int_0^\pi C_j^\lambda(\cos\varphi) C_k^\lambda(\cos\varphi) \sin^{2\lambda}\varphi \, d\varphi = \frac{2^{1-2\lambda}\pi\Gamma(j+2\lambda)}{[\Gamma(\lambda)]^2 (j+\lambda)\Gamma(j+1)} \delta_{jk}, \text{ for } \lambda > -\frac{1}{2}, \quad \lambda \ne 0. \tag{2.12}$$

To prove the completeness of $\bigoplus_{k=0}^{\infty} \mathcal{H}_k$ it suffices to show that if $f \in L^2(\Omega, d\Omega)$ is orthogonal to each $\mathcal{H}_k$, then $f = 0$ a. e. on Ω. Suppose, in fact, that

$$\int_\Omega f(\varphi,\omega)\sin^{\frac{\ell}{2}}\varphi C_m^{\frac{\ell}{2}+\frac{n}{4}}(\cos\varphi)Y_{\ell,j}(\omega)d\Omega = 0$$

for $j = 1, 2, \ldots, d_\ell$, ℓ and m in $\mathbb{N}\cup\{0\}$. By Fubini's theorem, we infer

$$\int_0^\pi u_{\ell,j}(\varphi)C_m^{\frac{\ell}{2}+\frac{n}{4}}(\cos\varphi)(\sin\varphi)^{\ell+\frac{n}{2}}d\varphi = 0,$$

where

$$u_{\ell,j}(\varphi) = \sin^{-\frac{\ell}{2}}\varphi\int_{S^{n-1}} f(\varphi,\omega)Y_{\ell,j}(\omega)d\omega.$$

One recognizes that $u_{\ell,j}\in L^2([0,\pi],(\sin\varphi)^{\ell+\frac{n}{2}}d\varphi)$. By the completeness of $\{C_m^\lambda(\cos\varphi)\}_{m=0}^\infty$ in the space $L^2([0,\pi],(\sin\varphi)^{2\lambda}d\varphi)$, we conclude $u_{\ell,j}(\varphi) = 0$ for a. e. $\varphi\in[0,\pi]$. Using the fact that the surface harmonics form a complete system in $L^2(S^{n-1})$ [SW], we finally have $f(\varphi,\omega) = 0$ for a. e. (φ,ω). This concludes the proof of the lemma. □

We now let

$$P_k : L^2(\Omega, d\Omega)\to\mathcal{H}_k \tag{2.13}$$

denote the projection operator onto the $(k+1)$-th eigenspace of $\mathcal{L}_\sigma$ in (1.13).

For $k,\ell\in\mathbb{N}\cup\{0\}$, we introduce the normalization constants $b_{k,\ell}$ by the formula

$$b_{k,\ell}^2 = \frac{(k+\frac{n}{2})2^{\ell+\frac{n}{2}-2}[\Gamma(\frac{\ell}{2}+\frac{n}{4})]^2\Gamma(\frac{k-\ell}{2}+1)}{\pi\Gamma(\frac{k+\ell}{2}+\frac{n}{2})}. \tag{2.14}$$

It follows from (2.12) that

$$\begin{aligned}\int_\Omega & b_{k,\ell_1}\sin^{\frac{\ell_1}{2}}\varphi C_{\frac{k-\ell_1}{2}}^{\frac{\ell_1}{2}+\frac{n}{4}}(\cos\varphi)Y_{\ell_1,j_1}(\omega)\\ &\cdot b_{k,\ell_2}\sin^{\frac{\ell_2}{2}}\varphi C_{\frac{k-\ell_2}{2}}^{\frac{\ell_2}{2}+\frac{n}{4}}(\cos\varphi)\overline{Y}_{\ell_2,j_2}(\omega)d\Omega = \delta_{\ell_1\ell_2}\delta_{j_1j_2},\end{aligned} \tag{2.15}$$

where $0\le\ell_i\le k$, $\ell_i\equiv k(\bmod 2)$ and $1\le j_i\le d_{\ell_i}$ for $i = 1,2$. Thus, we can write

$$P_k(g)(\varphi,\omega) = \int_0^\pi\int_{S^{n-1}} G_k(\varphi,\omega,\theta,\eta)g(\theta,\eta)(\sin\theta)^{\frac{n}{2}}d\theta\, dS^{n-1}(\eta) \tag{2.16}$$

where

$$G_k(\varphi,\omega,\theta,\eta) = \sum_{\substack{0\le l\le k\\ l\equiv k(\mathrm{mod}\,2)}} b_{k,\ell}^2 \sin^{\frac{\ell}{2}}\varphi C_{\frac{k-\ell}{2}}^{\frac{\ell}{2}+\frac{n}{4}}(\cos\varphi) \cdot \sin^{\frac{\ell}{2}}\theta C_{\frac{k-\ell}{2}}^{\frac{\ell}{2}+\frac{n}{4}}(\cos\theta)\sum_{j=1}^{d_\ell} Y_{\ell,j}(\omega)\overline{Y}_{\ell,j}(\eta). \tag{2.17}$$

It is known that, for $n \geq 2$,

$$\sum_{j=1}^{d_\ell} Y_{\ell,j}(\omega)\overline{Y}_{\ell,j}(\eta) = \frac{d_\ell}{|S^{n-1}|}\cdot\frac{C_\ell^{\frac{n-2}{2}}(\cos\zeta)}{C_\ell^{\frac{n-2}{2}}(1)} \tag{2.18}$$

where ζ is the angle between ω and η on S^{n-1}, and $0 \leq \zeta \leq \pi$ (see [E], v. 2, p. 243).

In the next two sections, we will study the mapping properties of the projection operator P_k.

3. $L^1 - L^\infty$ estimates.

Our goal in this section is to prove the following :

THEOREM 3.1. — *There exists $C > 0$ such that*

$$\|P_k(g)\|_{L^\infty(\Omega,d\Omega)} \leq C(k+1)^{n-1}\|g\|_{L^1(\Omega,d\Omega)}$$

for every $g \in L^1(\Omega, d\Omega)$.

The proof of Theorem 3.1 relies on the following lemma.

LEMMA 3.2. — *There exists $C > 0$ such that*

$$\sum_{\substack{0\le l\le k\\ l\equiv k(\mathrm{mod}\,2)}} b_{k,\ell}^2\left[\sin^{\frac{\ell}{2}}\varphi C_{\frac{k-\ell}{2}}^{\frac{\ell}{2}+\frac{n}{4}}(\cos\varphi)\right]^2(\ell+1)^{n-2} \leq C(k+1)^{n-1}$$

for every $k \in \mathbb{N}\cup\{0\}$.

Taking Lemma 3.2 for granted, we give

Proof of Theorem 3.1. — By (2.16), it suffices to show that

$$|G_k(\varphi,\omega,\theta,\eta)| \leq C(k+1)^{n-1} \text{ for } 0 \leq \varphi,\theta \leq \pi \text{ and } \omega,\eta \in S^{n-1}. \tag{3.3}$$

It follows from (2.7) and Stirling's formula for the Gamma function that

$$d_\ell \leq C(\ell+1)^{n-2}. \tag{3.4}$$

Thus, by (2.17), (2.18) and Lemma 3.2, we have

$$\begin{aligned}|G_k(\varphi,\omega,\varphi,\omega)| &\leq C \sum_{\substack{0\leq l\leq k\\ l\equiv k(\mathrm{mod}\,2)}} b_{k,\ell}^2\left[\sin^{\frac{\ell}{2}}\varphi C_{\frac{k-\ell}{2}}^{\frac{\ell}{2}+\frac{n}{4}}(\cos\varphi)\right]^2(\ell+1)^{n-2} \\ &\leq C(k+1)^{n-1}.\end{aligned} \tag{3.5}$$

The desired estimate (3.3) then follows from the Schwarz inequality and (3.5).

The proof is completed. □

To prove Lemma 3.2, we need the following

LEMMA 3.6. — *Let $0<\lambda<1$. There exists $C=C(\lambda)>0$ such that for each $k\in\mathbb{N}\cup\{0\}$ and $0<\theta<\pi$, we have*

$$\left|\frac{1}{(2-2\cos\theta)^\lambda}-\sum_{j=0}^{k}C_j^\lambda(\cos\theta)\right| \leq \frac{C(k+1)^{\lambda-1}}{(\sin\theta)^{\lambda+1}}.$$

Proof. — We have the following integral representation of $C_j^\lambda(\cos\theta)$, for $0<\theta<\pi$, $0<\lambda<1$

$$\begin{aligned}C_j^\lambda(\cos\theta) = \frac{2}{\pi}\sin(\pi\lambda)\,\mathrm{Im}\Bigg\{&\mathrm{e}^{\mathrm{i}[2\lambda\theta+(\frac{1}{2}-\lambda)\pi]} \\ &\cdot\int_0^1 (te^{i\theta})^j t^{2\lambda-1}(1-t)^{-\lambda}(1-te^{2i\theta})^{-\lambda}dt\Bigg\}\end{aligned}$$

(see [Sz], p. 90).

Recall that for $|r|<1$,

$$\frac{1}{(1-2r\cos\theta+r^2)^\lambda}=\sum_{j=0}^{\infty}r^jC_j^\lambda(\cos\theta),$$

(see [Sz], p. 82).

A limiting argument shows

$$\frac{1}{(2-2\cos\theta)^{\lambda}} - \sum_{j=0}^{k} C_j^{\lambda}(\cos\theta) = \left(\frac{2}{\pi}\right)\sin(\pi\lambda)$$
$$\operatorname{Im}\left\{ e^{i[2\lambda\theta+(\frac{1}{2}-\lambda)\pi]} \int_0^1 \frac{(te^{i\theta})^{k+1}}{1-te^{i\theta}} t^{2\lambda-1}(1-t)^{-\lambda}(1-te^{2i\theta})^{-\lambda}dt \right\}.$$

It follows that

(3.7)
$$\left|\frac{1}{(2-2\cos\theta)^{\lambda}} - \sum_{j=0}^{k} C_j^{\lambda}(\cos\theta)\right| \le C\int_0^1 \frac{t^{k+2\lambda}}{|1-te^{i\theta}|\,|1-t|^{\lambda}|1-te^{2i\theta}|^{\lambda}}dt.$$

One sees easily that, for $0<t<1$ and $0<\theta<\pi$,

$$|1-te^{i\theta}| \ge t\sin\theta,$$
$$|1-te^{2i\theta}| \ge ct\sin\theta.$$

Substitution in (3.7) yields

$$(3.8)\quad \left|\frac{1}{(2-2\cos\theta)^{\lambda}} - \sum_{j=0}^{k} C_j^{\lambda}(\cos\theta)\right| \le \frac{C}{(\sin\theta)^{\lambda+1}}\int_0^1 t^{k+\lambda-1}(1-t)^{-\lambda}dt$$
$$= \frac{C}{(\sin\theta)^{\lambda+1}}\cdot\frac{\Gamma(k+\lambda)\Gamma(1-\lambda)}{\Gamma(k+1)}.$$

The conclusion of Lemma 3.6 now easily follows from (3.8) and Stirling's formula for the Gamma function.

