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HARMONIC SYNTHESIS OF SOLUTIONS OF ELLIPTIC EQUATION WITH PERIODIC COEFFICIENTS

by Victor P. PALAMODOV

0. Introduction.

Let $p = p(x, D)$ be a $s \times t$-matrix, whose entries are linear differential operators on $\mathbf{R}^n$ with n-periodic coefficients, *i.e.* $p(x + q, D) = p(x, D)$ for any $q \in \mathbf{Z}^n$, where we denote $D = (i\partial/\partial x_1, \ldots, i\partial/\partial x_n)$, $i = \sqrt{-1}$. Assuming that p is an elliptic operator, we develop any solution of the system

$$p(x, D)u = 0, \tag{0.1}$$

which satisfies for some $a > 0$ the condition

$$u(x) = O(\exp(a|x|)), \quad |x| \to \infty, \tag{0.2}$$

in an integral over a variety of Floquet solutions. This development is similar to the exponential representation of solutions of (0.1) in the case of constant coefficients [1], [2]. A decomposition of this type was given by P. Kuchment [4] for the case $s = t$. Our approach gives a decomposition of solutions of (0.1) in a global integral over a series of holomorphic families $L_k = \{L_k(\lambda),\ \lambda \in N_k\}$, $k = 1, 2, \ldots$ of finite-dimensional representations $L_k(\lambda)$ of the translation group $\mathbf{Z}^n$. Here for each k the parameter λ runs

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over an irreducible analytic subset N_k of the variety Λ of all characters of the group in the space (0.2). Each representation $L_k(\lambda)$ consists of Floquet solutions of (0.1) with quasi-impulse λ and contains only one Bloch solution. It may be thought as a Jordan cell for the given representation of the group $\mathbf{Z}^n$ in the space of solutions of (0.1).

To get such a decomposition we use a *global Noether operator* for the characteristic sheaf of the system (0.1). Note that a similar decomposition obtained in [4] is more involved, since there only the local Noether operators [9] were used. We prove in § 3 that any coherent analytic sheaf on arbitrary Stein space admits a global Noether operator. In § 5 we state an analog of Malgrange's approximation theorem.

1. Main result.

Let $\mathbf{C}^n$ the complex dual to $\mathbf{R}^n$ and Z^n be the subgroup of integer vectors in $\mathbf{C}^n$. Then $\Lambda := \mathbf{C}^n/Z^n$ is the dual to $\mathbf{Z}^n$ complex Lie group and it is a Stein variety. There is a bilinear form $\Lambda \times \mathbf{Z}^n \to \mathbf{C}/\mathbf{Z}$, which is written as $\lambda \cdot q = \sum \zeta_j \cdot q_j$, where $\zeta := (\zeta_1, \dots, \zeta_n)$ is any pre-image of λ under the canonical surjection $\chi : \mathbf{C}^n \to \Lambda$. For any $\lambda \in \Lambda$ the function $q \rightsquigarrow \exp(2\pi i \lambda \cdot q)$ is a character of the group $\mathbf{Z}^n$. This group is represented in the space $D'(\mathbf{R}^n)$ by translation operators $T_q f(x) = f(x+q)$, $q \in \mathbf{Z}^n$. Choose an euclidean norm $|\cdot|$ in $\mathbf{R}^n$ and denote by $\|\cdot\|$ the dual hermitian norm on $\mathbf{C}^n$. For any positive a we denote by Λ_a the image in Λ of the strip $\|\operatorname{Im}\zeta\| < a/2\pi$, $\zeta \in \mathbf{C}^n$.

We assume that operator p is included in an elliptic differential complex :

$$(1.1) \qquad 0 \to L_0 \xrightarrow{p_0} L_1 \xrightarrow{p_1} L_2 \longrightarrow \cdots \xrightarrow{p_m} L_{m+1} \longrightarrow \cdots, \qquad p_0 = p,$$

where L_i, $i = 0, 1, \dots$ are sheaves of C^∞-sections of some finite-dimensional trivial bundles on $\mathbf{R}^n$ and $p_0, p_1, \dots$ are differential operators with n-periodic coefficients.

THEOREM 1.1. — *Any solution u of (0.1), which is defined on $\mathbf{R}^n$ and satisfies (0.2) for some $a > 0$, admits for any $b > a$ the following representation*

$$(1.2) \qquad u(x) = \sum_k \sum_{j=1}^{r(k)} \int_{N_k} f_{kj}(\lambda, x)\mu_{kj}(\lambda),$$

where

i) N_k, $k = 1, 2, \ldots$ *are closed irreducible analytic subsets of* Λ, *associated to the characteristic sheaf* M *of (0.1) (see § 3),*

ii) *for any* k, $f_{kj}(\lambda, x)$, $1, \ldots, r(k)$ *are smooth functions on* $N_k \times \mathbf{R}^n$, *which are holomorphic on* $\lambda \in N_k$ *and satisfy the equation (0.1) on* x;

iii) *for any* k, $\lambda \in N_k$ *the linear span* $L_k(\lambda)$ *of functions* $f_{kj}(\lambda, \cdot)$, $j = 1, \ldots, r(k)$ *is* $\mathbf{Z}^n$*-invariant and contains a unique invariant one-dimensional subspace; its character is equal to* $\exp(2\pi i \lambda \cdot x)$;

iv) μ_{kj} *are* C^∞*-densities on the set* $\operatorname{reg} N_k$ *of regular points of* N_k; $\operatorname{supp} \mu_{kj} \subset \operatorname{reg} N_k \cap \Lambda_b$. *Moreover for arbitrary proper closed analytic subsets* $\Omega_k \subset N_k$, $k = 1, \ldots$ *there exist densities* μ_{kj}, *which satisfies (1.2) such that* $\operatorname{supp} \mu_{kj} \subset N_k \cap \Lambda_b \setminus \Omega_k$ *for any* j *and* k.

Remark 1.2. — Inversely for any densities μ_{kj}, which fulfil iv), the second term of (1.2) is equal to $O(\exp(b|x|))$ at infinity and satisfies (0.1).

Remark 1.3. — The space E_p of solutions of (0.1), which satisfy (0.2), is generally an infinite-dimensional non-unitary representation of $\mathbf{Z}^n$. The equation (1.2) may be considered as a decomposition of E_p in an integral over the family of finite-dimensional subrepresentations $L_k(\lambda)$. But the densities μ_{kj} are far from being unique unlike the Stone-Naimark-Ambrose-Godement theorem for an unitary representation, where the spectral measure is unique.

