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ZEROS OF BOUNDED HOLOMORPHIC FUNCTIONS IN STRICTLY PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

by Jim ARLEBRINK

1. Introduction and statement of the results.

Let D be a bounded domain in $\mathbb{C}^2$, and let X be a positive divisor in D. This paper is concerned with the problem of finding conditions on X such that X is defined by a bounded holomorphic function f on D, *i.e.* f vanishes with given multiplicity on each branch of X.

In the case when D is the unit ball, Berndtsson [Be] proved that a sufficient condition is that X has finite area. Our aim is to extend this result to strictly pseudoconvex domains. More precisely, we will prove the following result :

THEOREM 1.1. — *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with C^3 boundary. If X is a positive divisor of D with finite area and the canonical cohomology class of X in $H^2(D,\mathbb{Z})$ is zero, then there exists a bounded holomorphic function that defines X.*

We remark that this result was proved in [Ar] under the additional assumption that the boundary of D is real analytic. Note also that Theorem 1.1 is not true in higher dimensions, and that the conditions on X is not necessary, see [Be] and [Sk1].

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When D is a strictly pseudoconvex domain in $\mathbb{C}^n$, a complete characterization of functions belonging to the Nevanlinna class was found independently by Henkin [He1] and Skoda [Sk2]. They showed that a positive divisor is a zero set of a function in the Nevanlinna class if and only if the divisor satisfies a generalization of the classical Blaschke condition. Assuming a stronger size condition on X, Varopoulos [Va] has proved that X is the zero set of a function in $H^p(D)$, for some p.

As observed by Lelong [Le1], [Le2], these problems are closely connected to the equation

$$i\partial\bar{\partial}u = \theta, \tag{1}$$

where θ is a closed and positive $(1,1)$-current associated to the zero set X. Now, if $H^1(D,\mathbb{C}) = 0$, then every solution of (1) can be written as $u = \log|f|$, where f is a holomorphic function that defines X. If $H^1(D,\mathbb{C}) \neq 0$, then u has to be modified slightly, see section 4. Thus, in order to prove results of this kind one has to solve equation (1) with control of the growth of u. In particular, if u is a negative solution of (1), then X is the zero set of a bounded holomorphic function.

Theorem 1.1 is obtained as a consequence of

THEOREM 1.2. — *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with C^3 boundary, and let θ be a closed positive $(1,1)$-form with coefficients in $C^\infty(\bar{D})$. Assume that the cohomology class of θ in $H^2(D,\mathbb{C})$ is zero. Then there exists a negative solution of*

$$i\partial\bar{\partial}u = \theta,$$

such that

$$\int_{\partial D} |u| d\sigma \leq C \int_D \operatorname{tr}\theta, \tag{2}$$

where C is independent of θ.

The proof of Theorem 1.2 follows, initially, the classical method of Lelong to solve equation (1). First, the assumption on θ implies that one can solve

$$idw = \theta, \tag{3}$$

where $w = w_{1,0} + w_{0,1}$. Note that (3) implies that $\bar{\partial}w_{0,1} = 0$. Then one solves

$$\bar{\partial}u = w_{0,1}.$$

Then, $2\operatorname{Re} u$ is a solution of (1), provided that w is chosen so that $w_{1,0} = -\bar{w}_{0,1}$.

By subtracting a pluriharmonic function p we obtain a solution of (1) with negative boundary values. The resulting solution depends on the choice of w, and is in fact not linearly dependent on θ.

The ideas we use are similar to those employed by Berndtsson [Be]; the principal difference is that in the ball case one can choose p in such a way that the final solution depends only on θ, in fact linearly.

The paper is organized as follows. In Section 2 we give, for motivation, an outline of the proof of Theorem 1.2, which is proved in detail in Section 3. Section 4 is devoted to the proof of Theorem 1.1. In Section 5 we solve the d-equation and obtain estimates for the solution. In Section 6 we obtain a solution to the $\bar{\partial}$-equation by means of integral formulas. Finally, in Section 7 we prove a lemma which is crucial for the proof of Theorem 1.2.

The notation $x \lesssim y$ means that there is a constant C, independent of x and y such that $x \leq Cy$. Further, $x \sim y$ is equivalent to $x \lesssim y$ and y $\lesssim x$. The surface measure on ∂D is denoted by $d\sigma$ and $d\lambda$ is the Lebesgue measure in $\mathbb{C}^2$. The trace of θ is denoted by $\operatorname{tr}\theta$.

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2. Idea of the proof of Theorem 1.2.

Let $D = \{\zeta \in \mathbb{C}^2 : \rho(\zeta) < 0\}$ be a strictly pseudoconvex domain in $\mathbb{C}^2$. Assume that there is a function $v(z,\zeta)$ defined on $\bar{D} \times \bar{D}$ satisfying the following conditions :

$$v(\zeta,\zeta) = \rho(\zeta), \tag{1}$$

$$\bar{\partial}_z v(z,\zeta) = \partial_\zeta v(z,\zeta) = 0, \tag{2}$$

$$v(\zeta,z) = \overline{v(z,\zeta)}, \tag{3}$$

and

$$v(z,\zeta) \neq 0 \ \text{ when } \ z \neq \zeta. \tag{4}$$

It is easy to see that if such a v exists, then it is unique. Let

$$S(z,\zeta) = (S_1, S_2) : \bar{D} \times \bar{D} \longrightarrow \mathbb{C}^2$$

be a smooth map, holomorphic in z for each $\zeta \in \partial D$, such that

$$v(z,\zeta) = \langle S, z-\zeta\rangle \quad \text{if} \quad (z,\zeta) \in \partial D \times \partial D. \tag{5}$$

Here we write $\langle \xi, \eta\rangle = \sum \xi_j \eta_j$ when $\xi, \eta \in \mathbb{C}^2$. With S we associate the $(1,0)$-form $\sum S_j d\zeta_j$, which is also denoted by S. We define another map Q by $Q(z,\zeta) = S(\zeta, z)$. Note that according to (3) and (5), Q satisfies

$$\bar{v} = \langle Q, \zeta - z\rangle \quad \text{for} \quad (z,\zeta) \in \partial D \times \partial D, \tag{6}$$

and that Q is holomorphic in ζ when $z \in \partial D$.