We are now in a position to give the

Proof of Lemma 3.2. — We first consider the case when k is even. In this case we can write $\ell = 2j, \quad j=0,1,\dots,k/2$.

Thus, by (2.14),

$$(3.9)\quad \sum_{\substack{0\le l\le k\\ l\equiv k(\mathrm{mod}\,2)}} b_{k,\ell}^2\left[\sin^{\frac{\ell}{2}}\varphi C_{\frac{k-\ell}{2}}^{\frac{\ell}{2}+\frac{n}{4}}(\cos\varphi)\right]^2(\ell+1)^{n-2}$$
$$\le C(k+1)\sum_{j=0}^{\frac{k}{2}} \frac{[\Gamma(j+\frac{n}{4})]^2\Gamma(\frac{k}{2}-j+1)2^{2j}}{\Gamma(\frac{k}{2}+j+\frac{n}{2})}\cdot\left[\sin^j\varphi C_{\frac{k}{2}-j}^{j+\frac{n}{4}}(\cos\varphi)\right]^2(j+1)^{n-2}.$$

We now recall the following addition formula for Gegenbauer polynomials (see [E], vol 2, p. 178) :

$$\begin{aligned}(3.10)\qquad & C_m^\lambda(\cos\varphi\cos\psi+\sin\varphi\sin\psi\cos\theta)\\ =&\sum_{j=0}^{m}\frac{[\Gamma(j+\lambda)]^2\Gamma(m-j+1)2^{2j}}{\Gamma(m+j+2\lambda)}\sin^j\varphi C_{m-j}^{j+\lambda}(\cos\varphi)\sin^j\psi C_{m-j}^{j+\lambda}(\cos\psi)\\ &\cdot\frac{\Gamma(2\lambda-1)(2j+2\lambda-1)}{[\Gamma(\lambda)]^2}\cdot C_j^{\lambda-\frac12}(\cos\theta).\end{aligned}$$

In (3.10), when $\lambda=1/2$, we must replace

$$\frac{\Gamma(2\lambda-1)(2j+2\lambda-1)}{[\Gamma(\lambda)]^2}C_j^{\lambda-\frac12}(\cos\theta)$$

by $[\Gamma(1/2)]^{-2}2\cos(j\theta)$ when $j\neq0$, or by $[\Gamma(1/2)]^{-2}$ when $j=0$.

We now let $\lambda=n/4$, $m=k/2$, $\varphi=\psi$ and $\theta=0$ in (3.10). It follows that the right-hand side of (3.9) is bounded by

$$C(k+1)\cdot(k+1)^{n-2-(\frac n2-1)}\cdot C_{\frac k2}^{\frac n4}(1)\le C(k+1)^{n-1}$$

where we have used $n-2\ge(n/2)-1$ when $n\ge2$ and

$$(3.11)\qquad C_j^\lambda(1)=\frac{\Gamma(j+2\lambda)}{\Gamma(2\lambda)\Gamma(j+1)}\sim(j+1)^{2\lambda-1}$$

(see [E], v. 2, p. 174).

This proves the lemma when k is even.

Suppose now that k is odd, and write $\ell=2j+1$, $j=0,1,\ldots,(k-1)/2$. We need to show

$$\begin{aligned}(3.12)\qquad & \sin\varphi\sum_{j=0}^{\frac{k-1}{2}}\frac{[\Gamma(j+\frac n4+\frac12)]^2\Gamma(\frac{k-1}{2}-j+1)2^{2j}}{\Gamma(\frac{k-1}{2}+j+\frac n2+1)}\\ &\left[\sin^j\varphi C_{\frac{k-1}{2}-j}^{j+\frac n4+\frac12}(\cos\varphi)\right]^2\cdot(j+1)^{n-2}\le C(k+1)^{n-2}\end{aligned}$$

where $C>0$ is independent of k and φ.

To this purpose we let $m=(k-1)/2$, $\lambda=(n/4)+(1/2)$ and $\varphi=\psi$ in (3.10), obtaining

$$
\begin{aligned}
&C^{\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}}(\cos^2\varphi+\sin^2\varphi\cos\theta)\\
(3.13)\quad &=\sum_{j=0}^{\frac{k-1}{2}}\frac{[\Gamma(j+\frac{n}{4}+\frac{1}{2})]^2\Gamma(\frac{k-1}{2}-j+1)2^{2j}}{\Gamma(\frac{k-1}{2}+j+\frac{n}{2}+1)}\left[\sin^j\varphi C^{j+\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}-j}(\cos\varphi)\right]^2\\
&\quad\cdot\frac{\Gamma(\frac{n}{2})(2j+\frac{n}{2})}{[\Gamma(\frac{n}{4}+\frac{1}{2})]^2}C^{\frac{n}{4}}_j(\cos\theta).
\end{aligned}
$$

If $n\ge 4$, then $n-2\ge(n/2)$. Setting $\theta=0$ in (3.13), we see that the left-hand side of (3.12) is bounded by

$$C\sin\varphi(k+1)^{n-2-\frac{n}{2}}C^{\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}}(1)\le C(k+1)^{n-2},$$

where we have also used (3.11).

Finally, we consider the case when $n=2$ or 3 (and k is odd).

We multiply both sides of (3.13) by $C^{\frac{n}{4}}_j(\cos\theta)\sin^{\frac{n}{2}}\theta$, $j=0,1,\ldots,(k-1)/2$, and integrate on $[0,\pi]$ with respect to θ. Using the orthogonality relation (2.12), we obtain for $j=0,1,\ldots,(k-1)/2$,

$$
\begin{aligned}
&\int_0^\pi C^{\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}}(\cos^2\varphi+\sin^2\varphi\cos\theta)C^{\frac{n}{4}}_j(\cos\theta)\sin^{\frac{n}{2}}\theta\,d\theta\\
&=C\cdot\frac{[\Gamma(j+\frac{n}{4}+\frac{1}{2})]^2\Gamma(\frac{k-1}{2}-j+1)2^{2j}}{\Gamma(\frac{k-1}{2}+j+\frac{n}{2}+1)}\left[\sin^j\varphi C^{j+\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}-j}(\cos\varphi)\right]^2\cdot\frac{\Gamma(j+\frac{n}{2})}{\Gamma(j+1)}.
\end{aligned}
$$

Summing in $j=0,1,\ldots,(k-1)/2$, we see that the left-hand side of (3.12) is bounded by

$$
\begin{aligned}
&C(k+1)^{\frac{n}{2}-1}\int_0^\pi C^{\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}}(\cos^2\varphi+\sin^2\varphi\cos\theta)\sin\varphi\sum_{j=0}^{\frac{k-1}{2}}C^{\frac{n}{4}}_j(\cos\theta)\cdot\sin^{\frac{n}{2}}\theta\,d\theta\\
&\quad=I+II.
\end{aligned}
$$

Here, I is that part of the integral performed on the set $A=\left\{0\le\theta\le\pi\,\middle|\,0\le\sin\theta\le\dfrac{1}{(k+1)\sin\varphi}\right\}$, whereas II is that part of the integral on the set $B=\left\{0\le\theta\le\pi\,\middle|\,\dfrac{1}{(k+1)\sin\varphi}\le\sin\theta\le 1\right\}$.

We recall the following asymptotic estimates for the Gegenbauer polynomials ([Sz], p. 172) :

(3.14)
$$C_j^\lambda(\cos\theta) = \begin{cases} \sin^{-\lambda}\theta\, O((j+1)^{\lambda-1}) & \text{if } \dfrac{1}{j+1} \leq \theta \leq \pi - \dfrac{1}{j+1} \\ O((j+1)^{2\lambda-1}), & \text{if } 0 \leq \theta \leq \dfrac{1}{j+1} \text{ or } \pi - \dfrac{1}{j+1} \leq \theta \leq \pi. \end{cases}$$

The estimate of I will follow from (3.14) and the following

$$\left| \sum_{j=0}^{\frac{k-1}{2}} C_j^\lambda(\cos\theta) \right| \leq \frac{C}{(\sin\theta)^{2\lambda}} \tag{3.15}$$

for $0 < \theta < \pi$ and $0 < \lambda < 1$, where C depends only on λ.

To see (3.15), suppose first $k \leq 1/(\sin\theta)$. Then,

$$\begin{aligned} \left| \sum_{j=0}^{\frac{k-1}{2}} C_j^\lambda(\cos\theta) \right| &\leq \sum_{j=0}^{\frac{k-1}{2}} C_j^\lambda(1) \leq C \sum_{j=0}^{\frac{k-1}{2}} (j+1)^{2\lambda-1} \\ &\leq C(k+1)^{2\lambda} \leq \frac{C}{(\sin\theta)^{2\lambda}}. \end{aligned}$$

When $k\sin\theta > 1$, (3.15) follows easily from Lemma 3.6. Using (3.15), we have

$$\begin{aligned} |I| &\leq C(k+1)^{\frac{n}{2}-1} C_{\frac{k-1}{2}}^{\frac{n}{4}+\frac{1}{2}}(1) \sin\varphi \text{ (measure of A)} \\ &\leq C(k+1)^{n-2}. \end{aligned} \tag{3.16}$$

We now turn to estimating II. For a fixed $\varphi \in [0,\pi]$ we define $\zeta \in [0,\pi]$ by

$$\cos\zeta = \cos^2\varphi + \sin^2\varphi\cos\theta. \tag{3.17}$$

We claim that

$$\frac{\sin\varphi\sin\theta}{\sin\zeta} \leq \sqrt{2}. \tag{3.18}$$

In fact, if $0 \leq \zeta \leq \frac{\pi}{2}$,

$$\sin\zeta \geq \sqrt{1-\cos\zeta} = \sin\varphi\sqrt{1-\cos\theta} \geq \frac{1}{\sqrt{2}}\sin\varphi\sin\theta.$$