Note that the representation $L_k(\lambda)$ is reducible, except for the case $\dim L_k(\lambda) = 1$, but is not decomposible, *i.e.* $L_k(\lambda)$ is not equal to a direct sum of some invariant subspaces.

Remark 1.4. — It follows from iii) that for any k there exists an integer d such that the identity

$$\left[T_q - \exp(2\pi i \lambda \cdot q)\right]^d f_{kj}(\lambda, \cdot) = 0, \quad \forall q \in \mathbf{Z}^n, \tag{1.3}$$

holds for any $\lambda \in N_k$ and $j = 1, \ldots r(k)$. This identity implies that for any k, j

$$f_{kj}(\lambda, x) = \sum_{|s|<d} x^s h_s(\lambda, x) \exp(2\pi i \lambda \cdot x),$$

where all the functions $h_s(\lambda, x)$ are n-periodic on x. Hence $f_{kj}(\lambda, x)$ is a *Floquet-solution* of (0.1) with quasi-impulse λ. Any generator, say f_{k1}, of the unique invariant one-dimensional subspace satisfies (1.3) with $d = 1$. It is called *Bloch-solution*.

2. Analytic lemmas.

Fix an integer k and consider the following family of elliptic complexes with n-periodic coefficients (*cf.* [4]) :

$$W_*^k : 0 \longleftarrow W_0^{k(0)} \overset{p_0'(\zeta)}{\longleftarrow} W_1^{k(1)} \overset{p_1'(\zeta)}{\longleftarrow} W_2^{k(2)} \longleftarrow \cdots \tag{2.1}$$

where W_i^k denotes the Sobolev space of sections of L_i on the torus T^n, which are square-summable with its derivatives up to k-th degree,

$$p_i'(\zeta) := {}^t p_i(x, D + 2\pi\zeta),$$

where ${}^t p$ means the formally adjoint operator to p, m_i is the order of p_i, $k(0) = 0$, $k(i) = k + m_0 + \cdots + m_{i-1}$, $i > 0$ and ζ runs over $\mathbf{C}^n$. This is a holomorphic family of Fredholm complexes since (1.1) is elliptic. Denote by $\mathcal{W}_i^k$ the sheaf of germs of holomorphic functions $f : \mathbf{C}^n \to \mathcal{W}_i^k$. Then (2.1) generates the following complex of analytic sheaves on $\mathbf{C}^n$

$$0 \longleftarrow \mathcal{W}_0^{k(0)} \overset{p_0'}{\longleftarrow} \mathcal{W}_1^{k(1)} \overset{p_1'}{\longleftarrow} \cdots \tag{2.2}$$

We define an action of the group $\mathbf{Z}^n$ on the sheaf $\mathcal{W}_i^k$ by the formula

$$T_\vartheta \psi(\zeta, x) = \exp(-2\pi i \vartheta \cdot x)\psi(\zeta + \vartheta, x), \quad \vartheta \in Z^n.$$

Let $\chi : \mathbf{C}^n \to \Lambda$ be the canonical projection; consider the sheaf $I_i^k|\Lambda$ of *invariant* sections of $\mathcal{W}_i^k$; a section of I_i^k on an open set $V \subset \Lambda$ is identified with a section φ of $\mathcal{W}_i^k$ on $\chi^{-1}(V)$ such that

$$\psi(\zeta + \vartheta, x) = \exp(2\pi i \vartheta \cdot x)\psi(\zeta, x), \forall \vartheta \in Z^n. \tag{2.3}$$

The following evident operator identity

$$p'(\zeta + \vartheta) = \exp(2\pi i \vartheta \cdot x)p'(\zeta)\exp(-2\pi i \vartheta \cdot x)$$

implies that (2.2) generates for any k a sheaf complex

$$I_*^k : 0 \longleftarrow I_0^{k(0)} \overset{p_0'}{\longleftarrow} I_1^{k(1)} \overset{p_1'}{\longleftarrow} I_2^{k(2)} \longleftarrow \cdots$$

Denote by $H_* = \sum H_i$ the homology of this complex. All H_i are coherent analytic sheaves on Λ, since (2.1) is a holomorphic Fredholm family (*cf.* [4]). This follows, for example, from [13, Lemma 4.3]. The embedding $I_*^{k+1} \to I_*^k$ induces for any k sheaf morphisms $h^k : H_*^{k+1} \to H_*^k$.

LEMMA 2.1. — *For any k, h^k is bijective.*

Proof. — Fix ζ and choose a parametrix r for the complex (2.1). This is a pseudodifferential operator in the graded space of (2.1) of degree

1 and of order $-m_i$, when acting on the term $W_i^{k+\cdots}$, which satisfies the equation

$$p(\zeta)r + rp(\zeta) = \mathrm{id} + q\,,$$

where q is a pseudodifferential operator of order -1 and of degree 0 as an endomorphism of the complex. It defines a morphism of complexes $q : \mathcal{W}_*^k \to \mathcal{W}_*^{k+1}$. This equation means that the compositions eq and qe are homotopic to the identity morphisms, where $e : \mathcal{W}_*^{k+1} \to \mathcal{W}_*^k$ is the natural embedding. This implies Lemma 2.1.

From now on we abbreviate the notation of H_i^k to H_i.

LEMMA 2.2. — *For any k and $a > 0$ there is a natural isomorphism*

$$H(\Gamma(\Lambda_a, I_*^k)) \cong \Gamma(\Lambda_a, H_*), \tag{2.4}$$

where $H(K)$ means the homology group of a complex K.

Proof. — Consider two spectral sequences for the functor $\Gamma(\Lambda_a, \cdot)$ and the complex I_*^k; both converge to the hyperhomology. For the first one we have $E^{pq} = H^p(H^q(\Lambda_a, I^k))$. This term vanishes for $q > 0$, since $H^q(\Lambda_a, I^k) = 0$, because I_*^k is a holomorphic Banach sheaf [5]. Hence the hyperhomology is isomorphic to the left-hand side of (2.4). For the second spectral sequence we find $E_2^{pq} = H^p(\Lambda_a, H_q)$. These groups vanish for $p > 0$ as well, since H_* is a coherent sheaf on a Stein space and $\Gamma(\Lambda_a, H_*) \cong E_2^{0*} \cong E_\infty$. This implies (2.4).

Now we pass in the spectrum $\Gamma_* := \Gamma(\Lambda_a, I_*^k)$ to the projective limit on k.