Now, let $w = w_{1,0} + w_{0,1}$ be a solution of $idw = \theta$ with $\|w\|_{L^1(\partial D)} \lesssim \|\theta\|_{L^1(D)}$ where θ is a smooth, positive and closed $(1,1)$ form. Then, as we will prove later, the function

$$u(z) = \frac{1}{4\pi^2} \int_{\partial D} \frac{Q \wedge S \wedge w_{0,1}}{\langle S, z-\zeta\rangle \langle Q, \zeta - z\rangle}$$

represents the boundary values of a solution to $\bar\partial u = w_{0,1}$. In view of (3), (5) and (6) we can write

$$u(z) = \frac{1}{4\pi^2} \int_{\partial D} \frac{Q \wedge S \wedge w_{0,1}}{|v|^2}.$$

It turns out that for our choice of S and Q, we can choose a pluriharmonic function p such that $\|p\|_{L^1(\partial D)} \lesssim \|\theta\|_{L^1(D)}$ and

$$2\operatorname{Re} u - p \le 2 \operatorname{Re} \frac{1}{4\pi^2} \int_{\partial D} \frac{\partial \bar v \wedge \bar\partial v \wedge w_{0,1}}{|v|^2}. \tag{7}$$

By Stokes' theorem the right hand side in (7) equals

$$2\operatorname{Re} \frac{1}{4\pi^2} \int_D \frac{\partial \bar v \wedge \bar\partial v \wedge \partial w_{0,1}}{|v|^2} = -\frac{1}{4\pi^2} \int_D \frac{i\partial \bar v \wedge \bar\partial v \wedge \theta}{|v|^2}, \tag{8}$$

since by (2) and (3), $(\partial \bar v \wedge \bar\partial v)/|v|^2$ is a closed form. Thus $2\operatorname{Re} u - p$ are the boundary values of a solution to equation (1.1) which are negative since $i\partial\bar v \wedge \bar\partial v \wedge \theta$ is a positive form.

It is easy to see that if v exists, then ρ is real analytic. Conversely, if the defining function ρ is real analytic, then v is obtained, locally, from the power series of $\rho(\zeta, \bar\zeta)$ by substituting z in place of ζ. In particular, when D is the unit ball B, then $v = z \cdot \bar\zeta - 1$ so that we can choose $S = \bar\zeta$ and $Q = \bar z$. The last term in (8) is then

$$-\frac{1}{4\pi^2} \int_B \frac{i d(\bar z \cdot \zeta) \wedge d(z \cdot \bar\zeta) \wedge \theta}{|z \cdot \bar\zeta - 1|^2},$$

which is the formula for the boundary values obtained by Berndtsson [Be]. In fact, in this case there is equality in (7) for an appropriate choice of p.

For an arbitrary strictly pseudoconvex domain, we can construct v with essentially all the required properties near the diagonal. This gives rise to certain error terms which, however, can be majorized by pluriharmonic functions and the scheme above still works.

3. Proof of Theorem 1.2.

Following the outline in Section 2, we will start by defining the function v. Let

$$v(z,\zeta) = -\Big[\rho(\zeta) + \sum \frac{\partial\rho(\zeta)}{\partial\zeta_j}(z_j-\zeta_j) + \frac{1}{2}\sum \frac{\partial^2\rho(\zeta)}{\partial\zeta_j\partial\zeta_k}(z_j-\zeta_j)(z_k-\zeta_k)\Big].$$

Then $v(z,\zeta)+\rho(\zeta)$ is just the Levi polynomial of ρ and hence

$$2\operatorname{Re} v(z,\zeta) \geq -\rho(\zeta) - \rho(z) + C|z-\zeta|^2, \tag{1}$$

for $|z-\zeta|$ sufficiently small. Clearly $v(z,\zeta)$ is holomorphic in z. We will use the following properties of v.

LEMMA 3.1.

$$\partial_\zeta v(z,\zeta) = O(|z-\zeta|^2) \tag{2}$$

$$\overline{v(z,\zeta)} = v(\zeta,z) + O(|z-\zeta|^3). \tag{3}$$

Proof. — Equality (2) follows by an easy calculation.

To prove (3) we expand $\bar v(z,\zeta)$ in a Taylor series in the ζ variable at the point z. Using (2) we get

$$\begin{aligned}\bar v(z,\zeta) &= \bar v(z,z) + \sum \frac{\partial \bar v}{\partial\zeta_j}(\zeta_j-z_j) + \frac{1}{2}\sum\frac{\partial^2\bar v}{\partial\zeta_j\partial\zeta_k}(\zeta_j-z_j)(\zeta_k-z_k) \\ &\qquad + O(|z-\zeta|^3) \\ &= -\rho(z) - \sum\frac{\partial\rho(z)}{\partial z_j}(\zeta_j-z_j) - \frac{1}{2}\sum\frac{\partial^2\rho(z)}{\partial z_j\partial z_k}(\zeta_j-z_j)(\zeta_k-z_k) \\ &\qquad + O(|z-\zeta|^3).\end{aligned}$$

Hence $\overline{v(z,\zeta)} = v(\zeta,z) + O(|z-\zeta|^3)$ as desired.

Remark. — A similar proof of (3) can be found in [KS].

In order to construct the map S we first observe that by the definition of v one can find a map $\bar S$ such that

$$\langle \bar S, z-\zeta\rangle = v(z,\zeta) + \rho(\zeta).$$

By (1) we can choose ε so small that $\operatorname{Re} v(z,\zeta) > 0$ when $0 < |z-\zeta| < 2\varepsilon$. Let $\chi(z,\zeta)$ be a smooth function such that $0 \le \chi \le 1$, $\chi = 1$ when $|z-\zeta| \le \varepsilon$ and $\chi = 0$ when $|z-\zeta| \ge 2\varepsilon$. We now define S by

$$S(z,\zeta) = \chi \bar{S}(z,\zeta) + (1-\chi)(\bar{z}-\bar{\zeta}).$$

Then

$$2\operatorname{Re}\langle S, z-\zeta\rangle \ge \rho(\zeta) - \rho(z) + C|z-\zeta|^2 \text{ when } (z,\zeta) \in D \times D.$$

It is obvious that, if $|z-\zeta| < \varepsilon$, $S(z,\zeta)$ is holomorphic in z and

$$\langle S, z-\zeta\rangle = v(z,\zeta) + \rho(\zeta) \tag{4}$$

holds.

We define the map Q by

$$Q(z,\zeta) = S(\zeta, z).$$

Then $\bar{\partial}_\zeta Q(z,\zeta) = 0$ for $|z-\zeta|$ small. By (3) and (4) we have

$$\langle Q, \zeta - z\rangle = \overline{v(z,\zeta)} + \rho(\zeta) + O(|z-\zeta|^3). \tag{5}$$

The following proposition is proved in section 6.

PROPOSITION 3.2. — *If $\bar{\partial} f = 0$, then there exists a function h on ∂D such that the function*

$$u(z) = \frac{1}{4\pi^2}\int_{\partial D} \frac{Q \wedge S \wedge f}{\langle Q, \zeta - z\rangle \langle S, z-\zeta\rangle} + h(z)$$

is the boundary values of a solution to $\bar{\partial} u = f$, and such that

$$\|h\|_{L^\infty(\partial D)} \lesssim \|f\|_{L^1(\partial D)} + \|f\|_{L^1(D)}.$$

We will need the following result about the solution of the d-equation but postpone its proof until Section 5.

PROPOSITION 3.3. — *Let D be a bounded domain in $\mathbb{C}^n$ with a C^2 boundary. Suppose that μ is a closed 2-form with coefficients in $C^\infty(\bar{D})$, and that the cohomology class of μ in $H^2(D,\mathbb{C})$ vanishes. Then there is a solution w of*

$$i dw = \mu$$

such that

$$\|w\|_{L^1(D)} + \|w\|_{L^1(\partial D)} \le C\|\mu\|_{L^1(D)},$$

where C is independent of μ.