If, on the other hand, $(\pi/2) \leq \zeta \leq \pi$, we have

$$\sin\zeta \geq \sqrt{1+\cos\zeta} = \sqrt{2-\sin^2\varphi + \sin^2\varphi\cos\theta}$$
$$\geq \sin\varphi\sqrt{1+\cos\theta} \geq \frac{1}{\sqrt{2}}\sin\varphi\sin\theta.$$

To finish the proof, we write

$$II = C(k+1)^{\frac{n}{2}-1}\int_B C^{\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}}(\cos\zeta)\sin\varphi\left[\sum_{j=0}^{\frac{k-1}{2}} C^{\frac{n}{4}}_j(\cos\theta) - \frac{1}{(2-2\cos\theta)^{\frac{n}{4}}}\right]$$
$$\sin^{\frac{n}{2}}\theta\,d\theta + C(k+1)^{\frac{n}{2}-1}\int_B C^{\frac{n}{4}+\frac{1}{2}}_{\frac{k-1}{2}}(\cos\zeta)\sin\varphi$$
$$\cdot\frac{1}{(2-2\cos\theta)^{\frac{n}{4}}}(\sin\theta)^{\frac{n}{2}}\,d\theta = II_1 + II_2.$$

Here, $B = \left\{0 \leq \theta \leq \pi \,\middle|\, \frac{1}{(k+1)\sin\varphi} \leq \sin\theta \leq 1\right\}$. It follows from Lemma 3.6, (3.14) and (3.18) that

$$|II_1| \leq C(k+1)^{n-\frac{5}{2}}(\sin\varphi)^{-\frac{n}{4}+\frac{1}{2}}\int_B(\sin\theta)^{-\frac{3}{2}}\,d\vartheta \tag{3.19}$$
$$\leq C(k+1)^{n-2}(\sin\varphi)^{1-\frac{n}{4}} \leq C(k+1)^{n-2}.$$

If $n=3$, by (3.14) and (3.18), we have

$$|II_2| \leq C(k+1)^{\frac{3}{4}}(\sin\varphi)^{-\frac{1}{4}}\int_B(\sin\theta)^{-\frac{5}{4}}\,d\theta \tag{3.20}$$
$$\leq C(k+1).$$

In the case when $n=2$, we need an integration by parts argument to estimate II_2.

By (3.17), we have

$$\frac{d\zeta}{d\theta} = \frac{\sin^2\varphi\sin\theta}{\sin\zeta}. \tag{3.21}$$

Thus,

$$C^1_{\frac{k-1}{2}}(\cos\zeta) = \frac{\sin(\frac{k+1}{2}\zeta)}{\sin\zeta} = -\frac{2\frac{d}{d\theta}\cos(\frac{k+1}{2}\zeta)}{(k+1)\sin^2\varphi\sin\theta} \tag{3.22}$$

(see [Sz], p. 80). It follows that

$$
\begin{aligned}
II_2 &= -\frac{C}{(k+1)\sin\varphi}\int_B \frac{d}{d\theta}\cos\left(\frac{k+1}{2}\zeta\right)\cdot\frac{1}{(2-2\cos\theta)^{\frac{1}{2}}}d\theta\\
&= -\frac{C}{(k+1)\sin\varphi}\left[\cos\left(\frac{k+1}{2}\zeta\right)\cdot\frac{1}{(2-2\cos\theta)^{\frac{1}{2}}}\bigg|_{\sin\theta=[(k+1)\sin\varphi]^{-1}}\right.\\
&\qquad\left.+\int_B \cos\left(\frac{k+1}{2}\zeta\right)\frac{\sin\theta}{(2-2\cos\theta)^{\frac{3}{2}}}d\theta\right],
\end{aligned}
$$

where we have used integration by parts. The desired estimate for II_2 then follows easily. This completes the proof of Lemma 3.2.

4. Weighted $L^2 - L^2$ estimates.

In this section we establish weighted $L^2 - L^2$ estimates for the projection operator P_k in (2.13).

THEOREM 4.1. — (a) *If n is even and $0 \le \alpha < 1/2$, there exists a constant $C > 0$ depending only on α and n, such that, for every $g \in L^2(\Omega, d\Omega)$,*

$$
\int_\Omega \left|\sin^{-\alpha}\varphi P_k\Big(\sin^{-\alpha}(\cdot)g\Big)(\varphi,\omega)\right|^2 d\Omega \le C\int_\Omega |g|^2 d\Omega. \tag{4.2}
$$

(b) *If n is odd, (4.2) holds provided* $0 \le \alpha < 3/8$.

Theorem 4.1 is a consequence of the following lemma :

LEMMA 4.3. — (a) *If n is even and $0 \le \alpha < 1/2$, there exists a constant $C > 0$ depending only on α and n, such that, for every $g \in \mathcal{H}_k$,*

$$
\|\sin^{-\alpha}(\cdot)g\|_{L^2(\Omega,d\Omega)} \le C\|g\|_{L^2(\Omega,d\Omega)}. \tag{4.4}
$$

(b) *If n is odd, (4.4) holds provided* $0 \le \alpha < 3/8$.

We will postpone the proof of Lemma 4.3, and show how Lemma 4.3 yields Theorem 4.1.

Proof of Theorem 4.1. — Let

$$
T_{\alpha,k}(g)(\varphi,\omega) = \sin^{-\alpha}\varphi P_k(g)(\varphi,\omega).
$$

It follows from Lemma 4.3 that

$$\|T_{\alpha,k}(g)\|_{L^2(\Omega,d\Omega)} \leq C\|P_k(g)\|_{L^2(\Omega,d\Omega)} \leq C\|g\|_{L^2(\Omega,d\Omega)} \tag{4.5}$$

for $0 \leq \alpha < 1/2$ when n is even, and $0 \leq \alpha < 3/8$ when n is odd. Note that the adjoint operator of $T_{\alpha,k}$ is given by

$$T^*_{\alpha,k}(g)(\varphi,\omega) = P_k(\sin^{-\alpha}(\cdot)g)(\varphi,\omega).$$

Since P_k is a projection operator, we may write

$$\begin{aligned}\sin^{-\alpha}\varphi P_k(\sin^{-\alpha}(\cdot)g)(\varphi,\omega) &= \sin^{-\alpha}\varphi P_k \circ P_k(\sin^{-\alpha}(\cdot)g)(\varphi,\omega)\\ &= T_{\alpha,k}\circ T^*_{\alpha,k}(g)(\varphi,\omega).\end{aligned}$$

Theorem 4.1 then follows from (4.5) and a duality argument. □

It remains to prove Lemma 4.3. To do so, we need to establish an estimate on ultraspherical polynomials.

LEMMA 4.6. — *Let $0 < \lambda \leq 1$ and $0 \leq \alpha < \min(1/2, (\lambda/2) + (1/4))$. Then, for $0 \leq j \leq k$,*

$$\int_0^\pi (\sin\varphi)^{2\lambda-2\alpha}d\varphi \int_0^\pi C_k^\lambda(\cos^2\varphi + \sin^2\varphi\cos\theta)$$
$$\cdot \frac{C_j^{\lambda-\frac12}(\cos\theta)}{C_j^{\lambda-\frac12}(1)}\cdot(\sin\theta)^{2\lambda-1}d\theta \leq \frac{C}{k+1}$$

where C is a constant depending only on α and λ.

Assuming Lemma 4.6 for a moment, we give the

Proof of Lemma 4.3. — Fix an integer $k \geq 0$, and let

$$h_\ell(\varphi) = b_{k,\ell}\sin^{\frac{\ell}{2}}\varphi C^{\frac{\ell}{2}+\frac{n}{4}}_{\frac{k-\ell}{2}}(\cos\varphi) \text{ for } 0 \leq \ell \leq k, \qquad \ell \equiv k (\mathrm{mod}\ 2), \tag{4.7}$$

where $b_{k,\ell}$ is the normalization constant given in (2.14).

By (2.15),

$$\left\{h_\ell(\varphi)Y_{\ell,j}(\omega)\,\middle|\, 0 \leq \ell \leq k, \ell \equiv k(\mathrm{mod}\ 2), \quad 1 \leq j \leq d_\ell\right\} \tag{4.8}$$

is an orthonormal basis for $\mathcal{H}_k \subset L^2(\Omega, d\Omega)$. Notice that (4.8) is also an orthogonal set in $L^2(\Omega, \sin^{-2\alpha}\varphi d\Omega)$. Thus, it is not difficult to see that the estimate in Lemma 4.3 will follow if we can show

$$\int_0^\pi |h_\ell(\varphi)|^2 (\sin\varphi)^{\frac{n}{2}-2\alpha} d\varphi \le C \tag{4.9}$$

for $0 \le \alpha < 1/2$ when n is even, and $0 \le \alpha < 3/8$ when n is odd.

To establish (4.9), we have to distinguish two cases. First, consider the case when k is even. In this case, we may write $\ell = 2j$, $j = 0, 1, 2, \ldots, k/2$. We need to show

$$\begin{aligned}(k+1)\cdot \frac{[\Gamma(j+\frac{n}{4})]^2\Gamma(\frac{k}{2}-j+1)2^{2j}}{\Gamma(\frac{k}{2}+j+\frac{n}{2})} \\ \int_0^\pi \left[\sin^j \varphi C_{\frac{k}{2}-j}^{j+\frac{n}{4}}(\cos\varphi)\right]^2 (\sin\varphi)^{\frac{n}{2}-2\alpha} d\varphi \le C.\end{aligned} \tag{4.10}$$

To this end, let γ_1 be the integer such that $(n/4) - (5/4) < \gamma_1 \le (n/4) - (1/4)$. We let $\lambda = \lambda_1 = (n/4) - \gamma_1$, $m = (k/2) + \gamma_1$ and $\varphi = \psi$ in the addition formula (3.10). We then multiply both sides of (3.10) by $C_{j+\gamma_1}^{\lambda_1 - \frac{1}{2}}(\cos\theta)(\sin\theta)^{2\lambda_1 - 1}/C_{j+\gamma_1}^{\lambda_1-\frac{1}{2}}(1)$, and integrate on $[0, \pi]$ with respect to θ, to obtain that the left-hand side of (4.10) equals

$$\begin{aligned}C(k+1)\int_0^\pi (\sin\varphi)^{2\lambda_1 - 2\alpha} d\varphi \int_0^\pi C_{\frac{k}{2}+\gamma_1}^{\lambda_1}(\cos^2\varphi + \sin^2\varphi\cos\theta) \\ \cdot \frac{C_{j+\gamma_1}^{\lambda_1-\frac{1}{2}}(\cos\theta)}{C_{j+\gamma_1}^{\lambda_1-\frac{1}{2}}(1)} \cdot (\sin\theta)^{2\lambda_1 - 1} d\theta.\end{aligned} \tag{4.11}$$

Clearly, $(1/4) \le (n/4) - \gamma_1 < (5/4)$. Since γ_1 is an integer, we have $(1/4) \le \lambda = (n/4) - \gamma_1 \le 1$. Moreover, if n is even, we get $(1/2) \le \lambda \le 1$. Thus, by (4.11) and Lemma 4.6, (4.10) holds for $0 \le \alpha < 1/2$ when n is even, and $0 \le \alpha < 3/8$ when n is odd.

