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BLOCK-DISTRIBUTION IN RANDOM STRINGS

by **Peter J. GRABNER**

1. Introduction.

We investigate some properties of infinite sequences of independent random variables, which take the values 0 and 1 with probabilities p and q respectively (Bernoulli's scheme). It is one of the basic results of probability theory that the limit relation

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N-k : x_n x_{n+1} \ldots x_{n+k} = a_1 \ldots a_k\}}{N} = \mu_k(A)$$

holds in probability for all blocks $A = a_1 \ldots a_k$ of a given constant length k ($\mu_k(A)$ is the k-fold product measure generated by $\mu(\{0\}) = p$ and $\mu(\{1\}) = q$). This result can also be naturally imbedded into ergodic theory : consider the infinite product space $X = \{0,1\}^{\mathbb{N}}$ equipped with the infinite product measure μ_∞ generated by μ. Then the shift operator S (Bernoulli shift) on X defined by $S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ is an ergodic transformation on X (cf. e.g. [Wa]) and the above relation is a consequence of Birkhoff's ergodic theorem.

It is now natural to ask how fast (depending on N) k could grow such that this relation persists. In order to answer this question we introduce a

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special notion of discrepancy (*cf.* [Hl], [KN]) :

$$D_N^k(x_1,\dots,x_N) = \max_{A\in\{0,1\}^k} \frac{1}{\sqrt{p^k\mu_k(A)}}\left|\frac{\#\{1\le n\le N-k : x_nx_{n+1}\dots x_{n+k}=a_1\dots a_k\}}{N}-\mu_k(A)\right|. \tag{1.1}$$

The following calculations will show that this is a proper measure for the distribution behaviour of the sequence $x_1, x_2, \dots$. Note that this definition agrees with the definition in [FKT] for $p = q = \frac{1}{2}$.

DEFINITION. — *A sequence $x_1, x_2, \dots$ is called $k(N)$-distributed with respect to μ if*

$$\lim_{N\to\infty} D_N^{k(N)}(x_1,\dots,x_N) = 0.$$

Our Theorem will show under which conditions almost all sequences are $k(N)$-distributed. Without loss of generality assume that $p \le q$. The notation $\operatorname{lp} n$ is the logarithm to base $\frac{1}{p}$: $\operatorname{lp} n = \log_{\frac{1}{p}} n$.

THEOREM. — *Let $k(N)$ be a non-decreasing sequence of positive integers. Then the following 0-1-law holds*

$$\mu_\infty\left(\lim_{N\to\infty} D_N^{k(N)}(x_1,\dots,x_N) = 0\right) = \begin{cases} 1 & \textit{if } \operatorname{lp} n - \operatorname{lp}\operatorname{lp} n - k(n) \to \infty \\ 0 & \textit{otherwise.}\end{cases}$$

It clearly follows from Kolmogoroff's 0-1-law or the fact that the set

$$\left\{\lim_{N\to\infty} D_N^{k(N)}(x_1,\dots,x_N) = 0\right\}$$

is invariant under the (ergodic) shift S, that the only possible values for the above probability are 0 and 1. The proof of this theorem will use bivariate correlation polynomials, which are a generalization of Guibas' and Odlyzko's correlation polynomials in one variable (cf. [GO]). Using these polynomials we are able to compute the probability generating functions of the events we are interested in.

2. Generating Functions.

Throughout this section let $A = a_1a_2\dots a_k$ be a 0-1-string of length k. We are interested in the cardinalities of the following subsets of the set

$\mathcal{S}_{r,s}$ of strings containing r digits 0 and s digits 1 :
(2.1)

$f_A(r,s) = \#\{B \in \mathcal{S}_{r,s} : B \text{ contains } A \text{ only at the end}\}$

$g_A(r,s) = \#\{B \in \mathcal{S}_{r,s} : B \text{ contains } A \text{ only at the beginning and at the end}\}$

$h_A(r,s,) = \#\{B \in \mathcal{S}_{r,s} : B \text{ does not contain } A\}$.