Now let θ be a closed, positive and smooth $(1,1)$-form. Applying Proposition 3.3 we obtain a solution $w = w_{0,1} + w_{1,0}$ in $L^1(\partial D) \cap L^1(D)$ to $idw = \theta$, with $\bar{\partial} w_{0,1} = 0$. Thus, by Proposition 3.2 and the discussion in Section 1,

$$2\,\mathrm{Re}\,\frac{1}{4\pi^2}\int_{\partial D}\frac{Q\wedge S\wedge w_{0,1}}{\langle Q,\zeta-z\rangle\langle S,z-\zeta\rangle} + h(z) \tag{6}$$

is the boundary values of a solution to $i\partial\bar{\partial}u = \theta$, where h can be estimated by $C\|\theta\|_{L^1(D)}$.

Next, consider the integral

$$\int_{\partial D}\frac{Q\wedge S\wedge w_{0,1}}{\langle Q,\zeta-z\rangle\langle S,z-\zeta\rangle}. \tag{7}$$

Choose $\psi \in C^\infty(\mathbb{C}^2\times\mathbb{C}^2)$, $0\le\psi\le 1$, which is equal to 1 for $|z-\zeta|\le\varepsilon/2$ and $\psi = 0$ for $|z-\zeta|\ge\varepsilon$. Then (7) equals

$$\int_{\partial D}\psi\frac{Q\wedge S\wedge w_{0,1}}{v(\bar{v}+O(|z-\zeta|^3)} + \int_{\partial D}(1-\psi)\frac{Q\wedge S\wedge w_{0,1}}{\langle Q,\zeta-z\rangle\langle S,z-\zeta\rangle} \tag{8}$$

in view of (4) and (5). The second integral in (8) has no singularity since its denominator is $\neq 0$ on the support of $1-\psi$. Hence it can be estimated by $C\|w_{0,1}\|_{L^1(\partial D)}$.

To deal with the first term we need

LEMMA 3.4. — *There exists a pluriharmonic function $p(z)$ such that*

$$\int_{\partial D}\psi\frac{|z-\zeta|^2|w_{0,1}|}{|v|^2}\lesssim p(z)$$

and

$$\|p\|_{L^1(\partial D)}\lesssim\|w_{0,1}\|_{L^1(\partial D)}. \tag{9}$$

The proof will be found at the end of this section.

Now observe that since

$$\frac{1}{\bar{v}+O(|z-\zeta|^3)}-\frac{1}{\bar{v}}=\frac{O(|z-\zeta|^3)}{\bar{v}(\bar{v}+O(|z-\zeta|^3)}$$

we can write the first term in (8) as

$$\int_{\partial D}\psi\frac{Q\wedge S\wedge w_{0,1}}{|v|^2}+\int_{\partial D}\psi O(|z-\zeta|^3)\frac{Q\wedge S\wedge w_{0,1}}{|v|^2(\bar{v}+O(|z-\zeta|^3)}. \tag{10}$$

We claim that the last term in (10) is dominated by a pluriharmonic function.

To establish the claim, note that if $|z-\zeta|$ is small, then by (1)

$$|v| \gtrsim |z-\zeta|^2 \text{ when } (z,\zeta) \in \partial D \times \partial D. \tag{11}$$

Moreover, since $Q = S$ when $z = \zeta$,

$$|Q \wedge S| = O(|z-\zeta|). \tag{12}$$

Hence (11) and (12) imply that

$$|Q \wedge S| \lesssim \sqrt{|v|}.$$

Thus

$$\frac{|z-\zeta|^3 |Q \wedge S|}{|v|^2(\bar{v} + O(|z-\zeta|^3))} \lesssim \frac{|z-\zeta|^2}{|v|^2},$$

and by Lemma 3.4, the claim is proved.

Consider now the first term in (10) :

$$\int_{\partial D} \psi \frac{Q \wedge S \wedge w_{0,1}}{|v|^2}.$$

We will need a result whose proof is somewhat tedious, so we have postponed it until Section 7.

LEMMA 3.5.

$$2\,\mathrm{Re} \int_{\partial D} \psi \frac{Q \wedge S \wedge w_{0,1}}{|v|^2} \lesssim 2\,\mathrm{Re} \int_{\partial D} \psi \frac{\partial \bar{v} \wedge \bar{\partial} v \wedge w_{0,1}}{|v|^2} + \int_{\partial D} \psi \frac{|z-\zeta|^2 |w_{0,1}|\, d\sigma}{|v|^2}.$$

Hence by Lemma 3.4, there is a pluriharmonic function p such that

$$2\,\mathrm{Re} \int_{\partial D} \psi \frac{Q \wedge S \wedge w_{0,1}}{|v|^2} - p(z) \lesssim 2\,\mathrm{Re} \int_{\partial D} \psi \frac{\partial \bar{v} \wedge \bar{\partial} v \wedge w_{0,1}}{|v|^2}. \tag{13}$$

We will now show that the last term in (13) is negative, modulo a function that can be estimated by a pluriharmonic function. Note first that

$$d\bar{v} \wedge dv = \partial \bar{v} \wedge \bar{\partial} v + \partial \bar{v} \wedge \partial v + \bar{\partial} \bar{v} \wedge \partial v + \bar{\partial} \bar{v} \wedge \bar{\partial} v, \tag{14}$$

and by (2) the last 3 terms in (14) are at least $O(|z-\zeta|^2)$. Thus by Lemma 3.4 we have

$$\int_{\partial D} \psi \frac{\partial \bar{v} \wedge \bar{\partial} v \wedge w_{0,1}}{|v|^2} + g(z) = \int_{\partial D} \psi \frac{d\bar{v} \wedge dv \wedge w_{0,1}}{|v|^2}, \tag{15}$$

where g can be estimated by a pluriharmonic function. Applying Stokes' theorem to the integral on the right side of (15) one obtains

$$\int_{\partial D} \psi \frac{d\bar{v} \wedge dv \wedge w_{0,1}}{|v|^2} = \int_D d\psi \frac{d\bar{v} \wedge dv \wedge w_{0,1}}{|v|^2} + \int_D \psi \frac{d\bar{v} \wedge dv \wedge w_{0,1}}{|v|^2}. \tag{16}$$

since $\bar\partial w_{0,1} = 0$. The first integral on the right hand side of (16) is non-singular and can be estimated by $C\|w_{0,1}\|_{L^1(D)}$.

