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AN EFFECTIVE MATSUSAKA BIG THEOREM

by Yum-Tong SIU (*)

Dedicated to Professor Bernard Malgrange

0. Introduction.

In this paper we will prove an effective form of the following Matsusaka Big Theorem ([M1, M2, LM]). Let $P(k)$ be a polynomial whose coefficients are rational numbers and whose values are integers at integral values of k. Then there is a positive integer k_0 depending on $P(k)$ such that, for every compact projective algebraic manifold X of complex dimension n and every ample line bundle L over X with $\sum_{\nu=0}^{n}(-1)^\nu \dim H^\nu(X, k\,L) = P(k)$ for every k, the line bundle $k\,L$ is very ample for $k \geq k_0$. Here ampleness means that the holomorphic line bundle admits a smooth Hermitian metric whose curvature form is positive definite everywhere. Very ampleness means that global holomorphic sections separate points and give local homogeneous coordinates at every point. By a result of Kollár and Matsusaka on Riemann-Roch type inequalities [KM], the positive integer k_0 can be made to depend only on the coefficients of k^n and k^{n-1} in the polynomial $P(k)$ of degree n. The known proofs of Matsusaka's Big Theorem depend on the boundedness of numbers calculated for some varieties and

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divisors in a bounded family and thus the positive integer k_0 from such proofs cannot be effectively computed from $P(k)$. To state our effective Matsusaka Big Theorem, we use the following standard notation. For holomorphic line bundles $L_1, \cdots, L_\ell$ over a compact complex manifold X of complex dimension n with Chern classes $c(L_1), \cdots, c(L_\ell)$ and for positive integers $k_1, \cdots, k_\ell$ with $k_1 + \cdots + k_\ell = n$, we denote the Chern number $c(L_1)^{k_1} \cdots c(L_\ell)^{k_\ell}$ by $L_1^{k_1} \cdots L_\ell^{k_\ell}$.

Our effective version of Matsusaka's Big Theorem is the following :

THEOREM (0.1). — *Let L be an ample holomorphic line bundle over a compact complex manifold X of complex dimension n with canonical line bundle K_X. Then $m\,L$ is very ample for*

$$m \geq \frac{(2^{3^{n-1}} 5n)^{4^{n-1}} (3(3n-2)^n L^n + K_X \cdot L^{n-1})^{4^{n-1} 3n}}{(6(3n-2)^n - 2n - 2)^{4^{n-1} n - \frac{2}{3}} (L^n)^{4^{n-1} 3(n-1)}}.$$

Theorem (0.1) is a consequence of the following Theorem (0.2) when the numerically effective holomorphic line bundle B in Theorem (0.2) is specialized to the trivial line bundle. Here the numerical effectiveness of B means that the value of the first Chern class of the line bundle B evaluated at any complex curve is nonnegative.

THEOREM (0.2). — *Let X be a compact complex manifold of complex dimension n and L be an ample line bundle over X and B be a numerically effective holomorphic line bundle over X. Let $H = 2(K_X + 3(3n-2)^n L)$. Then $mL - B$ is very ample for*

$$m \geq \frac{(n(H^n)^2 (H^{n-1} \cdot B + \frac{5}{2} H^n))^{4^{n-1}}}{(6(3n-2)^n - 2n - 2)^{4^{n-1} n - \frac{2}{3}}}.$$

The result of Demailly [D2] that $12\,n^n L + 2\,K_X$ is very ample for any ample line bundle L over X can also be regarded as an effective version of Matsusaka's Big Theorem. The difference between Demailly's result and Theorem (0.2) is that in Theorem (0.2) $2\,K_X$ is no longer needed and in its place $-B$ can be used for any ample line bundle B. On the other hand the coefficient $12\,n^n$ of L in Demailly's result depends only on n and is far sharper, whereas in Theorem (0.2) the coefficient for L depends on L^n, $L^{n-1} \cdot K_X$, $L^{n-1} \cdot B$, and $L^{n-2} \cdot B \cdot K_X$ as is expected. (The condition in Theorem (0.2) is expressible in terms of L^n, $L^{n-1} \cdot K_X$, $L^{n-1} \cdot B$, and $L^{n-2} \cdot B \cdot K_X$ by using inequalities of Chern numbers of numerically effective line bundles (see [D2, Prop. 5.2(b)] and the end of §4.) Kollár

$[K]$ gave an algebraic geometric proof of a result similar to but weaker than Demailly's criterion for very ampleness. For example, Kollár's result gives the very ampleness of $(n+3)!\,2(n+1)\,((n+2)L+K_X)+K_X$ for an ample line bundle L over X (by replacing L by $(n+2)L+K_X$ and setting $a=1$ in Theorem (1.1) and setting $N=K_X$ in Lemma (1.2) in $[K]$). We would like to mention also the following result in this area due to Ein and Lazarsfeld [EL]: For a big and numerically effective divisor L on a smooth complex projective threefold X, if $L\cdot C \geq 3$ and $L^2\cdot S \geq 7$ for any curve C in X and any surface S in X, then $L+K_X$ is generated by global holomorphic sections on X.

An earlier version of this paper gave the following bound for Theorem (0.1) : mL is very ample for $m \geq (24n^nC(1+C)^n)^{n(6n^3)^n}$ with $C = ((n+2)L+K_X)L^{n-1}$, which is by several order of magnitude not as sharp as the bound stated in the present Theorem (0.1). The earlier version of Theorem (0.2) gave the very ampleness of $mL - B$ for $m \geq (24n^nC(1+C)^n)^{n(6n^3)^n}$ with $C = ((n+2)L+B+K_X)L^{n-1}$. After the earlier version was circulated, Demailly told me a way to simplify the original proof and the simplication yielded the much sharper bound for Theorem (0.1). The simplification is to verify directly the numerical effectiveness of some line bundle by using its holomorphic sections on subvarieties of decreasing dimensions and to bypass the step, in the earlier version, of extending first those sections to the ambient manifold. I would like to thank Demailly for the simplication of the proof and the sharpening of the bound. One lemma in the proof of our effective Matsusaka Big Theorem uses the strong Morse inequality of Demailly which Demailly obtained by analytic methods. Lawrence Ein and Robert Lazarsfeld told me a simple algebraic proof of that lemma which avoids the use of the strong Morse inequality of Demailly. F. Catanese also has a simple algebraic proof of that lemma similar to the proof of Ein and Lazarsfeld. Both simple algebraic proofs of that lemma are reproduced here. I would like to thank Ein, Lazarsfeld, and Catanese for their simple algebraic proofs of that lemma.