Next, we consider the case when k is odd. Write $\ell = 2j + 1$, $j = 0, 1, \ldots, (k-1)/2$. We need to prove

$$\begin{aligned}(k+1)\frac{[\Gamma(j+\frac{n+2}{4})]^2\Gamma(\frac{k-1}{2}-j+1)2^{2j}}{\Gamma(\frac{k-1}{2}+j+\frac{n+2}{2})} \\ \int_0^\pi \left[\sin^{j+\frac{1}{2}} \varphi C_{\frac{k-1}{2}-j}^{j+\frac{n}{4}+\frac{1}{2}}(\cos\varphi)\right]^2 (\sin\varphi)^{\frac{n}{2}-2\alpha} d\varphi \le C.\end{aligned} \tag{4.12}$$

To do this, let γ_2 be the integer such that $(n/4)-(3/4)<\gamma_2\leq(n/4)+(1/4)$. We let $\lambda=\lambda_2=(n+2)/4-\gamma_2$, $m=(k-1)/2+\gamma_2$, and $\varphi=\psi$ in the addition formula (3.10). As in the case of k even, we multiply both sides of (3.10) by $C_{j+\gamma_2}^{\lambda_2-\frac{1}{2}}(\cos\theta)(\sin\theta)^{2\lambda_2-1}\Big/C_{j+\gamma_2}^{\lambda_2-\frac{1}{2}}(1)$ and integrate on $[0,\pi]$ with respect to θ. We then see that the left-hand side of (4.12) equals

$$C(k+1)\int_0^\pi(\sin\varphi)^{2\lambda_2-2\alpha}d\varphi\int_0^\pi C_{\frac{k-1}{2}+\gamma_2}^{\lambda_2}(\cos^2\varphi+\sin^2\varphi\cos\theta)\cdot\frac{C_{j+\gamma_2}^{\lambda_2-\frac{1}{2}}(\cos\theta)}{C_{j+\gamma_2}^{\lambda_2-\frac{1}{2}}(1)}\cdot(\sin\theta)^{2\lambda_2-1}d\theta. \tag{4.13}$$

Note that, by definition, $(1/4)\leq(n+2)/4-\gamma_2<(5/4)$. Hence, $(1/4)\leq\lambda_2=(n+2)/4-\gamma_2\leq 1$. Furthermore, if n is even, $(1/2)\leq\lambda_2\leq 1$. Thus, as before, by (4.13) and Lemma 4.6, (4.12) holds for $0\leq\alpha<1/2$ when n is even, and $0\leq\alpha<3/8$ when n is odd.

This completes the proof of Lemma 4.3. □

We close this section, by giving the

Proof of Lemma 4.6. — It follows from the addition formula (3.10), (2.12) and a familiar argument, that

$$\begin{aligned}&\int_0^\pi C_k^\lambda[\cos^2\varphi+\sin^2\varphi\cos\theta]\cdot\frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)}\cdot(\sin\theta)^{2\lambda-1}d\theta\\&=2^{2\lambda-1}\cdot\frac{[\Gamma(j+\lambda)]^2\Gamma(k-j+1)2^{2j}}{\Gamma(k+j+2\lambda)}\left[\sin^j\varphi C_{k-j}^{j+\lambda}(\cos\varphi)\right]^2\geq 0.\end{aligned}$$

Also, by (2.12), one sees easily that the estimate in Lemma 4.6 holds for $\alpha=0$. Thus, it suffices to show that

$$\begin{aligned}I=&\int_{\{0\leq\varphi\leq\pi;\sin\varphi\leq\frac{1}{2}\}}(\sin\varphi)^{2\lambda-2\alpha}d\varphi\\&\int_0^\pi C_k^\lambda(\cos^2\varphi+\sin^2\varphi\cos\theta)\cdot\frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)}\cdot(\sin\theta)^{2\lambda-1}d\theta\end{aligned}$$

is bounded by $C/(k+1)$ where C is independent of k and j.

We may assume $\lambda \neq 2\alpha$. We write $I = I_1 + I_2 + I_3$, where

$$I_1 = \int_{0 \le \sin\varphi \le \frac{1}{k+1}} (\sin\varphi)^{2\lambda-2\alpha} d\varphi$$
$$\int_0^\pi C_k^\lambda(\cos\zeta) \cdot \frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)} \cdot (\sin\theta)^{2\lambda-1} d\theta,$$
$$I_2 = \int_{\frac{1}{k+1} \le \sin\varphi \le \frac{1}{2}} (\sin\varphi)^{2\lambda-2\alpha} d\varphi$$
$$\int_{\sin\theta \le \frac{1}{(k+1)\sin\varphi}} C_k^\lambda(\cos\zeta) \cdot \frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)} \cdot (\sin\theta)^{2\lambda-1} d\theta,$$
$$I_3 = \int_{\frac{1}{k+1} \le \sin\varphi \le \frac{1}{2}} (\sin\varphi)^{2\lambda-2\alpha} d\varphi$$
$$\int_{\sin\theta \ge \frac{1}{(k+1)\sin\varphi}} C_k^\lambda(\cos\zeta) \cdot \frac{C^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)} \cdot (\sin\theta)^{2\lambda-1} d\theta.$$

In I_1, I_2, I_3 above, we have, as in (3.17), let $\cos\zeta = \cos^2\varphi + \sin^2\varphi\cos\theta$ for $\zeta \in [0, \pi]$.

We start with I_1. Since $\lambda > 0$,

$$\left| C_k^\lambda(\cos\zeta) \right| \le C_k^\lambda(1) \le C(k+1)^{2\lambda-1}. \tag{4.14}$$

It follows that

$$|I_1| \le C \int_0^\pi \left| \frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)} \right| (\sin\theta)^{2\lambda-1} d\theta$$
$$\int_{0 \le \sin\varphi \le \frac{1}{k+1}} (k+1)^{2\lambda-1} (\sin\varphi)^{2\lambda-2\alpha} d\varphi$$
$$\le C(k+1)^{2\alpha-2} \int_0^\pi \left| \frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)} \right| (\sin\theta)^{2\lambda-1} d\theta.$$

If $1 \geq \lambda \geq 1/2$, $|C_j^{\lambda-\frac{1}{2}}(\cos\theta)| \leq C_j^{\lambda-\frac{1}{2}}(1)$, and we have $|I_1| \leq C(k+1)^{2\alpha-2} \leq C(k+1)^{-1}$, since $\alpha < 1/2$. If $0 < \lambda < 1/2$ using (3.14), we obtain

$$\begin{aligned}
&\int_0^\pi \left|\frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)}\right| (\sin\theta)^{2\lambda-1} d\theta \\
&\quad \leq C \int_{\sin\theta \leq \frac{1}{j+1}} (\sin\theta)^{2\lambda-1} d\theta + C(j+1)^{-\lambda+\frac{1}{2}} \int_{\sin\theta \geq \frac{1}{j+1}} (\sin\theta)^{\lambda-\frac{1}{2}} d\theta \\
&\quad \leq C(j+1)^{-\lambda+\frac{1}{2}}.
\end{aligned}$$

Thus,

$$|I_1| \leq C(k+1)^{2\alpha-2}(j+1)^{-\lambda+\frac{1}{2}} \leq C(k+1)^{2\alpha-\lambda-\frac{3}{2}} \leq C(k+1)^{-1},$$

where we have used the assumption that $\alpha < (\lambda/2) + (1/4)$.

Next, we turn to the estimate of I_2. We have, by (4.14),

$$\begin{aligned}
|I_2| \leq C(k+1)^{2\lambda-1} &\int_{\frac{1}{k+1} \leq \sin\varphi \leq \frac{1}{2}} (\sin\varphi)^{2\lambda-2\alpha} d\varphi \\
&\int_{\sin\theta \leq \frac{1}{(k+1)\sin\varphi}} \left|\frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)}\right| (\sin\theta)^{2\lambda-1} d\theta.
\end{aligned}$$

If $\lambda \geq 1/2$, as in the case of I_1, we have

$$\begin{aligned}
|I_2| &\leq C(k+1)^{2\lambda-1} \int_{\frac{1}{(k+1)} \leq \sin\varphi \leq \frac{1}{2}} (\sin\varphi)^{2\lambda-2\alpha} d\varphi \\
&\qquad \int_{\sin\theta \leq \frac{1}{(k+1)}\sin\varphi} (\sin\theta)^{2\lambda-1} d\theta \\
&\leq C(k+1)^{-1} \int_{\frac{1}{(k+1)} < \sin\varphi \leq \frac{1}{2}} (\sin\varphi)^{-2\alpha} d\varphi \\
&\leq C(k+1)^{-1}.
\end{aligned}$$

If $\lambda < \frac{1}{2}$, we use (3.14) to obtain

$$
\begin{aligned}
|I_2| \le C(k+1)^{2\lambda-1} \int_{\frac{1}{(k+1)} \le \sin\varphi \le \frac{1}{2}} &(\sin\varphi)^{2\lambda-2\alpha} d\varphi \\
&\int_{\substack{\sin\theta \le \frac{1}{j+1} \\ \sin\theta \le \frac{1}{(k+1)}\sin\varphi}} (\sin\theta)^{2\lambda-1} d\theta \\
&+ C(k+1)^{2\lambda-1} \int_{\frac{1}{k+1} \le \sin\varphi \le \frac{1}{2}} (\sin\varphi)^{2\lambda-2\alpha} d\varphi \\
&\int_{\substack{\sin\theta \ge \frac{1}{j+1} \\ \sin\theta \le \frac{1}{(k+1)}\sin\varphi}} (j+1)^{\frac{1}{2}-\lambda} (\sin\theta)^{\lambda-\frac{1}{2}} d\theta \\
&\le C(k+1)^{-1} + C(k+1)^{\lambda-\frac{3}{2}} (j+1)^{\frac{1}{2}-\lambda} \\
&\int_{\sin\varphi \ge \frac{1}{k+1}} (\sin\varphi)^{\lambda-2\alpha-\frac{1}{2}} d\varphi \\
&\le C(k+1)^{-1}.
\end{aligned}
$$

Here, again, we used the assumption $\alpha < (\lambda/2) + (1/4)$.