LEMMA 2.3. — *There is an isomorphism*

$$H(\Gamma(\Lambda_a, I_*)) \cong \Gamma(\Lambda_a, H_*), \quad I_* := I_*^\infty. \tag{2.5}$$

Proof. — A formal scheme is the same as in the previous lemma. We compare two standard spectral sequences for the hyperhomology of the functor Pr of projective limit and of the spectrum Γ_*. The term $E_2^{pq} = \mathrm{Pr}^p(H_q(\Gamma_*))$ vanishes for $p > 0$ since the spectrum $H_q(\Gamma_*)$ is constant in virtue of Lemma 2.2. The term $E_2^{0*} = E_\infty$ is equal to the right-hand side of (2.5).

For the second spectral sequence we have $E_2^{pq} = H_p(\mathrm{Pr}^q(\Gamma_*))$. These groups vanish for $q > 1$, since $\mathrm{Pr}^q = 0$. Evidently $\mathrm{Pr}^0(\Gamma_*) = \Gamma(\Lambda_a, I_*)$ hence E_2^{*0} coincides with the left-hand side of (2.5). Now we verify that

$\mathrm{Pr}^1(\{\Gamma(\Lambda_a, I_*^k)\} = 0$. For this we need to show that the embedding of Fréchet spaces $\Gamma(\Lambda_a, I_*^{k+1}) \to \Gamma(\Lambda_a, I_*^k)$ has a dense image ([6]). This density property is easy to check, if we develop an arbitrary section of I_*^k in Fourier series on x. Therefore the sequence E_2 degenerates to E_2^{*0}, which completes the proof of Lemma 2.3.

For an arbitrary positive b we consider the space S_b of C^∞-functions φ on $\mathbf{R}^n$, which satisfy the inequality

$$|D^i\varphi(x)| \leq C_{b',i} \exp(-b'|x|) \tag{2.6}$$

for any $i = (i_1, \dots, i_n)$ and $b' < b$. For given b' and i take the minimal constant $C_{b',i} = C_{b',i}(\varphi)$. For any $b' > 0$ the functional $C_{b',i}(\varphi)$ is a norm on S_b. This family of norms makes S_b a Fréchet space.

The cube $P = \{\xi \in \mathbf{R}^n,\ 0 \leq \xi_j < 1,\ j = 1, \dots, n\}$ is a fundamental domain for the group $\mathbf{Z}^n$.

LEMMA 2.4 (*cf.* [3]). — *The formula*

$$\varphi(x) = \int_P \exp(-2\pi i\, \xi \cdot x)\psi(\xi, x) d\xi, \tag{2.7}$$

defines for any $b > 0$ an operator $S : \Gamma(\Lambda_b, I) \to S_b$, which is a topological isomorphism, where the sheaf I corresponds to the trivial line bundle L. The inverse operator S^{-1} can be written as follows :

$$\psi(\xi, x) = \sum_{q \in \mathbf{Z}^n} \exp(2\pi i\, \xi \cdot (x + q))\varphi(x + q). \tag{2.8}$$

It follows that for any differential operator r with n-periodic coefficients there is a commutative diagram :

$$\begin{array}{ccc} S_b & \xrightarrow{{}^t r} & S_b \\ {\scriptstyle S}\uparrow & & {\scriptstyle S}\uparrow \\ \Gamma(\Lambda_b, I) & \xrightarrow{r'} & \Gamma(\Lambda_b, I) \end{array} \tag{2.9}$$

where the operator r', generated by the family $r'(\zeta) = {}^t r(x, D + 2\pi\zeta)$ as above.

Proof of Lemma 2.4. — The integral (2.7) is evidently a bounded function of x and moreover for arbitrary $\eta \in \mathbf{R}^n$, $\|\eta\| < b/2\pi$ we have

$$\varphi(x) = \int_{P+i\eta} \exp(-2\pi i\, \zeta \cdot x)\psi(\zeta, x) d\zeta,$$

because of Cauchy theorem and of (2.3). Hence $\varphi(x) = O(\exp(2\pi\eta \cdot x))$ for $x \to \infty$, which implies (2.6) for $i = 0$. The same conclusion is valid for any derivative of φ, hence φ is an element of S_b and the operator S is continuous.

For any function $\varphi \in S_b$ its inverse Fourier transform $\hat{\varphi}(\zeta)$ is a holomorphic function in the strip Λ_b, which decreases as fast as $O(|\zeta|^{-q})$, when $|\zeta| \to \infty$, for any q. Hence the inverse Fourier transformation may be written as follows :

$$\varphi(x) = \int_{\mathbf{R}^n} \exp(-2\pi i\,\xi \cdot x)\hat{\varphi}(\xi)d\xi = \sum_{\vartheta \in Z^n} \int_{P+\vartheta} \exp(-2\pi i\,\xi \cdot x)\hat{\varphi}(\xi)d\xi$$
$$= \int_P \exp(-2\pi i\,\xi \cdot x)\psi(\xi, x)d\xi,$$

where

(2.10) $$\psi(\xi, x) = \sum_{\vartheta \in Z^n} \exp(-2\pi i\vartheta \cdot x)\hat{\varphi}(\xi + \vartheta).$$

It is easy to see that $\psi \in \Gamma(\Lambda_b, I)$ and the operator $R : \varphi \rightsquigarrow \psi$ is continuous. This formula means that R is a right inverse to S. If we prove that S is injective, Lemma will follow. Suppose that $S\psi = 0$ for an element $\psi \in \Gamma(\Lambda_b, I)$ and develop ψ into a Fourier series on x :

$$\psi(\zeta, x) = \sum_{k \in Z^n} \exp(2\pi i k \cdot x)\psi_k(\zeta).$$

Condition (2.3) implies that $\psi_k(\zeta + \vartheta) = \psi_{k-\vartheta}(\zeta)$, hence $\psi_k(\zeta) = \psi_0(\zeta - k)$. Therefore

$$0 \equiv \int_P \exp(-2\pi i\,\xi \cdot x)\psi(\xi, x)d\xi = \sum_k \int_P \exp(-2\pi i(\xi - k) \cdot x)\psi_0(\xi - k)d\xi$$
$$= \int_{\mathbf{R}^n} \exp(-2\pi i\,\xi \cdot x)\psi_0(\xi)d\xi.$$

It follows that $\psi_0(x) \equiv 0$ and therefore $\psi = 0$, q.e.d.