The method of proof of Theorem (0.2) uses the strong Morse inequality and the numerical criterion of very ampleness of Demailly [D1], [D2]. The earlier version used also Nadel's vanishing theorem [N] in order to extend some holomorphic sections of a line bundle from a subvariety to the ambient manifold. Kollár's result (whose proof is algebraic geometric) can be used instead of the very ampleness criterion of Demailly [D2] whose proof uses analysis. With the use of Kollár's result and the simple algebraic proofs of the lemma mentioned above due to Ein-Lazarsfeld and Catanese, the proof of our effective Matsusaka Big Theorem in this paper can be done purely algebraically, though the bound of Kollár's result is weaker than that of Demailly and therefore results in a bound that is less sharp.

The main idea of the proof of the effective Matsusaka Big Theorem in this paper is the following. Because of the very ampleness criterion of Demailly and Kollár, it suffices to show that for a given numerically effective line bundle B and an ample line bundle L there is an effective lower bound m such that $mL - B$ is numerically effective. This is done in three steps. The first step is a lemma on the existence of nontrivial holomorphic sections of a multiple of the difference of two ample line bundles whose Chern classes satisfy a certain inequality. This is the lemma for which the strong Morse inequality of Demailly is used and for which Ein-Lazarsfeld and Catanese gave simple algebraic proofs. Such a nontrivial holomorphic section enables us to construct a closed positive current which is a curvature current of the line bundle and the curvature current will be used in an application of L^2 estimates of $\bar\partial$ to construct holomorphic sections. The second step is to produce, for any d-dimensional irreducible subvariety Y of X and any very ample line bundle H of X, a nontrivial holomorphic section over Y of the homomorphism sheaf from the sheaf of holomorphic d-forms of Y to $(3\lambda(\lambda-1)/2 - d - 1)H|Y$ for $\lambda \geq H^d \cdot Y$. The sheaf of holomorphic d-forms of Y is defined from the presheaf of holomorphic p-forms on the regular part of Y which are L^2. This second step is obtained by representing Y as a branched cover over a complex projective space and the section is obtained by constructing explicitly a global sheaf-homomorphism by solving linear equations by Cramer's rule. The key point is that since the canonical line bundle of the complex projective space of complex dimension d is equal to $-(d+1)$ times the hyperplane section line bundle, by representing a compact complex space as a branched cover of the complex projective space, one can relate its canonical line bundle to a very ample line bundle on it. The third step is to get the numerical effectiveness of $mL - B$. The earlier version of this paper used Nadel's vanishing theorem for multiplier ideal sheaves [N] to produce global holomorphic sections of high multiples of $mL-B$ so that the dimension of the common zero-sets of these sections is inductively reduced to zero. A new stratification of unreduced structure sheaves by multiplier ideal sheaves was introduced in the earlier version to carry out the induction process. Demailly's simplication verifies the numerically effectiveness of $mL - B$ by producing holomorphic sections of high multiples of $mL - B$ on subvarieties of decreasing dimensions. The bypassing of the extension of those holomorphic sections (called $\sigma_{d,j}$ in §4) to the ambient manifold makes it unnecessary to use Nadel's vanishing theorem for multiplier ideal sheaves and the new way of stratifying unreduced structure sheaves. Though not used here, such a new stratification of unreduced structure sheaves by multiplier ideal sheaves can be applied in other context, *e.g.* to get a Demailly type very ampleness criterion for $m\,L + 2\,K_X$ without using the analytic tools of the Monge-Ampère equation, but we will not discuss it here.

1. Effectiveness of multiples of differences of ample line bundles.

First we state the strong Morse inequality due to Demailly [D1]. Let X be a compact complex manifold of complex dimension n, E be a holomorphic line bundle over X with smooth Hermitian metric along its fibers whose curvature form is $\theta(E)$, $X(\leq q)$ be the open subset of X consisting of all points of X where the curvature form $\theta(E)$ has no more than q negative eigenvalues. Demailly proved the following strong Morse inequality [D1] :

$$\sum_{j=0}^{q}(-1)^{q-j}\dim H^j(X,\,k\,E) \leq \frac{k^n}{n!}\int_{X(\leq q)}(-1)^q(\theta(E))^n + o(k^n).$$

Here $o(k^n)$ means the Landau symbol denoting a term whose quotient by k^n goes to 0 as $k \to \infty$. We now apply Demailly's strong Morse inequality to the case where $E = F - G$ with F and G being Hermitian holomorphic line bundles over a compact Kähler manifold X with semipositive curvature forms.

LEMMA (1.1). — *Let F and G be holomorphic line bundles over a compact Kähler manifold X with Hermitian metrics so that their curvature forms are semipositive. Then for every $0 \leq q \leq \dim X$,*

$$\sum_{0\leq j\leq q}(-1)^q\dim H^j(X,\,k(F-G)) \leq \frac{k^n}{n!}\sum_{0\leq j\leq q}(-1)^{q-j}\binom{n}{j}F^{n-j}G^j + o(k^n)$$

as $k \to \infty$.

Proof. — Let $\theta(F)$ and $\theta(G)$ be respectively the curvature forms of F and G which are smooth and semipositive. Let ω be the Kähler form of X. For $\epsilon > 0$, let $\theta_\epsilon(F) = \theta(F) + \epsilon\,\omega$ and $\theta_\epsilon(G) = \theta(G) + \epsilon\,\omega$. Then

$\theta(F-G) = \theta_\epsilon(F) - \theta_\epsilon(G)$. Let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ be the eigenvalues of $\theta_\epsilon(G)$ with respect to $\theta_\epsilon(F)$. Then $X(\leq q)$ is precisely the set of all x in X such that $\lambda_{q+1}(x) < 1$. Demailly's strong Morse inequality [D1] now reads

(1.1.1)
$$\sum_{0\leq j\leq q} (-1)^q \dim H^j(X, k(F-G)) \leq \frac{k^n}{n!}\int_{X(\leq q)} (-1)^q \prod_{1\leq j\leq n} (1-\lambda_j)\theta_\epsilon(F)^n + o(k^n).$$