Finally, we need to estimate I_3 which is the essential part of I.

Write

$$
\begin{aligned}
I_3 = \int_{\sin\theta \ge \frac{2}{k+1}} &\frac{C_j^{\lambda-\frac{1}{2}}(\cos\theta)}{C_j^{\lambda-\frac{1}{2}}(1)} (\sin\theta)^{2\lambda-1} d\theta \\
&\int_{\frac{1}{2} \ge \sin\varphi \ge \frac{1}{(k+1)\sin\theta}} C_k^{\lambda}(\cos\zeta)(\sin\varphi)^{2\lambda-2\alpha} d\varphi.
\end{aligned}
$$

We claim that, if $0 < \lambda \le 1$, $0 \le \alpha < \min(1/2, (\lambda/2) + (1/4))$ and $\lambda \ne 2\alpha$,

$$
\begin{aligned}
\left| \int_{\frac{1}{2} \ge \sin\varphi \ge \frac{1}{(k+1)\sin\theta}} C_k^{\lambda}(\cos\zeta)(\sin\varphi)^{2\lambda-2\alpha} d\varphi \right| \\
\le C(k+1)^{\lambda-2}(\sin\theta)^{-\lambda-1} + C(k+1)^{2\alpha-2}(\sin\theta)^{2\alpha-2\lambda-1}
\end{aligned}
\tag{4.15}
$$

and

$$
\begin{aligned}
\left| \int_{\frac{1}{(k+1)\sin\varphi} \le \sin\theta \le \frac{1}{j+1}} C_k^{1}(\cos\zeta) C_j^{\frac{1}{2}}(\cos\theta) \sin\theta d\theta \right| \\
\le C(k+1)^{-1}(\sin\varphi)^{-2}.
\end{aligned}
\tag{4.16}
$$

We assume (4.15) and (4.16) for a moment and give the estimate for I_3.

If $1/2 \le \lambda < 1$, by (4.15), we have

$$\begin{aligned} |I_3| &\le C(k+1)^{\lambda-2}\int_{\sin\theta\ge\frac{2}{k+1}}(\sin\theta)^{\lambda-2}d\theta \\ &\quad + C(k+1)^{2\alpha-2}\int_{\sin\theta\ge\frac{2}{k+1}}(\sin\theta)^{2\alpha-2}d\theta \\ &\le C(k+1)^{-1} \qquad \text{since } \alpha < 1/2. \end{aligned}$$

If $0 < \lambda < 1/2$, it follows from (4.15) and (3.14), that,

$$\begin{aligned} |I_3| &\le C(k+1)^{\lambda-2}\int_{\sin\theta\ge\frac{1}{k+1}}\left|\frac{C_j^{\lambda-\frac12}(\cos\theta)}{C_j^{\lambda-\frac12}(1)}\right|(\sin\theta)^{\lambda-2}d\theta \\ &\quad + C(k+1)^{2\alpha-2}\int_{\sin\theta\ge\frac{1}{k+1}}\left|\frac{C_j^{\lambda-\frac12}(\cos\theta)}{C_j^{\lambda-\frac12}(1)}\right|(\sin\theta)^{2\alpha-2}d\theta \\ &\le C(k+1)^{-1} + C(k+1)^{\lambda-2}(j+1)^{\frac12-\lambda}\int_{\sin\theta\ge\frac{1}{j+1}}(\sin\theta)^{-\frac32}d\theta \\ &\quad + C(k+1)^{2\alpha-2}(j+1)^{\frac12-\lambda}\int_{\sin\theta\ge\frac{1}{j+1}}(\sin\theta)^{2\alpha-\lambda-\frac32}d\theta \\ &\le C(k+1)^{-1}, \end{aligned}$$

where we have used $\alpha < (\lambda/2) + (1/4)$.

In the case $\lambda = 1$, we use (4.15) and (4.16) to obtain

$$\begin{aligned} |I_3| &\le C\int_{\sin\theta\ge\frac{1}{j+1}}|C_j^{\frac12}(\cos\theta)|\sin\theta d\theta \\ &\quad \left|\int_{\frac12\ge\sin\varphi\ge\frac{1}{(k+1)\sin\theta}}C_k^1(\cos\zeta)(\sin\varphi)^{2-2\alpha}d\varphi\right| + C\int_{\sin\varphi\ge\frac{1}{k+1}}(\sin\varphi)^{2-2\alpha}d\varphi \\ &\quad \left|\int_{\frac{1}{(k+1)\sin\varphi}\le\sin\theta\le\frac{1}{j+1}}C_k^1(\cos\zeta)C_j^{\frac12}(\cos\theta)\sin\theta d\theta\right| \\ &\le C(k+1)^{-1}\int_{\sin\theta\ge\frac{1}{j+1}}\left|C_j^{\frac12}(\cos\theta)\right|(\sin\theta)^{-1}d\theta \\ &\quad + C(k+1)^{-1}\int_{\sin\varphi\ge\frac{1}{k+1}}(\sin\varphi)^{-2\alpha}d\varphi \\ &\le C(k+1)^{-1}, \end{aligned}$$

where in the last inequality we have used (3.14) and $\alpha < 1/2$.

Thus, all that remains to prove is (4.15) and (4.16).

We recall an asymptotic formula for ultraspherical polynomials :

$$\begin{aligned}(4.17)\quad C_k^\lambda(\cos\zeta) &= \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(k+2\lambda)}{\Gamma(2\lambda)\Gamma(k+\lambda+\frac{1}{2})} P_k^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(\cos\zeta) \\ &= \frac{C\Gamma(k+2\lambda)}{\Gamma(k+\lambda+\frac{1}{2})\sqrt{k}}(\sin\zeta)^{-\lambda}\cos\left[(k+\lambda)\zeta - \frac{\pi\lambda}{2}\right] \\ &\quad + (k+1)^{\lambda-2}(\sin\zeta)^{-\lambda-1}O(1),\end{aligned}$$

uniformly for $\zeta \in [c/k, \pi - (c/k)]$ as $k \to \infty$ (see [Sz], p. 198).

Note that, if $\sin\varphi \geq 1/(k+1)\sin\theta$, then $\sin\zeta \geq C/(k+1)$ by (3.18). Substituting (4.17) into the left-hand side of (4.15), one sees that the integral which contains the remainder $(k+1)^{\lambda-2}(\sin\zeta)^{-\lambda-1}O(1)$ can be handled easily. We will give details for the estimate of the integral which contains the main term in (4.17). Consider

$$(4.18)\qquad \begin{aligned}&(k+1)^{\lambda-1}\int_{\frac{1}{2}\geq\sin\varphi\geq\frac{1}{(k+1)\sin\theta}}(\sin\zeta)^{-\lambda} \\ &\cos\left[(k+\lambda)\zeta-\frac{\pi\lambda}{2}\right](\sin\varphi)^{2\lambda-2\alpha}d\varphi.\end{aligned}$$

By (3.17),

$$(4.19)\qquad \frac{d\zeta}{d\varphi} = \frac{2\sin\varphi\cos\varphi(1-\cos\theta)}{\sin\zeta}.$$

It follows that

$$\begin{aligned}(4.20)\qquad &\cos\left[(k+\lambda)\zeta-\frac{\pi\lambda}{2}\right] \\ &= \frac{\sin\zeta}{2(k+\lambda)\sin\varphi\cos\varphi(1-\cos\theta)}\frac{d}{d\varphi}\left\{\sin\left[(k+\lambda)\zeta-\frac{\pi\lambda}{2}\right]\right\}.\end{aligned}$$

Hence, (4.18) equals

$$(4.21)\quad \frac{(k+1)^{\lambda-1}}{2(k+\lambda)(1-\cos\theta)}\int_{\frac{1}{2}\geq\sin\varphi\geq\frac{1}{(k+1)\sin\theta}}\frac{d}{d\varphi}\left\{\sin\left[(k+\lambda)\zeta-\frac{\pi\lambda}{2}\right]\right\} \cdot(\sin\zeta)^{-\lambda+1}(\sin\varphi)^{2\lambda-2\alpha-1}(\cos\varphi)^{-1}d\varphi.$$

Note that

$$(4.22)\quad \begin{aligned}\sin\zeta\leq\sqrt{2}\sqrt{1-\cos\zeta}&=\sqrt{2}\sqrt{1-\cos^2\varphi-\sin^2\varphi\cos\theta}\\ &=\sqrt{2}\sin\varphi\sqrt{1-\cos\theta}.\end{aligned}$$

It follows from integration by parts, (3.18) and (4.22) that (4.21) is bounded in absolute value by

$$\begin{aligned}&C(k+1)^{\lambda-2}(\sin\theta)^{-\lambda-1}+C(k+1)^{2\alpha-2}(\sin\theta)^{2\alpha-2\lambda-1}\\ &\quad+\frac{C(k+1)^{\lambda-2}}{(1-\cos\theta)}\int_{\frac{1}{2}\geq\sin\varphi\geq\frac{1}{(k+1)\sin\theta}}\\ &\quad\left|\frac{d}{d\varphi}\left\{(\sin\zeta)^{1-\lambda}(\sin\varphi)^{2\lambda-2\alpha-1}(\cos\varphi)^{-1}\right\}\right|d\varphi\\ &\leq C(k+1)^{\lambda-2}(\sin\theta)^{-\lambda-1}+C(k+1)^{2\alpha-2}(\sin\theta)^{2\alpha-2\lambda-1}\\ &\quad+C(k+1)^{\lambda-2}(\sin\theta)^{-\lambda-1}\int_{\frac{1}{2}\geq\sin\varphi\geq\frac{1}{(k+1)\sin\theta}}(\sin\varphi)^{\lambda-2\alpha-1}d\varphi\\ &\leq C(k+1)^{\lambda-2}(\sin\theta)^{-\lambda-1}+C(k+1)^{2\alpha-2}(\sin\theta)^{2\alpha-2\lambda-1}.\end{aligned}$$

This proves (4.15).