In order to compute the generating functions of these quantities we introduce the bivariate autocorrelation polynomial $[AA](z,w)$:

$$[z^r w^s][AA](z,w) = \begin{cases} 1 & \text{if } a_1 a_2 \dots a_{k-r-s} = a_{r+s+1} a_{r+s+2} \dots a_k \text{ and the} \\ & \text{string } a_1 a_2 \dots a_{r+s} \text{ contains } r \text{ digits 0 and } s \\ & \text{digits 1} \\ 0 & \text{otherwise,} \end{cases}$$

where $[z^r w^s]P(z,w)$ as usual denotes the coefficient of $z^r w^s$ in $P(z,w)$. We are now ready to formulate

PROPOSITION 1. — *The generating functions of the combinatorial expressions (2.1) are given by*

$$F_A(z,w) = \sum_{r,s=0}^{\infty} f_A(r,s) z^r w^s = \frac{z^{0(A)} w^{1(A)}}{z^{0(A)} w^{1(A)} + (1-z-w)[AA](z,w)}$$

$$G_A(z,w) = z^{0(A)} w^{1(A)} + \frac{(z+w-1) z^{0(A)} w^{1(A)}}{z^{0(A)} w^{1(A)} + (1-z-w)[AA](z,w)}$$

$$H_A(z,w) = \frac{[AA](z,w)}{z^{0(A)} w^{1(A)} + (1-z-w)[AA](z,w)},$$

where $0(A)$ and $1(A)$ denote the number of 0's and 1's in A respectively.

The proof of this proposition is analogous to the proof of the corresponding results for ordinary generating functions (*cf.* [GO]).

Remark 1. — Obviously these results can be generalized to any finite alphabet.

As in [FKT] we use these functions to compute the probability generating function (p.g.f.) of all strings containing the substring A exactly r times :

$$\Phi_A^{(r)}(z) = \frac{z^{-kr}}{\mu_k(A)} F_A(pz,qz)^2 G_A(pz,qz)^{r-1} \text{ for } r \geq 1$$

$$\Phi_A^{(0)}(z) = H_A(pz,qz).$$

Inserting the results of Proposition 1 and setting

$$P(z) = \frac{1}{\mu_k(A)}[AA](pz,qz) \tag{2.2}$$

yields

$$\Phi_A^{(r)}(z) = \frac{z^k}{\mu_k(A)} \frac{\left((1-z)(P(z)-\frac{1}{\mu_k(A)})+z^k\right)^{r-1}}{((1-z)P(z)+z^k)^{r+1}}$$
$$\Phi_A^{(0)}(z) = \frac{P(z)}{(1-z)P(z)+z^k}.$$

3. Proof of the Theorem.

We split the proof into two parts; first we show that almost all sequences are $k(N)$ distributed if $\operatorname{lp} n - \operatorname{lp}\operatorname{lp} n - k(n) \to \infty$. Using our p.g.f. results we can write

(3.1)
$$\mu_\infty\left(\#\{0 \le n \le N-k : x_{n+1}\dots x_{n+k} = a_1 \dots a_k\} = r\right) = p_A^{(r)}(N) =$$
$$[z^N]\Phi_A^{(r)}(z) = \frac{1}{2\pi i}\oint_C \Phi_A^{(r)}(z)\frac{dz}{z^{N+1}}.$$

In order to be able to estimate the integral we need information on the the zeros of the polynomial $(1-z)P(z)+z^k$.

LEMMA 1. — *The zero of smallest modulus z_0 of $(1-z)P(z)+z^k$ is real and positive and satisfies the estimate*

$$z_0 > 1 + C\mu_k(A)$$

for a positive constant C only depending on p.

Proof. — As $F_A(pz, qz)$ is a p.g.f. and $(1-z)P(z)+z^k$ is the denominator of this rational function the zero of smallest modulus has to be positive and ≥ 1. Investigation of the derivative shows the existence of the constant C.

