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M. S. BAOUENDI
L. P. ROTHSCHILD

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HARMONIC FUNCTIONS SATISFYING WEIGHTED SIGN CONDITIONS ON THE BOUNDARY

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0. Introduction.

In this paper we prove a local unique continuation theorem for harmonic functions defined in an open set of the half space, $\mathbb{R}^n_+$. These results involve a local boundary sign condition on the product of the function and a homogeneous polynomial. Classical results in this direction go back to the global theorems of G. Giraud and E. Hopf, concerning the nonvanishing of the normal derivative at a boundary point where an extremum is reached. (See e.g. Miranda [8] and the references given there.)

For $X \in \mathbb{R}^n$, we write $X = (x, y)$, with $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$ and denote by $\mathbb{R}^n_+$ the half space $\{y > 0\}$. Let $\mathcal{O}$ be an open neighborhood of 0 in $\mathbb{R}^n$ and put $\Omega = \mathbb{R}^n_+ \cap \mathcal{O}$ and $V = \mathbb{R}^{n-1} \cap \mathcal{O}$. We shall assume that Ω is connected. If u is a continuous function in Ω, we say that u is *flat* at 0 if for every positive integer N there is a constant C_N such that $|u(X)| \leq C_N |X|^N$.

We now state our main results.

THEOREM 1. — *Let $u \in C^0(\bar{\Omega})$ be harmonic in Ω and satisfy the following conditions:*

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(1) *There exists $P(x)$, a homogeneous polynomial in $n-1$ variables such that $P(x)u(x,0) \geq 0$ for $x \in V$.*

(2) *For every positive integer N , the function $|x|^{-N}u(x,0)$ is integrable in V.*

(3) *For every multi-index α in n variables with $|\alpha| \leq d$, where d is the degree of $P(x)$, the function $y \mapsto (\partial_X^\alpha u)(0,y)$ is flat at $y=0$.*

Then $u(x,0) \equiv 0$ in a neighborhood of 0 in V, and hence u extends as a real analytic function in a neighborhood of 0 in $\mathbb{R}^n$.

For applications, a more precise version of Theorem 1 will be useful.

THEOREM 1′. — *The conclusion of Theorem 1 still holds if condition (3) is replaced by the weaker condition:*

(3') *$y \mapsto Q\Big(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\Big)\Big(\dfrac{u(x,y)}{y}\Big)\Big|_{x=0}$ is flat at $y=0$, where Q is the homogeneous polynomial of degree d in n variables satisfying $P(x) = |X|^{n+2d}Q\Big(\dfrac{\partial}{\partial X}\Big)\Big(\dfrac{1}{|X|^n}\Big)$.*

Note that for an arbitrary polynomial $P(x)$ the existence of the polynomial $Q(X)$ in condition (3′) follows from Proposition 1.4 below. We now state some consequences of this theorem.

COROLLARY 1. — *Let $u \in C^0(\bar{\Omega})$ be harmonic in Ω and satisfy the following conditions:*

(i) *There exists $P(x)$, a homogeneous polynomial in $n-1$ variables such that $P(x)u(x,0) \geq 0$ for $x \in V$.*

(ii) *For every multi-index α in n variables with $|\alpha| \leq d$, where d is the degree of $P(x)$, $\partial_X^\alpha u(X)$ is flat at 0.*

Then $u(X) \equiv 0$ in Ω.

In particular, for $n=2$, we have the following corollaries:

COROLLARY 2. — *Assume $n=2$. Let $u \in C^0(\bar{\Omega})$ be harmonic in Ω and flat at 0. Then $u \equiv 0$ in Ω if one of the following conditions holds:*

(a) *$u(x,0)$ does not change sign in V.*

(b) *$xu(x,0)$ does not change sign in V and $y \mapsto \dfrac{\partial u}{\partial x}(0,y)$ is flat at $y=0$.*

(c) *$xu(x,0)$ does not change sign in V and $y \mapsto v(0,y)$ is flat at $y=0$, where $v(x,y)$ is a harmonic conjugate of u near 0 in Ω.*

The following is an immediate consequence of Corollary 2.