Now, if we use (14) on the last term in (16), we first note that for bidegree reasons

$$\partial\bar v \wedge \partial v \wedge \partial w_{0,1} = \bar\partial\bar v \wedge \bar\partial v \wedge \partial w_{0,1} = 0 \text{ on } D.$$

Next, we observe that by (2), $\bar\partial\bar v \wedge \partial v = O(|z-\zeta|^4)$, and so the integral

$$\int_D \psi \frac{\bar\partial\bar v \wedge \partial v \wedge \partial w_{0,1}}{|v|^2}$$

can be estimated by $C\|\partial w_{0,1}\|_{L^1(D)}$. Hence

$$\operatorname{Re}\int_D \psi \frac{d\bar v \wedge dv \wedge \partial w_{0,1}}{|v|^2} + C \le \operatorname{Re}\int_D \psi \frac{\partial\bar v \wedge \bar\partial v \wedge \partial w_{0,1}}{|v|^2}.$$

Finally, we note that since $\bar w_{0,1} = -w_{0,1}$ and $dw = i\theta$, we have

$$2\operatorname{Re}\int_D \psi \frac{\partial\bar v \wedge \bar\partial v \wedge \partial w_{0,1}}{|v|^2} = -\int_D \psi \frac{i\partial\bar v \wedge \bar\partial v \wedge \theta}{|v|^2},$$

where obviously the last term is negative.

Summarizing the results above, we have found that there is a pluri-harmonic function p and a bounded function h so that

$$(17)\quad 2\operatorname{Re}\left[\frac{1}{4\pi^2}\int_{\partial D} \frac{Q \wedge S \wedge w_{0,1}}{\langle Q, \zeta-z\rangle\langle S, z-\zeta\rangle} + h\right] + p \le -\frac{1}{4\pi^2}\int_D \frac{i\partial v \wedge \bar\partial v \wedge \theta}{|v|^2},$$

where the left side of (17) is the boundary values of a solution to $i\partial\bar\partial u = \theta$. Since θ is a positive form, it follows that u is plurisubharmonic and by the maximum principle for plurisubharmonic functions, this implies that $u \le 0$ in D.

In order to show estimate (1.2) it remains to prove that

$$\left\|\int_{\partial D} \frac{Q \wedge S \wedge w_{0,1}}{\langle Q, \zeta-z\rangle\langle S, z-\zeta\rangle}\right\|_{L^1(D)} \lesssim \|w_{0,1}\|_{L^1(\partial D)},$$

since $\|w_{0,1}\|_{L^1(\partial D)} \le \|\theta\|_{L^1(D)}$, and since by Proposition 3.2 and Lemma 3.4,

$$\|h\|_{L^\infty(\partial D)} + \|p\|_{L^1(\partial D)} \lesssim \|\theta\|_{L^1(D)}.$$

For this it is sufficient by Fubini's theorem to show that

$$(18)\qquad \int_{\partial D} \frac{|Q \wedge S|d\sigma(z)}{|\langle Q, \zeta-z\rangle\langle S, z-\zeta\rangle|} \le C,$$

where C is independent of $\zeta \in \partial D$.

To prove (18) it is enough to assume that ζ is close to z. In this case $v = \langle S, z-\zeta\rangle$ when $(z,\zeta) \in \partial D \times \partial D$. Moreover it is obvious that $|\langle Q, \zeta-z\rangle| = |\langle S, z-\zeta\rangle|$. Hence, by using (11) and (12) we find that

$$\frac{|Q \wedge S|}{|\langle Q, \zeta-z\rangle||\langle S, z-\zeta\rangle|} \lesssim \frac{1}{|\langle S, z-\zeta\rangle|^{3/2}}.$$

Next, observe that by (2) and (4) we have

$$d_\zeta \langle S, z-\zeta\rangle\big|_{z=\zeta} = -\partial\rho(z).$$

Estimate (18) is now a consequence of the following lemma which we recall from [Ra], (Lemma VII.1.5).

LEMMA 3.6. — *Let $H(z,\zeta)$ be defined in a neighborhood of $\bar{D} \times \partial D$, $H(z,\zeta) \neq 0$, if $z \neq \zeta$ and suppose that*

$$2\,\mathrm{Re}\, H(z,\zeta) \geq -\rho(\zeta) + \delta|z-\zeta|^2 \text{ when } |z-\zeta| < \varepsilon$$

and

$$d_\zeta H(z,\zeta) = -\partial\rho(z).$$

Then

$$\int_{\partial D} \frac{d\sigma(\zeta)}{|H(z,\zeta)|^\alpha} \leq C$$

if $\alpha < 2$, where C is independent of $\zeta \in \bar{D}$.

This finishes the proof of Theorem 1.2.

Proof of Lemma 3.4. — First we note that by Fornaess embedding theorem [Fo], there exists a C^1-function $\Phi(z,\zeta)$, defined on $\bar{D} \times \bar{D}$ and constants $\delta, \varepsilon > 0$ such that

(a) $\Phi(z,\zeta)$ is holomorphic in z

(b) $\Phi(z,\zeta) \neq 0$ when $z \neq \zeta$

(c) $2\,\mathrm{Re}\,\Phi(z,\zeta) \geq \rho(\zeta) - \rho(z) + \delta|z-\zeta|^2$ if $|z-\zeta| < \varepsilon$

and

(d) $d_\zeta \Phi(z,\zeta)\big|_{z=\zeta} = -\partial\rho(z)$.

Now let

$$p(z) = 2\,\mathrm{Re} \int_{\partial D} \frac{|w_{0,1}|\, d\sigma}{\Phi(z,\zeta)} = \int_{\partial D} \frac{2\,\mathrm{Re}\,\overline{\Phi(z,\zeta)}|w_{0,1}|\, d\sigma}{|\Phi(z,\zeta)|^2}.$$

Then $p(z)$ is clearly pluriharmonic. By (c) above we have for z close to ζ

$$2\,\mathrm{Re}\,\Phi(z,\zeta) \gtrsim |z-\zeta|^2 \text{ when } (z,\zeta) \in \partial D \times \partial D.$$

We claim that

$$|\Phi(z,\zeta)| \sim |v(z,\zeta)| \text{ when } (z,\zeta) \in \partial D \times \partial D$$

and $|z-\zeta|$ is small. Assuming the claim for the moment we have

$$p(z) \gtrsim \int_{\partial D} \psi \frac{|z-\zeta|^2|w|}{|v|^2}\, d\sigma.$$

Estimate (9) follows by Fubini's theorem and the fact that

$$\int_{\partial D} \frac{d\sigma(z)}{|\Phi(z,\zeta)|} \leq C,$$

where C is independent of ζ, by Lemma 3.6.

In order to prove the claim, we introduce real coordinates x_j, $1 \leq j \leq 3$, for $\zeta \in \partial D$ in a neighborhood of $z = p \in \partial D$ such that $x_j(p) = 0$, $1 \leq j \leq 3$, $dx_1\big|_p = d^c\rho\big|_p$ and x_2, x_3 arbitrary. We will show that if $H(z,\zeta)$ is a function satisfying

$$2\,\mathrm{Re}\, H(z,\zeta) \geq |z-\zeta|^2 \tag{19}$$

$$d\,\mathrm{Re}\, H = -d\rho, \quad d\,\mathrm{Im}\, H\big|_p = d^c\rho\big|_p \tag{20}$$

then

$$|H| \sim |x_1| + x_2^2 + x_3^2 \text{ on } \partial D. \tag{21}$$

Assuming this, the claim is proved, since both v and Φ satisfy (19) and (20).