Let σ_k^j be the j^{th} elementary symmetric function in $\lambda_1, \cdots, \lambda_k$. We use the convention that $\sigma_k^j = 1$ when $j = 0$ and $\sigma_k^j = 0$ when $j < 0$. Then

$$\binom{n}{j}\theta_\epsilon(F)^{n-j}\theta_\epsilon(G)^j = \sigma_n^j \theta_\epsilon(F)^n. \tag{1.1.2}$$

We claim that

$$\sum_{0\leq j\leq q} (-1)^{q-j}\sigma_n^j > (-1)^q \prod_{1\leq j\leq n} (1-\lambda_j)$$

for $\lambda_{q+1} < 1$. We verify the claim by induction on n. Assume that it is true when n is replaced by $n-1$. From

$$\sigma_n^j = \sigma_{n-1}^j + \sigma_{n-1}^{j-1}\lambda_n$$

it follows that

$$\begin{aligned}
\sum_{0\leq j\leq q} (-1)^{q-j}\sigma_n^j &= \sum_{0\leq j\leq q} (-1)^{q-j}(\sigma_{n-1}^j + \sigma_{n-1}^{j-1}\lambda_n) \\
&= \sum_{0\leq j\leq q} (-1)^{q-j}\sigma_{n-1}^j - \lambda_n \sum_{0\leq j\leq q-1} (-1)^{q-j}\sigma_{n-1}^j \\
&= (1-\lambda_n) \sum_{0\leq j\leq q} (-1)^{q-j}\sigma_{n-1}^j + \lambda_n\sigma_{n-1}^q \\
&\geq (1-\lambda_n) \sum_{0\leq j\leq q} (-1)^{q-j}\sigma_{n-1}^j > (-1)^q \prod_{1\leq j\leq n} (1-\lambda_j).
\end{aligned}$$

This proves the claim. The lemma now follows from the claim and (1.1.1) and (1.1.2) when we let $\epsilon \to 0$. □

In the earlier version of this paper the argument for the lemma was carried out only for $q = 1$ and was applied to get a nontrivial holomorphic section of $k(F-G)$ over X for k sufficiently large under the assumption

$$(F-G)^n - \sum_{q=1}^{[n/2]} \binom{n}{2q} F^{n-2q}G^{2q} > 0,$$

where the square bracket $[\cdot]$ means the integral part. There the same method of simultaneous diagonalization of curvature forms was used but with less precise analysis of the inequality concerning the eigenvalues $\lambda_1, \cdots, \lambda_n$. This more precise form for a general q was suggested to me by Demailly. In this paper the lemma will be applied only to the case $q = 1$ in the form of the following corollary.

COROLLARY (1.2). — *Let X be a compact projective algebraic manifold of complex dimension n and F and G be numerically effective line bundles over X such that $F^n > n\,F^{n-1}G$. Then for k sufficiently large there exists a nontrivial holomorphic section of $k(F-G)$ over X.*

Ein-Lazarsfeld and Catanese independently obtained similar algebraic geometric proofs of this corollary. We present them here.

The proof of Ein-Lazarsfeld. — For a coherent sheaf $\mathcal{F}$ over X, let $h^p(\mathcal{F})$ denote $\dim_{\mathbb{C}} H^p(X, \mathcal{F})$. By replacing F and G by $\ell\,F + L$ and $\ell\,G + L$ for some ample line bundle L over X and some sufficiently large ℓ we can assume without loss of generality that both F and G are ample. By multiplying both F and G by the same large positive number, we can assume without loss of generality that F and G are both very ample. Choose a smooth irreducible divisor D in the linear system $|G|$. Consider the exact sequence

$$0 \to \mathcal{O}_X(k(F-G)) \to \mathcal{O}_X(k\,F) \to \mathcal{O}_{kD}(k\,F) \to 0.$$

It is enough to show that for k sufficiently large $h^0(\mathcal{O}_X(k\,F)) > h^0(\mathcal{O}_{kD}(k\,F))$. There is a natural filtration of the sheaf $\mathcal{O}_{kD}(k\,F)$ whose quotients are sheaves of the form $\mathcal{O}_{kD}(k\,F - j\,D)$ for $0 \le j \le k-1$. So

$$h^0(\mathcal{O}_{kD}(k\,F)) \le \sum_{j=0}^{k-1} h^0(\mathcal{O}_{kD}(k\,F - j\,G)).$$

On the other hand, since G is very ample, we can choose a second smooth irreducible divisor D' in the linear system $|G|$ which meets D in normal crossing. Since $\mathcal{O}_{kD}(k\,F - j\,G)$ is isomorphic to $\mathcal{O}_{kD}(k\,F - j\,D')$ which is a subsheaf of $\mathcal{O}_{kD}(k\,F)$, it follows that $h^0(\mathcal{O}_{kD}(k\,F - j\,G)) \le h^0(\mathcal{O}_{kD}(k\,F))$. Thus $h^0(\mathcal{O}_{kD}(k\,F)) \le k \cdot h^0(\mathcal{O}_D(k\,F))$. By Kodaira's vanishing theorem and the theorem of Riemann-Roch

$$k \cdot h^0(\mathcal{O}_D(k\,F)) = \frac{k \cdot (k\,F)^{n-1} \cdot D}{(n-1)!} + o(k^n) = \frac{k^n F^{n-1} \cdot G}{(n-1)!} + o(k^n),$$

whereas $h^0(\mathcal{O}_X(k\,F)) = \dfrac{k^n F^n}{n!} + o(k^n)$, from which we obtain the conclusion.

Catanese's proof. — For a line bundle H over X, let $h^p(X, H)$ denote $\dim_{\mathbf{C}} H^p(X,\, H)$. As in the proof of Ein-Lazarsfeld we can assume without loss of generality that F and G are both very ample. Let k be any positive integer. We select k smooth members G_j, $1 \le j \le k$ in the linear system $|G|$ and consider the exact sequence

$$(1.2.1)\ \ 0 \to H^0\Big(X,\, k\,F - \sum_j G_j\Big) \to H^0(X,\, k\,F) \to \bigoplus_{j=1}^{k} H^0(G_j,\, k\,F|G_j).$$

By (1.2.1) and Kodaira's vanishing theorem and the theorem of Riemann-Roch

$$\begin{aligned} h^0(X,\, k(F-G)) &\ge \frac{k^n}{n!}F^n - o(k^n) - \sum_{j=1}^{k}\Big(\frac{k^{n-1}}{(n-1)!}F^{n-1}\cdot G_j - o(k^{n-1})\Big) \\ &\ge \frac{k^n}{n!}(F^n - n\,F^{n-1}\cdot G) - o(k^n). \end{aligned}$$

So for k sufficiently large there exists nontrivial holomorphic section of $k(F-G)$ over X.