COROLLARY 3. — *Assume $n=2$. Let h be a holomorphic function in Ω, flat at 0. If $u=\Re h$ is continuous in $\bar{\Omega}$ and the function $V \ni x \mapsto u(x,0)$ does not change sign, except perhaps at 0, then $h \equiv 0$.*

Theorem 1 was first proved for the case $P(x) \equiv 1$ by the authors in [5], both for open sets in the half space and also for open sets in the ball. It is likely that unique continuation results with weighted sign conditions similar to condition (1) of Theorem 1 can be found for more general open sets with analytic boundaries. Unique continuation for holomorphic functions with nonnegative real part on the boundary was given by the authors in [4], and was used to prove a generalized Schwarz reflection principle for holomorphic functions mapping the real line into a C^1 totally real manifold or into a real analytic set. Earlier results in this direction were obtained by the authors jointly with S. Alinhac [3], as well as by Bell-Lempert [6], Alexander [1], and Huang–Krantz [7]. We should also mention here more recent work of Alexander [2], in which the sign boundary condition for holomorphic functions is weakened in a different direction.

1. Some properties of the Poisson integral in a half space.

Let $f(x)$ be a continuous function with compact support in $\mathbb{R}^{n-1}$. We denote by $w(X)=\mathcal{P}f(X)$ the solution of the Dirichlet problem $\Delta w=0$, $w(x,0)=f(x), x\in\mathbb{R}^{n-1}$, given by the Poisson integral,

$$(1.1)\quad w(x,y)=\mathcal{P}f(x,y)=c_n\int_{\mathbb{R}^{n-1}}\frac{yf(x')}{(y^2+|x-x'|^2)^{n/2}}dx',\quad (x,y)\in\mathbb{R}^n_+,$$

where $c_n=\dfrac{\Gamma(n/2)}{\pi^{n/2}}$ (see e.g. [10]). We start with the following proposition:

PROPOSITION 1.2. — *Let $f\in C^0(\mathbb{R}^{n-1})$ with compact support and $w(X)$ given by (1.1). Assume that for every positive integer N, $|x|^{-N}f(x)\in L^1(\mathbb{R}^{n-1})$. Then for every $\alpha\in\mathbb{N}^n$, the function $y\mapsto\partial_X^\alpha w(0,y)$ is in $C^\infty(\overline{\mathbb{R}_+})$.*

Proof. — For $y>0$, we may differentiate (1.1) under the integral sign and then set $x=0$. Then $\partial_X^\alpha w(0,y)$ is a finite sum of terms of the form

$$\int_{\mathbb{R}^{n-1}} \frac{R(x', y) f(x')}{(y^2 + |x'|^2)^k} dx', \tag{1.3}$$

where $R(x', y)$ is a polynomial in n variables, and $k > 1$. Since $|x|^{-N} f(x) \in L^1(\mathbb{R}^{n-1})$ for every N, we may use the dominated convergence theorem to conclude that the function given by (1.3) is infinitely differentiable for $y \in \mathbb{R}$. □

We shall also need the following:

PROPOSITION 1.4. — *Let $\mathcal{P}_d(\mathbb{R}^n)$ be the space of all real valued polynomials in n variables homogeneous of degree d. Then the linear mapping $\Phi : \mathcal{P}_d(\mathbb{R}^n) \to \mathcal{P}_d(\mathbb{R}^n)$ given by*

$$P(X) = \Phi(Q)(X) = |X|^{n+2d} Q\Big(\frac{\partial}{\partial X}\Big)\Big(\frac{1}{|X|^n}\Big), \tag{1.5}$$

is a vector space isomorphism.