Let now

$$k(n) = \operatorname{lp} n - \operatorname{lp}\operatorname{lp} n - \operatorname{lp}\psi(n),$$

where $\psi(n) \to \infty$. We need estimates for the probability that the number of occurrences $Z_N(A)$ of a block A deviates too far from the mean value :

$$\begin{aligned} L_N(\delta_A) &= \mu_\infty(Z_N(A) < N\mu_k(A)(1-\delta_A)) \quad \text{and} \\ U_N(\delta_A) &= \mu_\infty(Z_N(A) > N\mu_k(A)(1+\delta_A)). \end{aligned} \tag{3.2}$$

These probabilities are sums of the $p_A^{(r)}(N)$ defined in (3.1) :

$$\begin{aligned} L_N(\delta_A) &= \sum_{r<N\mu_k(A)(1-\delta_A)} p_A^{(r)}(N) \quad \text{and} \\ U_N(\delta_A) &= \sum_{r>N\mu_k(A)(1+\delta_A)} p_A^{(r)}(N). \end{aligned} \tag{3.3}$$

We will use the integral representation (3.1) to estimate these quantities.

For convenience we now introduce some notations

$$\begin{aligned} &Q(z) = (1-z)P(z) + z^k \\ &a(z) = \frac{z^k}{Q(z)^2}, \quad b(z) = 1 + \frac{z-1}{\mu_k(A)Q(z)}. \end{aligned} \tag{3.4}$$

This gives

$$\Phi_A^{(r)}(z) = \frac{1}{\mu_k(A)} a(z) b(z)^{r-1}$$

for $r \geq 1$. Observe further that

$$\begin{aligned} a(1\pm\varepsilon) &= 1 + O\left(\frac{1}{\mu_k(A)}\varepsilon\right) \\ b(1\pm\varepsilon) &= 1 \pm \frac{\varepsilon}{\mu_k(A)} + O\left(\frac{\varepsilon^2}{\mu_k(A)^2}\right) \\ b^j(1\pm\varepsilon) &= \exp\left(\pm\frac{\varepsilon j}{\mu_k(A)} + O\left(\frac{\varepsilon^2 j}{\mu_k(A)^2}\right)\right) \\ (1\pm\varepsilon)^{-n} &= \exp\left(\mp n\varepsilon + O(n\varepsilon^2)\right). \end{aligned} \tag{3.5}$$

We can now write

$$U_N(\delta_A) = \frac{1}{2\pi i}\oint_C \frac{1}{\mu_k(A)} a(z) \frac{b^j(z)}{1-b(z)} \frac{dz}{z^{N+1}},$$

where $j = [N\mu_k(A)(1+\delta_A)]$. As all the power series involved have positive coefficients and because of Lemma 1 we can estimate

$$U_N(\delta_A) \leq \frac{1}{\mu_k(A)} a(1-\varepsilon) \frac{b^j(1-\varepsilon)}{1-b(1-\varepsilon)} (1-\varepsilon)^{-N}$$

for every positive $\varepsilon < C\mu_k(A)$. Using (3.5) yields

$$U_N(\delta_A) \leq \frac{1}{\varepsilon} \frac{1+O\left(\frac{\varepsilon}{\mu_k(A)}\right)}{1+O\left(\frac{\varepsilon}{\mu_k(A)}\right)} \exp\left(\left(N - \frac{j}{\mu_k(A)}\right)\varepsilon + O\left(\frac{\varepsilon^2 j}{\mu_k(A)^2}\right) + O(N\varepsilon^2)\right).$$

Inserting $\varepsilon = \left(\mu_k(A)\frac{\operatorname{lp} N}{N}\right)^{\frac{1}{2}}$ into the above inequality yields

$$U_N(\delta_A) \leq \exp(-\delta_A \left(N\mu_k(A) \operatorname{lp} N\right)^{\frac{1}{2}} + C_1 \log N). \tag{3.6}$$

In the same way we treat the lower tail. Let now $j = [N\mu_k(A)(1-\delta_A)]$. Thus we obtain

$$L_N(\delta_A) = \frac{1}{2\pi i}\oint_C \left(\frac{P(z)}{Q(z)} + \frac{a(z)}{\mu_k(A)}\frac{b^j(z)-1}{b(z)-1}\right)\frac{dz}{z^{N+1}}.$$