To prove (4.16), we recall that

$$C_k^1(\cos\zeta)=\frac{\sin[(k+1)\zeta]}{\sin\zeta}=-\frac{\frac{d}{d\theta}\cos[(k+1)\zeta]}{(k+1)\sin^2\varphi\sin\theta}$$

(see (3.21) and (3.22)). Thus, the left-hand side of (4.16) equals

$$\frac{1}{(k+1)\sin^2\varphi}\left|\int_{\frac{1}{(k+1)\sin\varphi}\leq\sin\theta\leq\frac{1}{j+1}}\frac{d}{d\theta}\{\cos[(k+1)\zeta]\}\cdot C_j^{\frac{1}{2}}(\cos\theta)d\theta\right|.$$

The desired estimate then follows from integration by parts, (3.14) and the fact

$$\frac{d}{d\theta}[C_j^{\frac{1}{2}}(\cos\theta)]=-C_{j-1}^{\frac{3}{2}}(\cos\theta)\sin\theta$$

(see [Sz], p. 81).

The estimate (4.16) is proved and the proof of Lemma 4.6 is finally complete. □

5. Carleman estimates.

Recall that $\rho = (|z|^4 + 4t^2)^{\frac{1}{4}}$ and $\sin\varphi = |z|^2/\rho^2$. In this section we prove the following Carleman estimates for the Grushin operator $\mathcal{L}$ in (0.1).

THEOREM 5.1. — *Let $0 < \varepsilon < 1/4$, $s > 100$ and $\delta = \operatorname{dist}(s, \mathbb{N}) > 0$. Suppose that $p = 2n/(n-1)$, $q = 2n/(n+1)$ (i.e., $1/p + 1/q = 1$ and $1/p = 1/q - 1/n$). Then there exists a constant $C > 0$ depending only on ε, δ and n, such that for $f \in C_0^\infty(\mathbb{R}^{n+1} \setminus \{0\})$*

$$\left\| \rho^{-s} (\sin\varphi)^{\varepsilon} f \right\|_{L^p(\mathbb{R}^{n+1}, \frac{dz\,dt}{\rho^{n+2}})} \leq C \left\| \rho^{-s+2} (\sin\varphi)^{-\varepsilon} \mathcal{L}(f) \right\|_{L^q(\mathbb{R}^{n+1}, \frac{dz\,dt}{\rho^{n+2}})}, \tag{5.2}$$

if $n \geq 2$ is even, and

$$\left\| \rho^{-s} (\sin\varphi)^{\frac{1}{4p}+\varepsilon} f \right\|_{L^p(\mathbb{R}^{n+1}, \frac{dz\,dt}{\rho^{n+2}})} \leq C \left\| \rho^{-s+2} (\sin\varphi)^{-\frac{1}{4p}-\varepsilon} \mathcal{L}(f) \right\|_{L^q(\mathbb{R}^{n+2}, \frac{dz\,dt}{\rho^{n+2}})}, \tag{5.3}$$

if $n \geq 3$ is odd.

Our proof of Theorem 5.1 follows the idea of D. Jerison in [J]. The key ingredient is a $L^q - L^p$ estimate for the projection operator P_k in (2.13).

THEOREM 5.4. — *Let $p = 2n/(n-1)$ and $q = 2n/(n+1)$.*

(a) *If $n \geq 2$ is even and $0 \leq \alpha < 1/p$, there exists a constant $C > 0$ depending only on α and n, such that for $g \in L^q(\Omega, d\Omega)$,*

$$\| \sin^{-\alpha}\varphi P_k(\sin^{-\alpha}(\cdot) g) \|_{L^p(\Omega, d\Omega)} \leq C (k+1)^{\frac{n-1}{n}} \|g\|_{L^q(\Omega, d\Omega)}. \tag{5.5}$$

(b) *If $n \geq 3$ is odd, (5.5) holds provided $0 \leq \alpha < 3/(4p)$.*

Theorem 5.4 follows from Theorem 3.1 ($L^1 - L^\infty$ estimates) and Theorem 4.1 ($L^2 - L^2$ estimates), by a standard complex interpolation (see [SW]). We omit the details.

We are now in a position to give the proof of Theorem 5.1. As we mentioned earlier, the argument is similar to that in [J].

Proof of Theorem 5.1. — We will only give the proof of (5.2) in the case of n even. (5.3) follows from part (b) of Theorem 5.4 in the same manner.

First, suppose $f(\rho,\varphi,\omega) = h(\rho)g_k(\varphi,\omega)$ where $h \in C_0^\infty(\mathbb{R}_+)$ and $g_k \in \mathcal{H}_k$. Using (1.12) and (2.1), it is not difficult to see that

$$\rho^{-s+2}\mathcal{L}(\rho^s f) = \sin\varphi g_k(\varphi,\omega)\{\rho^2 h''(\rho) + (n+2s+1)\rho h'(\rho) \\ + [s(n+s) - k(n+k)]h(\rho)\}.$$

Recall that the Mellin transform of h is defined by

$$\widetilde{h}(\eta) = \int_0^\infty h(\rho)\rho^{-i\eta-1}d\rho, \qquad \eta \in \mathbb{R}. \tag{5.6}$$

Now, let

$$\mathcal{L}_s(f) = \rho^{-s+2}\mathcal{L}(\rho^s f). \tag{5.7}$$

We have, if $f(\rho,\varphi,\omega) = h(\rho)g_k(\varphi,\omega)$, $g_k \in \mathcal{H}_k$,

$$(\mathcal{L}_s(f))^\sim(\eta,\varphi,\omega) = \sin\varphi a_s(\eta,k)\widetilde{h}(\eta)g_k(\varphi,\omega)$$

where

$$a_s(\eta,k) = -\eta^2 + i(n+2s+1)\eta + [s(n+s) - k(n+k)]. \tag{5.8}$$

It then follows that, for $f \in C_0^\infty(\mathbb{R}^{n+1} \setminus \{0\})$,

$$\{\mathcal{L}_s(\mathcal{P}_k(f))\}^\sim(\eta,\varphi,\omega) = \sin\varphi \cdot a_s(\eta,k) \cdot \{P_k(f)\}^\sim(\eta,\varphi,\omega).$$

Hence, for $f \in C_0^\infty(\mathbb{R}^{n+1} \setminus \{0\})$,

$$\{\mathcal{L}_s(f)\}^\sim(\eta,\varphi,\omega) = \sin\varphi \sum_{k=0}^\infty a_s(\eta,k)P_k(\widetilde{f}(\eta,\cdot,\cdot))(\varphi,\omega). \tag{5.9}$$

This implies that, at least formally,

$$\{\mathcal{L}_s^{-1}(f)\}^{\sim}(\eta,\varphi,\omega)=\sum_{k=0}^{\infty}\frac{1}{a_s(\eta,k)}P_k\left(\frac{\widetilde{f}(\eta,\cdot,\cdot)}{\sin(\cdot)}\right)(\varphi,\omega).$$

We shall show that, for $f\in C_0^\infty(\mathbb{R}_+\times\Omega)$,

$$\|\mathcal{L}_s^{-1}(f)\|_{L^p(\mathbb{R}_+\times\Omega,\frac{(\sin\varphi)^{-1+\varepsilon p}d\rho d\Omega}{\rho})}\le C\|f\|_{L^q(\mathbb{R}_+\times\Omega,\frac{(\sin\varphi)^{-1-\varepsilon q}d\rho d\Omega}{\rho})}. \tag{5.10}$$

Clearly, (5.10) yields the estimate (5.2) because

$$\frac{dz\,dt}{\rho^{n+2}}=\frac{(\sin\varphi)^{\frac{n-2}{2}}}{2\rho}d\rho d\varphi d\omega=\frac{(\sin\varphi)^{-1}}{2\rho}d\rho d\Omega$$

(see (1.7) and (2.10)).

To prove (5.10), let $g(y,\varphi,\omega)=f(e^y,\varphi,\omega)$ for $y\in\mathbb{R}$ and

$$R_s(g)(y,\varphi,\omega)=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}e^{i(y-x)\eta}\sum_{k=0}^{\infty}\frac{Q_k}{a_s(\eta,k)}d\eta\right)g(x,\varphi,\omega)dx \tag{5.11}$$

where Q_k is the operator defined by

$$Q_k(g)(x,\varphi,\omega)=P_k\left(\frac{g(x,\cdot,\cdot)}{\sin(\cdot)}\right)(\varphi,\omega). \tag{5.12}$$

Then, it is not hard to see that (5.10) is equivalent to

$$\|R_s(g)\|_{L^p(\mathbb{R}\times\Omega,(\sin\varphi)^{-1+\varepsilon p}dy\,d\Omega)}\le C\|g\|_{L^q(\mathbb{R}\times\Omega,(\sin\varphi)^{-1-\varepsilon q}dy\,d\Omega)} \tag{5.13}$$

for $g\in C_0^\infty(\mathbb{R}\times\Omega)$.