What we need from Corollary (1.2) is a curvature current for the difference of the two numerically effective line bundles. Let L be a holomorphic line bundle on a compact complex manifold X of complex dimension n. Let X be covered by a finite open cover $\{U_j\}$ so that $L|U_j$ is trivial and let g_{jk} on $U_j \cap U_k$ be the transition function for L. A curvature current $\theta(L)$ for L is a closed positive (1,1)-current on X which is given by $\dfrac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_j$ on U_j where φ_i is a plurisubharmoinc function on U_j such that $e^{-\varphi_j}|g_{jk}|^2 = e^{-\varphi_k}$ on $U_j \cap U_k$. The collection $\{e^{-\varphi_j}\}$ defines a (possibly nonsmooth) Hermitian metric along the fibers of L. A closed positive (1,1)-current θ means a current of type (1,1) which is closed and is positive in the sense that $\theta \wedge \prod_{j=1}^{n-1}(\sqrt{-1}\alpha_j \wedge \bar\alpha_j)$ is a nonnegative measure on X for any smooth (1,0)-forms $\alpha_j (1 \le j \le n-1)$ with compact support. A closed positive (1,1)-current is characterized by the fact that locally it is of the form $\sqrt{-1}\partial\bar\partial\psi$ for some plurisubharmonic function ψ. The Lelong number, at a point P, of a closed positive (1,1)-current θ on some open subset of $\mathbf{C}^n$ is defined as the limit of $(\pi r^2)^{1-n}\theta \wedge \Big(\dfrac{\sqrt{-1}}{2}\partial\bar\partial|z|^2\Big)^{n-1}$ over the ball of radius r centered at P as $r \to 0$, where $z = (z_1, \cdots, z_n)$ is the coordinate of $\mathbf{C}^n$.

The normalization is such that the Lelong number of $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|z|^2$ is 1 at 0. The Lelong number is independent of the choice of local coordinates. The function $e^{-\psi}$ is locally integrable at P if the closed positive (1,1)-current $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$ has Lelong number less than 1 at P. The function $e^{-\psi}$ is not locally integrable at P if the closed positive (1,1)-current $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$ has Lelong number at least n at P. See *e.g.* [L] and [S] for more detailed information on closed positive currents and Lelong numbers.

COROLLARY (1.3). — *Let X be a compact projective algebraic manifold of complex dimension n and F and G be numerically effective line bundles over X such that $F^n > n\,F^{n-1}G$. Then there exists a closed positive current which is the curvature current for $F - G$.*

Proof. — Let X be covered by a finite open cover $\{U_j\}$ so that $(F - G)|U_j$ is trivial and let $g_{j\ell}$ on $U_j \cap U_\ell$ be the transition function for $F - G$. Let s be a nontrivial holomorphic section for $k(F - G)$ for some sufficiently large k. The section s is given by a collection $\{s_j\}$ where s_j is a holomorphic function on U_j with $s_j = g_{j\ell}^k s_\ell$ on $U_j \cap U_\ell$. Define $\varphi_j = \frac{1}{k}\log|s_j|^2$. Then the closed positive current on X defined by $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_j$ on U_j is a curvature current for $F - G$. □

2. The use of the very ampleness criterion of Demailly.

In §1 we obtain nontrivial sections for high multiples of the difference of ample line bundles, but we cannot control yet the effective bound for the multiple. In this section we are going to use closed positive currents and the very ampleness criterion of Demailly [D2] to get an effective bound for the multiple. The very ampleness criterion of Demailly [D2] states that for any ample line bundle L over a compact complex manifold X of complex dimension n, the line bundle $12n^nL+2K_X$ is a very ample line bundle over X. We will use the more general result of Demailly on the global generation of s-jets to further improve on the bound. The argument works with the very ampleness of $12n^nL+2K_X$ but with a slightly worse bound. The result of Demailly on the global generation of s-jets states that, if L is an ample line bundle over a compact complex manifold X of complex dimension n, the global holomorphic sections of $m\,L + 2K_X$ generate the s-jets at every point of X if $m \geq 6(n+s)^n$.

Let X be a compact projective algebraic manifold and Y be an irreducible subvariety of complex dimension d in X. Let ω_Y be the sheaf on Y defined by the presheaf which assigns to an open subset U of Y the space of all holomorphic d-forms on $U \cap \operatorname{Reg} Y$ which are L^2 on $U \cap \operatorname{Reg} Y$, where $\operatorname{Reg} Y$ is the set of all regular points of Y. The sheaf ω_Y is coherent and is equal to the zeroth direct image of the sheaf of holomorphic d-forms on a desingularization Y' of Y under the desingularization map $Y' \to Y$.

PROPOSITION (2.1). — *Let L be an ample line bundle over a compact complex manifold X of complex dimension n and let B be a numerically effective line bundle over X. Let Y be an irreducible subvariety of complex dimension d in X. Let $C_n = 3(3n-2)^n$. Then for $m > d \dfrac{L^{d-1}\cdot B\cdot Y}{L^d\cdot Y}$ there exists a nontrivial holomorphic section of the sheaf $\omega_Y \otimes \mathcal{O}(mL - B + K_X + C_n L)|Y$ over Y.*

Proof. — Let $\pi : Y' \to Y$ be a desingularization of Y. Let E be an ample line bundle of Y'. We apply Corollary (1.3) to the numerically effective line bundles $F = k\,\pi^*(m\,L)$ and $G = E + k\,\pi^* B$ over Y' for some sufficiently large k. The choice of m implies that $F^d > d\,F^{d-1}\cdot G$ for k sufficiently large. Hence there exists a closed positive (1,1)-current θ on Y' which is a curvature current for $k(F-G)-E$. Choose a point P in Y' such that $\pi(P)$ is in the regular part of Y and the Lelong number of θ at P is zero. Since the global holomorphic sections of $p\,L + 2\,K_X$ over X generate the $2d$-jets at every point of X, for $p \geq 6(n+2d)^n$, there exists a Hermitian metric along the fibers of $p\,L + 2\,K_X$ which is smooth on $X - \pi(P)$ and whose Lelong number at $\pi(P)$ is 2d. Thus for $p \geq 3(n+2d)^n$ there exists a Hermitian metric along the fibers of $p\,L + K_X$ which is smooth on $X - \pi(P)$ and whose Lelong number at $\pi(P)$ is d. Since $d \leq n-1$, we can use $p = C_n$.