Proof. — It is easily checked that if $Q(X) \in \mathcal{P}_d(\mathbb{R}^n)$, then the right hand side of (1.5) is again a homogeneous polynomial of degree d. To prove the proposition, it suffices to show that the mapping Φ is injective. For this, assume that $Q \in \mathcal{P}_d(\mathbb{R}^n)$ satisfies

$$Q\Big(\frac{\partial}{\partial X}\Big)\Big(\frac{1}{|X|^n}\Big) \equiv 0, \quad X \in \mathbb{R}^n \backslash \{0\}. \tag{1.6}$$

Let T be the tempered distribution on $\mathbb{R}^n$ given by the finite part of $\dfrac{1}{|X|^n}$. Then its Fourier transform $\hat{T}$ is of the form $\hat{T} = c_1 \log|\xi| + c_2$, where c_1 and c_2 are constants. (See e.g. [9].) On the other hand, from (1.6) we obtain

$$Q\Big(\frac{\partial}{\partial X}\Big) T = S, \tag{1.7}$$

where S is a distribution in $\mathbb{R}^n$ supported at the origin. Taking the Fourier transform of (1.7) and using the form of $\hat{T}$ we obtain

$$Q(\xi)(c_1 \log|\xi| + c_2) = R(\xi), \tag{1.8}$$

with $R(\xi)$ a polynomial in n variables. Since $c_1 \neq 0$, (1.8) can hold only if $Q \equiv 0$. This completes the proof of Proposition 1.4. □

PROPOSITION 1.9. — *Let f and w be as in Proposition 1.2 and P and Q be homogeneous polynomials of degree d related by Proposition 1.4. Suppose that $P(x, y)$ is independent of y. Then for every integer $j \geq 0$ the following holds:*

$$\frac{1}{(2j)!} \partial_y^{2j} Q\Big(\frac{\partial}{\partial X}\Big)\Big(\frac{w(X)}{y}\Big)\Big|_{X=0} = (-1)^j c_n M_j \int_{\mathbb{R}^{n-1}} \frac{P(x) f(x)}{|x|^{n+2d+2j}} dx, \tag{1.10}$$

with $M_j = (-1)^j \dfrac{(n/2+d)(n/2+d+1)\dots(n/2+d+j-1)}{j!}$.

Proof. — We shall apply the operator $Q\Big(\dfrac{\partial}{\partial X}\Big)$ to both sides of (1.1) for $X \in \mathbb{R}^n_+$. By Proposition 1.4 and the fact that $P(x,y)$ is independent of y, we obtain

(1.11)

$$Q(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\Big(\frac{w(x,y)}{y}\Big) = c_n \int_{\mathbb{R}^n} \frac{P(x'-x)f(x')}{(y^2+|x-x'|^2)^{n/2+d}}dx', \quad (x,y)\in\mathbb{R}^n_+.$$

Define $v(y)$ by

$$v(y) = Q\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big)\Big(\frac{w(x,y)}{y}\Big)\Big|_{x=0}, \quad y>0. \tag{1.12}$$

By Proposition 1.2, $v(y)$ is smooth for $y>0$, up to $y=0$, hence admits a Taylor expansion at 0. We claim that for every positive integer N we have

$$v(y) = \sum_{j=0}^{N} a_j y^{2j} + O(|y|^{2N+2}), \tag{1.13}$$

where the a_j are given by

$$a_j = (-1)^j c_n M_j \int_{\mathbb{R}^n} \frac{P(x)f(x)}{|x|^{n+2d+2j}}dx. \tag{1.14}$$

Indeed, we have, for every integer $N>0$ and for $k>0$,

(1.15)

$$\begin{aligned}&\frac{1}{(y^2+|x|^2)^k}\\ &= \sum_{j=0}^{N}(-1)^j\frac{(k)(k+1)\dots(k+j-1)}{j!}\frac{y^{2j}}{|x|^{2j+2k}}\\ &+(-1)^{N+1}\frac{(k)(k+1)\dots(k+N)}{N!}\Big(\int_0^1\frac{(1-\tau)^N}{(\tau y^2+|x|^2)^{k+N}}d\tau\Big)y^{2N+2}.\end{aligned}$$