To find out an inverse formula we start from (2.10) and change the integration variables y to $y + x$:

$$\psi(\xi, x) = \sum_{\vartheta \in Z^n} \exp(-2\pi i\vartheta \cdot x) \int \exp(2\pi i(\xi + \vartheta) \cdot y)\varphi(y)dy$$
$$= \sum_{\vartheta} \int \exp(2\pi i(\xi \cdot (y + x) + \vartheta \cdot y)\varphi(y + x)dy = \sum \hat{\varphi}_{x\xi}(\vartheta),$$

where $\varphi_{x\xi}(y) := \exp(2\pi i\xi \cdot (y + x))\varphi(y + x)$. The right-hand side is equal to

$$\sum_{q \in Z^n} \varphi_{x\xi}(q) = \sum_q \exp(2\pi i\,\xi \cdot (x + q))\varphi(x + q),$$

since of the Poisson summation formula. The proof is complete.

3. Noether operators.

Let A be a commutative algebra, M be a A-module. A prime ideal $\mathfrak{p}$ of A is called *associated to* M ([7], [8]), if there exists an element $m \in M$, whose annulet ideal coincides with $\mathfrak{p}$. M is called $\mathfrak{p}$-*coprimary*, if $\mathfrak{p}$ is the only ideal, associated to M; we denote it $\mathfrak{p}(M)$. The *Lasker-Noether decomposition* of M is a representation of the zero submodule of M in the form

$$0 = M_1 \cap \ldots \cap M_J, \tag{3.1}$$

where all $M/M_1, \ldots, M/M_J$ are coprimary A-modules. This decomposition is called *irreductible*, if no one of modules M_j in (3.1) can be omitted and all the prime ideals $\mathfrak{p}_j = \mathfrak{p}(M/M_j)$, $j = 1, \ldots, J$ are different. If A is a Noetherian algebra, any A-module M of finite type admits an irreducible Lasker-Noether decomposition. The set of prime ideals $\{\mathfrak{p}_j, \ldots, \mathfrak{p}_J\}$ is defined uniquely.

Let K be a field and A be a commutative K-algebra, M, N be A-modules. A K-linear mapping $\delta : M \to N$ is called a *differential operator of order* $\leq d$, if $(\operatorname{ad} b)^{d+1}\delta = 0$ for any $b \in A$, where $(\operatorname{ad} b)\gamma := \gamma b - b\gamma$.

DEFINITION 3.1 [9]. — *Let $\mathfrak{p}$ be an ideal associated to a n A-module M; we call $\nu : M \to [A/\mathfrak{p}]^r$ a $\mathfrak{p}$-Noether operator, if*

i) *ν is a differential operator in A-modules and $r < \infty$,*

ii) $\operatorname{Ker}\nu$ *is a submodule of M and $\mathfrak{p}$ is not associated to* $\operatorname{Ker}\nu$.

A Noether operator for a module M is a direct sum

$$\nu = \sum \nu_j : M \to \sum_j [A/\mathfrak{p}_j]^{r(j)},$$

where ν_j is a $\mathfrak{p}_j$-Noether operator, $j = 1, \ldots, J$ and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_J\} =$ $\operatorname{Ass}(M)$.

PROPOSITION 3.1. — *If A is noetherian, then any Noether operator is injective.*

Proof. — We have $\operatorname{Ker}\nu = \cap \operatorname{Ker}\nu_j$ and $\operatorname{Ass}(\operatorname{Ker}\nu) \subset \operatorname{Ass}(M)$, since $\operatorname{Ker}\nu$ is a submodule of M. No one of ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_J$ is associated to $\operatorname{Ker}\nu$, according to Definition 3.1 ii). Hence $\operatorname{Ass}(\operatorname{Ker}\nu)$ is empty. This means that $\operatorname{Ker}\nu = 0$, because of existence of Lasker-Noether decomposition.

Recall that *analytic algebra* A is a $\mathbf{C}$-algebra, which is isomorphic to a quotient of the algebra $\mathbf{C}\{z_1, \dots, z_n\}$ for some n. It is a Noetherian algebra.

THEOREM 3.2. — *Let A be an analytic algebra, M, K, L be A-modules of finite type, $\lambda : M \to L$, $\kappa : M \to K$ be A-differential operators such that* $\operatorname{Ker} \kappa$ *is an A-submodule, L is $\mathfrak{p}$-coprimary and λ vanishes on* $\operatorname{Ker} \kappa$. *Then there exist an element $s \in A \setminus \mathfrak{p}$ and a differential operator $\sigma : K \to L$ such that $s\lambda = \sigma\kappa$.*

To check this statement we choose a Noether operator $\nu : K \to N$ for K and apply the Unicity theorem of [9] to the composition $\nu\kappa : M \to N$.

Let $(X, O(X))$ be a complex analytic space, M, N are $O(X)$-sheaves; a sheaf morphism $\delta : M \to N$ is called a *differential operator*, if for any point $x \in X$ the fibre morphism $\delta_x : M_x \to N_x$ is a differential operator over algebra $O_x(X)$; $\operatorname{ord} \delta := \sup \operatorname{ord} \delta_x$.

Fix a point $x \in \mathbf{C}^n$ and consider the analytic algebra $A := O_x(\mathbf{C}^n)$ of germs at x of holomorphic functions. Let G be an irreducible germ at x of analytic set in $\mathbf{C}^n$. The ideal $I(G) \subset A$, consisting of function germs, which vanish on G, is prime; vice versa, any prime ideal in A is equal to $I(G)$ for some irreducible germ G. We call such a germ G *associated to* an A-module M, if so is the ideal $I(G)$. For example, for any analytic germ Y in $\mathbf{C}^n$ the germs $G_1, \dots, G_K$ associated to A-module $O(Y)$ are all the irreducible components of Y.

Now we pass to the global case and operate with closed irreducible analytic sets in X instead of germs. Recall that a closed analytic subset Y in a complex space X is *irreducible*, if there is no proper open and closed analytic subset $Z \subset Y$.

DEFINITION 3.2. — *Let $(X, O(X))$ be a complex space, M be a coherent analytic sheaf on X. We call an analytic subset $Y \subset X$ associated to M, if*

i) *Y is closed and irreducible,*

ii) *for any point $x \in Y$ its germ Y_x is an union of some irreducible germs $G_1, \dots, G_K$, $K > 0$ associated to the $O_x(X)$-module M_x.*

A collection of all analytic sets associated to M is denoted $\operatorname{Ass}(M)$. If X is a Stein space, for any point $x \in X$ any germ G associated to M_x is a germ of a set $Y \in \operatorname{Ass}(M)$. This fact is contained in the following theorem for semi-local situation and in [10] for the general case :

THEOREM 3.3 [9]. — *For any coherent sheaf M on a complex space X and any point $x \in X$ there exists a neighbourhood U such that the set* $\mathrm{Ass}(M|U)$ *is finite and for any $Y \in \mathrm{Ass}(M|U)$ there exists an $O(X)$-differential operator*

$$\nu_Y : M \to \sum O(Y)$$

where the direct sum is finite such that the operator

$$\nu := \prod \nu_Y : M \longrightarrow \prod_{\mathrm{Ass}(M)} \sum O(Y)$$

is a Noether operator for M_x for each $x \in U$.