We can now estimate

$$L_N(\delta_A) \le \frac{P(1+\varepsilon)}{Q(1+\varepsilon)}(1+\varepsilon)^{-N} + \frac{1}{\mu_k(A)} j b^j(1+\varepsilon)a(1+\varepsilon)(1+\varepsilon)^{-N}.$$

Using the same value for ε as above yields

$$L_N(\delta_A) \le \exp\left(-\delta_A\,(N\mu_k(A)\operatorname{lp} N)^{\frac{1}{2}} + C_2\log N\right). \tag{3.7}$$

Combining this with (3.6) yields

$$\mu_\infty\left(\left|\frac{Z_N(A)}{N} - \mu_k(A)\right| > \delta_A\mu_k(A)\right) \le \exp(-\delta_A\,(N\mu_k(A)\operatorname{lp} N)^{\frac{1}{2}} + C_3\log N). \tag{3.8}$$

Let now $\delta_A = \delta\left(\frac{p^k}{\mu_k(A)}\right)^{\frac{1}{2}}$ and observe that $p^k = \frac{\operatorname{lp} N}{N}\psi(N)$. Therefore we have

$$\begin{aligned}\mu_\infty(D_N^{k(N)}(\omega) > \delta) &\le 2^{k(N)}\exp(-\delta\psi(N)^{\frac{1}{2}}\operatorname{lp} N + C_3\log N)\\ &\le \exp(-\delta\psi(N)^{\frac{1}{2}}\operatorname{lp} N + C'\log N).\end{aligned} \tag{3.9}$$

We now choose δ as a function of N

$$\delta = \psi(N)^{-\frac{1}{4}}$$

and observe that

$$\sum_{N=1}^{\infty}\exp(-\psi(N)^{\frac{1}{4}}\operatorname{lp} N + C'\log N) < \infty.$$

Thus by the Borel-Cantelli lemma (*cf.* [Fe]), we obtain the first part of our Theorem.

We now have to prove that almost no series are $k(N)$-distributed if $\operatorname{lp} n - \operatorname{lp}\operatorname{lp} n - k(n) \not\to \infty$ (we confine ourselves to the case $p < \frac{1}{2}$, because the case $p = \frac{1}{2}$ has been treated by Grill [Gr]). We introduce a set $\mathcal{A}$ of strings of length k, which have only trivial autocorrelation and do not overlap each other :

$$\mathcal{A} = \left\{\underbrace{0\ldots0}_{l}\ \underbrace{A}_{l+d(k)-2}\ \underbrace{1\ldots1}_{l}\right\},$$

where $l = \left\{\frac{k}{3}\right\}+1$ and $d(k) = k \bmod 3$. We need the p.g.f. $\varphi(z)$ of all strings not containing an element of $\mathcal{A}$. This function satisfies the equations

$$\varphi(z) + \varphi_{A_1}(z) + \ldots + \varphi_{A_m}(z) = z\varphi(z) + 1$$
$$\varphi_{A_1}(z) = z^k \mu_k(A_1)\varphi(z)$$
$$\ldots$$
$$\varphi_{A_m}(z) = z^k \mu_k(A_m)\varphi(z),$$

where $A_1, \ldots, A_m$ are the elements of $\mathcal{A}$ and $\phi_{A_l}(z)(l = 1, \ldots, m)$ is the p.g.f. of the blocks ending with A_l but containing no further occurrence of any element of $\mathcal{A}$. Solving these equations yields

$$\varphi(z)\frac{1}{1 - z + \mu_k(\mathcal{A})z^k}. \tag{3.10}$$

Note that the simplicity of these equations comes from the trivial overlap structure of the elements of $\mathcal{A}$.