To prove (21) we note first that since $d\,\mathrm{Im}\, H\big|_p = dx_1\big|_p$, we have $\mathrm{Im}\, H = x_1 + O(|x|^2)$. Hence (19) implies

$$|H| \sim |\mathrm{Im}\, H| + |\mathrm{Re}\, H| \gtrsim |x_1| + x_2^2 + x_3^2. \tag{22}$$

Next, since $d\rho = 0$ on ∂D, $\mathrm{Re}\, H = O(|x|^2)$. Thus

$$|H| \sim |\mathrm{Im}\, H| + |\mathrm{Re}\, H| \lesssim |x_1| + x_2^2 + x_3^2. \tag{23}$$

By combining (22) and (23) we obtain (21) and the proof is complete.

4. Proof of Theorem 1.1.

Let θ be the closed and positive $(1,1)$-current associated to the zero set X. We assume first that θ is defined in a neighborhood $\widetilde{D}$ of $\bar{D}$. Then there is a sequence θ_j of smooth, positive and closed $(1,1)$-forms that converges weakly to θ such that

$$\int_D \operatorname{tr}\theta_j \, d\lambda \lesssim \int_{\widetilde{D}} \operatorname{tr}\theta \, d\lambda.$$

Applying Theorem 1.2, we can find smooth real valued functions $U_j \leq 0$, such that

$$\frac{i}{\pi}\partial\bar{\partial}U_j = \theta_j,$$

with $\|U_j\|_{L^1(\partial D)} \lesssim \|\theta\|_{L^1(D)}$.

Since in particular

$$\Delta U_j = C \operatorname{tr}\theta_j, \tag{1}$$

the solution of (1) with boundary values u_j is given by the formula

$$U_j(z) = \int_{\partial D} P(z,\zeta)u_j(\zeta)\, d\sigma(\zeta) - C\int_D G(z,\zeta)\operatorname{tr}\theta_j \, d\lambda(\zeta), \tag{2}$$

where $G(z,\zeta)$ and $P(z,\zeta)$ are the Green's function and the Poisson kernel respectively.

Using the fact that $\|u_j\|_{L^1(\partial D)} \leq C$, we can find a subsequence of u_j that converges weakly to a bounded measure $u \leq 0$. Hence we get a negative solution U of

$$\frac{i}{\pi}\partial\bar{\partial}U = \theta$$

by taking limits in (2).

Now let h be a holomorphic function that defines X. Then

$$\partial\bar{\partial}(U - \log|h|) = 0,$$

so that $\alpha = U - \log|h|$ is pluriharmonic. Assume for the moment that $H^1(D,\mathbb{C}) = 0$. Then there is a real solution β to

$$d\beta = i(\bar{\partial}\alpha - \partial\alpha) = d^c\alpha,$$

so that $\alpha + i\beta$ is holomorphic. Thus $f = h\exp(\alpha + i\beta)$ is a bounded holomorphic function with the same zero set as h.

If $H^1(D,\mathbb{C}) \neq 0$, then we can no longer assume that $d^c\alpha$ is exact. However, we can get around this difficulty by means of the next lemma, which is a modification of an idea from [AC].

LEMMA 4.1. — *Let D be a bounded domain in $\mathbb{C}^n$ with a C^2 boundary. If $\alpha \in C^\infty(\bar{D})$ is pluriharmonic, then there exist a closed and* *smooth 1-form φ and a smooth real valued function β defined modulo $2\pi\mathbb{Z}$ such that $|\varphi| \leq C$ and*

$$d^c\alpha - \varphi = d\beta,$$

where C only depends on D and is stable under small C^2-perturbations of ∂D.

The proof will be found in the next section.

Using Lemma 4.1 on $\alpha = U - \log|h|$ we get φ and β so that $d^c\alpha - \varphi = d\beta$, and $|\varphi| \leq C$. In particular, $\bar\partial(\alpha + i\beta) = \varphi_{0,1}$. Since there are uniform estimates for the $\bar\partial$-equation in strictly pseudoconvex domains, there is a function γ with $|\gamma| \leq C'$ such that $\bar\partial\gamma = \varphi_{0,1}$. Hence $f = h\exp(\alpha+i\beta-\gamma-C')$ is a holomorphic function that defines X. Observe that

$$\log|f| = U - \operatorname{Re}\gamma - C' \leq 0$$

and

$$\int_{\partial D} -\log|f| \, d\sigma \leq \int_{\partial D} (-U + 2C'') \, d\sigma \leq C'',$$

where C'' again is a constant that only depends on D and is stable under small C^2–perturbations of ∂D.

Assume now that θ is only defined in D. Let $\rho_\varepsilon = \rho + \varepsilon$, where $\varepsilon > 0$ is small and let $D_\varepsilon = \{\rho_\varepsilon < 0\}$. We can now assume that θ is defined in a neighborhood of $\bar D_\varepsilon$. By the argument above we can find a bounded holomorphic function f^ε that defines $X \cap D_\varepsilon$. Since the sequence f^ε is uniformly bounded, we can extract a subsequence which converges to a bounded holomorphic function f. However, we need to verify that f is not identically equal to zero.

Since $V^\varepsilon = \log|f^\varepsilon| \in L^1(\partial D_\varepsilon)$ and V^ε is negative we have

$$0 \leq \int_{\partial D_\varepsilon} -V^\varepsilon d\sigma \leq C.$$

We will apply Green's formula to the functions V^ε and ρ_ε. Noting that $\rho_\varepsilon = 0$ on ∂D_ε we get

$$\int_{\partial D_\varepsilon} V^\varepsilon \partial_\nu \rho_\varepsilon \, d\sigma = \int_{D_\varepsilon} (V^\varepsilon \Delta\rho_\varepsilon - \rho_\varepsilon \Delta V^\varepsilon) \, d\lambda,$$

where ν is the unit outward normal vector field on ∂D_ε. Now, observe that $\Delta\rho_\varepsilon > 0$, since ρ_ε is strictly plurisubharmonic. Hence

$$0 \leq \int_{D_\varepsilon} -V^\varepsilon \Delta\rho_\varepsilon \, d\lambda = \int_{D_\varepsilon} -\rho_\varepsilon \Delta V^\varepsilon \, d\lambda + \int_{\partial D_\varepsilon} -V^\varepsilon \partial_\nu \rho_\varepsilon \, d\sigma \leq C.$$

Thus

$$0 \le \int_{D_\varepsilon} -V^\varepsilon \, d\lambda \le C,$$

and letting $\varepsilon \to 0$ we obtain

$$0 \le \int_{D} -V \, d\lambda \le C.$$

This implies that $f \not\equiv 0$ and thus f is the desired function. This proves the theorem.

5. Proof of Proposition 3.3.

Since the statement of the proposition does not involve the complex structure of $\mathbb{C}^n$, we may assume that D is a bounded domain in $\mathbb{R}^{2n}$ with a C^2-boundary.