DEFINITION 3.3. — *If Y is an analytic set associated to M, we call Y-Noether operator for M any differential operator $\nu : M \to \sum O(Y)$, where the direct sum is finite, such that for any point $x \in Y$ and any irreducible component G of Y_x the composition ρ_G is a G-Noether operator, where $\rho_G : \sum O(Y) \to \sum O(G)$ is the restriction morphism.*

In fact the operators ν_Y in Theorem 3.2 are Noetherian. Now we prove the following

THEOREM 3.4. — *For any Stein space X, arbitrary coherent analytic sheaf M on X and any set $Y \in \mathrm{Ass}(M)$ there exists an Y-Noether operator*

$$\nu_Y : M \to \sum O(Y),$$

which possesses the following property : there exists a holomorphic function $s \not\equiv 0$ on X such that for any element $a \in \Gamma(Y, O(Y))$ there is an $O(Y)$-endomorphism b of $\sum O(Y)$, which satisfies the equation

$$s(\operatorname{ad} a)\nu_Y = b\nu_Y. \tag{3.2}$$

LEMMA 3.5. — *Let M be a coherent analytic sheaf on a Stein space X, Y be an irreducible component of* $\operatorname{supp} M$ *and $\delta : M \to \sum O(Y)$ be an $O(X)$-differential operator. Suppose that there exists a point $y \in Y$ and an irreducible component W of Y_y such that the composition $\partial := \rho_W \delta_y$ is an W-Noether operator with the following property : for any element $a \in O_y(Y)$ there exists an $O_y(Y)$-endomorphism b of $\sum O_y(Y)$ such that*

$$\partial a = b\partial. \tag{3.3}$$

Then δ is a Y-Noether operator for M.

Proof of Lemma 3.5. — First we verify that the sheaf $K = \operatorname{Ker}\delta$ is an $O(X)$-subsheaf of M. For this we consider the sheaf $D(Y)$ of differential operators $e : M \to O(Y)$ of order $\leq \operatorname{ord}(\delta)$. It is a coherent sheaf ([9, Prop. 11.3]). Let $\delta_i : M \to O(Y)$, $i = 1, \dots, r$ be the components of the operator δ and I be the subsheaf of $D(Y)$, generated by $\delta_1, \dots, \delta_r$. Take an arbitrary holomorphic function a on X and consider the subsheaf A in $D(Y)$, generated by I and operators $\delta_i a$, $i = 1, \dots, r$, where a is considered as an endomorphism of M. The sheaf A/I is coherent and its support is contained in Y. The germ of $\operatorname{supp}(A/I)$ at y does not contain the germ W since of (3.3). Hence $\operatorname{supp}(A/I)$ is a proper analytic subset of Y. Choose arbitrary holomorphic function s, which belongs to the annulet ideal of A/I, but does not vanish identically on Y. All the operators $s\delta_i a$ are sections of the sheaf I, hence for any i and any point $x \in Y$

$$s\delta_i a = \sum b_j \delta_j$$

with some functions germs b_j at the point x. Therefore the equation $\delta(f) = 0$ for $f \in M_x$, $x \in Y$ implies that $s\delta(af) = 0$. This implies the equation $\delta(a'f) = 0$ for any $a' \in O_x(X)$, since functions $a \in \Gamma(X, O(X))$ are dense in $O_x(X)$ in $\mathfrak{m}_x$-adic topology and any differential operator is continuous with respect to this topology. Therefore K is $O(X)$-sheaf.

This sheaf is coherent in virtue of [9, Th. 2]. Hence $\operatorname{supp} K$ is a closed analytic subset of $\operatorname{supp} M$. We need only to check that $\operatorname{supp} K$ does not contain Y. Since Y is irreducible, it is sufficient to show that the germ of $\operatorname{supp} K$ at y does not contain W. We have $(\operatorname{supp} K)_y = \operatorname{supp} K_y \subset \operatorname{supp} M_y$. At the other hand $\operatorname{supp} K_y$ is the union of all germs V, associated to the $O_y(X)$-module K_y. It follows from the condition of Lemma that the germ W is not associated to the $O_y(X)$-module K_y. There is no other germ $V \supset W$ associated to $\operatorname{supp} K_y$, since W is an irreducible component of $\operatorname{supp} M_y$. Hence $\operatorname{supp} K_y$ does not contain the germ W, q.e.d.

Proof of Theorem 3.4. — We may assume that $X = \operatorname{supp} M$. Otherwise we shrink X to $\operatorname{supp} M$. Let X_j, $j \in J$ be the irreducible components of the space X (*cf.* [11], ch. V]). Each of them is a Stein space and the covering $X = \cup X_j$ is locally finite. Therefore it is sufficient to prove Theorem for each sheaf $M \otimes O(X_j)|X_j$, $j \in J$ and we may suppose that $\operatorname{supp} X$ is an irreducible Stein space.

Fix a point $x \in X$ and an irreducible component Y of the germ X_x. Since of Theorem 3.3 there exists a Y-Noether operator $\mu : M_x \to \sum O(Y)$.