By substituting (1.15) into (1.11) with x replaced by $x-x'$ and with $k=n/2+d$, we obtain the desired expansion (1.13) for $v(y)$, by putting $x=0$, provided we show that the integral

$$\int_{\mathbb{R}^{n-1}}\int_0^1 \frac{(1-\tau)^N}{(\tau y^2+|x|^2)^{n/2+d+N+1}}P(x)f(x)d\tau dx \tag{1.16}$$

is bounded independently of $y \geq 0$. The latter follows from the fact that the integrand in (1.16) is dominated by the function $|P(x)f(x)|$ $|x|^{-(n+2d+2N+2)}$, which is integrable by the assumption on f. The proof of Proposition 1.9 is complete. □

2. Proofs of Theorem 1′ and its corollaries.

Let Ω and V be as in the introduction, and $u \in C^0(\bar{\Omega})$ and harmonic in Ω. Let $\chi \in C_0^\infty(V)$ with $\chi(x) \equiv 1$ near 0 and $\chi(x) \geq 0$, and put $f(x) = \chi(x)u(x,0)$. We denote by $w(X)$ the solution of the Dirichlet problem $\Delta w = 0$, $w(x,0) = f(x), x \in \mathbb{R}^{n-1}$, given by the Poisson integral (1.1). We write

$$u(X) = (u-w)(X) + w(X), \quad X \in \Omega. \tag{2.1}$$

Note first that the function $u - w$ is harmonic in Ω, continuous in $\bar{\Omega}$ and vanishes in a neighborhood of 0 in V, by the choice of the cut-off function χ. Hence by the classical local regularity of the Dirichlet problem in the real analytic setting, $u - w$ extends to be real analytic in a neighborhood of 0 in V. Using also Proposition 1.2, we may conclude the following:

PROPOSITION 2.2. — *Let $u \in C^0(\bar{\Omega})$ be harmonic in Ω, and assume that for every positive integer N , the function $|x|^{-N}u(x,0)$ is integrable in V. Then if $\eta > 0$ is sufficiently small, for every $\alpha \in \mathbb{N}^n$, the function $y \mapsto \partial_X^\alpha u(0,y)$ is in $C^\infty([0,\eta])$.*

Similarly, from Proposition 1.9 and (2.1), using again the real analyticity of $u - w$ near the origin, we have:

PROPOSITION 2.3. — *Let u be as in Proposition 2.2. If P and Q are two homogeneous polynomials in n variables related by Proposition 1.4 with P independent of y, there exists $C > 0$ such that for every integer $j \geq 0$ the following holds:*

(2.4)
$$\left|\frac{1}{(2j)!}\partial_y^{2j} Q\Big(\frac{\partial}{\partial X}\Big)\Big(\frac{u(X)}{y}\Big)\Big|_{X=0} - (-1)^j c_n M_j \int_V \frac{P(x)u(x,0)}{|x|^{n+2d+2j}}dx\right| \leq C^{j+1},$$

with M_j as in Proposition 1.9.

We may now complete the proof of Theorem 1. Let $P(x)$ be the polynomial (in $n-1$ variables) given in assumption (1) of Theorem 1. By regarding P as a polynomial in $X = (x,y)$, but independent of y, and applying Proposition 1.4, one has a unique polynomial $Q(x,y)$, homogeneous of degree d satisfying (1.5). Put

$$g(y) = Q\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big)\Big(\frac{u(x,y)}{y}\Big)\Big|_{x=0} \tag{2.5}$$