First, let D be contractible and let μ be a smooth and closed 2-form on $\bar{D}$. If $\psi(x,t) : \bar{D} \times [0,1] \to \bar{D}$ is a smooth homotopy between the identity map and the constant map $x \to p$, then

$$w = \int_0^1 \psi^* \mu$$

is a smooth solution of

$$dw = \mu, \tag{1}$$

where ψ^* is the pullback of ψ. For instance, if D is convex and $0 \in D$, then one can take $\psi(x,t) = tx$. In this case it is easy to see that the estimate

$$\int_{\partial D} |w| \, d\sigma + \int_D |w| \, d\lambda \le C \int_D |\mu| \, d\lambda \tag{2}$$

holds, where C is independent of μ.

In the general case $\bar{D}$ is locally C^2-diffeomorphic to a convex domain and one can locally solve (2) with the required estimate. In order to complete the proof we need to piece together these local solutions into a global one such that (2) holds.

Let $\mathcal{F}_0$ be the sheaf of smooth and closed 1-forms on $\bar{D}$, $\mathcal{F}_1$ the sheaf of smooth 1-forms on $\bar{D}$ and $\mathcal{F}_2$ the sheaf of smooth and closed 2-forms on $\bar{D}$. Then

$$0 \longrightarrow \mathcal{F}_0 \xrightarrow{i} \mathcal{F}_1 \xrightarrow{d} \mathcal{F}_2 \longrightarrow 0$$

is a short exact sequence of sheaves. By standard cohomology arguments if follows that there are canonical isomorphisms such that

$$H^0(\bar{D},\mathcal{F}_2)/dH^0(\bar{D},\mathcal{F}_1) \simeq H^1(\bar{D},\mathcal{F}_0) \simeq H^2(\bar{D},\mathbb{C}). \tag{3}$$

The rest of the argument consists in using the isomorphisms in (3) to trace the problem back to $H^2(\bar{D},\mathbb{C})$. By means of Čech-cohomology this is then transformed into a coboundary problem, which turns out to be a finite dimensional linear problem.

Let $\mathcal{U} = \{\Omega_j\}$ be a finite covering of $\bar{D}$, such that $\bar{\Omega}_j$, $\bar{\Omega}_{jk} = \bar{\Omega}_j \cap \bar{\Omega}_k$ and $\bar{\Omega}_{jk\ell} = \bar{\Omega}_j \cap \bar{\Omega}_k \cap \bar{\Omega}_\ell$ are all diffeomorphic to convex sets. Let w_j be a smooth solution in Ω_j of $dw_j = \mu$ such that

$$\int_{\partial D\cap\Omega_j} |w_j|\, d\sigma + \int_{D\cap\Omega_j} |w_j|\, d\lambda \le C\int_D |\mu|\, d\lambda. \tag{4}$$

If $\Omega_{j,k} \neq \emptyset$, then $a_{jk} = w_j - w_k$ is a closed 1-form on Ω_{jk} and thus $a = (a_{jk})$ defines a 1-cocycle with values in $\mathcal{F}_0$. Note that

$$\int_{\partial D\cap\Omega_{jk}} |a_{jk}|\, d\sigma + \int_{D\cap\Omega_{jk}} |a_{jk}|\, d\lambda \le C\int_D |\mu|\, d\lambda, \tag{5}$$

where C is independent of μ. In the same way we can solve $db_{jk} = a_{jk}$ on Ω_{jk} so that the estimate (5) also holds for b_{jk}.

Now,

$$c_{jk\ell} = b_{jk} + b_{k\ell} + b_{\ell j} = (\delta b_{jk}))_{jk\ell}$$

is by definition a cocycle with values in $\mathbb{C}$, since

$$dc_{jk\ell} = (\delta(db_{jk}))_{jk\ell} = (\delta(a_{jk}))_{jk\ell} = 0.$$

Hence $(c_{jk\ell})$ defines an element in $H^2(\bar{D},\mathbb{C})$, and by the isomorphisms in (3), this element must, in fact, be zero since μ is assumed to be exact. Now, since $H^p(\Omega_I,\mathbb{C}) = 0$, for each multi index I, $\mathcal{U}$ is a Leray cover with respect to the sheaf $\mathbb{C}$. Therefore $H^2(\bar{D},\mathbb{C}) = H^2(\mathcal{U},\mathbb{C})$, see [Gu]. Hence $(c_{jk\ell})$ is a coboundary, which means that there is a 1-cochain c_{jk} such that $\delta(c_{jk}) = (c_{jk\ell})$.

Observe that, since this is a finite dimensional linear problem and $|c_{jk\ell}| \le C$, we can choose (c_{jk}) such that $|c_{jk}| \lesssim C$. If $f_{jk} = b_{jk} - c_{jk}$, then

$$\delta f_{jk} = \delta b_{jk} - \delta c_{jk} = c_{jk\ell} - c_{jk\ell} = 0$$

and

$$df_{jk} = db_{jk} - dc_{jk} = db_{jk} = a_{jk},$$

since c_{jk} are locally constant.

Let $\{\varphi_k\}$ be a partition of unity subordinated to the covering $\{\Omega_k\}$ and let

$$g_j = d\sum_k \varphi_k f_{jk}.$$

Obviously g_j is a closed 1-form on Ω_j and we have

$$g_j - g_\ell = d\sum_k \varphi_k(f_{jk} - f_{j\ell}) = d\sum_k \varphi_k f_{j\ell} = df_{j\ell} = d(b_{j\ell} - c_{j\ell}) = db_{j\ell} = a_{j\ell}.$$

Moreover g_j satisfies estimate (4) since

$$g_j = d\sum_k \varphi_k f_{jk} = \sum_k [(d\varphi_k) f_{jk} + \varphi_k a_{jk}]$$

and both f_{jk} and a_{jk} satisfy (4).

Finally, since

$$w_j - w_\ell = a_{j\ell} = g_j - g_\ell$$

we can define a global solution w on D by letting $w = w_j - g_j$ on Ω_j. Then

$$dw = dw_j - dg_j = dw_j = \mu \text{ on } \Omega_j$$

and w satisfies (2) as desired.

Proof of Lemma 4.1. — We use the same notation as above. First we solve $d\alpha_j = d^c\alpha$ in Ω_j. Then $b_{jk} = \alpha_j - \alpha_k$ is a cocycle with values in $\mathbb{C}$. Next, we choose $c_{jk} \in 2\pi\mathbb{Z}$ such that

$$|b_{jk} - c_{jk}| \leq 2\pi \text{ for all } j, k.$$

Then $\delta(b_{jk} - c_{jk})$ is a cocycle with values in $2\pi\mathbb{Z}$. Moreover we have

$$|\delta(b_{jk} - c_{jk})| \leq 6\pi.$$

Consider the coboundary equation

$$\delta e_{jk} = \delta(b_{jk} - c_{jk}).$$

The right hand side is $2\pi\mathbb{Z}$-valued, and is bounded by C. Moreover there is a solution with values in $2\pi\mathbb{Z}$. In fact, $-c_{jk}$ is a solution.