Consider the coherent sheaf $D(X)$ of germs of differential operators $\delta : M \to O(X)$ of order $\le \operatorname{ord}(\mu)$. Let δ_i, $i = 1, \dots, r$ be its sections on X, which generate this sheaf at x. Consider the following operator

$$\delta : M \to [O(X)]^r\ ; \quad \delta(f) = (\delta_1(f), \dots, \delta_r(f)).$$

LEMMA 3.6. — *The composition $\delta_Y := \rho_Y \delta$ is an Y-Noether operator.*

Proof. — First we check that $\operatorname{Ker} \delta_Y$ is an O_x-submodule of M_x. Take arbitrary element $m \in \operatorname{Ker} \delta_x$ and function germ $a \in O_x$. Choose an embedding of the germ X_x in $(\mathbf{C}^n, 0)$ and a function b on the germ $(\mathbf{C}^n, 0)$ such that $\pi(b) = a$, where $\pi : O(\mathbf{C}^n) \to O_x(X)$ is the canonical surjection. Then we can write according to the Leibnitz formula (*cf.* [9, Prop. 3.1])

$$\delta_Y(am) = \sum (i!)^{-1} \pi(D_z^i b)(\operatorname{ad} z)^i \delta_Y(m)$$

where $z = (z_1, \dots, z_n)$ are coordinates on $\mathbf{C}^n$ and $(\operatorname{ad} z)^i$ means

$$(\operatorname{ad} z_1)^{i_1} \cdot \ldots \cdot (\operatorname{ad} z_n)^{i_n}.$$

We have $(\operatorname{ad} z)^i \delta_Y = r_Y (\operatorname{ad} z)^i \delta$ and $(\operatorname{ad} z)^i \delta : M \to \sum O(X)$ is a differential operator of order $\le \operatorname{ord}(\delta) \le \operatorname{ord}(\mu)$. Hence $(\operatorname{ad} z)^i \delta_Y(m) = 0$ for each i. This implies that $\delta_Y(am) = 0$ and our assertion follows.

We have the inclusion $\operatorname{Ker} \mu \supset \operatorname{Ker} \delta_Y$, which follows from the fact that any component of μ belongs to $O_x(X)$-envelope of the set $\{\delta_i,\ i = 1, \dots, r\}$. This inclusion implies that the germ Y is not associated to the $\operatorname{Ker} \delta_Y$, hence δ_Y is a Y-Noether operator for the module M_x.

Lemma 3.6 implies that δ is a X-Noether operator for M. Set $K := \operatorname{Ker} \delta$; this is a coherent subsheaf of M and for any point x, $O_x(X)$-module K_x has no associated germs Z, $\dim Z = \dim X$. Now we argue using the induction on the number $\dim \operatorname{supp} F$, hence may suppose that Theorem 3.4 is true for the sheaf K.

LEMMA 3.7. — *The equation* $\operatorname{Ass}(M) = \{X\} \cup \operatorname{Ass}(K)$ *holds.*

Proof. — One has the trivial inclusion $\operatorname{Ass}(K) \subset \operatorname{Ass}(M)$. For any point $x \in X$ any component of the germ X_x belongs to $\operatorname{Ass}(M_x)$, since δ is a X-Noether operator for M. Hence it remains to check that for any point $x \in X$ any germ $Y \in \operatorname{Ass}(M_x)$, which is not a component of the germ X_x, belongs to $\operatorname{Ass}(K_x)$. Choose an element $m \in M_x$, whose annulet ideal is equal to $I(Y)$. We claim that $m \in K_x$. In fact the equation $am = 0$ implies in view of Leibnitz formula that $a^k\delta(m) = 0$, for $k = \operatorname{ord}(\delta) + 1$. It follows that $\delta(m) = 0$, since a^k is not a zero-divisor in $O_x(X)$. This means that $m \in K_x$, which implies that $Y \in \operatorname{Ass}(K_x)$, q.e.d.

To prove Theorem 3.4 we choose for any $Y \in \operatorname{Ass}(M)$ a regular point $y \in Y$ and a Y_y-Noether operator ν_Y for M. Consider the sheaf $D(Y)$ of all differential operators $M \to O(Y)$ of order $\leq k := \operatorname{ord}(\nu_Y)$. It is a coherent sheaf on Stein space X. Choose a finite set $\{\varepsilon_1, \dots \varepsilon_r\} \subset \Gamma(X, D(Y))$, which $O_y(Y)$-envelope is equal to the stalk $D(Y)_y$. Consider the differential operator $\varepsilon(Y) : M \to \sum O(Y)$ with the components $\varepsilon_1, \dots, \varepsilon_r$, and the sheaf $G = \operatorname{Ker} \varepsilon(Y)$. The stalk G_y is an $O_y(Y)$-module of finite type and the set $\operatorname{Ass}(G_y)$ does not contain the germ Y_y. This can be proved by the arguments of Lemma 3.6. Hence the germ of $\varepsilon(Y)$ at y is a Y_y-Noether operator, satisfying (3.3). Lemma 3.5 implies that $\varepsilon(Y)$ is a Y-Noether operator for M.

To check the property (3.2) we choose an arbitrary function $a \in \Gamma(Y, O(Y))$ and for any $i = 1, \dots, r$ consider the operator $\varepsilon_i a : M \to O(Y)$. It belongs to the O_y-envelope of the operators $\varepsilon_1, \dots, \varepsilon_r$ since of the construction. It follows that there exists a function $s \in \Gamma(Y, O(Y))$ such that the operator $s\varepsilon_i a$ belongs to $\Gamma(Y, O(Y))$-envelope of $\varepsilon_1, \dots, \varepsilon_r$ (see proof of Lemma 3.5). Therefore the operator $s(\operatorname{ad} a)\varepsilon_i = s\varepsilon_i a - sa\varepsilon_i$ belongs to this envelope as well. This prove (3.2).

THEOREM 3.8. — *Let M be a coherent sheaf on a Stein space X, ν_N for each $N \in \operatorname{Ass}(M)$ be a N-Noether operator for M and $S(N)$ be an arbitrary proper closed analytic subset of N such that $\operatorname{sing} N \subset S(N)$. Then for any open set $U \subset X$ the linear operator*

$$\nu := \prod \nu_N : \Gamma(U, M) \longrightarrow \prod \{ \textstyle\sum \Gamma(U \cap N \setminus S(N), O(N)),\ N \in \operatorname{Ass}(M)\}$$

is an isomorphism onto its image, when the second space is endowed with the topology induced from the distribution spaces $D'(U \cap N \setminus S(N))$.

Proof. — Firstly we suppose that U is a closed subspace of the open unit polydisc Δ in a coordinate space $\mathbf{C}^n$ and there is a morphism $\alpha : K \to L$ of free coherent $O(\mathbf{C}^n)$-sheaves on Δ such that $\operatorname{Cok} \alpha \cong M$. This implies the following exact sequence

$$\Gamma(\Delta, K) \xrightarrow{\alpha} \Gamma(\Delta, L) \xrightarrow{\pi} \Gamma(\Delta, M) \longrightarrow 0,$$

where the canonical surjection π is an open operator. For any $N \in \operatorname{Ass}(M)$ the composition

$$\nu_N \pi : \Gamma(\Delta, L) \longrightarrow \prod_N \sum \Gamma(\Delta \cap N, O(N))$$

is a differential operator on Δ. It may be written in the following explicit form :

$$\nu_N \pi(u) = \sum a_i D^i u,$$

where $a_i \in \Gamma(\Delta, L')$, $L' := \mathrm{Hom}(L, \sum O(N))$ and $a_i = 0$ if $|i| > \deg \nu_N$ (*cf.* [9]). Each operator D^i acting on $\mathbf{C}^n$ is continuous since of the Cauchy inequality. Therefore $\nu_N \pi$ is continuous. The same is true for ν_N, because π is open.