We give E a smooth metric with positive definite curvature form on Y'. By putting together the nonsmooth Hermitian metrics of $k(C_n L + K_X)$ and $k(F-G)-E$ and the smooth Hermitian metric of E, we get a nonsmooth Hermitian metric h of $\pi^*(mL - B + K_X + C_n L)$ such that the curvature current θ of h has Lelong number d at P and has Lelong number zero at every point of $U - \{P\}$ for some open neighborhood U of P in Y'. Moreover, the curvature current θ is no less than $\dfrac{1}{k}$ times the positive definite curvature form of E on Y. We can assume without loss of generality that both line bundles $\pi^*(mL - B + K_X + C_nL)|U$ and $K_{Y'}|U$ are trivial. Let σ be a holomorphic section of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$ over U such that σ is nonzero at P. We give $K_{Y'}$ any smooth metric so that from h we have some nonsmooth metric h' of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$. We take a smooth function ρ with compact support on U so that ρ is identically 1 on some open neighborhood W of P in U. Since the $(\pi^*(mL - B + K_X + C_nL) + K_{Y'})$-valued $\bar\partial$-closed (0,1)-form $(\bar\partial\rho)\sigma$ on X which is supported on U is zero on W and since the Lelong number of the curvature current of the metric h' is 0 at every point of $U - \{P\}$, it follows that $(\bar\partial\rho)\sigma$ is L^2 with respect to the metric h' of $\pi^*(mL - B + K_X + C_nL) + K_{Y'}$. Since the curvature current θ of the line bundle $\pi^*(mL - B + K_X + C_nL)$ is no less than some positive definite smooth (1,1)-form on Y', by the L^2 estimates of $\bar\partial$ (see *e.g.* [D2, *p.*332, Prop.4.1]) there exists some L^2 section τ of $\pi^*(mL-B+K_X+C_nL)+K_{Y'}$ over Y' such that $\bar\partial\tau = (\bar\partial\rho)\sigma$ on Y'. Since the Lelong number of θ is d at P, it follows that τ vanishes at P and $\rho\,\sigma - \tau$ is a global holomorphic section of $\pi^*(mL-B+K_X+C_nL)+K_{Y'}$ over Y' which is nonzero at P. The direct image of $\rho\,\sigma - \tau$ with respect to $\pi : Y' \to Y$ is a global holomorphic section of $\omega_Y \otimes \mathcal{O}(mL - B + K_X + C_nL)|Y$ over Y which is non identically zero. □

3. Sections of the sum of a very ample line bundle and the anticanonical bundle.

Because of the very nature of the very ampleness criterion of Demailly, in §2 we could only construct holomorphic sections of line bundles containing the canonical line bundle as a summand. In this section we are going to construct holomorphic sections of the sum of a very ample line bundle and the anticanonical line bundle which will later be used to obtain holomorphic sections of line bundles without the canonical line bundle as a summand. The idea is that since the canonical line bundle of the complex projective space of complex dimension d is equal to $-(d+1)$ times the hyperplane section line bundle, by representing a compact complex space as a branched cover of the complex projective space, one can relate its canonical line bundle to a very ample line bundle on it.

Let Y be a proper irreducible d-dimensional subvariety in $\mathbf{P}_N$ and H_N be the hyperplane section line bundle of $\mathbf{P}_N$. Let λ be a positive number no less than the degree $H_N^d \cdot Y$ of Y. We use the notation introduced in §2 for ω_Y which is the sheaf on Y from the presheaf of local holomorphic

d-forms on the regular part of Y which are L^2 locally at the singular points of Y.

Let V be a linear subspace of $\mathbf{P}_N$ of complex dimension $N-d-1$ which is disjoint from Y and let W be a linear subspace of $\mathbf{P}_N$ of complex dimension d which is disjoint from V. Denote by π' the projection map from $\mathbf{P}_N - V$ to W defined by the linear subspaces V and W of $\mathbf{P}_N$, which means that for $x \in \mathbf{P}_N - V$ the point $\pi'(x)$ is the point of intersection of W and the linear span of x and V. We identify W with $\mathbf{P}_d$ and let $\pi : Y \to \mathbf{P}_d = W$ be the restriction of π' to Y. We denote the restriction of H_N to $\mathbf{P}_d = W$ by H_d so that H_d is the hyperplane section line bundle of $\mathbf{P}_d$. We denote the restriction of H_N to Y by H.

LEMMA (3.1). — *There exists a nontrivial holomorphic section of the sheaf*

$$\mathcal{H}om(\omega_Y,\ \mathcal{O}_Y((3\lambda(\lambda-1)/2-d-1)H)$$

over Y.

Proof. — Since there are nontrivial holomorphic sections of any positive multiple of H over Y, without loss of generality we can assume that λ is equal to the degree $H_N^d \cdot Y$ of Y. Let $Z \subset \mathbf{P}_d$ be the branching locus of $\pi : Y \to \mathbf{P}_d$. Let D be some hyperplane in $\mathbf{P}_d$ not contained entirely in Z. Let D^* be the hyperplane in $\mathbf{P}_N$ containing D and V so that D^* is the topological closure in $\mathbf{P}_N$ of $\pi'^{-1}(D)$. Let s_{D^*} be the holomorphic section of H_N over $\mathbf{P}_N$ whose divisor is D^*. Let f be a holomorphic section of H_N over $\mathbf{P}_N$ such that $h := \dfrac{f}{s_{D^*}}$ assumes λ distinct values at the λ points of $\pi^{-1}(P_1)$ for some P_1 in $\mathbf{P}_d$.

Let s_D be the holomorphic section of H_d over $\mathbf{P}_d$ whose divisor is D. Let $z_1, \cdots, z_d$ be the inhomogeneous coordinates of $\mathbf{P}_d - D$. Then $t = (s_D)^{d+1} dz_1 \wedge \cdots \wedge dz_d$ can be regarded as a nowhere zero $(d+1)[D]$ — valued holomorphic d-form on $\mathbf{P}_d$.