Because of this simple overlap structure it is easy to see that

(3.11)

$$\phi_{j_1\ldots j_m}(z) = \frac{(j_1 + \ldots + j_m)!}{j_1! \cdots j_m!}\mu_k(A_1)^{j_1} \cdots \mu_k(A_m)^{j_m} z^{k(j_1+\cdots+j_m)}\varphi(z)^{j_1+\cdots+j_m+1}$$

is the p.g.f. of all blocks containing A_l exactly j_l times $(l = 1 \ldots m)$. As in the first part of the proof we use

$$\begin{aligned} M_N(\delta) &= \mu_\infty\left(|Z_N(A_l) - N\mu_k(A_l)| \le N\mu_k(A_l)\delta_{A_l}, l = 1 \ldots m\right) \\ &= \frac{1}{2\pi i}\oint_C \sum_{\substack{|j_l - N\mu_k(A_l)| \le N\mu_k(A_l)\delta_{A_l} \\ l=1,\ldots,m}} \varphi_{j_1\ldots j_m}(z)\frac{dz}{z^{N+1}}, \end{aligned} \tag{3.12}$$

where $\delta_{A_l} = \delta\left(\frac{p^k}{\mu_k(A_l)}\right)^{\frac{1}{2}}$.

We want to treat (3.12) exactly like the corresponding expressions in the first part of the proof. For this purpose we need information on the zeros of the polynomial $1 - z + \mu_k(\mathcal{A})z^k$.

LEMMA 2. — *The zero of smallest modulus z_0 of $1 - z + \mu_k(\mathcal{A})z^k$ is real and satisfies*

$$z_0 > 1 + \mu_k(\mathcal{A}).$$

Proof. — The proof of the first statement is as in the proof of Lemma 1. For the proof of the inequality insert $z = 1 + \mu_k(\mathcal{A})$ into the polynomial.

Observe now that

$$\frac{1}{2\pi i}\oint_C \varphi(z)^{J+1}\frac{dz}{z^{N-kJ+1}} \le \varphi(1+\varepsilon)^{J+1}(1+\varepsilon)^{kJ-N}$$

for $\varepsilon \le \mu_k(\mathcal{A})$. Inserting $\varepsilon = \mu_k(\mathcal{A}) - \frac{J}{N}$ and performing similar calculations as in the first part of the proof yields
(3.13)

$$\frac{1}{2\pi i}\oint_C \varphi(z)^{J+1}\frac{dz}{z^{N-kJ+1}} \le \frac{1}{\mu_k(\mathcal{A})^{J+1}}\exp\left(-\frac{(J-N\mu_k(\mathcal{A}))^2}{2N\mu_k(\mathcal{A})} + O(k\mu_k(\mathcal{A})^2N)\right).$$

Let now $n = N\mu_k(\mathcal{A})$, $J = j_1 + \cdots + j_m$ and $p_l = \frac{\mu_k(A_l)}{\mu_k(\mathcal{A})}$ and insert (3.13) into (3.12) to obtain
(3.14)

$$M_N(\delta) \le \frac{1}{\mu_k(\mathcal{A})}\sum_{\substack{|j_l - np_l| \le np_l\delta_{A_l}\\ l=1,\dots,m}} \frac{J!}{j_1!\cdots j_m!}\prod_{l=1}^{m} p_l^{j_l}\exp\left(-\frac{(J-n)^2}{2n} + O(k\mu_k(\mathcal{A})n)\right),$$

where $\sum_{l=1}^{m} p_l = 1$. Thus we have arrived at an expression that we can treat by the normal approximation of the multinomial distribution.