In the general case the topology of $\Gamma(U, M)$ is the supremum of topologies, induced from the spaces $\Gamma(\Delta, \varphi_*(M|Y))$, where (Y, φ) runs over a set of analytic polyhedrons, which covers U, and $\varphi : Y \to \Delta$ is a closed embedding. Hence the general case is reduced to the case $X = \Delta$, $M \cong \mathrm{Cok}\,\alpha$. It is obvious that ν is still continuous in this case. To prove the openness of ν we use

LEMMA 3.9. — *For any point $z \in \Delta$ and its neighbourhood $U \subset\subset \Delta$ there exist a neighbourhood $V \subset U$ of z, a neighbourhood $W(N)$ of $S(N)$ and a constant C such that for any $f \in \Gamma(U, L)$ there exists a section $g \in \Gamma(V, K)$, which satisfies the inequality*

$$\begin{aligned} &\sup(|f + \alpha g|,\ z \in V) \\ &\le C \max\left\{ \sup(|\nu_N f(z)|,\ z \in U \cap N \setminus W(N)),\ N \in \mathrm{Ass}(M) \right\}. \end{aligned} \tag{3.4}$$

The maximum in the right-hand side is well-defined since $U \cup N$ is empty, except for a finite subset of $\mathrm{Ass}(M)$. Lemma 3.9 implies Theorem 3.8, since we can choose a polyhedral covering for X, consisting of neighbourhoods V, which satisfy (3.4) and the sup-norm in the right-hand side is majorized by the topology induced from the distribution space $D'(U \cap N \setminus S(N))$.

Proof of Lemma 3.9. — In fact it is proved in [1, ch. IV] for a special Noether operator λ, $\lambda_N : M \to \sum O(N)$. We have for any N

$$\lambda_N = s^{-1} \sum \sigma_{N\Lambda} \nu_\Lambda\,, \tag{3.5}$$

where according to Theorem 3.2 $\sigma_{N\Lambda} : \sum O(\Lambda) \to \sum O(N)$ is a differential operator in a neighbourhood of z and $\bar{s} \in O(X_z) \backslash I(N_z)$. Note that $\sigma_{N\Lambda} \neq 0$ only if $N \subset \Lambda$, since $\sigma_{N\Lambda}$ is a differential operator. Applying [1], we get the estimate

$$\begin{aligned} &\sup(|f + \alpha g|,\ z \in V) \\ &\le C \max\left\{ \sup(|\lambda_N f|,\ z \in U' \cap N \setminus W(N)),\ N \in \mathrm{Ass}(M) \right\} \end{aligned} \tag{3.6}$$

for a section g of the sheaf K, some neighbourhoods V, $U' \subset\subset U$ of z. We may assume that the set $S(N)$ contains $s^{-1}(0)$. Then we need to prove the inequality

$$\begin{aligned} (3.7) \qquad & \sup\left\{|\lambda_N f|,\ z \in U' \cap N \setminus W(N)\right\} \\ & \leq C' \max\left\{\sup(|\nu_\Lambda f|,\ z \in U \cap \Lambda \setminus W(\Lambda),\ \Lambda \supset N\right\}. \end{aligned}$$

Combining it with (3.6), we get (3.4). To prove (3.7) we use (3.5), the inequality $|s^{-1}| \leq$ const on the set $U \cap N \setminus W(N)$ and the estimate

$$\sup\left\{|\sigma_{N\Lambda} h|,\ z \in U' \cap N \setminus W(N)\right\} \leq C \sup\{|h|,\ z \in U \cap \Lambda \setminus W(\Lambda)\}$$

for any holomorphic function h on $U \cap \Lambda$. To check this estimate we apply the Cauchy inequality. Lemma 3.9 and hence Theorem 3.8 are proved.

4. End of the proof of Theorem 1.1.

Now we apply Theorem 3.4 to the sheaf $M := H_0 \equiv \operatorname{Cok} p_0'$, denoting $N_1, N_2, \ldots$ all the elements of $\operatorname{Ass}(M)$. Thus for any k there exists a N_k-Noether operator

$$\nu_k : \Gamma(\Lambda, M) \longrightarrow \sum \Gamma(N_k, O(N_k)),$$

possessing the property (3.2). Moreover Theorem 3.8 implies that the continuous operator

$$\nu = \prod \nu_k : \Gamma(\Lambda_b, M) \longrightarrow \prod_k \sum \Gamma(\Lambda_b \cap N_k, O(N_k))$$

is an open mapping onto its image, when the first space is equipped with the canonical topology and the second one is endowed with the topology induced from $\prod \sum D'(\Lambda_b \cap N_k \setminus \Omega_k)$, where $D'(\cdot)$ means the space of distributions and Ω_k is any proper closed analytic subset of N_k such that $\operatorname{sing} N_k \subset \Omega_k$. We may assume that for any k this set satisfies the condition : $s \neq 0$ on $N_k \setminus \Omega_k$, where s is holomorphic function, which appears in (3.2). Combining this mapping with the morphism $\pi : \Gamma(\Lambda_a, I_0) \to \Gamma(\Lambda_a, M)$, we get for any $a > 0$ the complex

$$(4.1) \qquad \Gamma(\Lambda_b, I_1) \xrightarrow{p_0'} \Gamma(\Lambda_b, I_0) \xrightarrow{\nu\pi} \prod_k \sum \Gamma(\Lambda_b \cap N_k, O(N_k)).$$

It is exact, since $\operatorname{Ker} \pi = \operatorname{Im} p_0'$, because of Lemma 2.3 and of Proposition 3.1. The composition $\nu\pi$ is an open operator onto its image, since π is open by the definition of the topology of $\Gamma(\Lambda_a, M)$ and ν is open, because of the aforesaid.