Now, since the cover of D is finite there are only a finite number of $\delta(b_{jk} - c_{jk})$ with values in $2\pi\mathbb{Z}$ and bounded by C. Hence there is a solution e_{jk} with $|e_{jk}| \leq C'$, where C' is a new constant, which only depends on the number of elements in the cover of D. If $f = c + e$, then f has values in $2\pi\mathbb{Z}$. Moreover $|b - f| \leq C$ and $\delta(b - f) = 0$.

Let $\{\psi_k\}$ be a partition of unity subordinated to $\{\Omega_k\}$ and let

$$h_j = \sum_k \psi_k(b_{jk} - f_{jk}) \text{ on } \Omega_j.$$

Then

$$h_j - h_k = b_{jk} - f_{jk} = \alpha_j - \alpha_k - f_{jk}.$$

Let $\beta = \alpha_j - h_j$ on Ω_j. It follows that

$$\beta = \alpha_j - h_j = \alpha_k - h_k + f_{jk} \quad \text{on} \quad \Omega_{jk},$$

so that β is well-defined modulo $2\pi\mathbb{Z}$. Finally,

$$d\beta = d\alpha - dh = d^c\alpha - dh.$$

If we put $\varphi = dh$, then φ is a 1-form, $|\varphi| \lesssim C$ and $d\beta = d^c\alpha - \varphi$. Thus the lemma is proved.

6. Solution of the $\bar{\partial}$-equation.

Our aim in the present section is to obtain the solution of the $\bar{\partial}$-equation needed in the proof of Theorem 1.2. Let us first recall some standard notation and facts about the Cauchy-Leray kernels, which will be used in the sequel. For the proof of these facts, we refer to e.g. [Ra].

As before let $S(z,\zeta)$ be a C^1 map satisfying

$$(1) \qquad |S| \lesssim |z-\zeta|$$

and

$$(2) \qquad |\langle S, z-\zeta\rangle| \gtrsim |z-\zeta|^2$$

uniformly for $\zeta \in \bar{D}$ and z in any compact subset of D. With S we identify the form $S = \sum S_j d(\zeta_j - z_j)$.

The Cauchy-Leray kernel associated to S in $\mathbb{C}^2$ is defined by

$$K[S] := \frac{S \wedge dS}{\langle S, z-\zeta\rangle^2}.$$

We shall often write K for $K[S]$. It is easy to see that

$$(3) \qquad dK = 0 \text{ for } \zeta \neq z,$$

and if φ is a scalar valued function, nonvanishing for $\zeta \neq z$, then

$$(4) \qquad K[\varphi S] = K[S].$$

Denote by $K_{p,q}$ the component of K of bidegree (p,q) in z and $(2-p, 1-q)$ in ζ. If f is a $(0,1)$-form, then Koppelman's formula holds :

$$(5) \qquad f = -\frac{1}{4\pi^2}\Big[\int_{\partial D} K_{0,1} \wedge f - \int_D K_{0,1} \wedge \bar{\partial} f - \bar{\partial} \int_D K_{0,0} \wedge f\Big].$$

Now suppose that $S(z,\zeta)$ is holomorphic in $z \in D$ when $\zeta \in \partial D$. Then it follows from (5) that the function

$$u(z) = \frac{1}{4\pi^2}\int_D K_{0,0}[S] \wedge f \tag{6}$$

is a solution of $\bar\partial u = f$ if $\bar\partial f = 0$.

The map that we constructed in Section 3 is only holomorphic in z near the diagonal, hence the function (6) is not a solution to the $\bar\partial$-equation. In fact, S does not satisfy (2) either. However, we can use instead

$$S' = \frac{\overline{\langle S, z-\zeta\rangle}}{|\langle \zeta, z-\zeta\rangle|^2} S - \rho(\zeta)(\bar z - \bar\zeta)$$

which by the dilation invariance (4) of the kernels, is equivalent to S for $\zeta \in \partial D$.

The following lemma shows that (6) is not far from being a solution.

LEMMA 6.1. — *Let f be a closed $(0,1)$-form with C^∞ coefficients and let*

$$u(z) = \frac{1}{4\pi^2}\int_D K_{0,0}[S] \wedge f.$$

Then there is a function $h \in C^\infty(\bar D)$ such that

$$\bar\partial(u - h) = f \tag{7}$$

and

$$\|h\|_{L^\infty(\partial D)} \lesssim \|f\|_{L^1(\partial D)}. \tag{8}$$

Proof. — Since S is holomorphic in z near the diagonal when $\zeta \in \partial D$, $K_{0,1}[S]$ is smooth for $\zeta \in \partial D$. Hence by a well known theorem, see [Ra], there is a function $h \in C^\infty(\bar D)$ such that

$$\bar\partial h = \frac{1}{4\pi^2}\int_{\partial D} K_{0,1}[S] \wedge f$$

and by Koppelman's formula (7) holds. Estimate (8) follows immediately.

Next, we give a relation between kernels with different sections.

LEMMA 6.2. — *Suppose that S and Q satisfy* (1) *and* (2). *Let f be a $(0,1)$-form such that $\bar\partial f = 0$. Then*

$$\int_{\partial D} \frac{Q \wedge S \wedge f}{\langle Q, \zeta - z\rangle\langle S, z-\zeta\rangle} = \int_D K_{0,0}[Q] \wedge f - \int_D K_{0,0}[S] \wedge f. \tag{9}$$

Proof. — First we claim that

$$d_\zeta\Big[\frac{Q\wedge S}{\langle Q,\zeta-z\rangle\langle S,z-\zeta\rangle}\Big]=K_{0,0}[Q]-K_{0,0}[S]. \tag{10}$$

To establish (10) consider the map

$$t_\lambda=\frac{\lambda S}{\langle S,z-\zeta\rangle}+\frac{(1-\lambda)Q}{\langle Q,\zeta-z\rangle}\text{ for }0\le\lambda\le 1.$$

Then by (4) $K[t_\lambda]=t_\lambda\wedge dt_\lambda$. Here d is taken with respect to λ and ζ. Now, (3) gives $dK[t_\lambda]=0$, so that if we write $K=K_\lambda+K'$ where K_λ is the component of K that contains $d\lambda$, we get $d_\zeta K=-d_\lambda K'$.

Define

$$H=\int_0^1 K_\lambda.$$

A simple calculation shows that

$$H=\frac{Q\wedge S}{\langle Q,\zeta-z\rangle\langle S,z-\zeta\rangle}$$

and

$$d_\zeta H=\int_0^1 d_\zeta K_\lambda=-\int_0^1 d_\lambda K'=K[Q]-K[S],$$

which proves the claim.

An application of Stokes' theorem gives

$$\int_D d_\zeta H\wedge f=\int_{\partial D}H\wedge f.$$

In view of (10), the result follows.

We remark that (9) can be considered as a limit case of solution formulas with weights, see [BA]. Similar formulas have also been used by Henkin [He2].