The nontrivial holomorphic section of $\mathcal{H}om(\omega_Y, \mathcal{O}_Y(3(\lambda(\lambda-1)/2 - d-1)H)$ over Y will be defined by using h and t. We will define it in the following way. We take a holomorphic section θ of ω_Y over an open neighborhood U of some point P of Y and we will define a holomorphic section of $\mathcal{O}_Y((3\lambda(\lambda-1)/2-d-1)H$ over an open neighborhood of P in U. Without loss of generality we can assume that U is of the form $\pi^{-1}(U')$ for some open subset U' of $\mathbf{P}_d$.

We consider the Vandermonde determinant defined for the λ values of $h|Y$ on the same fiber of $\pi : Y \to \mathbf{P}_d$. In other words, we take $P \in \mathbf{P}_d$ such that $\pi^{-1}(P)$ has λ distinct points $P^{(1)}, \cdots, P^{(\lambda)}$ and we form the determinant

$$h'_0 = \det((h(P^{(\mu)}))^{\nu-1})_{1 \le \mu,\nu \le \lambda}.$$

Let $h' = (h'_0)^2$. Then h' is a meromorphic function on $\mathbf{P}_d$ with pole only along D. By considering the pole order of h along $\pi^{-1}(D)$, we conclude that the pole order of h' along D is $2 \sum_{\mu=1}^{\lambda-1} \mu = \lambda(\lambda - 1)$. We write

$$\theta(P^{(j)}) = \sum_{\nu=0}^{\lambda-1} a_\nu(P)(h(P^{(j)}))^\nu (1 \le j \le \lambda) \tag{3.1.1}$$

for some d-form a_ν on $U'(0 \le \nu \le \lambda - 1)$. We solve the system (3.1.1) of equations by Cramer's rule. Let $c^{(\nu)}_{\rho\mu} = h(P^{(\mu)})^{\rho-1}$ for $\rho \neq \nu$ and $c^{(\nu)}_{\nu\mu} = \theta(P^{(\mu)})$. Let b_ν be the determinant of the $\lambda \times \lambda$ matrix $(c^{(\nu)}_{\rho\mu})_{1 \le \rho,\mu \le \lambda}$. Then $a_\nu(P) = b_\nu / h'_0 = b_\nu h'_0 / h'$. We are going to use the removable singularity for L^2 holomorphic top degree forms so that an L^2 holomorphic d-form on $\Omega - Z$ for some open subset Ω of $\mathbf{P}_d$ extends to a holomorphic d-form on Ω. Thus, since θ is an L^2 holomorphic d-form on $U \cap \operatorname{Reg} Y$, we conclude from the pole orders of h and h' that the restriction of $s_D^{3\lambda(\lambda-1)/2} h'\ a_\nu$ to $U' - (Z \cap D)$ is a holomorphic $(3(\lambda(\lambda - 1)/2)H_d$-valued d-form on $U' - (Z \cap D)$ and therefore $s_D^{3\lambda(\lambda-1)/2} h'\ a_\nu$ is a holomorphic $(3(\lambda(\lambda - 1)/2)H_d$-valued d-form on U' by removability of singularity of codimension two.

We have $\theta = \sum_{\nu=0}^{\lambda-1} (a_\nu \circ \pi)(h^\nu|Y)$. The section $\pi^*(s_D^{3\lambda(\lambda-1)/2} h')\theta \,/\, \pi^*(t)$ of $(3\lambda(\lambda - 1)/2 - d - 1)H$ over U is holomorphic, because

$$\pi^*(s_D^{3\lambda(\lambda-1)/2} h')\theta \,/\, \pi^*(t) = \sum_{\nu=0}^{\lambda-1} ((s_D^{3\lambda(\lambda-1)/2} h'\ a_\nu / t) \circ \pi)(h^\nu|Y).$$

The section of $\mathcal{H}om(\omega_Y, \mathcal{O}_Y((3\lambda(\lambda - 1)/2 - d - 1)H)$ over Y to be constructed is now given by the map which sends the holomorphic section θ of ω_Y over U to the holomorphic section $\pi^*(s_D^{3\lambda(\lambda-1)/2} h')\theta \,/\, \pi^*(t)$ of $(3\lambda(\lambda - 1)/2 - d - 1)H$ over U. □

PROPOSITION (3.2). — *Let L be an ample line bundle over a compact complex manifold X of complex dimension n and let B be a numerically effective line bundle over X. Let Y be a proper irreducible subvariety of complex dimension d in X. Let H be a very ample line bundle of Y. Let $C_n = 3(3n - 2)^n$. Then for $m > d \dfrac{L^{d-1} \cdot B \cdot Y}{L^d \cdot Y}$ there* *exists a nontrivial holomorphic section of the holomorphic line bundle* $(3\lambda(\lambda-1)/2-d-1)H+(mL-B+K_X+C_nL)|Y$ *over* Y.

Proof. — The proposition follows from Lemma (3.1) and Proposition (2.1) after we embed Y into a complex projective space by using the holomorphic sections of H over Y. □

COROLLARY (3.3). — *Let* L *be an ample line bundle over a compact complex manifold* X *of complex dimension* n *and let* B *be a numerically effective line bundle over* X. *Let* Y *be a proper irreducible subvariety of complex dimension* d *in* X. *Let* $C_n = 3(3n-2)^n$ *and* $H = 2\,C_nL + 2\,K_X$. *Then for* $m > d\dfrac{L^{d-1}\cdot B\cdot Y}{L^d\cdot Y}$ *there exists a nontrivial holomorphic section of the holomorphic line bundle* $((3\lambda(\lambda-1)/2)H + mL - B)|Y$ *over* Y.

Proof. — The case $d=0$ is trivial. Hence we can assume that $d\geq 1$ and $n\geq 2$. In that case $2C_n = 3(3n-2)^n \geq 12\,n^n$ and by Demailly's very ampleness criterion the line bundle H is very ample over X. Since $\left(d+\frac{1}{2}\right)H$ is very ample over X, the corollary follows from Proposition (3.2). □

We will need a statement similar to Corollary (3.3) for the case $Y=X$.

PROPOSITION (3.4). — *Let* L *be an ample line bundle over a compact complex manifold* X *of complex dimension* $n\geq 2$ *and let* B *be a numerically effective line bundle over* X. *Let* $C_n = 3(3n-2)^n$ *and* $H = 2\,C_nL + 2\,K_X$. *Then for* $m > n\,\dfrac{L^{n-1}\cdot B}{L^n}$ *there exists a nontrivial holomorphic section of the sheaf* $mL - B + H$ *over* X.