Now, fix $s>100$ such that $\operatorname{dist}(\mathrm{s},\mathbb{N})=\delta>0$. Suppose $2^N\le(s/10)<2^{N+1}$. Choose a partitionon of unity $\{\Phi_\beta\}_{\beta=0}^N$ for $\mathbb{R}_+$ such that

$$\begin{cases}\sum_\beta\Phi_\beta(r)=1\text{ for all }r>0\\ \mathrm{supp}\Phi_\beta\subset\{\mathrm{r}:2^{\beta-2}\le\mathrm{r}\le2^\beta\}, & \beta=1,2,\dots,\mathrm{N}-1\\ \mathrm{supp}\Phi_0\subset\{\mathrm{r}:0<\mathrm{r}\le1\}\\ \mathrm{supp}\Phi_{\mathrm{N}}\subset\left\{\mathrm{r}:\mathrm{r}\ge\frac{s}{40}\right\}\end{cases} \tag{5.14}$$

and

$$\left|\frac{d^\ell}{dr^\ell}\Phi_\beta(r)\right|\le C_\ell2^{-\beta\ell},\qquad\ell=0,1,2,\dots. \tag{5.15}$$

Note that

$$(5.16)\quad a_s(\eta,k) = -\left\{\eta - i\left[\left(s+\frac{n+1}{2}\right) - \sqrt{k(n+k)+s+\frac{(n+1)^2}{4}}\right]\right\}$$
$$\cdot\left\{\eta - i\left[\left(s+\frac{n+1}{2}\right) + \sqrt{k(n+k)+s+\frac{(n+1)^2}{4}}\right]\right\}.$$

So, for $0 \le \beta \le N$, we let
(5.17)

$$b_s^\beta(\eta,k) = \frac{1}{a_s(\eta,k)}\Phi_\beta\left(\left|\eta - i\left[\left(s+\frac{(n+1)}{2}\right) - \sqrt{k(n+k)+s+\frac{(n+1)^2}{4}}\right]\right|\right)$$

and

$$(5.18)\quad R_s^\beta(g)(y,\varphi,\omega) = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{i(y-x)\eta}\sum_{k=0}^{\infty} b_s^\beta(\eta,k)Q_k d\eta\right)g(x,\varphi,\omega)dx.$$

We first consider the case that $0 \le \beta \le N-1$. Note that, if $b^\beta(\eta,k) \neq 0$, then, by (5.14),

$$\delta 2^{\beta-2} \le \left|\eta - i\left[\left(s+\frac{n+1}{2}\right) - \sqrt{k(n+k)+s+\frac{(n+1)^2}{4}}\right]\right| \le 2^\beta.$$

It follows that $|\eta| \le 2^\beta$ and $|s-k| \le 2^{\beta+1}$. This implies that there are at most $2^{\beta+2}$ nonzero terms in the sum over k which defines R_s^β and the values of these k's are comparable to s. The above fact, together with (5.15), (5.16) and (5.17), also yields

$$(5.19)\quad \left|\left(\frac{\partial}{\partial\eta}\right)^j b_s^\beta(\eta,k)\right| \le C_j\cdot 2^{-\beta}\cdot s^{-1}\cdot 2^{-j\beta}.$$

By (5.5) in Theorem 5.4, we have

$$\|Q_k(g)\|_{L^p(\Omega,(\sin\varphi)^{-\alpha p}d\Omega)} \le C(k+1)^{\frac{n-1}{n}}\|g\|_{L^q(\Omega,(\sin\varphi)^{\alpha q-q}d\Omega)}$$

for $0 \le \alpha < 1/p$. Let $\varepsilon = 1/p-\alpha$, we see that

$$(5.20)\quad \|Q_k(g)\|_{L^p(\Omega,(\sin\varphi)^{-1+\varepsilon p}d\Omega)} \le C(k+1)^{\frac{n-1}{n}}\|g\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q}d\Omega)}.$$

It then follows from integration by parts, (5.19) and (5.20) that

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} e^{i(y-x)\eta} \sum_{k=0}^{\infty} b_s^\beta(\eta,k) Q_k(g)\, d\eta \right\|_{L^p(\Omega,(\sin\varphi)^{-1+\varepsilon p} d\Omega)} \\
&\quad \le \frac{C}{|y-x|^j} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left| \left(\frac{\partial}{\partial\eta}\right)^j b_s^\beta(\eta,k) \right| d\eta \|Q_k(g)\|_{L^p(\Omega,(\sin\varphi)^{-1+\varepsilon p} d\Omega)} \\
&\quad \le \frac{C}{|y-x|^j} \cdot 2^{-j\beta} \cdot 2^\beta \cdot s^{-1+\frac{n-1}{n}} \|g\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q} d\Omega)} \\
&\quad = \frac{C}{(2^\beta|y-x|)^j} \cdot 2^\beta \cdot s^{-\frac{1}{n}} \|g\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q} d\Omega)}.
\end{aligned}$$

Choosing $j=10$ and $j=0$, we see that

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} e^{i(y-x)\eta} \sum_{k=0}^{\infty} b_s^\beta(\eta,k) Q_k(g)\, d\eta \right\|_{L^p(\Omega,(\sin\varphi)^{-1+\varepsilon p} d\Omega)} \\
&\quad \le \frac{C}{(1+2^\beta|y-x|)^{10}} \cdot s^{-\frac{1}{n}} \cdot 2^\beta \|g\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q} d\Omega)}.
\end{aligned}$$

Thus, by (5.18), for $0 \le \beta \le N-1$,

$$\begin{aligned}
&\|R_s^\beta(g)\|_{L^p(\mathbb{R}\times\Omega,(\sin\varphi)^{-1+\varepsilon p} dy\, d\Omega)} \\
&\le C \cdot s^{-\frac{1}{n}} \cdot 2^\beta \left\| \int_{\mathbb{R}} \frac{1}{(1+2^\beta|y-x|)^{10}} \|g(x,\cdot,\cdot)\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q} d\Omega)} \right\|_{L^p(\mathbb{R},dy)} \\
&\le C \cdot s^{-\frac{1}{n}} \cdot 2^\beta \left\| \frac{1}{(1+2^\beta|\cdot|)^{10}} \right\|_{L^{\frac{n}{n-1}}(\mathbb{R},dy)} \|g\|_{L^q(\mathbb{R}\times\Omega,(\sin\varphi)^{-1-\varepsilon q} dy\, d\Omega)} \\
&\le C \cdot s^{-\frac{1}{n}} \cdot 2^{\frac{\beta}{n}} \|g\|_{L^q(\mathbb{R}\times\Omega,(\sin\varphi)^{-1-\varepsilon q} dy\, d\Omega)}
\end{aligned}$$

where we used Minkowski's inequality in the first inequality and Young's inequality in the second one.

It then follows that

$$\sum_{\beta=0}^{N-1} \|R_s^\beta(g)\|_{L^p(\mathbb{R}\times\Omega,(\sin\varphi)^{-1+\varepsilon p} dy\, d\Omega)} \le C \|g\|_{L^q(\mathbb{R}\times\Omega,(\sin\varphi)^{-1-\varepsilon q} dy\, d\Omega)}.$$

Finally, we need to estimate $R_s^N(g)$. To this end, one first observes that, on the support of $b_s^N(\eta, k)$,

$$|a_s(\eta, k)| \sim (|\eta| + s + k)^2.$$

Moreover,

$$\left|\left(\frac{\partial}{\partial\eta}\right)^j b_s^N(\eta, k)\right| \leq \frac{C_j}{(|\eta| + s + k)^{j+2}}. \tag{5.21}$$

It follows from integration by parts and (5.21) that

$$\left|\int_{\mathbb{R}} e^{i(y-x)\eta} b_s^N(\eta, k) d\eta\right| \leq \frac{C}{(k+s)[1 + |y-x|(k+s)]}. \tag{5.22}$$

Thus,

$$\begin{aligned}
&\left\|\sum_{k=0}^{\infty} \int_{\mathbb{R}} e^{i(y-x)\eta} b_s^N(\eta, k) Q_k(g) d\eta\right\|_{L^p(\Omega,(\sin\varphi)^{-1+\varepsilon p} d\Omega)} \\
&\leq C \sum_{k=0}^{\infty} \frac{(k+1)^{\frac{n-1}{n}}}{(k+s)[1 + |y-x|(k+s)]} \|g\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q} d\Omega)} \\
&\leq C \|g\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q} d\Omega)} \left\{ \sum_{k \leq \frac{1}{|y-x|}} (k+1)^{-\frac{1}{n}} + \sum_{k > \frac{1}{|y-x|}} (k+1)^{-1-\frac{1}{n}} |y-x|^{-1} \right\} \\
&\leq \frac{C}{|y-x|^{\frac{n-1}{n}}} \|g\|_{L^q(\Omega,(\sin\varphi)^{-1-\varepsilon q} d\Omega)}.
\end{aligned}$$

The desired estimate for R_s^N then follows from Minkowski's inequality and the well known theorem on fractional integration.

The proof is complete.

6. The strong unique continuation property.

In this section we apply the Carleman estimate (Theorem 5.1) to establish the strong unique continuation property for $-\mathcal{L} + V$ under certain L^p conditions on V.

For $\rho > 0$ and $t_0 \in \mathbb{R}$, let

$$B_\rho = B_\rho((0, t_0)) = \{(z, t) \in \mathbb{R}^{n+1} | (|z|^4 + 4|t - t_0|^2)^{\frac{1}{4}} < \rho\}. \tag{6.1}$$

We denote by $S^2(B_\rho)$ the closure of $C_0^\infty(B_\rho)$ under the norm
(6.2)

$$\|u\|_{S^2(B_\rho)} = \left\{ \int_{B_\rho} (|\nabla_z^2 u|^2 + |\,|z|^2 \partial_t^2 u|^2 + |\nabla_z u|^2 + |\,|z|\partial_t u|^2 + |u|^2) dz\, dt \right\}^{\frac{1}{2}}.$$

By the subelliptic estimates, if $u \in S^2(B_\rho)$, then $|\nabla_z u| + |\partial_t u| \in L^2(B_\rho)$. In particular, it follows that $u \in L^{q_0}(B_\rho)$ where $q_0 = 2(n+1)/(n-1)$ by Sobolev embedding.

We say that u vanishes of infinite order at the point $(0, t_0)$ in the L^p mean, if

$$\int_{B_\rho((0,t_0))} |u|^p dz\, dt = O(\rho^N), \text{ as } \rho \to 0 \text{ for all } N > 0. \tag{6.3}$$

We now state and prove the main result of this paper.

THEOREM 6.4. — *Suppose that $u \in S^2(B_{\rho_0}((0,t_0)))$ for some $\rho_0 > 0$ and $t_0 \in \mathbb{R}$. Also, assume that*

$$\left|\Delta_z u + |z|^2 \partial_t^2 u\right| \le |Vu| \qquad \text{in } B_{\rho_0} = B_{\rho_0}((0,t_0)) \tag{6.5}$$

for some potential $V \in L^r_{\rm loc}$ where $r > n$ when n is even and $r > 2n^2/(n+1)$ when n is odd. Then $u \equiv 0$ in B_{ρ_0} if u vanishes of infinite order at $(0, t_0)$ in the L^2 mean.

Proof. — The argument we will use to deduce Theorem 6.4 from the Carleman estimates (Theorem 5.1) is similar to that in the elliptic case (e.g. see [JK]).

We first consider the case of n even. Without loss of generality, we may assume $t_0 = 0$ and $\rho_0 = 1$.