By Proposition 2.2, $g \in C^\infty([0,\eta])$ for $\eta > 0$ and sufficiently small. Since by assumption (3′) of Theorem 1′, $g(y)$ is flat at 0, all its derivatives must vanish at 0. Hence we conclude from Proposition 2.3, noting that $M_j \geq 1$,

$$c_n \int_V \frac{P(x)u(x,0)}{|x|^{n+2d+2j}} dx \leq C^{j+1}. \tag{2.6}$$

As in [5], we reason now by contradiction. If $u(x,0)$ does not vanish in any neighborhood of 0, then since $P(x)u(x,0) \geq 0$ by assumption (1) of Theorem 1, for every positive ϵ sufficiently small we would have

$$\int_{|x|<\epsilon} P(x)u(x,0)dx > 0. \tag{2.7}$$

On the other hand it follows from (2.6) that we have for every $\epsilon > 0$ sufficiently small

$$\frac{c_n}{\epsilon^{n+2d+2j}} \int_{|x|<\epsilon} P(x)u(x,0)dx \leq C^{j+1}, \quad j \geq 0. \tag{2.8}$$

Taking the jth root of both sides of (2.8), making use of (2.7), and letting j go to infinity we obtain $1 \leq C\epsilon^2$. Since ϵ can be taken arbitrarily small, we reach a contradiction, which proves the vanishing of $u(x,0)$ in a neighborhood of 0 in $\mathbb{R}^{n-1}$. The real analyticity of u in a neighborhood of $\mathbb{R}^n$ then follows from the classical real analyticity up to the boundary of the Dirichlet problem. This completes the proof of Theorem 1′. □

Proof of Theorem 1. — We note that Proposition 1.4 guarantees the existence of the polynomial $Q(X)$ of condition (3′) of Theorem 1′. It is then clear that (3′) implies (3); hence Theorem 1 follows from Theorem 1′. □

Proof of Corollary 1. — It is not hard to see that conditions (i) and (ii) of the corollary imply conditions (1), (2), (3) of Theorem 1. Therefore, by applying the theorem, we conclude that $u(x,0)$ vanishes in a neighborhood of 0 in V, and hence u extends to be real analytic in a neighborhood of 0. This, together with the flatness of u, i.e. condition (ii), implies that u must vanish identically in the connected set Ω. □

Proof of Corollary 2.

(a) Take $P(X) = Q(X) = \pm 1$ and apply Corollary 1 to u.

(b) Take $P(X) = \pm x$ and hence $Q(X) = \mp x/2$ in Theorem 1′ to conclude that $u(x,0) \equiv 0$ for x in a neighborhood of 0. The desired conclusion then follows from the flatness of u as in the proof of Corollary 1.

(c) Take $P(X) = \pm x$ and hence $Q(X) = \mp x/2$. In order to apply Theorem 1′ , we must show that $\dfrac{\partial u}{\partial x}(0,y)$ is flat at $y = 0$. If v is a harmonic conjugate of u, by the Cauchy-Riemann equations, we have $\frac{\partial u}{\partial x}(0,y) = \frac{\partial v}{\partial y}(0,y)$ for $y > 0$ sufficiently small. Since by Proposition 2.2, $\frac{\partial u}{\partial x}(0,y)$ is smooth up to $y = 0$, the same is true for $\frac{\partial v}{\partial y}(0,y)$. The assumption of flatness of v at 0 implies that $\frac{\partial v}{\partial y}(0,y)$ (and hence also $\frac{\partial u}{\partial x}(0,y)$) is also flat. The rest of the proof is as in (b).

Remark 2.9. — Corollary 3 follows from Corollary 2 parts (a) and (c). However, we note that if the condition $u = \Re h \in C^0(\bar{\Omega})$ is replaced by the stronger condition $h \in C^0(\bar{\Omega})$, then the result can be proved using only Corollary 2 part (a). Indeed under this stronger condition if $u(x,0)$ changes sign at 0, i.e. $xu(x,0)$ does not change sign, then we can apply part (a) to the harmonic function $\Re zh(z)$ (with $z = x + iy$).

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M.S. BAOUENDI and L.P. ROCHSCHILD,
Department of Mathematics-0112
University of California
SAN DIEGO, LA JOLLA CA 92093-0112 (USA).