Proof. — By Corollary (1.3) there exists a nonsmooth Hermitian metric h_1 for $m\,L-B$ whose curvature current θ_1 is a closed positive (1,1)-current. Take a point P in X at which the Lelong number of θ_1 is 0. By [D2] (see the proof of Theorem 11.6 on p. 364 and the proof of Corollary 2 on p. 369) there exists a nonsmooth Hermitian metric h_2 for $2C_nL+K_X$ such that

(i) the Lelong number of the curvature current θ_2 of h_2 is at least n at P,

(ii) the Lelong number of θ_2 is < 1 at every point of $U-\{P\}$ for some open neighborhood U of P in X,

(iii) θ_2 is no less than some positive definite smooth (1,1)-form on X.

Now we put together the metrics h_1 and h_2 to form a nonsmooth metric h_3 for $m\,L - B + 2C_nL + K_X$. Without loss of generality we can assume that at every point of U the Lelong number of θ_1 is zero and that the holomorphic line bundles $(m\,L - B + 2C_nL)|U$ and $K_X|U$ are trivial. We give K_X any smooth metric so that from h we have some nonsmooth metric h' of $mL - B + H$. We take a smooth function ρ with compact support on U so that ρ is identically 1 on some open neighborhood W of P in U. Since the Lelong number of $\theta_1 + \theta_2$ is < 1 on $U - \{P\}$, it follows that the $(mL - B + H)$-valued $\bar{\partial}$-closed (0,1)-form $(\bar{\partial}\rho)\sigma$ on X is L^2. By applying the L^2 estimates of $\bar{\partial}$, we obtain an L^2 section τ of $mL - B + H$ over X such that $\bar{\partial}\tau = (\bar{\partial}\rho)\sigma$ on X. Since the Lelong number of $\theta_1 + \theta_2$ is at least n at P, it follows that τ vanishes at P and $\rho\,\sigma - \tau$ is a global holomorphic section of $mL - B + H$ over X which is nonzero at P. $\square$

4. Final step in the proof of the effective Matsusaka Big Theorem.

Since the effective Matsusaka Big Theorem is obviously true for the case when X is of complex dimension of one, we can assume that the complex dimension n of X is at least 2. To get Matsusaka's Big Theorem, it is enough to get an effective bound on m for $m\,L - (B + 2K_X + pL)$ to be numerically effective for some $p \geq 2(n+1)$, because then by Demailly's very ampleness criterion [D2, p. 370, Remark 12.7] the holomorphic line bundle $(12n^n + m - p)L - B = 12\,n^nL + 2\,K_X + m\,L - (B + 2K_X + pL)$ becomes very ample. We use $B + 2K_X + pL$ instead of just $B + 2K_X$, because we need Fujita's result [F] on the numerical effectiveness of $(n+1)L + K_X$.

The numerical effectiveness of $m\,L - (B + 2K_X + pL)$ is verified by evaluating it on an arbitrary compact curve. The main idea is to use Corollary (3.3). We can obtain nontrivial holomorphic sections of the line bundle of the form $m\,L - B$ over subvarieties and inductively we apply the argument to subvarieties which are the zero sets of such nontrivial holomorphic sections. This way we will get a numerical criterion for the numerical effectiveness of $m\,L - B$ and then afterwards we will replace B by $B + 2K_X + pL$.

Let $C_n = 3(3n-2)^n$ and $H = 2C_nL + 2\,K_X$. As in the outline of the argument in the preceding paragraph, to obtain the numerical effectiveness of $m\,L - B$, we are going to use Corollary (3.3) to construct inductively a sequence of (not necessarily irreducible) algebraic subvarieties

$X = Y_n \supset Y_{n-1} \supset \cdots \supset Y_1 \supset Y_0$ and positive integers m_d $(1 \leq d \leq n)$ with the following two properties :

(i) Y_d is d-dimensional.

(ii) For every irreducible component $Y_{d,j}$ of Y_d, there exists a nontrivial holomorphic section $\sigma_{d,j}$ of $m_d L - B$ over $Y_{d,j}$ such that Y_{d-1} is the union of the zero-sets of $\sigma_{d,j}$ when j runs through the set indexing the branches $Y_{d,j}$ of Y_d.

To start the induction, by Proposition (3.4) we can use $m_n > n\frac{L^{n-1} \cdot (B+H)}{L^n}$ and get a nontrivial holomorphic section σ_n of $m_n L - B$ over X. Let Y_{n-1} be the zero-set of σ_n. Suppose we have constructed Y_ν for $d \leq \nu \leq n$ and m_ν for $d < \nu \leq n$. We are going to construct Y_{d-1} and m_d. Let $C_n' = 2(C_n - (n+1))$. Then $H - C_n'\, L = 2((n+1)L + K_X)$ is numerically effective over X by Fujita's result [F]. To choose m_d we have to compute $Y_{d,j} \cdot H^d$ and $Y_{d,j} L^{d-1} B$ for every irreducible component $Y_{d,j}$ of Y_d. Now $Y_{d,j}$ is a branch of the zero-set of some nontrivial holomorphic section $\sigma_{d+1,k}$ of $m_{d+1} L - B$ over $Y_{d+1,k}$ for some branch $Y_{d+1,k}$ of Y_d. For any numerically effective line bundles $E_1, \cdots, E_d$ over X, we have

$$\begin{aligned} Y_{d,j} \cdot E_1 \cdots E_d &\leq Y_{d+1,k} \cdot (m_{d+1} L - B) \cdot E_1 \cdots E_d \\ &\leq Y_{d+1,k} \cdot m_{d+1} L \cdot E_1 \cdots E_d \\ &\leq Y_{d+1,k} \cdot \frac{m_{d+1}}{C_n'} H \cdot E_1 \cdots E_d \end{aligned}$$

(by the numerical effectiveness of $H - C_n'\, L$) which by induction on d yields

$$Y_{d,j} \cdot E_1 \cdots E_d \leq \frac{m_n}{C_n'} \cdots \frac{m_{d+1}}{C_n'} H^{n-d} E_1 \cdots E_d.$$

Hence

$$Y_{d,j} H^d \leq \frac{m_n}{C_n'} \cdots \frac{m_{d+1}}{C_n'} H^n,$$

$$Y_{d,j} L^{d-1} B \leq \frac{m_n}{C_n'} \cdots \frac{m_{d+1}}{C_n'} H^{n-d} L^{d-1} B.$$