Assume that N runs through a subsequence of $\mathbb{N}$ such that

$$\operatorname{lp} N - \operatorname{lp}\operatorname{lp} N - k(N) \to \limsup_{N\to\infty}(\operatorname{lp} N - \operatorname{lp}\operatorname{lp} N - k(N)) = \operatorname{lp} C < \infty.$$

It will suffice to prove our theorem for the case that $\limsup \operatorname{lp} N - \operatorname{lp}\operatorname{lp} N - k(N) > -\infty$, such that $0 < C < \infty$. Observe now that $N\mu_k(A_l)\delta_{A_l} = \delta\sqrt{CN\mu_k(A_l)\operatorname{lp} N}$ and use Stirling's formula

$$\left(\frac{n}{e}\right)^n\sqrt{2\pi n} \le n! \le \frac{11}{10}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}$$

to obtain

(3.15)

$$M_N(\delta) \le \frac{11}{10} \frac{\exp(O(k\mu_k(\mathcal{A})n))}{\mu_k(\mathcal{A})\sqrt{2\pi}^{m-1}} \sum_{|j_l - np_l| \le \delta\sqrt{CN\mu_k(A_l)\,\mathrm{lp}\,N}} \frac{\sqrt{J}}{\sqrt{j_1 \cdots j_m}}$$
$$\times \frac{J^J}{j_1^{j_1} \cdots j_m^{j_m}} \prod_{l=1}^{m} p_l^{j_l} \exp\left(-\frac{(J-n)^2}{2n}\right)$$
$$= \frac{11}{10} \frac{\exp(O(k\mu_k(\mathcal{A})n))}{n^{m-1 2}\mu_k(\mathcal{A})(2\pi)^{\frac{m-1}{2}}\sqrt{p_1 \cdots p_m}} \sum_{\substack{|x_l| \le \delta_{A_l} \\ l=1,\ldots,m}} \frac{\sqrt{1+\eta}}{\sqrt{(1+x_1)\cdots(1+x_m)}}$$
$$\times \exp\left(-\frac{n}{2}\sum_{l=1}^{m} p_l x_l^2 + O(n\eta^3) + O\left(n\sum_{l=1}^{m} p_l x_l^3\right)\right),$$

where $j_l = np_l(1+x_l)$ and $J = n(1+\eta)$. The terms in the last exponential come from $(1+x)^{1+x} = \exp\left(x + \frac{x^2}{2} + O(x^3)\right)$ for $x \to 0$ and the observation that $j_1 + \cdots + j_m = J$ transforms to $\sum p_l x_l = \eta$. In the following we will use $p < \frac{1}{2}$ which yields $\delta_{A_l} \to 0$ for our choice of $\mathcal{A}$ (in the case $p = \frac{1}{2}$ we have $\delta_{A_l} = \delta$ and the following arguments cannot be used).

Inserting the definition of δ_{A_l} yields the estimate

(3.16)

$$\left|n\sum_{l=1}^{m} p_l x_l^3\right| \le \frac{(\log N)^{\frac{3}{2}}}{\sqrt{n}} \sum_{l=1}^{m} \frac{1}{\sqrt{p_l}}$$
$$= O(N^{-\frac{1}{3}+\frac{1}{6}\frac{\log q}{\log p}+\frac{1}{3}\,\mathrm{lp}\left(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{q}}\right)}(\log N)^{\frac{4}{3}-\frac{1}{6}\frac{\log q}{\log p}-\frac{1}{3}\,\mathrm{lp}\left(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{q}}\right)})$$

and a similar estimate holds for $n\eta^3$. Using an exponential estimate yields

$$\frac{\sqrt{1+\eta}}{\sqrt{(1+x_1)\cdots(1+x_m)}}$$
$$= \exp(O(N^{-\frac{1}{3}+\frac{1}{6}\frac{\log q}{\log p}+\frac{1}{3}\,\mathrm{lp}\left(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{q}}\right)}(\log N)^{\frac{4}{3}-\frac{1}{6}\frac{\log q}{\log p}-\frac{1}{3}\,\mathrm{lp}\left(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{q}}\right)})).$$

Inserting these inequalites into (3.15) and setting $\alpha = \frac{1}{6}\frac{\log q}{\log p} + \frac{1}{3}\,\mathrm{lp}\left(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{q}}\right)$ yields

$$\frac{M_N(\delta) \le \exp(O(N^{-\frac{1}{3}+\alpha}(\log N)^{\frac{4}{3}-\alpha}))n^{\frac{m+1}{2}}\sqrt{p_1 \cdots p_m}}{\mu_k(\mathcal{A})(2\pi)^{\frac{m-1}{2}}}$$
$$\times \sum_{|x_l| \le \delta_{A_l}} \exp\left(-\frac{n}{2}\sum_{l=1}^{m} p_l x_l^2\right) \frac{1}{(np_1)\cdots(np_m)}.$$