Now observe that the map Q, constructed in Section 3, satisfies $\bar\partial_\zeta Q(z,\zeta)=0$ for $z\in\partial D$ in a neighborhood of the diagonal. Hence the kernel $K_{0,0}[Q]$ is nonsingular for $z\in\partial D$ and it follows that we have the estimate

$$\|\int_D K_{0,0}[Q]\wedge f\|_{L^\infty(\partial D)}\lesssim\|f\|_{L^1(D)}.$$

Combining Lemma 6.1 and Lemma 6.2, we obtain Proposition 3.2.

7. Proof of Lemma 3.5.

In this section it will be convenient to use the notation $\eta = z-\zeta$, $\check{\eta} = -\eta$, $\rho_j(z) = \dfrac{\partial\rho}{\partial\zeta_j}(z)$, $\rho_{jk}(z) = \dfrac{\partial^2\rho}{\partial z_j \partial z_k}(z)$, and so forth. We assume here that $|\eta| < \varepsilon$.

Let us first show that

$$Q \wedge S = -\partial\bar{v} \wedge \partial\rho + O(|\eta|^2). \tag{1}$$

To prove (1), note that since S and Q are $(1,0)$ forms,

$$Q \wedge S = (Q_1 S_2 - Q_2 S_1) d\zeta_1 \wedge d\zeta_2.$$

Then we have, disregarding terms of order ≥ 2 in η,

$$\begin{aligned}
Q_1 S_2 &= \left[\rho_1(z) + \frac{1}{2}(\rho_{11}(z)\check{\eta}_1 + \rho_{12}(z)\check{\eta}_2)\right]\left[\rho_2(\zeta) + \frac{1}{2}(\rho_{21}(\zeta)\eta_1 + \rho_{22}(\zeta)\eta_2)\right] + \cdots \\
&= \rho_1(z)\rho_2(\zeta) + \frac{1}{2}[(\rho_1(z)\rho_{21}(\zeta) - \rho_2(\zeta)\rho_{11}(z)]\eta_1 \\
&\qquad + \frac{1}{2}[(\rho_1(z)\rho_{22}(\zeta) - \rho_2(\zeta)\rho_{12}(z)]\eta_2 + \cdots,
\end{aligned}$$

and so, by symmetry, we obtain

$$\begin{aligned}
Q_1 S_2 - Q_2 S_1 &= \rho_1(z)\rho_2(\zeta) - \rho_1(\zeta)\rho_2(z) \\
&\quad + \frac{1}{2}[\rho_1(z)\rho_{21}(\zeta) + \rho_1(\zeta)\rho_{21}(z) - (\rho_2(\zeta)\rho_{11}(z) + \rho_2(z)\rho_{11}(\zeta))]\eta_1 \\
&\quad + \frac{1}{2}[\rho_1(z)\rho_{22}(\zeta) + \rho_1(\zeta)\rho_{22}(z) - (\rho_2(\zeta)\rho_{12}(z) + \rho_2(z)\rho_{12}(\zeta))]\eta_2 + \cdots \\
&= \rho_1(z)\rho_2(\zeta) - \rho_1(\zeta)\rho_2(z) + [\rho_1(\zeta)\rho_{21}(z) - (\rho_2(\zeta)\rho_{11}(z)]\eta_1 \\
&\quad + [\rho_1(\zeta)\rho_{22}(z) - \rho_2(\zeta)\rho_{12}(z)]\eta_2 + O(|\eta|^2).
\end{aligned} \tag{2}$$

On the other hand

$$\begin{aligned}
-\partial_\zeta \bar{v}(\zeta, z) \wedge \partial\rho(\zeta) &= [(\rho_1(z) + \rho_{11}(z)\check{\eta}_1 + \rho_{12}(z)\check{\eta}_2)\rho_2(\zeta) \\
&\quad - (\rho_2(z) + \rho_{12}(z)\check{\eta}_1 + \rho_{22}(z)\check{\eta}_2)\rho_1(\zeta)] d\zeta_1 \wedge d\zeta_2 \\
&= [\rho_1(z)\rho_2(\zeta) - \rho_1(\zeta)\rho_2(z) + (\rho_1(\zeta)\rho_{12}(z) - \rho_2(\zeta)\rho_{11}(z))\eta_1 \\
&\quad + (\rho_1(\zeta)\rho_{22}(z) - \rho_2(\zeta)\rho_{12}(z))\eta_2] d\zeta_1 \wedge d\zeta_2.
\end{aligned} \tag{3}$$

By comparing (2) and (3) we obtain (1).

Now observe that since $d\rho = 0$ on ∂D, we have

$$\int_{\partial D} \partial\bar{v} \wedge \partial\rho \wedge w_{0,1} = -\int_{\partial D} \partial\bar{v} \wedge \bar{\partial}\rho \wedge w_{0,1}. \tag{4}$$

It follows from the definition of v that

$$\bar{\partial} v = -\bar{\partial}\rho + \alpha_{0,1}, \tag{5}$$

where $|\alpha_{0,1}| = O(|\eta|)$. Inserting (5) into the last integral of (4) we get

$$(6)\quad -\int_{\partial D} \partial\bar{v} \wedge \bar{\partial}\rho \wedge w_{0,1} = \int_{\partial D} \partial\bar{v} \wedge \bar{\partial} v \wedge w_{0,1} - \int_{\partial D} \partial\bar{v} \wedge \alpha_{0,1} \wedge w_{0,1}.$$

Using (5) again on the last integral of (6) we can write

$$\int_{\partial D} \partial\bar{v} \wedge \alpha_{0,1} \wedge w_{0,1} = -\int_{\partial D} \partial\rho \wedge \alpha_{0,1} \wedge w_{0,1} + \int_{\partial D} \alpha_{1,0} \wedge \alpha_{0,1} \wedge w_{0,1},$$

where $|\alpha_{1,0} \wedge \alpha_{0,1}| = O(|\eta|)^2)$. Note that for bidegree reasons

$$\int_{\partial D} \partial\rho \wedge \alpha_{0,1} \wedge w_{0,1} = \int_{\partial D} d\rho \wedge \alpha_{0,1} \wedge w_{0,1} = 0.$$

Thus

$$(7)\quad \int_{\partial D} \partial\bar{v} \wedge \alpha_{0,1} \wedge w_{0,1} = \int_{\partial D} O(|\eta|^2)|w_{0,1}|\, d\sigma.$$

By combining (4), (6) and (7) we obtain

$$(8)\quad \int_{\partial D} \partial\bar{v} \wedge \partial\rho \wedge w_{0,1} = -\int_{\partial D} \partial\bar{v} \wedge \bar{\partial} v \wedge w_{0,1} + \int_{\partial D} O(|\eta|^2)|w_{0,1}|\, d\sigma.$$

(1) and (8) finally give

$$2\,\mathrm{Re}\int_{\partial D} \psi \frac{Q \wedge S \wedge w_{0,1}}{|v|^2} \lesssim 2\,\mathrm{Re}\int_{\partial D} \psi \frac{\partial\bar{v} \wedge \bar{\partial} v \wedge w_{0,1}}{|v|^2} + \int \psi \frac{|z-\zeta|^2 |w_{0,1}|\, d\sigma}{|v|^2}$$

and the proof is complete.

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