To apply Corollary (3.3) we impose on m_d $(1 \leq d < n)$ the condition that $m_d > \frac{d}{L^d \cdot Y_{d,j}} \Big(L^{d-1} \cdot Y_{d,j} \cdot \Big(B + \frac{3}{2}\lambda_d(\lambda_d - 1)H\Big)\Big)$ for some $\lambda_d \geq Y_{d,j} \cdot H^d$ for all j. Since $L^d \cdot Y_{d,j} \geq 1$ and $H - C_n'\, L$ is numerically effective, to get the condition we can let

$$\lambda_d \geq \frac{m_n}{C_n'} \cdots \frac{m_{d+1}}{C_n'} H^n$$

and set $m_d > \frac{d\lambda_d}{(C_n')^{d-1}} \Big(\frac{H^{n-1} \cdot B}{H^n} + \frac{3}{2}\lambda_d(\lambda_d - 1)\Big)$. In order to get a simpler closed expression later we replace the last inequality by the stronger inequality that m_d is no less than the integral part of $\frac{n\lambda_d^3}{(C'_n)^{d-1}}\Big(\frac{H^{n-1}\cdot B}{H^n}+\frac{3}{2}\Big)$. Once these inequalities are satisfied, by Corollary (3.3) we can find a nontrivial holomorphic section $\sigma_{d,j}$ of m_dL-B over $Y_{d,j}$. Our Y_{d-1} is now the union of the zero-sets of all $\sigma_{d,j}$ when j runs through the set indexing the branches $Y_{d,j}$ of Y_d. The above inequality for m_d $(1\le d<n)$ is satisfied if inductively m_d is no less than the integral part of

$$\frac{n}{(C'_n)^{d-1}}\Big(\frac{H^{n-1}\cdot B}{H^n}+\frac{3}{2}\Big)\Big(\frac{m_n}{C'_n}\cdots\frac{m_{d+1}}{C'_n}H^n\Big)^3.$$

To get a closed formula, we consider for $1\le d\le n$ the equations

$$q_d=\frac{n}{(C'_n)^{d-1}}\Big(\frac{H^{n-1}\cdot B}{H^n}+\frac{3}{2}\Big)\Big(\frac{q_n}{C'_n}\cdots\frac{q_{d+1}}{C'_n}H^n\Big)^3.$$

Then $q_d=\frac{1}{(C'_n)^2}q_{d+1}^4$ and $q_d=\frac{1}{(C'_n)^{4^{n-d}-\frac{2}{3}}}q_n^{4^{n-d}}$ for $1\le d<n$. Thus we can inductively define m_d to be the integral part of

$$\frac{\big(\frac{n}{(C'_n)^{n-1}}(H^n)^2(H^{n-1}\cdot B+\frac{3}{2}H^n)\big)^{4^{n-d}}}{(C'_n)^{4^{n-d}-\frac{2}{3}}}=\frac{(n(H^n)^2(H^{n-1}\cdot B+\frac{3}{2}H^n))^{4^{n-d}}}{(C'_n)^{4^{n-d}n-\frac{2}{3}}}.$$

Clearly $m_d\le m_1$ for $1\le d\le n$.

We now verify that the line bundle m_1L-B is numerically effective over X. Let Γ be an irreducible complex curve in X. We have to verify that $(m_1L-B)\cdot\Gamma$ is nonnegative. There exists an integer $1\le d\le n$ such that Γ is contained entirely in some branch $Y_{d,j}$ of Y_d but is not contained entirely in Y_{d-1}. There exists a nontrivial holomorphic section $\sigma_{d,j}$ of m_dL-B over $Y_{d,j}$ whose zero-set is contained in Y_{d-1}. Thus $\sigma_{d,j}$ does not vanish identically on Γ and we conclude that $(m_dL-B)\cdot\Gamma$ is nonnegative. Since $(m_1-m_d)L$ is ample, it follows that $(m_1L-B)\cdot\Gamma$ is nonnegative. This completes the proof that m_1L-B is numerically effective.

Since $2C_n\ge 12n^n$ for $n\ge 2$, by [D2, p. 370, Remark 12.7] from the numerical effectiveness of m_1L-B it follows that the line bundle m_1L-B+H is very ample. After replacing B by $B+H$, we conclude that $m\ L-B$ is very ample for

$$m\ge\frac{(n(H^n)^2(H^{n-1}\cdot B+\frac{5}{2}H^n))^{4^{n-1}}}{(C'_n)^{4^{n-1}n-\frac{2}{3}}}$$

and Theorem (0.2) is proved.

We now derive Theorem (0.1) from Theorem (0.2) by using the following inequalities of Chern numbers of numerically effective line bundles

[D2, Prop. 5.2(b)]. If $L_1, \cdots, L_n$ are numerically effective holomorphic line bundles over a compact projective algebraic manifold X of complex dimension n and $k_1, \cdots, k_\ell$ are positive integers with $k_1 + \cdots + k_\ell = n$, then

$$L_1^{k_1} \cdots L_\ell^{k_\ell} \geq (L_1^n)^{k_1/n} \cdots (L_\ell^n)^{k_\ell/n}.$$

We now apply the inequality to the case of $L_1 = F + G$ and $L_2 = F$. Then

$$(F+G)F^{n-1} \geq ((F+G)^n)^{1/n}(F^n)^{(n-1)/n}$$

or

$$(F+G)^n \leq \frac{((F+G)F^{n-1})^n}{(F^n)^{n-1}}.$$

Let $F = (C_n - (n+1))L$ and $G = (n+1)L + K_X$. Then $F + G = \frac{1}{2}H$ and

$$\frac{1}{2^n}H^n \leq \frac{((C_n L + K_X) \cdot L^{n-1})^n}{(L^n)^{n-1}}.$$

Thus

$$\left(n\frac{5}{2}(H^n)^3\right)^{4^{n-1}} \leq (2^{3n-1}5n)^{4^{n-1}}\left(\frac{((C_n L + K_X) \cdot L^{n-1})^n}{(L^n)^{n-1}}\right)^{3\cdot 4^{n-1}}$$

In the case of $B = 0$ the line bundle $m\ L$ is very ample if

$$m \geq \frac{(2^{3n-1}5n)^{4^{n-1}}((C_n L + K_X) \cdot L^{n-1})^{4^{n-1}3n}}{(C_n')^{4^{n-1}n-\frac{2}{3}}(L^n)^{4^{n-1}3(n-1)}}.$$

Thus Theorem (0.1) is proved.

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