Let $\beta \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\beta = 1$ when $\rho(z,t) \le 1/2$ and $\beta = 0$ when $\rho \ge 3/4$. Also, let $\chi_j(\rho) = \chi(j\rho)$ where $\chi = 1 - \beta$. A standard limiting argument shows that the Carleman estimate (5.2) holds for $f = \beta\chi_j u$. Thus, for $p = 2n/(n-1)$, $q = 2n/(n+1)$ and $s = k + 1/2$,

$$\begin{aligned} &\left\|\rho^{-s}(\sin\varphi)^\varepsilon \beta\chi_j u\right\|_{L^p(\mathbb{R}^{n+1}, \frac{dz\,dt}{\rho^{n+2}})} \\ &\le C\left\|\rho^{-s+2}(\sin\varphi)^{-\varepsilon} \mathcal{L}(\chi_j u)\right\|_{L^q(\rho<\rho_1, \frac{dz\,dz}{\rho^{n+2}})} \end{aligned} \tag{6.6}$$

$$+C\left\|\rho^{-s+2}(\sin\varphi)^{-\varepsilon}\mathcal{L}(\beta u)\right\|_{L^q(\rho\geq\rho_1,\frac{dz\,dz}{\rho^{n+2}})}$$
$$=I+II$$

where $0<\rho_1<1$ is a constant to be determined and $j\gg 1$.

Clearly, if ε is small enough, by Hölder inequality,

$$II\leq C\rho_1^{-s+2-\frac{n+2}{q}}\|\mathcal{L}(\beta u)\|_{L^2(B_1)}\leq C\rho_1^{-s+2-\frac{n+2}{q}}\|u\|_{S^2(B_1)}.$$

To estimate I, note that

$$L(\chi_j u)=L(\chi_j)u+2\nabla_z\chi_j\cdot\nabla_z u+2|z|^2\partial_t\chi_j\cdot\partial_t u+\chi_j L(u).$$

It follows that

$$\text{(6.7)}\quad I\leq C\|\rho^{-s+2}(\sin\varphi)^{-\varepsilon}Vu\|_{L^q(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})}+Cj^M\left(\int_{\rho<\frac{1}{j}}|u|^2dz\,dt\right)^{\frac{1}{2}}$$
$$+Cj^M\left(\int_{\rho<\frac{1}{j}}\left(|\nabla_z u|^2+|z|^2|\partial_t u|^2\right)dz\,dt\right)^{\frac{1}{2}}$$

where $M>0$ is a constant depending on s.

We claim that

$$\text{(6.8)}\qquad \int_{B_\rho}(|\nabla_z u|^2+|z|^2|\partial_t u|^2)dz\,dt=O(\rho^N)\text{ as }\rho\to 0\text{ for all }N>0.$$

In fact, by a variant of Caccioppoli's inequality and (6.5),

$$\text{(6.9)}\quad\int_{B_\rho}(|\nabla_z u|^2+|z|^2|\partial_t u|^2)dz\,dt\leq\frac{C}{\rho^2}\int_{B_{2\rho}}|u|^2dz\,dt+\int_{B_{2\rho}}|V|\,|u|^2dz\,dt.$$

By Hölder's inequality,

$$\text{(6.10)}\qquad\int_{B_{2\rho}}|V|\,|u|^2dz\,dt\leq\|V\|_{L^r(B_1)}\left(\int_{B_{2\rho}}|u|^{2r'}dz\,dt\right)^{\frac{1}{r'}}.$$

By assumption, $V\in L^r(B_1)$ and $r>n$. It follows that $2<2r'<2n/(n-1)<q_0=2(n+1)/(n-1)$. Since $u\in L^{q_0}(B_1)$, we obtain, by interpolation, that u vanishes of infinite order at the origin in the $L^{2r'}$ mean. The claim (6.8) then follows from (6.9) and (6.10).

Now, let $j \to +\infty$ in (6.6), using (6.7) and (6.8), we see that, if ε is small enough,

$$\begin{aligned}
&\|\rho^{-s}(\sin\varphi)^{\varepsilon}u\|_{L^p(\rho<\rho_1,\frac{dz\,dz}{\rho^{n+2}})}\\
&\le C\|\rho^{-s+2}(\sin\varphi)^{-\varepsilon}Vu\|_{L^q(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})} + C\rho_1^{-s+2-\frac{n+2}{q}}\|u\|_{S^2(B_1)}\\
&\le C\|(\sin\varphi)^{-2\varepsilon}V\|_{L^n(B_{\rho_1})}\|\rho^{-s}(\sin\varphi)^{\varepsilon}u\|_{L^p(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})} + C\rho_1^{-s+2-\frac{n+2}{q}}\|u\|_{S^2(B_1)}\\
&\le C\|V\|_{L^r(B_{\rho_1})}\|\rho^{-s}(\sin\varphi)^{\varepsilon}u\|_{L^p(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})} + C\rho_1^{-s+2-\frac{n+2}{q}}\|u\|_{S^2(B_1)}
\end{aligned}$$

where we have used the Hölder inequality and the assumption $1/p = 1/q - 1/n$, $r > n$.

Finally, we choose $\rho_1 > 0$ so small that $C\|V\|_{L^r(B_{\rho_1})} < 1/2$, to obtain

$$\left\|\left(\frac{\rho}{\rho_1}\right)^{-s}(\sin\varphi)^{\varepsilon}u\right\|_{L^p(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})} \le C\|u\|_{S^2(B_1)}.$$

Letting $s = k + \dfrac{1}{2} \to +\infty$, we get $u \equiv 0$ in $B_{\rho_1}((0,0))$. Hence, $u \equiv 0$ in $B_1((0,0))$ by the unique continuation results for the second order elliptic equation with C^∞ coefficients (see [H]).

To complete the proof, we now consider the case when $n \ge 3$ is odd. In this case, we use the Carleman estimate (5.3), the same argument as above, and the fact

$$\left\|(\sin\varphi)^{-\frac{1}{4p}-\varepsilon}v\right\|_{L^q(B_\rho)} \le C\|v\|_{L^2(B_\rho)}, \text{ for } \varepsilon \text{ small.}$$

We obtain

$$\begin{aligned}
&\left\|\rho^{-s}(\sin\varphi)^{\frac{1}{4p}+\varepsilon}u\right\|_{L^p(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})}\\
&\qquad\le C\left\|(\sin\varphi)^{-\frac{1}{2p}-2\varepsilon}V\right\|_{L^n(B_{\rho_1})}\left\|\rho^{-s}(\sin\varphi)^{\frac{1}{4p}+\varepsilon}\right\|_{L^p(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})}\\
&\qquad+ C\rho_1^{-s+2-\frac{n+2}{q}}\|u\|_{S^2(B_1)}.
\end{aligned}$$

Note that, by Hölder inequality,

$$\left\|(\sin\varphi)^{-\frac{1}{2p}-2\varepsilon}V\|\right\|_{L^n(B_{\rho_1})} \le \left\|(\sin\varphi)^{-\frac{1}{2p}-2\varepsilon}\right\|_{L^\tau(B_{\rho_1})}\|V\|_{L^r(B_{\rho_1})}$$

where $1/\tau = 1/n - 1/r$. Since $r > 2n^2/(n+1)$ by assumption, we have $\tau < 2n^2/(n-1)$. It follows that $\tau\left(-\frac{1}{2p} - 2\varepsilon\right) > -\frac{n}{2}$ is ε is small enough. Thus, $\left\|(\sin\varphi)^{-\frac{1}{2p}-2\varepsilon}\right\|_{L^\tau(B_{\rho_1})} < +\infty$ by (1.7). This implies that

$$\left\|\rho^{-s}(\sin\varphi)^{\frac{1}{4p}+\varepsilon}u\right\|_{L^p(\rho<\rho_1,\frac{dz\,dt}{\rho^{n+2}})} \leq C\rho_1^{-s+2-\frac{n+2}{q}}\|u\|_{S^2(B_2)}.$$

Hence, $u \equiv 0$ in $B_1((0,0))$ by the same argument as in the case of $n \geq 2$ even. The proof is complete. □

BIBLIOGRAPHY

[ABV] W. O. AMREIN, A.M. BERTHIER and V. GEORGESCU, L^p inequalities for the Laplacian and unique continuation, Ann. Inst. Fourier, Grenoble, 31-3 (1981), 153-168.

[B] H. BAHOURI, Non–prolongement unique des solutions d'opérateurs, "Somme de carrés", Ann. Inst. Fourier, Grenoble, 36-4 (1986), 137-155.

[C] T. CARLEMAN, Sur un problème d'unicité pour les systèms d'èquations aux derivées partielles à deux variables indépendantes, Ark. Mat., 26B (1939), 1-9.

[E] A. ERDELYI (Director), Higher transcendental functions, Bateman manuscript project, McGraw-Hill, New York, 1955.

[G] N. GAROFALO, Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension, J. Diff. Eq., 104 (1) (1993), 117-146.

[Gr] P. GREINER, Spherical harmonics on the Heisenberg group, Canad. Math. Bull., 23(4) (1980), 383-396.

[Gru1] V.V. GRUSHIN, On a class of hypoelliptic operators, Math. USSR Sbornik, 12(3) (1970), 458-476.

[Gru2] V.V. GRUSHIN, On a class of hypoelliptic pseudodifferential operators degenerate on a submanifold, Math. USSR Sbornik, 13(2) (1971), 155-186.

[H] L. HÖRMANDER, Uniqueness theorems for second order elliptic differential equations, Comm. P. D. E., 8 (1983), 21-64.

[J] D. JERISON, Carleman inequalities for the Dirac and Laplace operators and unique continuation, Adv. in Math., 63 (1986), 118-134.

[JK] D. JERISON and C. E. KENIG, Unique continuation and absence of positive eigenvalues for Schrödinger operators, Ann. of Math., 121 (1985), 463-494.

[K] C.E. KENIG, Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation, Lecture Notes in Math., 1384 (1989) 69-90.

[RS] L.P. ROTHSCHILD and E.M. STEIN, Hypoelliptic differential operators and nilpotent groups, Acta Math., 137 (1976), 247-320.

[SS] M. SCHECHTER and B. SIMON, Unique continuation for Schrödinger operators with unbounded potential, J. Math. Anal. Appl., 77 (1980), 482-492.

[S] C.D. SOGGE, Oscillatory integrals and spherical harmonics, Duke Math. J., 53 (1986), 43-65.

[SW] E.M. STEIN and G. WEISS, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, 1971.

[Sz] G. SZEGÖ, Orthogonal Polynomials, A. M. S. Colloq. Publ., 4th edition, 23, 1975.

Manuscrit reçu le 26 février 1993.

N. GAROFALO & Z. SHEN,
Purdue University
Department of Mathematics
Mathematical Sciences Building
West Lafayette IN 47907 (USA).