Now take an arbitrary solution u of (0.1), which satisfies (0.2). It may be considered as a functional on S_b for arbitrary $b > a$. This functional vanishes on $\operatorname{Im} {}^t p$. Let S^* be the adjoint to the operator S (see Lemma 2.4). Then $S^*(u)$ is a continuous functional on $\Gamma(\Lambda_b, I_0)$, which vanishes on the subspace $\operatorname{Im} p'_0 = \operatorname{Ker} \nu\pi$, because of (2.9). Consider the operator

$$\rho : \Gamma(\Lambda_b, I_0)/\operatorname{Ker}\nu\pi \longrightarrow \operatorname{Im}\nu\pi,$$

generated by $\nu\pi$. It is a topological isomorphism, since $\nu\pi$ is open, hence we may consider a continuous functional $v := (\rho^{-1})^* S^*(u)$ on $\operatorname{Im}\nu\pi$. Applying Hahn-Banach theorem, we take a continuous extension w of v to the space $\prod \sum \Gamma(\Lambda_b \cap N_k, O(N_k))$. It can be written as a finite sum

$$w = \sum_k \sum_{j=1}^{r(k)} w_{kj},$$

where w_{kj} is a continuous functional on $\Gamma(\Lambda_b \cap N_k, O(N_k))$. Then we use Hahn-Banach theorem once more to extend w_{kj} to a continuous functional $\tilde{w}_{kj}$ on the space $D'(\Lambda_b \cap N_k \setminus \Omega_k)$ and write it as an integral

$$\tilde{w}_{kj}(f) = \int f \mu_{kj}$$

with a smooth density μ_{kj} such that $\operatorname{supp}\mu_{kj} \subset \Lambda_b \cap N_k \setminus \Omega_k$. Hence

$$u(\varphi) = v(\psi) = w(\nu\pi(\psi)) = \sum_{k,j} \int \nu_{kj}\pi(\psi)\mu_{kj}, \tag{4.2}$$

where $\varphi \in S_b$, $\psi := S^{-1}(\varphi)$ and

$$\nu_{kj} : \Gamma(\Lambda, M) \longrightarrow \Gamma(N_k, O(N_k)), \quad j = 1, \dots, r(k)$$

are components of ν_k. The equality (4.2) coincides with (1.2) if we set

$$f_{kj}(\lambda, \varphi) := \delta_\lambda \delta_{kj}(\psi), \quad \delta_{kj} := \nu_{kj}\pi : \Gamma(\Lambda, I_0) \longrightarrow \Gamma(N_k, O(N_k)),$$

where $f_{kj}(\lambda, \varphi)$ means the value of the distribution $f_{kj}(\lambda, \cdot)$ on a test function φ and δ_λ denotes the delta-distribution supported by the point $\lambda \in N_k$. The distribution $f_{kj}(\lambda, \cdot)$ satisfies (0.1) since it may be written in the form (4.2) with $\mu_{kj} := \delta_\lambda$. This a smooth function on $x \in \mathbf{R}^n$, since the equation (0.1) is elliptic. This solution is weakly holomorphic on ζ and therefore is a smooth function on $N_k \times \mathbf{R}^n$. This implies ii). Properties i) and iv) were proved earlier.

To check iii) we choose arbitrary $q \in \mathbf{Z}^n$ and compute for arbitrary k and j

$$T_q f_{kj}(\lambda, \varphi) = f_{kj}(\lambda, T_{-q}(\varphi)) = \delta_\lambda \delta_{kj}(e_q \psi), \quad e_q(\lambda) := \exp(2\pi i \lambda \cdot q),$$

$$\delta_{kj}(e_q\psi) = e_q\delta_{kj}(\psi) + (\operatorname{ad} e_q)\delta_{kj}(\psi).$$

The operator $\gamma := s(\operatorname{ad} e_q)\delta_{kj}$ belongs to the linear span of operators δ_{ki}, $i = 1, \dots, r(k)$ over algebra $\Gamma(\Lambda, O(\Lambda))$ since of (3.2) and $\operatorname{ord}\gamma < \operatorname{ord}\delta_{kj}$. We have

$$s(\lambda)\delta_\lambda\delta_{kj}(e_q \cdot \psi) = s(\lambda)e_q(\lambda)\delta_\lambda\delta_{kj}(\psi) + \delta_\lambda\gamma(\psi),$$

therefore for any $\lambda \in N_k \setminus \Omega_k$ whe have

$$T_q f_{kj}(\lambda, \varphi) = e_q(\lambda) f_{kj}(\lambda, \varphi) + s(\lambda)^{-1}\delta_\lambda\gamma(\psi),$$

hence

$$\big[T_q - \exp(2\pi i\lambda \cdot q)\big] f_{kj}(\lambda, \cdot) = g(\cdot),$$

where g is an element of the linear span of functions $f_{ki}(\lambda, \cdot)$, $i = 1, \dots, r(k)$. Applying this computation to g and so on, we come to (1.3), which implies iii). The proof is complete.

5. Approximation.

THEOREM 5.1. — *Suppose that a set $\Phi \subset \Lambda$ has a non-empty intersection with N_k for each k. Then the set of Floquet solutions of (0.1) with quasi-impulses $\lambda \in \Phi$ is total in the space of all solutions, which satisfies (0.2) for some $a > 0$.*

Remark. — The similar result for differential equations with constant coefficients is due to Malgrange [12].

Proof. — The statement is equivalent to the following : for $\varphi \in \Gamma(\Lambda_a, I_0)$ the system of equations

$$\gamma_\lambda(\varphi) = 0, \qquad \lambda \in \Phi, \tag{5.1}$$

implies that $\varphi \in \operatorname{Im} p_0'$, if γ_λ runs over the set of linear functionals over $\Gamma(\Lambda, I_0)$ supported at λ, which vanish on the image of the operator p_0' in (4.1). To prove this implication we note that for any k, any $\lambda \in \Phi \cap N_k$ and any functional δ over $\sum \Gamma(N_k, O(N_k))$ supported at λ, the functional $\gamma_\lambda := \delta\nu_k\pi$ vanishes on $\operatorname{Im} p_0'$. Hence the system (5.1) implies the equation $\delta(\nu_k\pi\varphi) = 0$ for any δ. This means that the image of $\nu_k\pi(\varphi)$ in $\sum \widehat{O}_\lambda(N_k)$ vanishes, where the symbol $\widehat{\ }$ denote the completion in $\mathfrak{m}_\lambda$-adic topology. If follows that the germ at λ of the function $\nu_k\pi(\varphi)$ is equal to zero, since the canonical mapping $F_\lambda \to \widehat{F}_\lambda$ is an injection for any coherent sheaf F.

This implies the equalities $\nu_k\pi(\varphi) = 0$, $k = 1, \dots$, since N_k is irreducible for any k. Therefore $\varphi \in \operatorname{Im} p_0'$ because (4.1) is exact and Theorem 5.1 follows.

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