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ON INDUCED ACTIONS OF ALGEBRAIC GROUPS

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Let $H$ be a subgroup of an algebraic group $G$. Let $Y$ be an algebraic space with an action of $H$ (shortly an algebraic $H$-space). The aim of this note is to study some properties of $G \times_H Y$, defined as a quotient of $G \times Y$ by the action of $H$ determined by $h(g, y) = (gh^{-1}, hy)$, for all $h \in H$, $g \in G$ and $y \in Y$. Left translations by elements of $G$ on $G$ determine an action of $G$ on $G \times_H H$. The importance of the space $G \times_H Y$ follows from the fact that the map $Y \rightarrow G \times_H H$, which to $y \in Y$ attaches the image of $(1, y)$ in $G \times_H H$ solves the universal problem of $H$-equivariant morphisms of $Y$ into $G$-spaces. In analogy with the theory of modules and representations we can say that the space $G \times_H Y$ is induced from the $H$-space $Y$ by the group extension $H \subset G$ and that the action of $G$ on $G \times_H Y$ is induced by the action of $H$ on $Y$. In applications the notion is used for constructing a space with an action of $G$, when a space with an action of its subgroup $H$ is given. Properties of $G \times_H Y$ in the case where $Y$ is quasi-projective were studied in the classical paper [Se]. Though results presented here are perhaps predictable or even known, we hope that the paper will be useful as a reference.

In order to make our arguments more lucid, we are going to start with considering more general situations.

1. Let $X$ and $Y$ be two algebraic $H$-spaces. Then $X \times_H Y$ is defined as a quotient of the product $X \times Y$ by the action of $H$ defined by $h(x, y) = (hx, hy)$, for $h \in H$. In general, neither the meaning of the notion

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of quotient, nor its existence (when the meaning of the quotient has been already fixed) is clear. In the note we consider only the case when $H$ is affine and $X$ is a principal locally isotrivial $H$-fibration in the category of algebraic spaces. In this case we require the quotient $X \times_H Y$ to be an algebraic space, and the map $X \times Y \to X \times_H Y$ to be affine and a geometric quotient in the sense of [GIT]. If $X = G$, where $G$ is an affine algebraic group containing $H$ as its subgroup with an action of $H$ by right translations, then by [Se] the above assumptions concerning $H$ and $X$ are satisfied.

**Theorem 1.** Let $H$, $X$, $Y$ be as above. Moreover assume that $X$ is normal. Then $S \times_H Y$ exists in the category of algebraic spaces. If, moreover, $X$ is an algebraic variety, $Y$ is normal and can be covered by $H$-invariant open quasi-projective subsets, then $X \times_H Y$ is an algebraic variety.

**Proof.** The theorem will be proved in several steps.

1st step. Assume that the $H$-fibration on $X$ is trivial i.e. $X = H \times U$, where $U$ is an algebraic space. Then $X \times Y = H \times U \times Y \to U \times Y$ defined by $(h, x, y) \mapsto (x, h^{-1}y)$ satisfies desired conditions. Thus $X \times_H Y = U \times Y$.

2nd step. Assume that the $H$-fibration on $X$ is isotrivial with the base space $U$. Since $X$ is normal, $U$ is normal and then there exists a Galois ramified cover $Z \to U$ such that $X \times_U Z$ is trivial. It follows from the 1st step that there exists $(X \times_U Z) \times_H Y$ and by Deligne’s theorem [K] p. 183-4 there exists its quotient by the action of the Galois group (induced by the action on $Z$) in the category of algebraic spaces. The quotient can be identified with $X \times_H Y$. If moreover $Y$ is normal quasi-projective and $U$ is affine, then $(X \times_U Z) \times_H Y$ is normal quasi-projective. Because the quotient of a normal quasi-projective variety by an action of a finite group is quasi-projective, hence $X \times_H Y$ is also quasi-projective.

3rd step. Assume that $X$ and $Y$ are covered by $H$-invariant open subsets $\{U_i, i \in I\}$, $\{V_j, j \in J\}$, such that $U_i \times_H V_j$ exist, for all $i \in I$ and $j \in J$ (in the category of algebraic spaces). Then $X \times_H Y$ also exists (in the same category) and $\{U_i \times_H V_j\}$ form an open covering of $X \times_H Y$. Moreover if $U_i \times_H V_j$ are quasi-projective, then $X \times_H Y$ is an algebraic variety. Proof of this step is obvious.

4th step. Now we consider the general case. Notice first that the base space of the $H$-fibration given on $X$ can be covered by open subsets $\{W_k\}$,
$k \in K$, such that for every $k \in K$, the inverse image $U_k$ of $W_k$ in $X$, as a principal $H$-fibration, is isotrivial. Then it follows from the 2nd step that, for every $k \in K$, $U_k \times_H Y$ exists and from the 3rd step that $X \times_H Y$ exists in the category of algebraic spaces. Moreover, if $Y$ can be covered by $H$-invariant open quasi-projective subsets $V_j$, where $j \in J$, then by the second part of the 3rd step, we infer that $X \times_H Y$ is an algebraic variety.

**Corollary 2.** — Let $Y$ be an algebraic space with an action of an algebraic group $H$ and let $G$ be an affine algebraic group containing $H$ as its subgroup. Then $G \times_H Y$ is an algebraic space with an action of $G$ induced by left translations on $G$. Moreover, if $Y$ can be covered by $H$-invariant quasi-projective open subsets, then $G \times_H Y$ is an algebraic variety.

**Theorem 3.** — Let $G$ be a connected affine algebraic group and let $H$ be its subgroup. Let $Y$ be a normal algebraic space with an action of $H$. Then $G \times_H Y$ is an algebraic variety if and only if $Y$ can be covered by $H$-invariant quasi-projective open subsets.

Proof. — It follows from Corollary 2 that, if $Y$ can be covered by $H$-invariant open quasi-projective subsets, then $G \times_H Y$ is an algebraic variety. Let us assume now that $G \times_H Y$ is an algebraic variety. Since $G$ is connected, $G \times_H Y$ by Sumihiro Theorem [Su] can be covered by $G$-invariant open quasi-projective subsets. Intersecting these subsets with $H \times_H Y \subseteq G \times_H Y$ we obtain an $H$-invariant quasi-projective open covering of $H \times_H Y$. Since $Y \simeq H \times_H Y$ we obtain that $Y$ can be covered by open quasi-projective $H$-invariant subsets.

It follows from the above results and Sumihiro Theorem that whenever $H$ is connected, any induced $G$-space from an algebraic normal $H$-variety is also an algebraic (normal) variety. However in case where $H$ is a finite subgroup of a connected algebraic group and $Y$ is an algebraic $H$-variety which can not be covered by $H$-invariant open quasi-projective open subsets, then the induced algebraic $G$-space is not an algebraic variety. For example, if two element group $Z_2$ acts on a normal algebraic variety $Y$ in such a way that a $Z_2$-orbit is not contained in any affine open subset (see [H] or Chap. 4§ 3 in [GIT] for an example), then for any connected affine group $G$ containing $E_2$ as a subgroup, $G \times_{Z_2} Y$ is an algebraic space but not an algebraic variety.
**BIBLIOGRAPHY**


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