The sum in the last line can be interpreted as a lower Riemann sum for the integral

$$\int_{|x_l|\le\delta_{A_l}} \exp\left(-\frac{n}{2}\sum_{l=1}^{m} p_l x_l^2\right) dx_1\cdots dx_m$$

using the lattice

$$\left\{(x_1,\ldots,x_m) \mid x_l = \frac{j_l}{np_l}-1, \quad |x_l|\le\delta_{A_l}, \quad l=1,\ldots,m\right\}.$$

Thus we obtain

(3.17)

$$M_N(\delta)\le \exp(O(N^{-\frac{1}{3}+\alpha}(\log N)^{\frac{4}{3}-\alpha})+O(\log N)))(\Phi(\delta\sqrt{C\,\mathrm{lp}\,N}))^m,$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{t^2}{2}}\,dt \sim 1-\sqrt{\frac{2}{\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}$$

for $x\to\infty$. Therefore we can estimate

(3.18)

$$M_N(\delta) \le \exp\left(-\sqrt{\frac{2}{\pi}}\frac{m}{\delta\sqrt{C\,\mathrm{lp}\,N}e^{\frac{1}{2}\delta^2 C\,\mathrm{lp}\,N}}+O(N^{-\frac{1}{3}+\alpha}(\log N)^{\frac{4}{3}-\alpha})+O(\log N)\right).$$

Observe now

$$\begin{aligned} m &\asymp \left(\frac{N}{\log N}\right)^{\frac{1}{3}\mathrm{lp}\,2} \\ \mu_k(\mathcal{A}) &\asymp \left(\frac{\log N}{N}\right)^{-\frac{1}{3}\mathrm{lp}\,pq} \qquad (3.19)\\ n &\asymp N^{1+\frac{1}{3}\mathrm{lp}\,pq}(\log N)^{-\frac{1}{3}\mathrm{lp}\,pq}. \end{aligned}$$

Inserting these estimates into (3.18) yields

$$M_N(\delta)\le\exp\left(-D\frac{N^{\frac{1}{3}\mathrm{lp}\,2-\frac{1}{2}\delta^2 C\,\mathrm{lp}\,e}}{\delta(\log N)^{\frac{1}{2}+\frac{1}{3}\mathrm{lp}\,2}}+O(N^{-\frac{1}{3}+\alpha}(\log N)^{\frac{4}{3}-\alpha})+O(\log N)\right),$$

where $D>0$ is a constant implied by (3.19). The right hand side tends to 0 for sufficiently small $\delta>0$, because $\frac{1}{3}\mathrm{lp}\,2 > -\frac{1}{3}+\alpha$ holds for $p<\frac{1}{2}$.

Note that

$$\mu_\infty(D_N^{k(N)}(\omega)<\delta)\le M_N(\delta).$$

Thus the proof is complete. □

Remark 2. — Modifying (1.1) one can also investigate discrepancies $D_N^{k,\phi}(\omega)$

$$= \max_{A\in\{0,1\}^k} \sqrt{\frac{\phi(k)}{\mu_k(A)}} \left| \frac{\#\{1 \le n \le N-k : x_n x_{n+1} \ldots x_{n+k} = A\}}{N} - \mu_k(A) \right|,$$

where ϕ is a monotonically increasing function. Then the same calculations as above yield

$$\mu_\infty \left(\lim_{N\to\infty} D_N^{k(N),\phi}(\omega) = 0 \right) = \begin{cases} 1 & \text{if } \lim_{N\to\infty} \frac{N\phi(k(N))}{\log N} = \infty \\ 0 & \text{otherwise.} \end{cases}$$

This answers a question posed by Flajolet, Kirschenhofer and Tichy [FKT], Remark 2.

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